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# Particular cases for uniform attainability of quantum information in spin-half systems

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# 1 Introduction

Braunstein and Caves [3] showed that Helstrom's quantum information [5] constitutes an upper bound for classical Fisher information. They also proposed to adopt quantum information as a metric tensor on the space of density operators. Barndorff-Nielsen and Gill [1] gave an example of non-uniform attainability of quantum information in spin-$\frac{1}{2}$ systems. It follows that Helstrom information cannot be used to define a natural Riemannian metric on the space of density operators. The relation between classical and quantum information has been widely studied in Barndorff-Nielsen and Gill [2], Luati [7] and Luati and Guidotti [8]. However, because of Barndorff-Nielsen and Gill example, no emphasis has been devoted to the problem of uniform attainability. In this study, we analyse if some conditions exist such that Fisher information equals its maximum, quantum information. First, we provide a formal proof of non-uniform attainability in spin-$\frac{1}{2}$ systems, which are a general class of 2-dimensional quantum systems. We do that by showing that attainability implies that the measurement to be performed in general depends on the parameter of interest. Secondly, we find a fixed value of a constant parameter (special value) and some values of the parameter of interest (optimal values) such that the bound can be attained by a constant measurement, i.e. uniformly in the parameter space. In the first case, uniform attainability holds, in the second, the measurement is constant only at a single point of the parameter space.

In the following section some results on quantum statistics for spin-$\frac{1}{2}$ systems are briefly summarised. In section 3, the proof of non-uniform attainability is given. In section 4 we derive special and optimal values such that global and local uniformly attaining measurements can be constructed. Comments and open problems are in section 5. An appendix contains some calculations.

## 2 Statistical Inference on Spin-$\frac{1}{2}$ Systems

We summarise the statistical results of the above mentioned papers of Barndorff-Nielsen and Gill [1] and of Braunstein and Caves [3]. For a complete treatise on quantum statistical inference, we refer to the books of Holevo [6] and of Helstrom [4]; for an introduction on quantum theory, the reference is the book of Peres [9].

The physical space of a spin-$\frac{1}{2}$ particle parametrised by Euler angles $\phi$ and $\eta$ (Peres [9]) is

$$(\mathcal{H}_2, \rho(\phi, \eta)) \tag{1}$$

where $\mathcal{H}_2$ is a two-dimensional complex Hilbert space and $\rho(\phi, \eta)$ is a parametric density matrix. We assume that $\rho(\phi, \eta)$ is a pure state, i.e.

$$\rho(\phi, \eta) = |\psi(\phi, \eta)\rangle \langle \psi(\phi, \eta)| \tag{2}$$

where $|\psi(\phi, \eta)\rangle$ is a generic unit vector in $\mathcal{H}_2$

$$|\psi(\eta, \phi)\rangle = \begin{bmatrix} \cos\left(\frac{\eta}{2}\right) e^{-\frac{1}{2}i\phi} \\ \sin\left(\frac{\eta}{2}\right) e^{\frac{1}{2}i\phi} \end{bmatrix}. \tag{3}$$

and $\langle \psi(\phi, \eta)|$ is its Hermitian transposed, according to Dirac 'bra-ket' notation: a column vector $|\psi\rangle$ is called 'ket' and its complex conjugate row vector $\langle\psi|$ is called 'bra' and their inner product $\langle\psi| |\psi\rangle$ is a 'bra-ket'. In the present study we consider $\phi \in [0, 2\pi[$ as the unknown parameter of interest and $\eta \in ]0, \pi[$ as a known constant parameter.

On some measurable space $(\mathbb{X}, \mathcal{A})$ a generalised measurements of the form

$$M(A) = \int_A m(x)\mu(dx), \forall A \in \mathcal{A} \tag{4}$$



is defined, where $\{m(x)\}_{x\in\mathbb{X}}$ is a family of nonnegative and selfadjoint operators on $\mathcal{H}_2$, also called a measurement, and $\mu$ is a real $\sigma$-finite measure on $\mathbb{X}$ so that on a measurable space $(\Omega, \mathcal{F})$ the probability density of the random function $X : (\Omega, \mathcal{F}) \to (\mathbb{X}, \mathcal{A})$ is

$$p(x; \phi, \eta) = tr\{\rho(\phi, \eta)\, m(x)\}. \tag{5}$$

Since $\mathcal{H}_2$ is isomorphous to the complex vector space $\mathbb{C}^2$, it is equivalent to refer to selfadjoint operators or to Hermitian matrices.

Expected Fisher information on the parameter $\phi$ is

$$i(\phi, M) = E\left\{\left(l_{/\phi}\right)^2\right\} = \int_{\mathbb{X}} \left(l_{/\phi}\right)^2 p(x; \phi, \eta)\mu(dx), \tag{6}$$

where $l_\phi = \ln p(x; \phi, \eta)$ and $l_{/\phi} = \frac{\partial}{\partial\phi} \ln p(x; \phi, \eta)$. Note that in quantum statistics expected Fisher information depends on the measurement $M$. To derive an upper bound for Fisher information, Braunstein and Caves [3] used the symmetric logarithmic derivative $(SLD)$ or quantum score of $\rho(\phi, \eta)$ with respect to $\phi$ which is the self-adjoint operator $\rho_{//\phi}$, firstly introduced by Helstrom [4], implicitly defined by the relation

$$\rho_{/\phi} = \frac{1}{2}\left(\rho(\phi, \eta)\, \rho_{//\phi} + \rho_{//\phi}\rho(\phi, \eta)\right),$$

where the matrix $\rho_{/\phi}$ is obtained by differentiating each element of $\rho(\phi, \eta)$ with respect to $\phi$.

Helstrom information or expected quantum information on the parameter $\phi$ is defined as

$$I(\phi) = tr\left\{\rho(\phi, \eta)\left(\rho_{//\phi}\right)^2\right\}. \tag{7}$$

By writing classical Fisher information in terms of the $SLD$ operator, one gets

$$i(\phi; M) = \int_{\mathbf{X}} [p(x;\phi,\eta)]^{-1} \left[\mathrm{Re}\, tr \left\{\rho(\phi,\eta)\, \rho_{//\phi} m(x)\right\}\right]^2 \mu(dx). \tag{8}$$

By means of Cauchy-Schwarz inequality based on Hilbert-Schmidt inner product $\langle A, B\rangle := tr\left\{A^H B\right\}, A, B \in \mathcal{H}$, Braunstein and Caves [3] derived the information inequality

$$i(\phi; M) \le I(\phi). \tag{9}$$

Equality in (9) holds if and only if the two following conditions are fulfilled

$$\mathrm{Im}\, tr\left\{\rho(\phi,\eta)\, m(x)\, \rho_{//\phi}\right\} = 0 \tag{10}$$

$$m^{\frac{1}{2}}(x) \left\{k(x;\phi,\eta)^{\frac{1}{2}} \mathbf{I} - \rho_{//\phi}\right\} \rho^{\frac{1}{2}}(\phi,\eta) = \mathbf{0} \tag{11}$$

where $\mathbf{I}$ and $\mathbf{0}$ are the identity and null operator respectively and

$$k(x;\phi,\eta) = \frac{tr\left\{\rho(\phi,\eta)\, \rho_{//\phi} m(x)\, \rho_{//\phi}\right\}}{tr\left\{\rho(\varphi,\eta)\, m(x)\right\}}.$$

is a real quantity (see Luati [7]). By expressing the operators $\rho(\phi,\eta), \rho_{//\phi}, m(x)$ with respect of the orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ where $|\uparrow\rangle = |\psi(\eta,\phi)\rangle$ and $|\downarrow\rangle = |\psi(\pi-\eta, \pi+\phi)\rangle$ Barndorff-Nielsen and Gill [1] showed that the upper bound can be attained if the measurement is of the form

$$m(x) = |\xi(x)\rangle \langle \xi(x)| \tag{12}$$



with

$$|\xi(x)\rangle = \alpha_1(x)\, |\uparrow\rangle + \alpha_2(x)\, |\downarrow\rangle \tag{13}$$

and

$$\alpha_1(x), \alpha_2(x) \in \mathbb{R}. \tag{14}$$

Writing the components of the vector $|\xi(x)\rangle$ as functions of $\alpha_1(x)$ and $\alpha_2(x)$, Barndorff-Nielsen and Gill [1] showed that the vector $|\xi(x)\rangle$ depends on $\phi$. This implies that $m(x)$ is not constant with respect to $\phi$ and therefore that the bound cannot be uniformly achieved for $\phi$.

In the following section we provide a detailed proof of this statement.

## 3 Non-Uniform Attainability

Rather than considering the components of the vector $|\xi(x)\rangle$ as functions of $\alpha_1(x)$ and $\alpha_2(x)$, we consider $\alpha_1(x)$ and $\alpha_2(x)$ as functions of the components of the vector $|\xi(x)\rangle$: to satisfy attainability condition (14), non-constancy of $|\xi(x)\rangle$ with respect to $\phi$ naturally arises. It evidences that some particular values can be found such that (14) holds and $|\xi(x)\rangle$ is constant.

The vector $|\xi(x)\rangle = \alpha_1(x)\, |\uparrow\rangle + \alpha_2(x)\, |\downarrow\rangle$ can be written in matrix form as

$$|\xi(x)\rangle = \Psi\, |\alpha(x)\rangle \tag{15}$$

where

$$|\alpha(x)\rangle = \begin{bmatrix} \alpha_1(x) \\ \alpha_2(x) \end{bmatrix} \tag{16}$$

and

$$\Psi = \begin{bmatrix} \cos\left(\frac{\eta}{2}\right) e^{-i\frac{\phi}{2}} & \sin\left(\frac{\eta}{2}\right) e^{-i\left(\frac{\phi}{2}+\frac{\pi}{2}\right)} \\ \sin\left(\frac{\eta}{2}\right) e^{i\frac{\phi}{2}} & \cos\left(\frac{\eta}{2}\right) e^{i\left(\frac{\phi}{2}+\frac{\pi}{2}\right)} \end{bmatrix} \quad (17)$$

is the matrix whose columns are the vectors of the ordered basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. As that basis is orthonormal, the matrix $\Psi$ is unitary and relation (15) can be written as

$$|\alpha(x)\rangle = \Psi^H |\xi(x)\rangle \quad (18)$$

where $\Psi^H$ is for the Hermitian transpose of $\Psi$. Now let the vector

$$|\xi(x)\rangle = \begin{bmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{bmatrix} \quad (19)$$

be such that its components are represented in polar coordinates $r_1, r_2 \in \mathbb{R}, \theta_1, \theta_2 \in [0, 2\pi[$. Therefore relation (18) can be explicitly written as

$$\begin{aligned} \alpha_1(x) &= r_1 e^{i\theta_1} \cos\frac{\eta}{2} e^{i\frac{\phi}{2}} + r_2 e^{i\theta_2} \sin\frac{\eta}{2} e^{-i\frac{\phi}{2}} & (20) \\ \alpha_2(x) &= r_1 e^{i\theta_1} \sin\frac{\eta}{2} e^{i\left(\frac{\phi}{2}+\frac{\pi}{2}\right)} + r_2 e^{i\theta_2} \cos\frac{\eta}{2} e^{-i\left(\frac{\phi}{2}+\frac{\pi}{2}\right)}. & (21) \end{aligned}$$

We have now all the elements to state and prove the following proposition.

**Proposition 1** *Non-uniform attainability in a spin-$\frac{1}{2}$ system.*

With the above notations, let (12) and (19) with

$$\begin{aligned} r_1, r_2 &\in \mathbb{R} - \{0\} & (22) \\ \theta_1, \theta_2 &\in [0, 2\pi[ \end{aligned}$$



be constant with respect to

$$\phi \in [0, 2\pi[ - \{-2\theta_1\}.$$

Then, the system of equations

$$\begin{aligned} \mathrm{Im}\{\alpha_1(x)\} &= 0 & (23) \\ \mathrm{Im}\{\alpha_2(x)\} &= 0 & (24) \end{aligned}$$

where $\alpha_1(x), \alpha_2(x)$ are given by equations (20) – (21) is impossible.

**Proof** of Proposition 1.

By (20) and (21), system (23) – (24) becomes

$$\begin{aligned} \mathrm{Im}\left\{r_1 e^{i\theta_1} \cos\frac{\eta}{2} e^{i\frac{\phi}{2}} + r_2 e^{i\theta_2} \sin\frac{\eta}{2} e^{-i\frac{\phi}{2}}\right\} &= 0 & (25) \\ \mathrm{Im}\left\{r_1 e^{i\theta_1} \sin\frac{\eta}{2} e^{i\left(\frac{\phi}{2}+\frac{\pi}{2}\right)} + r_2 e^{i\theta_2} \cos\frac{\eta}{2} e^{-i\left(\frac{\phi}{2}+\frac{\pi}{2}\right)}\right\} &= 0. & (26) \end{aligned}$$

By collecting the exponential terms and by developing them according to the Euler formula $e^{\pm i\omega} = \cos\omega \pm i \sin\omega$

$$\begin{aligned} \mathrm{Im}\left\{ \begin{array}{l} r_1 \left[\cos\left(\theta_1 + \frac{\phi}{2}\right) + i \sin\left(\theta_1 + \frac{\phi}{2}\right)\right] \cos\frac{\eta}{2} + \\ + r_2 \left[\cos\left(\theta_2 - \frac{\phi}{2}\right) + i \sin\left(\theta_2 - \frac{\phi}{2}\right)\right] \sin\frac{\eta}{2} \end{array} \right\} &= 0 & (27) \\ \mathrm{Im}\left\{ \begin{array}{l} i r_1 \left[\cos\left(\theta_1 + \frac{\phi}{2}\right) + i \sin\left(\theta_1 + \frac{\phi}{2}\right)\right] \sin\frac{\eta}{2} + \\ - i r_2 \left[\cos\left(\theta_2 - \frac{\phi}{2}\right) + i \sin\left(\theta_2 - \frac{\phi}{2}\right)\right] \cos\frac{\eta}{2} \end{array} \right\} &= 0 & (28) \end{aligned}$$

which is true if and only if (see appendix)

$$\begin{aligned} r_1 \sin\left(\theta_1 + \frac{\phi}{2}\right) \cos\frac{\eta}{2} + r_2 \sin\left(\theta_2 - \frac{\phi}{2}\right) \sin\frac{\eta}{2} &= 0 & (29) \\ r_1 \cos\left(\theta_1 + \frac{\phi}{2}\right) \sin\frac{\eta}{2} - r_2 \cos\left(\theta_2 - \frac{\phi}{2}\right) \cos\frac{\eta}{2} &= 0. & (30) \end{aligned}$$

Let now:

$$a = \cos\left(\frac{\eta}{2}\right)\sin\left(\theta_1+\frac{\phi}{2}\right) \quad (31)$$
$$b = \sin\left(\frac{\eta}{2}\right)\sin\left(\theta_2-\frac{\phi}{2}\right) \quad (32)$$
$$d = \sin\left(\frac{\eta}{2}\right)\cos\left(\theta_1+\frac{\phi}{2}\right) \quad (33)$$
$$f = -\cos\left(\frac{\eta}{2}\right)\cos\left(\theta_2-\frac{\phi}{2}\right). \quad (34)$$

The system of equations (29) – (30) becomes then

$$ar_1 + br_2 = 0 \quad (35)$$
$$dr_1 + fr_2 = 0 \quad (36)$$

that for $a \neq 0$ *a.t.b.h.*[1] becomes

$$r_1 = -\frac{b}{a}r_2 \quad (37)$$
$$\left(\frac{af-bd}{-a}\right)r_2 = 0 \quad (38)$$

which admits a non-trivial solution if and only if

$$af - bd = 0. \quad (39)$$

If condition (39) is fulfilled, then equation (38) is true for any constant value of $r_2$. Supposing for the moment equation (39) is

[1] *a.t.b.h.* stands for always true by hypothesis.



meaningful (i.e. the upper bound can be achieved), then equation (37) becomes

$$\frac{r_1}{r_2} = -\frac{b}{a}. \quad (40)$$

Replacing the values (31) – (32), equation (40) becomes

$$\frac{r_1}{r_2} = -\tan\left(\frac{\eta}{2}\right)\frac{\sin\left(\theta_2-\frac{\phi}{2}\right)}{\sin\left(\theta_1+\frac{\phi}{2}\right)}. \quad (41)$$

According to the hypothesis of constancy of $r_1$ and $r_2$, the ratio $\frac{r_1}{r_2}$ should be constant with respect to $\phi$, but equation (41) tells that for fixed $\theta_1, \theta_2$ and $\eta$, $\frac{r_1}{r_2}$ depends on $\phi$, as in general

$$\phi_1 \neq \phi_2 \Rightarrow -\tan\left(\frac{\eta}{2}\right)\frac{\sin\left(\theta_2-\frac{\phi_1}{2}\right)}{\sin\left(\theta_1+\frac{\phi_1}{2}\right)} \neq -\tan\left(\frac{\eta}{2}\right)\frac{\sin\left(\theta_2-\frac{\phi_2}{2}\right)}{\sin\left(\theta_1+\frac{\phi_2}{2}\right)}. \quad (42)$$

This contradicts the hypothesis of constancy of $r_1$ and $r_2$ and proves that it is impossible to uniformly achieve the bound for any fixed value of the constants $\theta_1, \theta_2$ and $\eta$. ■

## 4 A Special Value for Uniform Attainability

We showed that in general it is not possible for expected Fisher information to equal its maximum, quantum information, uniformly in $\phi$. However, by equation (41) it is immediate to argue that particular values of $\theta_1, \theta_2$ and $\eta$ could exist such that the ratio $\frac{r_1}{r_2}$ does not depend on $\phi$. We will now state and prove the main result of this study.

**Proposition 2** *A condition for uniform attainability in a spin-$\frac{1}{2}$ system.*

With the above notations, let

$$\begin{aligned} r_2 &\in \mathbb{R}-\{0\}, r_1(\phi) \in \mathbb{R}-\{0\} \\ \theta_1, \theta_2 &\in [0, 2\pi[ \\ \eta &\in ]0, \pi[, \phi \in \Phi = [0, 2\pi[ - \{(\pi - 2\theta_1); (2\theta_2 - \pi); -2\theta_1\}. \end{aligned}$$

Then

$$i(\phi; M) = I(\phi), \forall \phi \in \Phi \Leftrightarrow \frac{\eta}{2} = \frac{\pi}{2} \text{ and } \theta_1 = -\theta_2. \tag{43}$$

**Proof** of Proposition 2.

To prove Proposition 2 it is sufficient to show that the system

$$\begin{aligned} ar_1 + br_2 &= 0 && (44) \\ dr_1 + fr_2 &= 0 && (45) \\ \frac{\partial}{\partial \phi} r_1 &= 0, \forall \phi \in \Phi && (46) \end{aligned}$$

admits the unique $\phi$-independent solution for $\theta_1 = -\theta_2$ if and only if $\eta = \frac{\pi}{2}$. Recall that equations (44) and (45) represent the condition for attainability, while equation (46) represents the condition for uniform attainability.

By (38) and (37), system (44) – (46) reduces to

$$\begin{aligned} af - bd &= 0 && (47) \\ \frac{\partial}{\partial \phi} r_1 &= 0, \forall \phi \in \Phi. && (48) \end{aligned}$$

Equation (39) is fulfilled if and only if (see appendix)

$$\tan\left(\theta_1 + \frac{\phi}{2}\right) = -\tan^2\frac{\eta}{2}\tan\left(\theta_2 - \frac{\phi}{2}\right). \tag{49}$$



Equation (46) is satisfied only for (see appendix)

$$\tan\left(\theta_1 + \frac{\phi}{2}\right) = -\tan\left(\theta_2 - \frac{\phi}{2}\right). \tag{50}$$

System (47) – (48) becomes then

$$\begin{aligned} \tan\left(\theta_1 + \frac{\phi}{2}\right) &= -\tan^2\frac{\eta}{2}\tan\left(\theta_2 - \frac{\phi}{2}\right) && (51) \\ \tan\left(\theta_1 + \frac{\phi}{2}\right) &= -\tan\left(\theta_2 - \frac{\phi}{2}\right) && (52) \end{aligned}$$

which admits the unique $\phi$-independent solution

$$\theta_1 = -\theta_2 \tag{53}$$

if and only if

$$\tan^2\frac{\eta}{2} = 1, \text{ i.e. } \eta = \frac{\pi}{2}. \tag{54}$$

We define $\eta = \frac{\pi}{2}$ a *special value* for uniform attainability. ■

We have just derived a special value of $\eta$ allowing uniform attainability according to a good choice of $\theta_1$ and $\theta_2$, under some conditions imposed by the derivation itself. We now examine what happens if some of such constraints are not fulfilled.

- $\phi = -2\theta_1 \Rightarrow \cos\left(\theta_1 + \frac{\phi}{2}\right) = 0 \Rightarrow a = 0.$

By equation (35) and (36) $a = 0 \Rightarrow b = 0 \Leftrightarrow \frac{\phi}{2} = \theta_2 \Rightarrow$

$$r_1 = r_2 \cot\frac{\eta}{2}. \tag{55}$$

Equation (55) implies that $\frac{\phi}{2} = \theta_1 = -\theta_2$ is an *optimal point* in the parameter space $\Phi$ as the bound is attainable and it is possible to construct a measurement whose elements do not depend on $\phi$, but it has to be stressed that it is a *local* optimal point as the constancy of the so constructed measurement does not hold globally in $\Phi$.

- $\phi = \pi - 2\theta_1 \Rightarrow d = 0 \Rightarrow f = 0 \Leftrightarrow \phi = 2\theta_2 - \pi \Leftrightarrow \theta_1 = \pi - \theta_2 \Rightarrow$

$$r_1 = r_2 \tan \frac{\eta}{2}. \tag{56}$$

Once again $\phi = \pi - 2\theta_1 = 2\theta_2 - \pi$, is a local optimal point.

## 5 Discussion

We summarise and comment the previously stated results:

1. it is not possible for Fisher information to $\phi$-uniformly attain the upper bound of quantum Fisher information for any value of the constants $\theta_1, \theta_2$ and $\eta$. In fact, in general, any attaining measurement turns out to depend on the parameter of interest and since Fisher information depends on such a measurement, uniform attainability is not possible;

2. it is possible however to uniformly achieve the bound for a special value of the constant parameter $\eta$, namely for $\eta = \frac{\pi}{2}$, and according to a special choice of the constants $\theta_1, \theta_2$, i.e. $\theta_1 = -\theta_2$. In this case, a constant measurement can be constructed such that equality in (9) holds for all $\phi$ in $\Phi$;

3. for some fixed optimal points in the parameter space, such as $\phi = -2\theta_1$ and $\phi = \pi - 2\theta_1$, it is possible to construct an attaining measurement which does not depend on some other values of $\phi$.

The relevance of these results lies on the fact that only for the special value $\eta = \frac{\pi}{2}$ of quantum Fisher information is a metric tensor which can be used to define a statistical distance on the space of density operators acting on 2-dimensional systems. The problem of finding conditions for attainability for $n$-dimensional quantum systems, or equivalently for spin-$j$ systems, with $n = 2j + 1$, is still completely open.



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## Appendix

**Proof** of equations (29) and (30).

$$\begin{aligned}\mathrm{Im}\left\{\alpha_1(x)\right\} &= 0\\ \mathrm{Im}\left\{\alpha_2(x)\right\} &= 0\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\left\{r_1 e^{i\theta_1}\cos\frac{\eta}{2}e^{i\frac{\phi}{2}} + r_2 e^{i\theta_2}\sin\frac{\eta}{2}e^{-i\frac{\phi}{2}}\right\} &= 0\\ \mathrm{Im}\left\{r_1 e^{i\theta_1}\sin\frac{\eta}{2}e^{i\left(\frac{\phi}{2}+\frac{\pi}{2}\right)} + r_2 e^{i\theta_2}\cos\frac{\eta}{2}e^{-i\left(\frac{\phi}{2}+\frac{\pi}{2}\right)}\right\} &= 0\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\left\{r_1 e^{i\left(\theta_1+\frac{\phi}{2}\right)}\cos\frac{\eta}{2} + r_2 e^{i\left(\theta_2-\frac{\phi}{2}\right)}\sin\frac{\eta}{2}\right\} &= 0\\ \mathrm{Im}\left\{r_1 e^{i\left(\theta_1+\frac{\phi}{2}+\frac{\pi}{2}\right)}\sin\frac{\eta}{2} + r_2 e^{i\left(\theta_2-\frac{\phi}{2}-\frac{\pi}{2}\right)}\cos\frac{\eta}{2}\right\} &= 0\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\left\{r_1 e^{i\left(\theta_1+\frac{\phi}{2}\right)}\cos\frac{\eta}{2} + r_2 e^{i\left(\theta_2-\frac{\phi}{2}\right)}\sin\frac{\eta}{2}\right\} &= 0\\ \mathrm{Im}\left\{r_1 e^{i\left(\theta_1+\frac{\phi}{2}\right)}e^{i\frac{\pi}{2}}\sin\frac{\eta}{2} + r_2 e^{i\left(\theta_2-\frac{\phi}{2}\right)}e^{-i\frac{\pi}{2}}\cos\frac{\eta}{2}\right\} &= 0\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\left\{r_1 e^{i\left(\theta_1+\frac{\phi}{2}\right)}\cos\frac{\eta}{2} + r_2 e^{i\left(\theta_2-\frac{\phi}{2}\right)}\sin\frac{\eta}{2}\right\} &= 0\\ \mathrm{Im}\left\{ir_1 e^{i\left(\theta_1+\frac{\phi}{2}\right)}\sin\frac{\eta}{2} - ir_2 e^{i\left(\theta_2-\frac{\phi}{2}\right)}\cos\frac{\eta}{2}\right\} &= 0\end{aligned}$$

$$\begin{aligned}\mathrm{Im}\left\{\begin{array}{l} r_1\left[\cos\left(\theta_1+\frac{\phi}{2}\right)+i\sin\left(\theta_1+\frac{\phi}{2}\right)\right]\cos\frac{\eta}{2}+\\ +r_2\left[\cos\left(\theta_2-\frac{\phi}{2}\right)+i\sin\left(\theta_2-\frac{\phi}{2}\right)\right]\sin\frac{\eta}{2}\end{array}\right\} &= 0\\ \mathrm{Im}\left\{\begin{array}{l} ir_1\left[\cos\left(\theta_1+\frac{\phi}{2}\right)+i\sin\left(\theta_1+\frac{\phi}{2}\right)\right]\sin\frac{\eta}{2}+\\ -ir_2\left[\cos\left(\theta_2-\frac{\phi}{2}\right)+i\sin\left(\theta_2-\frac{\phi}{2}\right)\right]\cos\frac{\eta}{2}\end{array}\right\} &= 0\end{aligned}$$

$$\begin{aligned}r_1\sin\left(\theta_1+\frac{\phi}{2}\right)\cos\frac{\eta}{2} + r_2\sin\left(\theta_2-\frac{\phi}{2}\right)\sin\frac{\eta}{2} &= 0\\ r_1\cos\left(\theta_1+\frac{\phi}{2}\right)\sin\frac{\eta}{2} - r_2\cos\left(\theta_2-\frac{\phi}{2}\right)\cos\frac{\eta}{2} &= 0\end{aligned}$$



**Proof** of equation (49).

By equations (31) – (34), $af = bd \Leftrightarrow$
$\Leftrightarrow -\cos^2\left(\frac{\eta}{2}\right)\sin\left(\theta_1+\frac{\phi}{2}\right)\cos\left(\theta_2-\frac{\phi}{2}\right) =$
$= \sin^2\left(\frac{\eta}{2}\right)\sin\left(\theta_2-\frac{\phi}{2}\right)\cos\left(\theta_1+\frac{\phi}{2}\right) \Leftrightarrow$
$\Leftrightarrow$ for $\left(\theta_1+\frac{\phi}{2}\right)\neq\frac{\pi}{2}a.t.b.h.$, $\left(\theta_2-\frac{\phi}{2}\right)\neq\frac{\pi}{2}a.t.b.h.$,
$\tan\left(\theta_1+\frac{\phi}{2}\right) = -\tan^2\frac{\eta}{2}\tan\left(\theta_2-\frac{\phi}{2}\right)$.■

**Proof** of equation (50).

$\frac{\partial}{\partial\phi}r_1 = 0 \overset{(37)}{\Leftrightarrow} \frac{\partial}{\partial\phi}\left(-\frac{b}{a}r_2\right) = 0 \Leftrightarrow$
$\Leftrightarrow \frac{\partial}{\partial\phi}\left(-\frac{b}{a}\right) = 0 \Leftrightarrow a'b = b'a, a\neq 0\ a.t.b.h.$.
By equations (31) – (31) and for
$\left(\theta_1+\frac{\phi}{2}\right)\neq\frac{\pi}{2}a.t.b.h.$, $\left(\theta_2-\frac{\phi}{2}\right)\neq\frac{\pi}{2}a.t.b.h.$
it becomes
$\frac{1}{2}\sin\left(\frac{\eta}{2}\right)\cos\left(\frac{\eta}{2}\right)\cos\left(\theta_1+\frac{\phi}{2}\right)\sin\left(\theta_2-\frac{\phi}{2}\right) =$
$= -\frac{1}{2}\sin\left(\frac{\eta}{2}\right)\cos\left(\frac{\eta}{2}\right)\sin\left(\theta_1+\frac{\phi}{2}\right)\cos\left(\theta_2-\frac{\phi}{2}\right)$
that is just equation (50).■