Accuracy Evaluation of a New Broad-band Vector Digital Spectrum Analyzer

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Abstract – *In this paper the theoretical expression of the accuracy of a previously introduced vector digital spectrum analyzer, based on a random asynchronous sampling strategy whose bandwidth limitation is due uniquely to the bandwidth of the Sample/Hold circuit, has been deduced and compared with the simulated results. In order to avoid the influence of the conventional time origin and of the measuring instant on the result the ratio between each spectral component of the signal an its fundamental one elevated to the same order of the considered harmonic is introduced. It is shown that the asymptotic mean value of this new parameter is independent from the measurement occasion if the number of samples is sufficiently great; besides it is characterized by a bias which depends on the approximation of the estimated fundamental frequency of the signal with respect to its true value. At last a procedure to estimate the fundamental frequency is proposed.*

I. INTRODUCTION

Recently we introduced a new vector spectrum analyzer [1] based on a random sampling strategy previously used for the realization of both a digital power meter [2,3] and a power spectrum analyzer [4,5]. To this end it is necessary to recall that, according to the Fourier series, a generic spectral component X_n of a periodic signal $x(t)$:

$$x(t) = \sum_{q=-M}^{+M} X_q e^{j2\pi q f_1 t} \qquad (1)$$

where $f_1 = 1/T_1$ is the fundamental frequency and M the practically finite maximum order of harmonics, can be deduced by the well known relationship:

$$X_n = \frac{1}{T_1} \int_{-T_1/2}^{+T_1/2} x(t) e^{-j2\pi n f_1 t} dt \qquad (2)$$

For the digital implementation of this equation, the integral can be approximated by using a sequence of $2N+1$ samples of the periodic signal and the estimate $\hat{X}_n$ of a generic spectral component can be expressed as follows:

$$\hat{X}_n = \frac{1}{2N+1} \sum_{i=p_0-N}^{p_0+N} x(t_i) e^{-j2\pi n \tilde{f}_1 t_i} \qquad (3)$$

where p_0 is a positive integer (greater than N) which marks

the centred value of the sequence of $2N+1$ samples, $\tilde{f}_1$ a suitable estimate of f_1 and t_i a generic sampling instant. The adopted random sampling strategy is defined as:

$$t_i = (i + Y_i) T_s + \tau \qquad (4)$$

where T_s represents an "average" sampling rate, Y_i a set of independent random variables with a probability density function uniformly distributed between $-1/2$ and $+1/2$, τ the unknown delay between the signal time origin and the sampling time generator. Therefore the estimator $\hat{X}_n$ becomes a random variable and, by substituting eqns.1 and 4 into eqn.3, it can be expressed as follows:

$$\hat{X}_n = h_{nn} X_n e^{j2\pi n(f_1-\tilde{f}_1)(\tau+p_0 T_s)} + \sum_{\substack{q=-M \\ q \neq n}}^{+M} h_{qn} X_q e^{j2\pi(q f_1 - n\tilde{f}_1)(\tau+p_0 T_s)} \qquad (5)$$

where:

$$h_{qn} = \frac{1}{2N+1} \sum_{i=-N}^{+N} e^{j2\pi(q f_1 - n\tilde{f}_1)(i+Y_{p_0+i})T_s} \qquad (6)$$

This equation is valid also for $q = n$. We can also observe that h_{qn} is independent from τ. The expected value of $\hat{X}_n$ with respect to the random variables $\{Y_{p_0+i}\}$ results:

$$E\{\hat{X}_n\} = E\{h_{nn}\} X_n e^{j2\pi n(f_1-\tilde{f}_1)(\tau+p_0 T_s)} + \sum_{\substack{q=-M \\ q \neq n}}^{+M} E\{h_{qn}\} X_q e^{j2\pi(q f_1 - n\tilde{f}_1)(\tau+p_0 T_s)} \qquad (7)$$

where (see eqn.7 in [1]):

$$E\{h_{qn}\} = \text{sinc}\left((q f_1 - n\tilde{f}_1)(2N+1)T_s\right) \qquad (8)$$

is a real quantity, independent of p_0 since the random variables $\{Y_{p_0+i}\}$ have a common distribution for any p_0 and i, and coincides with the Fourier Transform of a continuous rectangular window of length $(2N+1)$. By referring to the last expression of eqn.7 we observe the mean value of the estimate of X_n is characterized by the following errors: a) an amplitude error due to $E\{h_{nn}\}$; b) a phase error due to $n(f_1 - \tilde{f}_1)(\tau + p_0 T_s)$; c) an error due to the influence of the other spectral components into the sum. Because eqn.8 is not

periodic, the aliasing effect is not present in this random sampling strategy and consequently the last error is due uniquely to the leakage effect

The expected value of $\hat{X}_n$ is conditioned to particular values of the delay τ and the centred point p_0 of the sampled values. By interpreting as random variables both τ and p_0, the unknown delay τ (eqn.4) can be assumed as a continuous variable uniformly distributed in a generic time-interval T' and the centred value p_0 a discrete variable uniformly distributed in a generic discrete interval $(-m,+m)$. The random variables τ, p_0 and $\{Y_i\}$ are mutually independent. In order to avoid the influence of the conventional time origin on the instrument performance and of the measuring instant, the asymptotic behaviour can be obtained by imposing that the quantities T' and/or m are sufficiently large and theoretically tending to infinity. Obviously, when the period T_1 is known, the asymptotic behaviour can be obtained also if the time-interval T' coincides with T_1. The asymptotic expected value of $\hat{X}_n$ also with respect to these last quantities is expressed as in eqn.11 in [1]. From this equation we deduce that the asymptotic expected value is null. In order to overcome this cumbersome and also to avoid the phase error $n(f_1 - \tilde{f}_1)(\tau + p_0 T_s)$ we proposed to introduce the ratio $\hat{\rho}_n$ defined as follows [1]:

$$\hat{\rho}_n = \frac{\hat{X}_n}{(\hat{X}_1)^n} \qquad \text{for } n > 1 \qquad (9)$$

It allows to normalize for each n the expected value $\mathrm{E}\{\hat{X}_n\}$ with respect to $\mathrm{E}\{\hat{X}_1\}$.

it can be shown that, on the hypothesis of a leakage effect sufficiently low, it results (see eqn.8):

$$\mathrm{E}\{\hat{\rho}_n\} \cong \frac{X_n}{X_1^n} \frac{\mathrm{E}\{h_{nn}\}}{\mathrm{E}^n\{h_{11}\}} = \frac{X_n}{X_1^n} \frac{\mathrm{sinc}(n\Delta)}{\mathrm{sinc}^n(\Delta)} \qquad (10)$$

where:

$$\Delta = (f_1 - \tilde{f}_1)(2N+1)T_s \qquad (11)$$

From eqn.10 we observe that there is uniquely a relative amplitude error equal to $\mathrm{sinc}(n\Delta)/\mathrm{sinc}^n(\Delta) - 1$. This error is null only if the condition $\Delta = 0$, i.e. $\tilde{f}_1 = f_1$, is satisfied; when this condition is only approximately verified, it is necessary to evaluate the relative amplitude error. To this end each term of the function $\mathrm{sinc}(n\Delta)/\mathrm{sinc}^n(\Delta)$ must be developed in Taylor series around the value $\Delta = 0$. It can be shown that:

$$\mathrm{E}\{\hat{\rho}_n\} \cong \frac{X_n}{(X_1)^n} \frac{1 - \frac{(\pi n\Delta)^2}{6}}{1 - \frac{n(\pi\Delta)^2}{6}} \cong \frac{X_n}{(X_1)^n}(1+\varepsilon) \qquad (12)$$

where:

$$\varepsilon = -n(n-1)\frac{(\pi\Delta)^2}{6} \qquad (13)$$

represents the relative amplitude error on the hypothesis that it is sufficiently low. For a prefixed value of ε, when the order n of the estimated spectral component increases, the frequency of the signal must be estimated with greatest accuracy.

It can be shown that the asymptotic variance can be ex-

$$\mathrm{Var}\{\rho_n\} \cong \frac{1}{2N+1}\left\{\frac{1}{\mathrm{E}^{2n}\{h_{11}\}}\left[\sum_{q=-M}^{+M}\left(\frac{|X_q|}{|X_1|^n}\right)^2(1-\mathrm{E}_0^2\{h_{qn}\}) + (2N+1)\sum_{\substack{q=-M\\q\neq n}}^{+M}\left(\frac{|X_q|}{|X_n|}\right)^2 \mathrm{E}^2\{h_{qn}\}\right]\right.$$

$$+ n^2\frac{\mathrm{E}^2\{h_{nn}\}}{\mathrm{E}^{2(n+1)}\{h_{11}\}}\left[\sum_{q=-M}^{+M}\left(\frac{|X_q|}{|X_1|^n}\frac{|X_n|}{|X_1|}\right)^2(1-\mathrm{E}_0^2\{h_{q1}\}) + (2N+1)\sum_{\substack{q=-M\\q\neq 1}}^{+M}\left(\frac{|X_q|}{|X_1|^n}\frac{|X_n|}{|X_1|}\right)^2 \mathrm{E}^2\{h_{q1}\}\right]$$

$$- 2n\frac{\mathrm{E}\{h_{nn}\}}{\mathrm{E}^{2n+1}\{h_{11}\}}\frac{|X_n|^2}{|X_1|^{2n}}\left[\mathrm{E}\{h_{(n-1)(n-1)}\}\left(1-\frac{\mathrm{E}_0\{h_{nn}\}\mathrm{E}_0\{h_{11}\}}{\mathrm{E}_0\{h_{(n-1)(n-1)}\}}\right)\right.$$

$$+ \sum_{\substack{q=-M\\q\neq n}}^{+M}\left(\frac{X_q X_{q-n+1}^*}{X_n X_1^*} + \frac{X_q^* X_{q-n+1}}{X_n^* X_1}\right)\left(\mathrm{E}\{h_{(n-1)(n-1)}\}\left(1-\frac{\mathrm{E}_0\{h_{qn}\}\mathrm{E}_0\{h_{(q-n+1)1}\}}{\mathrm{E}_0\{h_{(n-1)(n-1)}\}}\right) + \mathrm{E}\{h_{qn}\}\mathrm{E}\{h_{(q-n+1)1}\}(2N+1)\right)\right]\right\} \qquad (14)$$

II. ACCURACY ANALYSIS

The ratio $\hat{\rho}_n$, defined by eqn.9, represents the new output of the instrument. By evaluating the asymptotic mean of $\hat{\rho}_n$,

pressed as follows (eqn. A31):

Taking into account eqns.8 and 11 and recalling that $|n\Delta|$ must be sufficiently small in order to reduce the relative amplitude error (eqn.10), we can write:

$$\mathrm{E}\{h_{qn}\} \cong \mathrm{sinc}[(q-n)(2N+1)f_1 T_s] \quad \text{for } q \neq n \qquad (15)$$

$$\mathrm{E}\{h_{nn}\} = \mathrm{sinc}[n\Delta] \cong 1 \qquad (16)$$

$$E_0\{h_{qn}\} \cong \text{sinc}[(q-n)f_1 T_s] \quad \text{for } q \neq n \qquad (17)$$

$$E_0\{h_{nn}\} = \text{sinc}\left[n\frac{\Delta}{2N+1}\right] \cong 1 \qquad (18)$$

Obviously the previous approximations are acceptable for estimating the variance; indeed, for the evaluation of the amplitude error the effect of $n\Delta$ must be taken in more significant account. By substituting into eqn.14 we obtain the final result:

$$
\begin{aligned}
\text{Var}\{\rho_n\} &\cong \frac{1}{2N+1}\left\{ \sum_{\substack{q=-M \\ q \neq n}}^{+M} \left(\frac{|X_q|}{|X_1|^n}\right)^2 \left[1 - \text{sinc}^2((q-n)f_1 T_s) + (2N+1)\text{sinc}^2((q-n)(2N+1)f_1 T_s)\right] \right. \\
&+ n^2 \sum_{\substack{q=-M \\ q \neq 1}}^{+M} \left(\frac{|X_q|}{|X_1|^n}\frac{|X_n|}{|X_1|}\right)^2 \left[1 - \text{sinc}^2((q-1)f_1 T_s) + (2N+1)\text{sinc}^2((q-1)(2N+1)f_1 T_s)\right] \qquad (19) \\
&\left. - n\sum_{\substack{q=-M \\ q \neq n}}^{+M} \left(\frac{X_q X_{q-n+1}^*}{X_n X_1^*} + \frac{X_q^* X_{q-n+1}}{X_n^* X_1}\right)\left[1 - \text{sinc}^2((q-n)f_1 T_s) + (2N+1)\text{sinc}^2((q-n)(2N+1)f_1 T_s)\right] \right\}
\end{aligned}
$$

From this equation we deduce that the variance of the estimated ratio $\hat{\rho}_n$ decreases with a quantity which is inversely proportional to $(2N+1)$, i.e. the number of samples of the window, while in the equally spaced one decreases with the square of it [3]; this quantity is multiplied by different terms
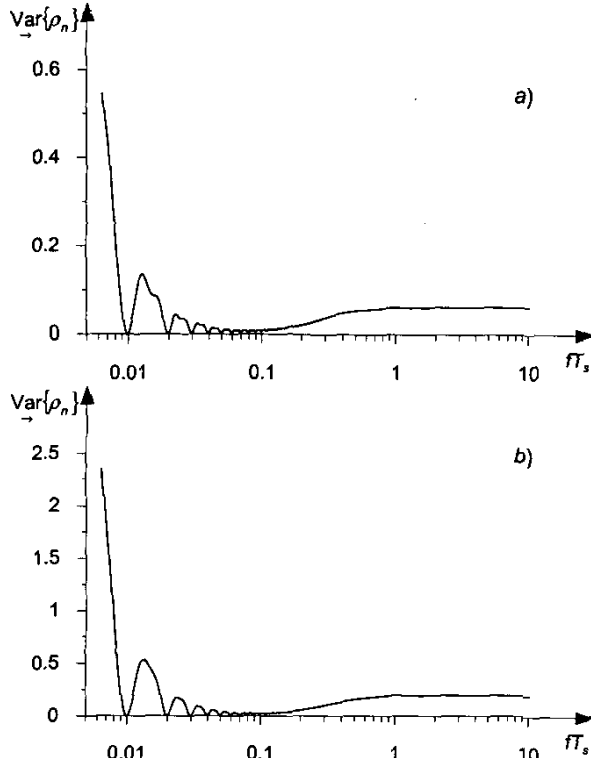

Fig. 1 – Shape of the theoretical expression
of $\text{Var}\{\rho_n\}$ as a function of fT_s

which depend on the relative amplitude of the other spectral components.

Figure 1 shows the shape of the theoretical expression of $\text{Var}\{\rho_n\}$ (eqn.19) $X_q = 1$, $N = 50$ and $M = 3$ respectively for $n = 2$ (a) and $n = 3$ (b). It is interesting to observe that the second term into eqn.19 predominates and tends to the value $n^2(M-1)/(2N+1)$.

III.   SIMULATED RESULTS

An estimate of the expected value of the ratio $\hat{\rho}_n$ of eqn.9 can be deduced by the following expression:

$$
\tilde{E}\{\hat{\rho}_n\} = \frac{1}{m}\sum_{p_0=N+1}^{N+m} \frac{1}{R}\sum_{r=1}^{R}\frac{1}{W}\sum_{w=1}^{W}(2N+1)^{n-1} \cdot
$$

$$
\cdot \frac{\sum_{i=p_0-N}^{p_0+N} x[(i+y_{ir})T_s + \tau_w]e^{-j2\pi n \tilde{f}_1[(i+y_{ir})T_s+\tau_w]}}{\left[\sum_{i=p_0-N}^{p_0+N} x[(i+y_{ir})T_s + \tau_w]e^{-j2\pi \tilde{f}_1[(i+y_{ir})T_s+\tau_w]}\right]^n} \qquad (20)
$$

where y_{ir} is generic value of the random variable Y_{ir}, uniformly distributed in the interval between zero an one, τ_w is a generic value of the random variable τ, uniformly distributed in a generic interval, for example between $-T'$ and $+T'$. In order to obtain the asymptotic value of this estimate, we can assume $T' = \tilde{T}_1$ because it is necessary to select $\tilde{f}_1 \cong f_1$ in order to have an amplitude error of the ratio $\hat{\rho}_n$ sufficiently small. From eqn.20 we deduce that the value of N must be selected sufficiently great in order to obtain a small asymptotic variance of the ratio $\hat{\rho}_n$ and not too great in order to have also a small amplitude error of the same ratio; for example we can impose $N = 10^4$.

An estimate of the variance of the ratio $\hat{\rho}_n$ of eqn.9 can be deduced by considering the last expression of eqn.A14:

$$\tilde{\text{Var}}\{\hat{\rho}_n\} = \tilde{E}\{\hat{\rho}_n\hat{\rho}_n^*\} - \tilde{E}\{\hat{\rho}_n\}\tilde{E}\{\hat{\rho}_n^*\} = \tilde{E}\{|\hat{\rho}_n|^2\} - |\tilde{E}\{\hat{\rho}_n\}|^2 \qquad (21)$$

To this end, by recalling eqn.18, we can deduce firstly the estimate of the expected value of the product $\hat{\rho}_n\hat{\rho}_n^*$:

$$\tilde{E}\{\hat{\rho}_n\hat{\rho}_n^*\}=\frac{1}{m}\sum_{p_0=N+1}^{N+m}\frac{1}{R}\sum_{r=1}^{R}\frac{1}{W}\sum_{w=1}^{W}(2N+1)^{2(n-1)}\cdot$$

$$\cdot\frac{\sum_{i=p_0-N}^{p_0+N}x\left[(i+y_{ir})T_s+\tau_w\right]e^{-j2\pi nf_1\left[(i+y_{ir})T_s+\tau_w\right]}}{\left[\sum_{i=p_0-N}^{p_0+N}x\left[(i+y_{ir})T_s+\tau_w\right]e^{-j2\pi f_1\left[(i+y_{ir})T_s+\tau_w\right]}\right]^n}\cdot \qquad (22)$$

$$\cdot\frac{\sum_{i'=p_0-N}^{p_0+N}x^*\left[(i'+y_{i'r})T_s+\tau_w\right]e^{j2\pi nf_1\left[(i'+y_{i'r})T_s+\tau_w\right]}}{\left[\sum_{i'=p_0-N}^{p_0+N}x^*\left[(i'+y_{i'r})T_s+\tau_w\right]e^{j2\pi f_1\left[(i'+y_{i'r})T_s+\tau_w\right]}\right]^n}$$

By introducing this equation into eqn.21 and by recalling eqn.20 we obtain the final expression of the estimate of the variance. In order to consider the asymptotic properties of this variance, we must also in this case increase the value of m till its value becomes a constant.

CONCLUSION

A vector spectrum analyzer based on a random asynchronous sampling strategy whose bandwidth limitation is due uniquely to bandwidth of the S/H circuit is described. In order to obtain a measurement result independent of the measurement occasion, the ratio between each spectral component and the fundamental one is introduced. It has been shown that the mean value of this new parameter is independent from the measurement occasion if the number of samples is sufficiently large and it can be considered an unbiased estimate of the generic spectral component if the frequency of the fundamental frequency is sufficiently near to its true value. To this end a procedure for a software "locking" of this frequency is proposed.

APPENDIX

In order to separate the terms with $q=n$ and $q\neq n$ in $\hat{\rho}_n$ defined by eqn.9 we introduce the parameter g_n defined as follows:

$$g_n=\sum_{\substack{q=-M\\q\neq n}}^{+M}\frac{X_q}{X_n}e^{j2\pi(q-n)f_1(\tau+p_0T_s)}h_{qn} \qquad (A1)$$

where h_{qn} is given by eqn.6. This parameter takes into account the leakage effect, i.e. the effect of the other spectral components in the estimate of the nth one. Consequently, by recalling eqn.5, eqn.9 can be rewritten as follows:

$$\hat{\rho}_n=\frac{X_n}{X_1^n}\frac{h_{nn}+g_n}{\left[h_{11}+g_1\right]^n} \qquad (A2)$$

In order to evaluate the expected value of the parameter g_n we have:

$$E\{g_n\}=\sum_{\substack{q=-M\\q\neq n}}^{+M}\frac{X_q}{X_n}E\{e^{j2\pi(q-n)f_1\tau}\}E\{h_{qn}e^{j2\pi(q-n)f_1p_0T_s}\} \qquad (A3)$$

This equation takes into account the independence among τ, p_0 and $\{Y_i\}$. Because both the unknown delay τ has been assumed random, uniformly distributed in a generic time-interval T', and the centred value p_0 random, uniformly distribute in a generic discrete interval $(-m,+m)$, we obtain:

$$E\{g_n\}=\sum_{\substack{q=-M\\q\neq n}}^{+M}\frac{X_q}{X_n}\text{sinc}\left[(q-n)f_1T'\right]E\{h_{qn}e^{j2\pi(q-n)f_1p_0T_s}\} \qquad (A4)$$

In order to avoid the influence on the instrument performance both of the conventional time origin and of the measuring instant, the quantity T' must be sufficiently large and theoretically must tend to infinity, or, when the period T_1 of the periodic signal $x(t)$ is known, by imposing simply $T'=kT_1$, with k a positive integer. From eqn.A4 we deduce that the asymptotic mean is null for each p_0:

$$E\{g_n\}=0 \qquad (A5)$$

Therefore asymptotically the leakage effect nulls.

In order to deduce the statistical parameters of the ratio $\hat{\rho}_n$ it is convenient to introduce the complex random errors of h_{nn} and h_{11} with reference to their respective expected values $E\{h_{nn}\}$ and $E\{h_{11}\}$; therefore $\hat{\rho}_n$ of eqn.A2 can be rewritten as follows:

$$\hat{\rho}_n=\frac{X_n}{X_1^n}\frac{E\{h_{nn}\}+\Delta h_{nn}+g_n}{\left[E\{h_{11}\}+\Delta h_{11}+g_1\right]^n} \qquad (A6)$$

where the complex random error Δh_{nn} is defined as follows:

$$\Delta h_{nn}=h_{nn}-E\{h_{nn}\} \qquad (A7)$$

with $E\{\Delta h_{nn}\}=0$. By assuming the variability of the amplitude error and the leakage effect sufficiently low so that at the denominator of eqn.A6 the successive powers $(\Delta h_{11}+g_1)^n$ with $n\geq 2$ can be neglected, we obtain:

$$\hat{\rho}_n\cong\frac{X_n}{X_1^n}\frac{E\{h_{nn}\}}{E^n\{h_{11}\}}\frac{1+\dfrac{\Delta h_{nn}+g_n}{E\{h_{nn}\}}}{1+n\dfrac{\Delta h_{11}+g_1}{E\{h_{11}\}}} \qquad (A8)$$

Now, by multiplying numerator and denominator of this ratio by the difference of the two denominator terms and, as usually, by neglecting the second order contributions of the errors we can write:

$$\hat{\rho}_n\cong\frac{X_n}{X_1^n}\frac{E\{h_{nn}\}}{E^n\{h_{11}\}}(1+\delta_n) \qquad (A9)$$

where:

$$\delta_n=\frac{\Delta h_{nn}+g_n}{E\{h_{nn}\}}-n\frac{\Delta h_{11}+g_1}{E\{h_{11}\}} \qquad (A10)$$

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It is useful to verify that $\delta_1 = 0$ because, for $n = 1$, eqn.9 becomes equal to one.

By considering the expected value of $\hat{\rho}_n$, we have:

$$E\{\hat{\rho}_n\} \cong \frac{X_n}{X_1^n} \frac{E\{h_{nn}\}}{E^n\{h_{11}\}}(1 + E\{\delta_n\})$$ (A11)

where:

$$E\{\delta_n\} = \frac{E\{g_n\}}{E\{h_{nn}\}} - n\frac{E\{g_1\}}{E\{h_{11}\}}$$ (A12)

with $E\{g_n\}$ given by eqn.A4 and $E\{h_{nn}\}$ by eqn.8 with $q = n$. From eqn.A5 we deduce that asymptotically $E\{\delta_n\}$ nulls:

$$\underset{\rightarrow}{E}\{\delta_n\} = 0$$ (A13)

Therefore the asymptotic expected value of $\hat{\rho}_n$ results:

$$\underset{\rightarrow}{E}\{\hat{\rho}_n\} \cong \frac{X_n}{X_1^n} \frac{E\{h_{nn}\}}{E^n\{h_{11}\}}$$ (A14)

From this equation we deduce that asymptotically both the leakage and the phase error have no effect on the estimate of the parameter $\hat{\rho}_n$. Taking into account eqn.A14, eqn.A9 can be rewritten as follows:

$$\hat{\rho}_n \cong \underset{\rightarrow}{E}\{\hat{\rho}_n\}(1 + \delta_n)$$ (A15)

In order to evaluate the variance of the estimator $\hat{\rho}_n$, it is necessary to take into account that it is a complex quantity and therefore defined as follows:

$$\text{Var}\{\hat{\rho}_n\} = E\{\hat{\rho}_n \hat{\rho}_n^*\} - E\{\hat{\rho}_n\}E\{\hat{\rho}_n^*\}$$ (A16)

By using the first expression of the variance, from eqns.A9 and A11 we deduce:

$$\text{Var}\{\hat{\rho}_n\} \cong \frac{|X_n|^2}{|X_1|^{2n}} \frac{E^2\{h_{nn}\}}{E^{2n}\{h_{11}\}}[E\{\delta_n\delta_n^*\} - E\{\delta_n\}E\{\delta_n^*\}]$$ (A17)

By taking into account eqn.A13, the asymptotic value of this variance results:

$$\underset{\rightarrow}{\text{Var}}\{\hat{\rho}_n\} = \frac{|X_n|^2}{|X_1|^{2n}} \frac{E^2\{h_{nn}\}}{E^{2n}\{h_{11}\}} \underset{\rightarrow}{E}\{\delta_n\delta_n^*\}$$ (A18)

By recalling eqn.A10 where $E\{h_{nn}\}$ is real (see eqn.8 for $q = n$), we can write:

$$\delta_n\delta_n^* = \frac{(\Delta h_{nn} + g_n)(\Delta h_{nn}^* + g_n^*)}{E^2\{h_{nn}\}} + n^2\frac{(\Delta h_{11} + g_1)(\Delta h_{11}^* + g_1^*)}{E^2\{h_{11}\}} +$$
$$-\frac{n}{E\{h_{nn}\}E\{h_{11}\}}[(\Delta h_{nn} + g_n)(\Delta h_{11}^* + g_1^*) + (\Delta h_{nn}^* + g_n^*)(\Delta h_{11} + g_1)]$$ (A19)

In order to deduce the expected value of $\delta_n\delta_n^*$ we must therefore evaluate the contributions of the successive terms of this equation. To this end, in analogy to eqn.A7, in general we impose:

$$\Delta h_{qn} = h_{qn} - E\{h_{qn}\}$$ (A20)

where $E\{\Delta h_{qn}\} = 0$. Besides, in analogy to eqn.A16 it results:

$$E\{\Delta h_{qn}\Delta h_{qn}^*\} = \text{Var}\{h_{qn}\} = E\{h_{qn}h_{qn}^*\} - E^2\{h_{qn}\}$$ (A21)

being $E\{h_{qn}\}$ real. On the other hand, by recalling eqn.6 we can evaluate the expected value conditioned to a particular value of p_0; that is:

$$E\{h_{qn}h_{qn}^* // p_o\} =$$
$$= \frac{1}{2N+1} + \frac{\text{sinc}^2[(qf_1 - n\tilde{f}_1)T_s]}{(2N+1)^2} \sum_{\substack{i=-N \\ i \neq i'}}^{+N} \sum_{i'=-N}^{+N} e^{j2\pi(qf_1 - n\tilde{f}_1)(i-i')T_s} =$$
$$= E\{h_{qn}h_{qn}^*\}$$ (A22)

where in the second passage we have taken into account the independence between Y_{p_o+i} and $Y_{p_o+i'}$ for $i \neq i'$, while in the third passage we recalled eqn.8 in [1], and in the last passage we point out that the conditioned expected value is independent from p_0. Moreover we can write:

$$\frac{1}{(2N+1)^2} \sum_{\substack{i=-N \\ i \neq i'}}^{+N} \sum_{i'=-N}^{+N} e^{j2\pi(qf_1 - n\tilde{f}_1)(i-i')T_s} =$$
$$= \frac{E^2\{h_{qn}\}}{E_0^2\{h_{11}\}} - \frac{1}{2N+1}$$ (A23)

where in the last passage we have taken into account eqn.9 in [1] and eqn.8; besides we have introduced the symbol E_0 to indicate the expected value of eqn.8 when $N = 0$:

$$E_0\{h_{qn}\} = \text{sinc}((qf_1 - n\tilde{f}_1)T_s)$$ (A24)

By substituting into eqn.A22, we obtain:

$$E\{h_{qn}h_{qn}^*\} = \frac{1}{2N+1}[1 - E_0^2\{h_{qn}\} + (2N+1)E^2\{h_{qn}\}]$$ (A25)

Therefore, from eqn.A21 it follows:

$$E\{\Delta h_{qn}\Delta h_{qn}^*\} = \frac{1}{2N+1}[1 - E_0^2\{h_{qn}\}]$$ (A26)

which is a real quantity.

Further, by recalling eqn.A1 we have:

$$g_n g_n^* = \sum_{\substack{q=-M \\ q \neq n}}^{+M} \sum_{\substack{q'=-M \\ q' \neq n}}^{+M} \frac{X_q}{X_n} \frac{X_{q'}^*}{X_n^*} e^{j2\pi(q-q')f_1(\tau+p_0T_s)} h_{qn}h_{q'n}^*$$ (A27)

As usually, we interpret the unknown delay τ and the centred value p_0 as random variables. The expected value of $g_n g_n^*$ must therefore be evaluated with respect both to the random vector $\{Y_i\}$ and these random variables; it results:

$$E\{g_n g_n^*\} = \sum_{\substack{q=-M \\ q \neq n}}^{+M} \sum_{\substack{q'=-M \\ q' \neq n}}^{+M} \frac{X_q}{X_n} \frac{X_{q'}^*}{X_n^*} E\{h_{qn}h_{q'n}^* e^{j2\pi(q-q')f_1 p_0 T_s}\}E\{e^{j2\pi(q-q')f_1\tau}\}$$ (A28)

whose asymptotic value with respect to τ ($T' \to \infty$ or $T' = kT_1$ when T_1 is known) is always null except for $q = q'$. Therefore, by recalling eqn.A25 we have:

$$\underset{\rightarrow}{E}\{g_n g_n^*\} = \frac{1}{2N+1} \sum_{\substack{q=-M\\q\neq n}}^{+M} \frac{|X_q|^2}{|X_n|^2}\left[1 - E_0^2\{h_{qn}\} + (2N+1)E^2\{h_{qn}\}\right]$$

(A29)

By considering all the terms of eqn.A19 it can be shown that the asymptotic mean value of $\delta_n \delta_n^*$ results:

$$\underset{\rightarrow}{E}\{\delta_n \delta_n^*\} = \frac{E\{\Delta h_{nn}\Delta h_{nn}^*\} + \underset{\rightarrow}{E}\{g_n g_n^*\}}{E^2\{h_{nn}\}} + n^2 \frac{E\{\Delta h_{11}\Delta h_{11}^*\} + \underset{\rightarrow}{E}\{g_1 g_1^*\}}{E^2\{h_{11}\}} - n\frac{2E\{\Delta h_{nn}\Delta h_{11}^*\} + \underset{\rightarrow}{E}\{g_n g_1^*\} + \underset{\rightarrow}{E}^*\{g_n g_1^*\}}{E\{h_{nn}\}E\{h_{11}\}} \cong$$

$$= \frac{1}{2N+1}\left\{ \frac{1}{E^2\{h_{nn}\}}\left[\sum_{q=-M}^{+M}\left(\frac{|X_q|}{|X_n|}\right)^2 \left(1 - E_0^2\{h_{qn}\}\right) + (2N+1)\sum_{\substack{q=-M\\q\neq n}}^{+M}\left(\frac{|X_q|}{|X_n|}\right)^2 E^2\{h_{qn}\}\right] \right.$$

$$+ \frac{n^2}{E^2\{h_{11}\}}\left[\sum_{q=-M}^{+M}\left(\frac{|X_q|}{|X_1|}\right)^2 \left(1 - E_0^2\{h_{q1}\}\right) + (2N+1)\sum_{\substack{q=-M\\q\neq 1}}^{+M}\left(\frac{|X_q|}{|X_1|}\right)^2 E^2\{h_{q1}\}\right]$$

$$- \frac{n}{E\{h_{nn}\}E\{h_{11}\}}\left[2E\{h_{(n-1)(n-1)}\}\left(1 - \frac{E_0\{h_{nn}\}E_0\{h_{11}\}}{E_0\{h_{(n-1)(n-1)}\}}\right) \right.$$

$$\left.\left. + \sum_{\substack{q=-M\\q\neq n}}^{+M}\left(\frac{X_q X_{q-n+1}^*}{X_n X_1^*} + \frac{X_q^* X_{q-n+1}}{X_n^* X_1}\right)\left(E\{h_{(n-1)(n-1)}\}\left(1 - \frac{E_0\{h_{qn}\}E_0\{h_{(q-n+1)1}\}}{E_0\{h_{(n-1)(n-1)}\}}\right) + (2N+1)E\{h_{qn}\}E\{h_{(q-n+1)1}\}\right)\right]\right\}$$

(A30)

It is important to emphasize that the variance becomes null for $n=1$ since, from eqn.9, $\hat\rho_1 = 1$.

By substituting into eqn.A18 we obtain the final result:

$$\underset{\rightarrow}{Var}\{\rho_n\} \cong \frac{1}{2N+1}\left\{ \frac{1}{E^{2n}\{h_{11}\}}\left[\sum_{q=-M}^{+M}\left(\frac{|X_q|}{|X_1|^n}\right)^2 \left(1 - E_0^2\{h_{qn}\}\right) + (2N+1)\sum_{\substack{q=-M\\q\neq n}}^{+M}\left(\frac{|X_q|}{|X_n|}\right)^2 E^2\{h_{qn}\}\right] \right.$$

$$+ n^2 \frac{E^2\{h_{nn}\}}{E^{2(n+1)}\{h_{11}\}}\left[\sum_{q=-M}^{+M}\left(\frac{|X_q|}{|X_1|^n}\frac{|X_n|}{|X_1|}\right)^2 \left(1 - E_0^2\{h_{q1}\}\right) + (2N+1)\sum_{\substack{q=-M\\q\neq 1}}^{+M}\left(\frac{|X_q|}{|X_1|^n}\frac{|X_n|}{|X_1|}\right)^2 E^2\{h_{q1}\}\right]$$

$$- 2n\frac{E\{h_{nn}\}}{E^{2n+1}\{h_{11}\}}\frac{|X_n|^2}{|X_1|^{2n}}\left[E\{h_{(n-1)(n-1)}\}\left(1 - \frac{E_0\{h_{nn}\}E_0\{h_{11}\}}{E_0\{h_{(n-1)(n-1)}\}}\right) \right.$$

$$\left.\left. + \sum_{\substack{q=-M\\q\neq n}}^{+M}\left(\frac{X_q X_{q-n+1}^*}{X_n X_1^*} + \frac{X_q^* X_{q-n+1}}{X_n^* X_1}\right)\left(E\{h_{(n-1)(n-1)}\}\left(1 - \frac{E_0\{h_{qn}\}E_0\{h_{(q-n+1)1}\}}{E_0\{h_{(n-1)(n-1)}\}}\right) + (2N+1)E\{h_{qn}\}E\{h_{(q-n+1)1}\}\right)\right]\right\}$$

(A31)

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