Properness results for constant mean curvature surfaces

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1 Introduction.

This paper is a preliminary version. Throughout this paper, we let $\mathcal{N}$ denote the set of homogeneously regular[1] three-manifolds; for a given $N \in \mathcal{N}$, we let $\mathcal{M}(N)$ denote the set of complete, embedded, constant mean curvature surfaces in N, and let $\mathcal{M}$ denote the union $\bigcup_{N\in\mathcal{N}} \mathcal{M}(N)$. We will study the geometry and topological properties of surfaces $M \in \mathcal{M}$. For the sake of simplicity, we will assume that both M and N are connected and orientable. We will call M minimal if its mean curvature is zero and will call M a CMC surface if its mean curvature is a positive constant.

Most of the results of this paper deal with the construction of interesting examples in $\mathcal{M}$ and with theoretical results related to the classical question of properness of the surfaces in $\mathcal{M}$ under certain geometric constraints. These results complement our paper [9] where we prove that certain complete, embedded CMC surfaces in locally homogeneous or in homogenously regular three-manifolds have bounded second fundamental form. Our first theorem below generalizes the properness results of Meeks and Rosenberg [7] for bounded second fundamental form CMC surfaces in $\mathcal{M}(\mathbb{R}^3)$ to certain constant mean curvature surfaces in other constant curvature three-manifolds.

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[1] A three-manifold N is *homogeneously regular* if it has positive injectivity radius and bounded sectional curvature.

Theorem 1.1 (Classical properness theorem) *Suppose $N \in \mathcal{N}$ has constant sectional curvature δ, $M \in \mathcal{M}(N)$ has locally bounded second fundamental form*[2]*, and M has mean curvature H_M. M is properly embedded in N if any of the following four conditions hold:*

1. $\delta > 0$;
2. $\delta = 0$ *and* $H_M \neq 0$;
3. $\delta = 0$, $H_M = 0$, *M has bounded second fundamental form and M is not totally geodesic;*
4. $\delta < 0$ *and* $H_M \geq |\delta|$.

By Theorem 4.1 in section 4, for any $H \in [0, 1)$, there exist nonproper, simply connected surfaces in $\mathcal{M}(\mathbb{H}^3)$ of constant mean curvature H and bounded second fundamental form. Thus, the properness result described in Theorem 1.1 is sharp for $\delta < 0$.

Motivated by our properness results in [9] for CMC surfaces of finite topology in $\mathcal{M}(\mathbb{R}^3)$ and the similar recent properness result in the minimal case by Colding and Minicozzi [1], it is natural to ask whether every $M \in \mathcal{M}(\mathbb{R}^3)$ is properly embedded. In the minimal case this classical question is referred to as the Calabi-Yau problem for complete, embedded minimal surfaces in $\mathbb{R}^3$. In regards to this question, Proposition 6.1 in section 6 states that there exist disconnected, complete, embedded CMC surfaces in $\mathbb{R}^3$ which are proper in an open slab but not proper in the entire space. The first author has conjectured that there exist connected, non-proper minimal surfaces in $\mathcal{M}(\mathbb{R}^3)$ (see Conjecture 15.23 in [6]).

This paper is organized as follows. In section 2, we define the notion of a CMC lamination and discuss some related background material. In section 3, we prove Theorem 1.1. In section 4, we prove that Theorem 1.1 is sharp by constructing for each $H \in [0, 1)$, a nonproper, simply connected surface $M_H \in \mathcal{M}(\mathbb{H}^3)$ with $H_M = H$. In section 5, we prove Theorem 5.1 which shows that a necessary and sufficient condition for a properly embedded, separating CMC surface in $\mathcal{M}$ to have a fixed sized one-sided neighborhood on its mean convex side is for it to have bounded second fundamental form. In section 6, we construct examples of disconnected, complete, embedded CMC surfaces in $\mathbb{R}^3$ which are contained in a slab.

2 Background on CMC laminations.

In order to help understand the results described in this paper, we make the following definitions.

[2] A surface M in a three-manifold N has *bounded second fundamental form* if $|A_M| = \sqrt{k_1^2 + k_2^2}$ is bounded, where k_1, k_2 are the principal curvatures; M has *locally bounded second fundamental form* if $|A_M|$ is bounded on compact sets of N.

Definition 2.1 Let M be a complete, embedded surface in a three-manifold N. A point $p \in N$ is a *limit point* of M if there exist points $\{p_n\}_n \subset M$ which diverge as $n \to \infty$ to infinity on M with respect to the intrinsic Riemannian topology on M but converge in N to p as $n \to \infty$. Let $L(M)$ denote the set of all limit points of M in N. In particular, $L(M)$ is a closed subset of N and $\overline{M} - M \subset L(M)$, where $\overline{M}$ denotes the closure of M.

Definition 2.2 A *CMC lamination* $\mathcal{L}$ of a three-manifold N is a collection of immersed surfaces $\{L_\alpha\}_{\alpha \in I}$ of constant positive mean curvature H called *leaves* of $\mathcal{L}$ satisfying the following properties.

1. $\mathcal{L} = \bigcup_{\alpha \in I} \{L_\alpha\}$ is a closed subset of N.
2. For each leaf L_α of $\mathcal{L}$, considered to be the zero section Z_α of its tangent bundle TL_α, there exists a one-sided neighborhood $N(Z_\alpha) \subset TL_\alpha$ of Z_α such that:
   (a) the exponential map $\exp\colon N(Z_\alpha) \to N$ is a submersion;
   (b) with respect to the pull-back metric on $N(Z_\alpha)$, $Z_\alpha \subset \partial N(Z_\alpha)$ is mean convex;
   (c) $\exp^{-1}(\mathcal{L}) \cap Z_\alpha$ is a lamination of $N(Z_\alpha)$.

The reader not familiar with the subject of minimal or CMC laminations should think about a geodesic on a Riemannian surface. If the geodesic is complete and embedded (a one-to-one immersion), then its closure is a geodesic lamination of the surface. When this geodesic has no accumulation points, then it is proper. Otherwise, there pass complete embedded geodesics through the accumulation points forming the leaves of the geodesic lamination of the surface. The similar result is true for a complete, embedded CMC surface of locally bounded second fundamental form (curvature is bounded in compact extrinsic balls) in a Riemannian three-manifold, i.e. the closure of a complete, embedded CMC surface of locally bounded second fundamental form has the structure of a CMC lamination. For the sake of completeness, we now give the proof of this elementary fact, see [8] for the proof in the minimal case.

Consider a complete, embedded CMC surface M of locally bounded second fundamental form in a three-manifold N. Consider a limit point p of M, i.e., p is a limit of a sequence of divergent points p_n in M. Since M has bounded curvature near p and M is embedded, for some small $\varepsilon > 0$, a subsequence of the intrinsic ε-disks $B_M(p_n, \varepsilon)$ converges to an embedded CMC disk $D(p, \varepsilon) \subset N$ of intrinsic radius ε, centered at p and of constant mean curvature H_M. Since M is embedded, any two such limit disks, say $D(p, \varepsilon)$, $D'(p, \varepsilon)$, do not intersect transversally. By the maximum principle for CMC surfaces, we conclude that if a second disk $D'(p, \varepsilon)$ exists, then $D(p, \varepsilon)$, $D'(p, \varepsilon)$ are the only such limit disks and they are oppositely oriented at p.

Now consider any sequence of embedded disks E_n of the form $B_M(q_n, \frac{\varepsilon}{4})$ or $D(q_n, \frac{\varepsilon}{4})$ such that q_n converges to a point in $D(p, \frac{\varepsilon}{2})$ and such that E_n locally lies on the mean convex side of $D(p, \varepsilon)$. For ε sufficiently small and for n, m large, E_n and E_m must be graphs over domains in $D(p, \varepsilon)$ such that when oriented as graphs, they have the same mean curvature (for a proof of this fact see the proof of item *1* of Theorem 5.1). By the maximum principle, the graphs E_n and E_m are disjoint. It follows that near p and on the mean convex side of $D(p, \varepsilon)$, $\overline{M}$ has the structure of a lamination with leaves with constant mean curvature H_M. This proves that $\overline{M}$ has the structure of a CMC lamination.

3 The proof of the classical properness theorem.

In this section, we prove Theorem 1.1 in the introduction.

Proof. Suppose M and N satisfy the hypothesis of the theorem. Since M has locally bounded second fundamental form in N, its closure $\overline{M}$ is a minimal or a CMC lamination of N with leaves of constant mean curvature H_M. In the special case that condition *3* holds, after lifting the minimal lamination $\overline{M}$ to the universal cover $\widetilde{N} = \mathbb{R}^3$, the result claimed in the theorem follows from the statement of the Structure Theorem for minimal laminations of $\mathbb{R}^3$ (Theorem 1.6 in [8]). This proves that M is properly embedded if condition *3* holds.

The proof of the Structure Theorem for minimal laminations of $\mathbb{R}^3$ demonstrates that if M were not proper in N, then $\overline{M}$ has a limit leaf L whose universal cover $\widetilde{L}$ is stable in the following sense: There exists a positive function $u\colon \widetilde{L} \to \mathbb{R}$ such that $\widetilde{\Delta} u + |A_{\widetilde{L}}|^2 u + \mathrm{Ric}(\widetilde{n}) u = 0$, where $\widetilde{n}$ is the unit normal vector field to $\widetilde{L}$.

If $\widetilde{L}$ were a sphere, then, after possibly lifting to a two-sheeted cover of N, L would likewise be a sphere. But if L were a sphere, then since it is a compact, simply connected limit leaf of the CMC lamination $\overline{M}$, one can lift it to nearby leaves of L in $\overline{M}$. This means that M would be a sphere which implies M is proper. Hence, assume $\widetilde{L}$ is not a sphere. If $H_M = H_{\widetilde{L}} > |\delta|$ or $\delta > 0$, then, by the Lawson correspondence [5], $\widetilde{L}$ isometrically immerses into some constant curvature three-sphere as a stable minimal surface. But this result contradicts a theorem of Fischer-Colbrie and Schoen [4] that any complete three-manifold with Ricci curvature bounded from below by a positive constant does not contain any complete, simply connected, stable minimal surfaces. In particular, it now follows that M is properly embedded if conditions *1*, *2*, or *3* hold and that case *4* holds if $H_M > |\delta|$.

It remains to prove the theorem when $H_M = |\delta|$ and $\delta < 0$. In this case, $\widetilde{L}$ isometrically immerses in $\mathbb{R}^3$ as a stable minimal surface via the Lawson correspondence. A classical result is that this stable minimal immersion of $\widetilde{L}$ into $\mathbb{R}^3$ is a plane (see [3, 4, 10]). It then follows from the Lawson correspondence [5] that $\widetilde{L}$ has a unique isometric immersion into $\mathbb{H}^3(\delta)$ with constant mean curvature $H_M = H_{\widetilde{L}}$ and the image of $\widetilde{L}$ is a horosphere in $\mathbb{H}^3(\delta)$. The upshot of this observation is that, after lifting the CMC lamination of $\overline{M} \subset N$ to a CMC lamination $\mathcal{L}$ of $\mathbb{H}^3(\delta)$, the set of limit leaves of $\mathcal{L}$ are horospheres in $\mathbb{H}^3(\delta)$.

We now check that M is not a limit leaf of $\overline{M}$. If it were a limit leaf, then our discussion above implies that its lifts to $\mathbb{H}^3(\delta)$ are horospheres. In this case, one easily obtains a contradiction. Hence, M is proper in $N - L(M)$, where $L(M)$ is the set of limit leaves of $\overline{M}$.

Let $\mathcal{L}$ be the lamination of $\mathbb{H}^3(\delta)$ obtained by lifting the CMC lamination $\overline{M}$ of N to its universal cover. Let $L(\mathcal{L})$ be the nonempty set of limit leaves of $\mathcal{L}$, which can be identified with the sublamination of $\mathcal{L}$ consisting of lifts of limit leaves of $\overline{M}$; note that $L(\mathcal{L})$ is a sublamination of $\mathcal{L}$ consisting entirely of horospheres. Let $M(\mathcal{L})$ denote the set of nonlimit leaves in $\mathcal{L} - L(\mathcal{L})$ and note that $M(\mathcal{L})$ consists of the lifts of M to $\mathbb{H}^3(\delta)$.

We first claim that no leaf L of $M(\mathcal{L})$ lies on the mean convex side of any of the horospheres in $L(\mathcal{L})$. Arguing by contradiction, suppose that L lies on the mean convex side of some horosphere H in $L(\mathcal{L})$. Let W be the component of $\mathbb{H}^3(\delta) - L(\mathcal{L})$ that contains L; note that W lies on the mean convex side of H. Clearly, W is simply connected with one or two horospheres as boundary components. Since L is properly embedded in W and W is simply connected, L separates W. If L were not properly embedded in $\mathbb{H}^3(\delta)$, then there exists a sequence of compact disks $D_n \subset L$ which are normal graphs over a compact disk D on one of the horosphere components of ∂W, say $H' \subset \partial W$. After orienting the leaves of $\mathcal{L}$ by their mean curvature vectors and noting that, by the separation property of L in W, the disks D_n can be chosen to have the opposite orientations of D, we obtain a contradiction to the fact that the disks D_n converge smoothly to D. This contradiction implies L is proper in $\mathbb{H}^3(\delta)$. A result in [2] states that given two, disjoint, properly embedded, constant mean curvature one surfaces L_1, L_2 in $\mathbb{H}^3(\delta)$ such that L_1 lies on the mean convex side of L_2, then L_1 and L_2 are parallel horospheres. Thus, since L is proper in $\mathbb{H}^3(\delta)$, L is not a horosphere and L lies on the mean convex side of the horosphere H, we obtain a contradiction to our assumption that L is not a horosphere. This proves that no leaf of $M(\mathcal{L})$ lies on the mean convex side of a horosphere in $L(\mathcal{L})$.

An immediate consequence of the fact that no leaf L of $M(\mathcal{L})$ lies in the mean convex side of a horosphere leaf of $L(\mathcal{L})$ is that two distinct horospheres of $L(\mathcal{L})$ are disjoint. In particular, each component of $\mathbb{H}^3(\delta) - L(\mathcal{L})$ is simply connected. Let W be a component of $\mathbb{H}^3(\delta) - L(\mathcal{L})$ with $\Sigma = M(\mathcal{L}) \cap W$ nonempty. Since a leaf L of Σ is proper in W and so separates W, an argument in the previous paragraph implies L is proper in $\mathbb{H}^3(\delta)$. Hence, after possibly replacing W by a different component of $\mathbb{H}^3(\delta) - L(\mathcal{L})$, we may assume that there exists a sequence of disks D_n in Σ which are normal graphs over a disk D contained in a horosphere $H \subset \partial N$, which lie on distinct leaves of Σ and which converge smoothly to D. Since these leaves do not lie on the mean convex side of H by the claim proved in the previous paragraph, we see that H lies on the mean convex side of some leaf L of Σ. Since L and H are disjoint properly embedded surfaces with constant mean curvature one and H lies on the mean convex side of L, the previously mentioned theorem in [2] implies L is a horosphere, which is a contradiction. This contradiction completes the proof of Theorem 1.1. $\square$

Remark 3.1 With small modifications, the arguments in the proof of Theorem 1.1 generalizes to prove the following general result: *Suppose X is a complete Riemannian three-manifold and $M \subset X$ is a complete, embedded surface with constant mean curvature H_M and locally bounded second fundamental form. If X does not contain any complete surfaces of constant mean curvature H_M which are stable and nonspherical, then M is properly embedded.*

4 Examples of nonproper simply connected surfaces in $\mathcal{M}(\mathbb{H}^3)$.

In Theorem 1.1, we proved that if $M \in \mathcal{M}(\mathbb{H}^3)$ has locally bounded second fundamental form and $H_M \geq 1$, then M is properly embedded in $\mathbb{H}^3$. The next theorem shows that this properness result is sharp.

Recall that a surface $M \in \mathcal{M}(\mathbb{H}^3)$ is stable if there exists a positive Jacobi function, that is a function $u\colon M \to \mathbb{R}$ such that $\widetilde{\Delta} u + |A_{\widetilde{L}}|^2 u + \mathrm{Ric}(\widetilde{n})u = 0$, where $\widetilde{n}$ is the unit normal vector field to M. Sometimes this notion of stability is referred to as strong stability.

Theorem 4.1 *For each $H \in [0,1)$, there exists a simply connected surface $M_H \in \mathcal{M}(\mathbb{H}^3)$ satisfying the following statements:*

1. *M_H has mean curvature H.*
2. *M_H is stable and has bounded second fundamental form.*
3. *M_H is not properly embedded in $\mathbb{H}^3$.*

Proof. For the constructions that we carry out in the proof of this theorem, we will use coordinates that arise in the ball model $\mathbb{B}(1) \subset \mathbb{R}^3$ of $\mathbb{H}^3$ with respect to the induced spherical coordinates (θ, ϕ, r) of $\mathbb{R}^3$, where $r \in [0,1)$ and θ is well-defined up to multiples of 2π.

Now fix an $H \in [0,1)$. For such a value of H, there exist properly embedded, pairwise disjoint, stable, constant mean curvature H annuli $A_H, B_H \in \mathcal{M}(\mathbb{H}^3)$, which are surfaces of revolution around the x_3-axis and which are invariant under reflection in $\mathbb{H}^3$ across the horizontal plane $\phi = \frac{\pi}{2}$; assume A_H lies outside of B_H as in figure 1. These surfaces have asymptotic boundaries which are circles $S_+(A_H), S_-(A_H)$ and $S_+(B_H), S_-(B_H)$, respectively; see figure 1. We can chose the surfaces A_H, B_H such that the nonsimply connected annular region R_H between them is foliated by similar annuli of revolution around the z-axis, with the same constant mean curvature H. In particular, the leaves of this foliation are stable constant mean curvature surfaces, with positive Jacobi function u given by taking the inner product of the variational vector field of the product foliation with the unit normal field on the leaf.

For $r \in (0,1)$ sufficiently close to one, $A_H(r) = A_H \cap \mathbb{B}(r)$ and $B_H(r) = B_H \cap \mathbb{B}(r)$ are annuli of revolution. Define embedded curves $\Gamma_{H,r}(t) = (t, \phi_r(t), r) \subset \partial\mathbb{B}(r)$ with $0 < \phi_r'(t) < 1$ which vary smoothly in r and such that $\lim_{t\to\infty} \phi_r(t) = \phi(S_+(A_H))$ and $\lim_{t\to-\infty} \phi_r(t) = \phi(S_+(B_H)))$. Let $-\Gamma_{H,r}(t) = (t, \pi - \phi_r(t), r)$ be the related reflected curves with $\lim_{t\to\infty} \phi_r(t) = \phi(S_-(A_H))$ and $\lim_{t\to-\infty} \phi(t) = \phi(S_-(B_H))$. Furthermore, we choose these embedded curves $\Gamma_{H,r}$ to converge C^1 to a similar embedded curves $\Gamma_{H,1}$ on $\mathbb{S}^2 = \partial\mathbb{B}(1)$ with limit set being corresponding asymptotic boundary circles of A_H, B_H (see figure 1).

We claim that for each r close to 1, there exists a complete, embedded, stable simply connected CMC strip $W_r \subset \mathbb{B}(r)$ diffeomorphic to $[0,1] \times \mathbb{R}$ such that:

1. $\partial W_r = \Gamma_{H,r} \cup -\Gamma_{H,r}$;

2. the limit set of W_r is $A_H(r) \cup B_H(r)$ and such that $W_r \cup A_H(r) \cup B_H(r)$ is a CMC lamination of $\mathbb{B}(r)$;

3. each W_r is a multigraph over its θ-projection to the disk $D_r = R_H \cap \mathbb{B}(r) \cap \{(x,y,z) | y > 0\}$.

Let $D_r \times \mathbb{R} \to R_H \cap \mathbb{B}(r)$ define by $(p,t) \to (p, t \bmod 2\pi)$.

After choosing a subsequence, the strips $W_{1-\frac{1}{n}}$ converge to a simply connected stable surface $M_H \in \mathcal{M}(\mathbb{H}^3)$ with mean curvature H and with limit set $A_H \cup B_H$. This completes the proof of the proposition. $\square$

The proof of Theorem 4.1 generalizes to prove the following similar theorem in $\mathbb{H} \times \mathbb{R}$. One of the outstanding conjectures in this subject is: *A surface $M \in \mathcal{M}(\mathbb{H} \times \mathbb{R})$ with constant mean curvature $H_M \geq \frac{1}{2}$ and with locally bounded second fundamental form is properly embedded in $\mathbb{H} \times \mathbb{R}$.* The next theorem show that if the conjecture holds, then it is sharp. We prove in [9] that under the stronger hypothesis $H_M > 1$, M is properly embedded.

Theorem 4.2 *Let $\mathbb{H}$ be the hyperbolic plane. For each $H \in [0, \frac{1}{2})$, there exists a simply connected surface $M_H \in \mathcal{M}(\mathbb{H} \times \mathbb{R})$ satisfying the following statements:*

1. *M_H has mean curvature H.*

2. *M_H is stable and has bounded second fundamental form.*

3. *M_H is not properly embedded in $\mathbb{H}^3$.*

5 The existence of one-sided regular neighborhood.

Meeks and Rosenberg [7] proved that a CMC surface $M \in \mathcal{M}(\mathbb{R}^3)$ of bounded second fundamental form has a fixed size regular neighborhood on its mean convex side. In other words, for such an M there exists an $\varepsilon > 0$ such that for any $p \in M$, the normal line segment l_p of length ε based at p and contained on the mean convex side of M, intersects M only at the point p. An immediate consequence of the existence of such a regular neighborhood is that the surface is properly embedded and for some $c > 0$, the area of the surface is at most cR^3 in any ball of radius R. The next theorem generalizes this result to CMC surfaces in other homogeneous three-manifolds. Actually, the theorem generalizes easily to show that a connected, codimension-one submanifold with nonzero constant mean curvature and bounded second fundamental form, of a homogenously regular n-manifold N, have a fixed one-sided regular neighborhood on its mean convex side.

Theorem 5.1 (One-sided regular neighborhood theorem) *Suppose $X \in \mathcal{N}$ with absolute sectional curvature bounded by a constant S_0 and with injectivity radius at least I_0. Suppose $M \subset X$ is a separating, properly embedded CMC surface with mean curvature H_0 and with $|A_M| \leq A_0$. Then the following statements hold.*

1. *There exists a $D \in (0, I_0)$, depending on A_0, H_0, I_0, S_0, such that M has a regular neighborhood of width D.*

2. *There exists a $c > 0$ such that the area of M in ambient balls of radius 1 is less than c.*

Proof. We will first prove the existence of a one-sided regular neighborhood of M in X with width $D \in (0, I_0)$. The uniform bound $|A_M| \leq A_0$ implies that there exists an $\varepsilon > 0$ sufficiently small so that for any $p \in M$, every component of $B_X(p, 2\varepsilon) \cap M$ is a graph over its projection to the disk of radius 3ε in T_pM; here we are considering the tangent plane to be a plane in normal coordinates and for the orthogonal projection map to be well defined in these coordinates. Moreover, we can choose ε sufficiently small and a smaller positive δ such that if a component of $B_X(p, 2\varepsilon) \cap M_n$ intersects $B_X(p, \delta)$, then such component contains a graph over the disk of radius ε in T_pM, denoted by $D(p, \varepsilon)$. Also, we may assume that in our coordinates, these graphs all have small gradient.

The theorem will follow from the observation that two disjoint graphs, u_1 and u_2 over $D(p, \varepsilon)$ of constant mean curvature H_0 which are oppositely oriented and such that u_2 lies on the mean convex side of u_1, cannot be too close at their centers (the difference of the graphs is a positive function and the Laplacian of this difference is bounded from above by a negative constant). Let $p \in M$ and let $u_1 \subset M$ be the graph over $D(p, \varepsilon)$ containing p. Let $\delta_1 \in (0, \delta)$ and suppose that $B_X(p, \delta_1) \cap M$ contains a connected component different from $u_1 \cap B_X(p, \delta_1)$ and which lies on the mean convex side of u_1. Let $u_2 \subset M$

be the graph over $D(p,\varepsilon)$ which contains such a component. We conclude that, by our observation, when δ_1 is chosen sufficiently small, u_1 and u_2 when oriented as graphs have the same mean curvature. However, by the separation hypothesis for M, if u_2 were to exist, then there would be another graph $u_3 \subset M$ between u_1 and u_2 that, when oriented as a graph over $D(p,\varepsilon)$, has oppositely signed mean curvature which is impossible by the same observation. Hence, when δ_1 is sufficiently small, $B_X(p,\delta_1)$ does not intersect M on the mean convex side of u_1. Letting D be such a small δ_1 proves item *1*.

Let $N(M,\delta)$ be the one-sided regular neighborhood of M in X given by item *1*. For a domain $E \subset M$, let $N(E,D) \subset N(M,D)$ be the associated one-sided regular neighborhood. Note that there exists a constant K such that the area of any compact domain E on M is less than K times the volume of $N(E,D)$. Since the volumes of balls in N of radius 1 are uniformly bounded, the area of M in such balls is also uniformly bounded. This proves item *2*, and thus completes the proof of the theorem.

□

Our proof of the Dynamics Theorem for CMC surfaces in [9] uses the following corollary to Theorem 5.1. For this application, we need to allow the CMC surface M in its statement to be *almost-embedded* in the sense that there exist arbitrarily small C^1 perturbations of M which are embedded; in particular, such surfaces only intersect tangentially. Although we did not state Theorem 5.1 for properly immersed CMC surfaces which are almost-embedded, its proof can be easily modified to this more general situation.

Corollary 5.2 *Suppose X is a complete, simply connected three-manifold with absolute sectional curvature bounded by S_0. Suppose $M \subset X$ is a properly immersed CMC surface which is almost-embedded, has constant mean curvature H_0 and satisfies $|A_M| \leq A_0$ for some $A_0 \geq 0$. Then:*

1. *M has a fixed size regular neighborhood on its mean convex side.*
2. *There exists a $c > 0$ depending on A_0, H_0, S_0 such that the area of M in ambient balls of radius one is at most c.*

6 The existence of CMC surfaces in $\mathcal{M}(\mathbb{R}^3)$ which are not properly embedded.

Proposition 6.1 *There exists a properly embedded, constant mean curvature one, doubly-periodic surface M_n in $\mathbb{R}^3$ which is contained in the slab $\{(\frac{1}{2})^{n+1} < x_3 < (\frac{1}{2})^n\}$. Hence, $M_\infty = \bigcup_{n\in\mathbb{N}} M_n$ is a complete, embedded CMC surface in the slab $\{0 < x_3 < \frac{1}{2}\}$ which is nonproper in $\mathbb{R}^3$.*

Proof. In 1970, Lawson [5] constructed an example of a properly embedded surface $M \subset \mathbb{R}^3$ of constant mean curvature one contained in a horizontal slab. Furthermore, M is invariant under a square lattice $\Lambda_M = \{(ma, na, 0) \mid m, n \in \mathbb{Z}\}$, for some $a > 0$ and the quotient surface M/Λ_M is compact with genus two. Using a modification of Lawson's construction, we will construct a family M_s, $s \in (0,1)$ of properly embedded, constant mean curvature one surfaces in horizontal slabs $S_s \subset \mathbb{R}^3$ which are invariant under square lattices Λ_s and such that M_s/Λ_s has genus two. We note that as in Lawson's examples, each of the surfaces M_s in this family are invariant under reflection in the $(x, y), (x, z), (y, z)$ and $x = y$ planes. We will prove that as s goes to zero, the widths of the slabs S_s converge to zero which will prove Proposition 6.1. □

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