THE VALUATION OF NEW VENTURES

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January 2006

Abstract

Traditional tools fail to capture the value of new ventures, such as R&D projects and start-up companies, because of their dependence on future events that are uncertain at the time of the initial decision. In the real options setting the value of these investments is the value of the follow-on opportunities they may create. Our aim is to study the multicompound option approach to value sequential investment opportunities taking into account multiple interactions among real options.

Key words: multicompound options; sequential investments; multiple real options.

JEL Classification: G 12; G 13; G 30; C 69

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1 Introduction

Many new business ventures have the characteristic of growth option, since investment decisions are made sequentially and in a particular order. Staging investment involves firms either with some degree of flexibility in proceeding with investment or when there is a maximum rate at which outlays or construction can proceed, that is, it takes time-to-build. Several researchers and practitioners have noted that traditional tools fail to capture the value of sequential investments, such as R&D projects and start-up ventures, because of their dependence on future events that are uncertain at the time of the initial decision. This notwithstanding, firms take projects with negative net present values because doing so allows them either to take on other projects with higher present values or to enter more profitable markets in the future. In the real options setting investment opportunities may be viewed as options; in particular, since each stage can be viewed as an option on the value of subsequent stage, the pricing formulae for compound options can be applied. Compound options have been extensively used in corporate finance; for example, Geske (1979) suggested that when a company has common stock and coupon bonds outstanding, the firm's stock can be viewed as a call option on a call option. Rubinstein (1992) generalized this result to all four possible combinations: call on a call, put on a call, call on a put and put on a put. Carr (1988) analyzed sequential compound options, which involve options to acquire subsequent options to exchange an underlying risky asset for another risky asset. Gukhal (2003) derives analytical valuation formulas for compound options when the underlying asset follows a jump-diffusion process, applying these results to value extendible options, American call options on stocks that pay discrete dividends and American options on assets that pay continuous proportional dividends. Agliardi and Agliardi study multicompound call option in the case of variable interest rate and volatility. Roll (1977), Whaley (1981), Geske and Johnson (1894) and Selby and Hodges (1987) also study compound options.

The objective of this paper is to study the multicompound option approach to value sequential investment opportunities taking into account multiple interactions among real options. The first contribution of this paper is to derive an analytical formula for multicompound call and put option to the valuation problem of option to continuously shut-down and restart operations. The second contribution is to value sequential expansion and contraction option integrating the first result with work on compound exchange options by Carr (1988).

The structure of our paper is as follows. Section 2 reviews the articles in the literature related to ours. This is followed by a description of the economic model in section 3. Section 4 derives an analytical valuation formula for multicompound call/put option applied to the valuation of the option to continuously shut-down and restart operations. Section 5 derives the valuation formula for multicompound exchange option applied to the valuation of sequential technology adoption and contraction. Section 6 concludes the paper.

2 Literature Review

A number of existing research contribution has previously analyzed various aspects of optimal sequential investment behaviour for firm facing multistage projects. In the real options setting, the value of these investments is the value of the follow-on opportunities they may create. In this sense, firms undertake these projects not so much for their own returns, but rather to get started a pilot that may eventually reach an operating stage. Take the example of developing a new drug. Investing in a new drug by pharmaceutical company is a multistage process, beginning with research that leads with some probability to a new compound; such a project continues with testing and concludes with the construction of a production facility and the marketing of the product. Thus, in order to draw the analogy with the valuation and exercising of financial option, an R&D venture by pharmaceutical company can be compared to multicompound option involving sequential decisions to exercise the options to invest only when the R&D outcomes are successful. Although the preceding analysis suggested the use of more suitable technique when we attempt to value new ventures, such investments are hard to value even with the real options approach. The main reason for this is that there are multiple sources of uncertainty in R&D investments and that they interact in complicated way. In practice, the bulk of the literature have dealt with the development of numerical simulation methods based on optimal stopping time problems. Similarly, start-up companies are very close to the ventures we consider earlier, because of the large investments they require. For example, engaging a pilot in Internet requires to the venture capitalist to continue making investments in up-dating its technology and marketing its product just to keep up. Thus, the pricing formulae for multicompound options can be applied. Internet is a young and dynamic industry, still in search of profitable business, creating market openings for competitors and potential entrants. This feature makes the risk of an Internet start-up much greater than risk faced by start-up in other industries, and then more valuable from the real options perspective. Unlikely the investments in R&D, for the Internet start-up the biggest source of uncertainty is represented by the evolving nature of Internet business, making the valuation problem more easy if we want to use the real options methodology.

Majd and Pindyck (1987) develop a continuous investment model with time-to-build. They solve an investment problem in which the project requires a fixed total investment to complete, with a maximum instantaneous rate of investment. Pindyck (1993) also takes into account market and technical uncertainty. Schwartz and Moon (2000) and Schwartz (2003) have studied R&D investment projects in the pharmaceutical industry using a real options framework. Berk, Green and Naik (2004) develop a dynamic model of multistage investment project that captures many features of R&D ventures and start-up companies. They assume different sources of risk and study their interaction in determining the value and risk premium of the venture. Closed-form solutions for important cases are obtained.

2.1 The Valuation of Multiple (Compound) Real Options

Most work in real options has focused on valuing individual options. However, many real investments often involve a collection of various options, which need to be valued together because their combined value may differ from the sum of their separate values. As it is well known, the operating flexibility and strategic value aspect of intangible investment projects cannot be properly captured by traditional tools, because of their discretionary nature. As new information arrives and uncertainty about market conditions and future cash flows is resolved, firms may have valuable flexibility to alter its initial operating strategy in order to capitalize on favourable future opportunities. For example, management may be able to temporarily shut-down, restart, expand or contract its project during construction.

For our purpose, we deal with the Internet start-up venture as N-nested series of compound options. Moreover, in order to capture the value of the multiple interacting options embedded, we will consider that the project may be continuously shut-down and restarted and that this opportunity can be seen as a sequence of three or more call or put options. When the venture capitalist exercises its option to get started the pilot, this fact yields two options: a call option to continue investing and a put to temporarily shut-down, if it exercises the put, it gets the option to invest again and so on. In our N-stages model we can work backwards to determine the value of the call/put option in each stage of the project; in practice, an analytic expression for the most outer option is simply derived according to Black-Scholes-Merton, while the next one can be defined following the method of Geske. Further, we repeatedly add a time step and solve the corresponding partial differential equation.

Brennan and Schwartz (1985) deal with interacting options in their analysis of the options to close and reopen a mine. Trigeorgis (1993) focuses explicitly on the nature of real option interactions, pointing out that the presence of subsequent options can increase the value of the effective underlying asset for earlier options, while the exercise of prior real options may alter the underlying asset itself, and hence the value of subsequent options on it. Thus, the combined value of a collection of real options may differ from the sum of separate option values. Kulatilaka (1995) examines the impact of interactions among such options on their optimal exercise schedules. Abel, Dixit, Eberly and Pindyck (1996) also analyzed multiple interacting options.

3 Model and Assumptions

Let us consider the investment decision by a venture capital fund that is evaluating the project of start-up company providing software tools in the Internet industry. We assume that the commercial phase of the project can not be launched before a pilot phase consisting on N-stages of investment is completed. Let be the amount of investment required for completion of any pilot stage. Furthermore, to make the analysis easier we will assume that the project is patent-protected.

Suppose the inverse demand function for the software, giving price in terms of quantity Q is $P = YD(Q)$, where Y is a stochastic shift variable. The risk free rate in our setting will be denoted by $r(t)$. Moreover, the investment project, once completed, produces one unit of output per year at zero operating costs. We assume the price for the software, P, follows a stochastic differential equation of the form:

$$dP = \alpha(t) P dt + \sigma(t) P dz,$$

where dz is the increment of the standard Wiener process; $\sigma(t)$ is the instantaneous standard deviation of the spot price at time t and $\alpha(t)$ is the trend rate in the price. The assumption of time-dependent volatility and interest rate seems more suitable due to the sequential nature of start-up projects (see Agliardi and Agliardi, 2003). Let V the expected present value of the project when the current price is P, in this case V, being a constant multiple of P, also follows a geometric Brownian motion with the same parameters $\alpha(t)$ and $\sigma(t)$.

3.1 Valuing the Option to Continuously Shut - Down and Restart Operations

Unlike most compound options in the financial market, it is perfectly possible for the firm to suspend investment on the pilot at a certain time T_k, $k = 1, .., N$, if, for instance, market conditions are not favourable, and resume investment at a later point in time.

Let $F_1(V, t; \varsigma_1)$ denote the value of a European call/put option with exercise price I_1 and expiration date T_1. Let us now define inductively a sequence of call/put options, with value F_k, on the call/put option whose value is F_{k-1}, with exercise price I_k and expiration date T_k, $k = 1, .., N$, where we assume $T_1 \geq T_2 \geq ... \geq T_N$.

Because all the calls and puts are function of the value of the firm V and the time t, the following partial differential equation holds for F_k:

$$\frac{\partial F_k}{\partial t} = r(t) F_k - r(t) V \frac{\partial F_k}{\partial V} - \frac{1}{2}\sigma^2(t) V^2 \frac{\partial^2 F_k}{\partial V^2}, \quad t \leq T_k, k = 1, .., N,$$

$T_1 \geq T_2 \geq ... \geq T_N$.The boundary condition is:

$$F_k(F_{k-1}(V, T_k; \varsigma_1, .., \varsigma_{k-1}), T_k; \varsigma_1, .., \varsigma_k) =$$

$$\max\left(\varsigma_k F_{k-1}\left(V, T_k; \varsigma_1, .., \varsigma_{k-1}\right) - \varsigma_k I_k, 0\right),$$

where $F_{k-1}\left(V, T_k; \varsigma_1, .., \varsigma_{k-1}\right)$ stands for the price of the underlying compound option and the binary option operator $\varsigma_k = \pm 1$, $k = 1, .., N$ when the k^{th}-compound option is a call/put and the operator $\varsigma_{k-1} = \pm 1$, when the $(k-1)^{th}$-underlying compound option is a call/put. Naturally, if $k = 1$ the well-known pricing formulae for simple options are obtained.

In order to solve the partial differential equations above subject to their boundary conditions we need to use the following notation: let V_k^* denote the value of V such that $F_{k-1}\left(V, T_k; \varsigma_1, .., \varsigma_{k-1}\right) - I_k = 0$ if $k > 1$, and $V_1^* = I_1$. Let us define now:

$$b_k(t) = \frac{\ln\left(\frac{V}{V_k^*}\right) + \int_t^{T_k} r(\tau) - \frac{\sigma^2(\tau)}{2} d\tau}{\left(\int_t^{T_k} \sigma^2(\tau)\, d\tau\right)^{\frac{1}{2}}}, \tag{1}$$

and:

$$a_k(t) = b_k(t) + \left(\int_t^{T_k} \sigma^2(\tau)\, d\tau\right)^{\frac{1}{2}}; \tag{2}$$

moreover, we set:

$$\rho_{ij}(t) = \left(\frac{\int_t^{T_j} \sigma^2(\tau)\, d\tau}{\int_t^{T_i} \sigma^2(\tau)\, d\tau}\right)^{\frac{1}{2}}, \text{ for } 1 \leq i < j \leq k, \quad t \leq T_k. \tag{3}$$

For any k, $1 \leq k \leq N$, let $\Sigma_k^{(N)}(t)$ denote the k-dimension symmetric correlation matrix with typical element $\rho_{ij}(t) = \rho_{N-k+i,N-k+j}(t)$ if $i < j$.

Since we want to derive a valuation formula for the price of N-fold multi-compound call/put option, that is for $F_N\left(V, t; \varsigma_1, .., \varsigma_N\right)$, $0 \leq t \leq T_N$, let V_N^* denote the value of V such that $F_{N-1}\left(V, T_N; \varsigma_1, .., \varsigma_{N-1}\right) - I_N = 0$. Then, for V greater than V_N^* the N^{th}-compound call option will be exercised, otherwise the project will be temporarily suspended.

As usual, it is desirable to transform the partial differential equation for F_N into the diffusion equation. First, we adopt the following change of variables:

$$F_N(V, t) = e^{-\int_t^{T_N} r(\tau) d\tau}\, \tilde{F}_N(u, z),$$

where:

$$u = -\ln\left(\frac{V}{V_N^*}\right) - \int_t^{T_N} r(\tau) - \frac{\sigma^2(\tau)}{2} d\tau,$$

and:

$$z = \frac{1}{2}\int_t^{T_N} \sigma^2(\tau)\, d\tau.$$

In term of the new independent variables the fundamental equation for F_N becomes:

$$\frac{\partial \tilde{F}_N}{\partial z} = \frac{\partial^2 \tilde{F}_N}{\partial u^2}, \quad -\infty < u < +\infty, \quad z \geq 0.$$

The partial differential equation above subject to the initial value condition $\tilde{F}_N(u,0)$, has a unique solution which we use to write F_N as follows:

$$F_N(V,t) = e^{-\int_t^{T_N} r(\tau)d\tau} \int_{-\infty}^{+\infty} \tilde{F}_N(\xi,0)\, \frac{1}{2\sqrt{\pi z}} e^{-(u-\xi)^2/4z} d\xi.$$

Substituting the solution for F_{N-1} into this expression and changing the variable u with $\varsigma_N..\varsigma_1 b_N(t)$, gives the following identity:

$$\begin{aligned}
&F_N(V,t;\varsigma_1,..,\varsigma_N) = \\
&\varsigma_N..\varsigma_1 V e^{-\int_t^{T_N} r(\tau)d\tau} \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b_N + \xi/\sqrt{2z}\right)^2/2} \\
&N_{N-1}\left(\varsigma_{N-1}..\varsigma_1 a_{N-1}(T_N), .., \varsigma_1 a_1(T_N); \Xi_{N-1}^{(N-1)}(T_N)\right) d\xi + \\
&-\sum_{j=1}^{N-1} \varsigma_N..\varsigma_j I_j e^{-\int_t^{T_j} r(\tau)d\tau} \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b_N + \xi/\sqrt{2z}\right)^2/2} \\
&N_{N-j}\left(\varsigma_{N-1}..\varsigma_1 b_{N-1}(T_N), .., \varsigma_j b_j(T_N); \Xi_{N-j}^{(N-1)}(T_N)\right) d\xi + \\
&-\varsigma_N I_N e^{-\int_t^{T_N} r(\tau)d\tau} \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b_N + \xi/\sqrt{2z}\right)^2/2} d\xi;
\end{aligned} \tag{4}$$

where $N_k(\varsigma_k..\varsigma_1 b_k, .., \varsigma_1 b_1; \Xi_k)$ denotes the k-dimension multinormal cumulative distribution function with upper limits of integration $\varsigma_1 b_1, .., \varsigma_k..\varsigma_1 b_k$ and $\Xi_k^{(N-1)}(T_N)$ denotes the k-dimension modified symmetric correlation matrix:

$$\Xi_k^{(N-1)}(t) = \begin{bmatrix} 1 & \varsigma_2\rho_{12} & \cdots & \varsigma_{N-1}..\varsigma_2\rho_{1,N-1} \\ \varsigma_2\rho_{12} & 1 & & \vdots \\ \vdots & & \ddots & \varsigma_{N-1}\rho_{N-2,N-1} \\ \varsigma_{N-1}..\varsigma_2\rho_{1,N-1} & \cdots & \varsigma_{N-1}\rho_{N-2,N-1} & 1 \end{bmatrix}, \tag{5}$$

with the entries $\rho_{ij}(T_N) = \rho_{N-1-k+i,N-1-k+j}(T_N)$, $i < j$, defined as above.

The third term can be easily written in the form:

$$-\varsigma_N I_N e^{-\int_t^{T_N} r(\tau)d\tau} N_1(\varsigma_N..\varsigma_1 b_N(t)).$$

In order to solve the remaining integrals, let us set $x = \varsigma_N..\varsigma_1 a_N(t) + \xi/\sqrt{2z}$ in the integral of the first term above and $x = \varsigma_N..\varsigma_1 b_N(t) + \xi/\sqrt{2z}$ in the second; further, we replace any element $\rho_{ij}(T_N)$ in the matrix $\Xi_k^{(N-1)}(T_N)$ with a function of t, according to the following rule:

$$\rho_{ij}(T_N) = \frac{\left(\rho_{ij}(t) - \rho_{iN}(t)\rho_{jN}(t)\right)}{\sqrt{\left(1-\rho_{iN}^2(t)\right)\left(1-\rho_{jN}^2(t)\right)}}, \text{ for } 1 \le i < j \le N, t \le T_N. \tag{6}$$

The first term can be written in the form:

$$\varsigma_N..\varsigma_1 V \int_{-\infty}^{\varsigma_N..\varsigma_1 a_N(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times$$

$$N_{N-1}(\varsigma_{N-1}..\varsigma_1 a_{N-1}(t) - x\varsigma_N\rho_{N-1,N}(t)/\sqrt{\left(1-\rho_{N-1,N}^2(t)\right)}, ..$$
$$.., \varsigma_1 a_1(t) - x\varsigma_N..\varsigma_2\rho_{1,N}(t)/\sqrt{\left(1-\rho_{1,N}^2(t)\right)}; \tilde{\Xi}_{N-1}^{(N-1)}(t))dx+$$

and the second term:

$$-\sum_{j=1}^{N-1} \varsigma_N..\varsigma_j I_j e^{-\int_t^{T_j} r(\tau)d\tau} \int_{-\infty}^{\varsigma_N..\varsigma_1 b_N(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times$$

$$N_{N-j}(\varsigma_{N-1}..\varsigma_1 b_{N-1}(t) - x\varsigma_N \rho_{N-1,N}(t) / \sqrt{\left(1-\rho_{N-1,N}^2(t)\right)}, ..$$

$$.., \varsigma_j b_j(t) - x\varsigma_N..\varsigma_{j+1}\rho_{j,N}(t) / \sqrt{\left(1-\rho_{j,N}^2(t)\right)}; \tilde{\Xi}_{N-j}^{(N-1)}(t))dx;$$

Lemma 1 (generalized) *Let $\Xi_k^{(N)}(t)$ denote the k-dimension correlation matrix with entries $\varsigma_j \rho_{ij}(t)$ for $i < j$ and let $\tilde{\Xi}_k^{(N-1)}$ be the matrix obtained from $\Xi_k^{(N-1)}$ replacing any element ρ_{ij} with*

$$\left(\rho_{ij} - \rho_{iN}\rho_{jN}\right) / \sqrt{(1-\rho_{iN}^2)(1-\rho_{jN}^2)}$$

for $1 \leq i < j \leq N$. Moreover, let $\varsigma_k = \pm 1$, $k = 1, .., N$, if the k^{th}-compound option is a call/put and $\varsigma_{k-1} = \pm 1$ if the $(k-1)^{th}$-underlying compound option is a call/put. Then, the following identity holds:

$$\int_{-\infty}^{\varsigma_N..\varsigma_1 b_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$N_k\left(\frac{\varsigma_{N-1}..\varsigma_1 b_{N-1} - x\varsigma_N\rho_{N-1,N}}{\sqrt{\left(1-\rho_{N-1,N}^2\right)}}, .., \frac{\varsigma_{N-k} b_{N-k} - x\varsigma_N..\varsigma_{N-k+1}\rho_{N-k,N}}{\sqrt{\left(1-\rho_{N-k,N}^2\right)}}; \tilde{\Xi}_k^{(N-1)}\right) dx =$$

$$= N_{k+1}\left(\varsigma_N..\varsigma_1 b_N, .., \varsigma_{N-k} b_{N-k}; \Xi_{k+1}^N\right).$$

Proof. by induction. ■

Applying this argument to the first and the second terms, we have the following result for the value of a multicompound call/put option.

Proposition 2 *The value of the multicompound call/put option F_N with maturity T_N and strike price I_N written on a compound call/put option F_{N-1} with maturity T_{N-1} and strike price I_{N-1} is given by:*

$$F_N(V, t; \varsigma_1, .., \varsigma_N) = \varsigma_N..\varsigma_1 V N_N\left(\varsigma_N..\varsigma_1 a_N(t), .., \varsigma_1 a_1(t); \Xi_N^{(N)}(t)\right) +$$

$$-\sum_{j=1}^{N} \varsigma_N..\varsigma_j I_j e^{-\int_t^{T_j} r(\tau)d\tau} N_{N+1-j}\left(\varsigma_N..\varsigma_1 b_N(t), .., \varsigma_j b_j; \Xi_{N+1-j}^{(N)}(t)\right),$$

$$0 \leq t \leq T_N,$$

where the a_is, the b_is and the ρ_{ij}s are as defined previously.

This proposition is the main result of this paper and forms the basis for the valuation of sequential expansion and contraction option.

4 Considering the Value of Future Expansion and Contraction Option

When future returns are uncertain, these features yield two options. First, venture capitalists sometimes engage the pilot either to make further investments or to enter other markets in the future. On the other hand, when a firm gets started a project or installs a new technology that it may later abandon, it acquires a put option. Once again, the opportunities for future expansion or contraction are examples of the strategic dimension of the Internet start-up venture. The firms' ability to later contract or expand capacity is clearly more valuable for more volatile business with higher returns on project, such as computer software or biotechnology, than it is for traditional business, as real estate or automobile production. Next, we recast the main assumptions allowing the firm to face the opportunity to continuously upgrade or contract the technology in use. The sequential technological expansion/contraction decision can be viewed to be similar to the exercise of a multicompound exchange option.

Carr (1988) obtained a closed form solution to a compound exchange option integrating work on compound option pricing by Geske (1979) with work on exchange option pricing by Margrabe (1978). Exercise of this instrument involves delivering one asset in return for an exchange option. The option received upon delivery may then be used to make another exchange at a later date.

As before, we assume the inverse demand function for the software, giving price in terms of quantity Q is $P = YD(Q)$, where Y is a stochastic shift variable. Once again, the variable costs of production are assumed to be zero. Let V_i, $i = 1,..,N$, the price of i–th underlying asset which we could interpret as the value of the operating project with current technology. As before, we assume that the underlying risky assets pay no dividends, and that they follow standard diffusion processes as:

$$dV_i = \alpha_i(t) V_i dt + \sigma_i(t) V_i dz_i, \quad i = 1,..,N,$$

where dz_i, $i = 1,..,N$, are Wiener processes. They are correlated:

$$E[dz_i, dz_j] = \rho_{ij} dt, \quad i,j = 1,..,N, \quad i \neq j,$$

with $\rho_{ij} = \rho_{ji}$, $\rho_{ii} = 1$, $i,j = 1,..,N$, and ρ_{ij} is the correlation coefficient between V_i and V_j. Let $F(V,t)$ the function $F(V_1, V_2, .., V_N; t)$, according to the generalized Ito lemma we can determine that F will satisfy the following partial differential equation:

$$\frac{\partial F}{\partial t}+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{ij}\sigma_i\left(t\right)\sigma_j\left(t\right)V_iV_j\frac{\partial^2 F_k}{\partial V_i\partial V_j}+r\left(t\right)\sum_{i=1}^{N}V_i\frac{\partial F}{\partial V_i}-r\left(t\right)F=0,$$

$0\leq V_i$, $i=1,..,N$, $0\leq t\leq T$.

5 The Valuation of Sequential Technology Adoption - Contraction

The firm's ability to later contract or expand its technology represents a critical component of the Internet industry's investment decisions. Abel, Dixit, Eberly and Pindyck (1996) shows how opportunities for future expansion or contraction can be valued as options, how their valuation relates to the q theory of investment, and their effect on the incentive to invest. Moreover, a number of existing research contribution has previously analyzed various aspects of optimal sequential investment behaviour for firm facing multi-stage projects. For example, Alvarez and Stenbacka (2000) develops a real options approach in order to characterize the optimal timing of when to adopt an incumbent technology, incorporating as an embedded option a technologically uncertain prospect of opportunities for updating the technology to future superior versions.

Our study differs from those mentioned above in several crucial respects. In fact, we emphasized that sequential technology adoption and contraction policies can be valued using the techniques of real options through the analysis of sequential exchange opportunities by Carr (1988). Particularly, in light of the prospect of interrelated generations of new technologies, the earlier of which is prerequisite for those to follow, the Internet start-up venture can be seen as a multicompound call/put option to exchange two or more assets, where the arrival times are assumed to be known in advantage.

Following the analysis in the previous stage we attempt to evaluate whether or not venture capitalists should update its technology to superior new versions depending on the dynamic of the assets underlying the underlying option. The vehicle for analysis is the concept of a multicompound call/put option, whose a closed form solution is provided previously. A valuation formula for sequential technology adoption-contraction options is derived.

5.1 Notation and Result

Let $F(V_1, V_2, t; \varsigma_1)$ denote the value of a European call/put option to exchange asset one for asset two which can be exercised at T_1. This option is simultaneously a call/put option on asset two with exercise price V_1 and a put/call option on asset one with exercise price V_2. Taking V_1 as numeraire, the option to exchange asset one for asset two is a call/put option on asset two with exercise price equal to unity and interest rate equal to zero. The option sells for:

$$F(V_1, V_2, t; \varsigma_1) / V_1 = W_1(V, t; \varsigma_1),$$

where $\varsigma_1 = \pm 1$ if is a call/put and $V = V_2/V_1$. An analytic expression for $W_1(V, t)$ was found in Margrabe (1978).

Let us now define inductively a sequence of call/put option, with value W_k, on the call/put option whose value is W_{k-1}, with exercise price q_k and expiration date T_k, $k = 1, .., N$, where we assume $T_1 \geq T_2 \geq ... \geq T_N$. Because all the calls and puts are function of the value of the firm V and the time t, the following partial differential equation holds for W_k:

$$\frac{\partial W_k}{\partial t} + \frac{1}{2}\sigma^2(t) V^2 \frac{\partial^2 W_k}{\partial V^2} = 0, \quad t \leq T_k, \quad k = 1, .., N,$$

$T_1 \geq T_2 \geq ... \geq T_N$. The boundary conditions can be written in the form:

$$W_k(W_{k-1}(V, T_k; \varsigma_1, .., \varsigma_{k-1}), T_k; \varsigma_1, .., \varsigma_k) =$$
$$\max(\varsigma_k W_{k-1}(V, T_k; \varsigma_1, .., \varsigma_{k-1}) - \varsigma_k q_k, 0),$$

where $\varsigma_k = \pm 1$, $k = 1, .., N$, if the k^{th}-compound option is a call/put and $\varsigma_{k-1} = \pm 1$, if the $(k-1)^{th}$-underlying compound option is a call/put. Naturally, if $k = 1$ the well-known pricing formula for simple exchange option is obtained.

In order to solve the partial differential equations above subject to their boundary conditions we need to use the following notation: let V_k^* denote the value of V such that $W_{k-1}(V, T_k; \varsigma_1, .., \varsigma_{k-1}) - q_k = 0$ if $k > 1$, and $V_1^* = q_1$. Let us define now:

$$b'_k(t) = \frac{\ln\left(\frac{V}{V_k^*}\right) + \int_t^{T_k} \frac{\sigma^2(\tau)}{2} d\tau}{\left(\int_t^{T_k} \sigma^2(\tau) d\tau\right)^{\frac{1}{2}}}, \tag{7}$$

and:

$$\acute{a}_k(t) = \acute{b}_k(t) + \left(\int_t^{T_k} \sigma^2(\tau)\, d\tau \right)^{\frac{1}{2}}; \tag{8}$$

moreover, we set $\rho_{ij}(t)$ as in (3). For any k, $1 \leq k \leq N$, let $\Sigma_k^{(N)}(t)$ denote the k-dimension symmetric correlation matrix with typical element $\rho_{ij}(t) = \rho_{N-k+i,N-k+j}(t)$ if $i < j$.

Since we want to derive a valuation formula for the price of N-fold multicompound call/put option to exchange asset one for asset two, that is for $W_N(V, t; \varsigma_1, .., \varsigma_N)$, $0 \leq t \leq T_N$, let V_N^* denote the value of V such that $W_{N-1}(V, T_N; \varsigma_1, .., \varsigma_{N-1}) - q_N = 0$. Then, for V greater than V_N^* the N^{th}-compound call option will be exercised, that is, the firm will update to a superior, new technology, otherwise it will contract it.

As usual, it is desirable to transform the partial differential equation for W_N into the diffusion equation. First, we adopt the following change of variables:

$$W_N(V, t) = \tilde{W}_N(u, z),$$

where:

$$u = -\ln\left(\frac{V}{V_N^*}\right) - \int_t^{T_N} \frac{\sigma^2(\tau)}{2} d\tau,$$

and:

$$z = \frac{1}{2} \int_t^{T_N} \sigma^2(\tau)\, d\tau.$$

In term of the new independent variables the fundamental equation for F_N becomes:

$$\frac{\partial \tilde{W}_N}{\partial z} = \frac{\partial^2 \tilde{W}_N}{\partial u^2}, \quad -\infty < u < +\infty, \quad z \geq 0.$$

The partial differential equation above subject to the initial value condition $\tilde{W}_N(u, 0)$, has a unique solution which we use to write W_N as follows:

$$W_N(V, t) = \int_{-\infty}^{+\infty} \tilde{W}_N(\xi, 0) \frac{1}{2\sqrt{\pi z}} e^{-(u-\xi)^2/4z} d\xi.$$

Substituting the solution for W_{N-1} into this expression and changing the variable u with $\varsigma_N..\varsigma_1 b'_N(t)$, gives the following identity:

$$W_N(V,t;\varsigma_1,..,\varsigma_N)=$$

$$\varsigma_N..\varsigma_1 V \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b'_N+\xi/\sqrt{2z}\right)^2/2}$$

$$N_{N-1}\left(\varsigma_{N-1}..\varsigma_1 a'_{N-1}(T_N),..,\varsigma_1 a'_1(T_N);\Xi_{N-1}^{(N-1)}(T_N)\right)d\xi+$$

$$-\sum_{j=1}^{N-1}\varsigma_N..\varsigma_j q_j \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b'_N+\xi/\sqrt{2z}\right)^2/2}$$

$$N_{N-j}\left(\varsigma_{N-1}..\varsigma_1 b'_{N-1}(T_N),..,\varsigma_j b'_j(T_N);\Xi_{N-j}^{(N-1)}(T_N)\right)d\xi+$$

$$-\varsigma_N q_N \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi z}} e^{-\left(\varsigma_N..\varsigma_1 b'_N+\xi/\sqrt{2z}\right)^2/2} d\xi; \tag{9}$$

where $N_k(\varsigma_k..\varsigma_1 b_k,..,\varsigma_1 b_1;\Xi_k)$ denotes the k-dimension multinormal cumulative distribution function with upper limits of integration $\varsigma_1 b'_1,..,\varsigma_k..\varsigma_1 b'_k$ and $\Xi_k^{(N-1)}(T_N)$ denotes the k-dimension modified symmetric correlation matrix with typical element $\rho_{ij}(T_N)=\rho_{N-1-k+i,N-1-k+j}(T_N)$ for $i<j$, as defined in (5). The third term can be easily written in the form:

$$-\varsigma_N q_N N_1(\varsigma_N..\varsigma_1 b'_N(t)).$$

In order to solve the remaining integrals, let us set $x=\varsigma_N..\varsigma_1 a'_N(t)+\xi/\sqrt{2z}$ in the integral of the first term above and $x=\varsigma_N..\varsigma_1 b'_N(t)+\xi/\sqrt{2z}$ in the second; further, we replace any element $\rho_{ij}(T_N)$ in the matrix $\Xi_k^{(N-1)}(T_N)$ with a function of t, according to (6).

The first term can be written in the form:

$$\varsigma_N..\varsigma_1 V \int_{-\infty}^{\varsigma_N..\varsigma_1 a'_N(t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$N_{N-1}(\varsigma_{N-1}..\varsigma_1 a'_{N-1}(t)-x\varsigma_N\rho_{N-1,N}(t)/\sqrt{\left(1-\rho^2_{N-1,N}(t)\right)},..$$

$$..,\varsigma_1 a'_1(t)-x\varsigma_N..\varsigma_2\rho_{1,N}(t)/\sqrt{\left(1-\rho^2_{1,N}(t)\right)};\tilde{\Xi}_{N-1}^{(N-1)}(t))dx+$$

and the second term:

$$-\sum_{j=1}^{N-1}\varsigma_N..\varsigma_j q_j \int\limits_{-\infty}^{\varsigma_N..\varsigma_1 b'_N(t)} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

$$N_{N-j}(\varsigma_{N-1}..\varsigma_1 b'_{N-1}(t) - x\varsigma_N\rho_{N-1,N}(t)/\sqrt{\left(1-\rho^2_{N-1,N}(t)\right)},..$$
$$..,\varsigma_j b'_j(t) - x\varsigma_N..\varsigma_{j+1}\rho_{j,N}(t)/\sqrt{\left(1-\rho^2_{j,N}(t)\right)};\tilde{\Xi}^{(N-1)}_{N-j}(t))dx.$$

Again, in light of the identity obtained before, we finally obtain the following result for the value of a sequential exchange option.

Proposition 3 *Let W_N the value of the multicompound call/put option with maturity T_N and strike price I_N written on a compound call/put option W_{N-1} with maturity T_{N-1} and strike price I_{N-1} , whose value is given by:*

$$W_N(V,t;\varsigma_1,..,\varsigma_N) = \varsigma_N..\varsigma_1 V N_N\left(\varsigma_N..\varsigma_1 a'_N(t),..,\varsigma_1 a'_1(t);\Xi^{(N)}_N(t)\right)+$$
$$-\sum_{j=1}^{N}\varsigma_N..\varsigma_j q_j N_{N+1-j}\left(\varsigma_N..\varsigma_1 b'_N(t),..,\varsigma_j b'_j;\Xi^{(N)}_{N+1-j}(t)\right),\quad 0\le t\le T_N,$$

The value of a multicompound call/put option to switch is:

$$F_N(V_1,V_2,t;\varsigma_1,..,\varsigma_N) = \varsigma_N..\varsigma_1 V_2 N_N\left(\varsigma_N..\varsigma_1 a'_N(t),..,\varsigma_1 a'_1(t);\Xi^{(N)}_N(t)\right)+$$
$$-V_1\sum_{j=1}^{N}\varsigma_N..\varsigma_j q_j N_{N+1-j}\left(\varsigma_N..\varsigma_1 b'_N(t),..,\varsigma_j b'_j;\Xi^{(N)}_{N+1-j}(t)\right),\quad 0\le t\le T_N,$$

where the a_is, the b_is and the ρ_{ij}s are as defined previously.

6 Conclusion

The pricing formulas we found earlier are special cases of the multicompound call options result, incorporating the value of the multiple interacting among real options; these, represent more suitable techniques for the venture capital fund who is evaluating the project to get started a multi-stage business, since involve the flexibility to continuously shut-down, restart, expand or contract it.

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