CMC surfaces in locally homogenous three-manifolds

William H. Meeks, III* Giuseppe Tinaglia

April 28, 2007

1 Introduction.

This paper is a preliminary version. Throughout this paper, we let $\mathcal{N}$ denote the set of complete, locally homogeneous[1] three-manifolds; for a given $N \in \mathcal{N}$, we let $\mathcal{M}(N)$ denote the set of complete, embedded, constant mean curvature surfaces in N, and let $\mathcal{M}$ denote the union $\bigcup_{N\in\mathcal{N}} \mathcal{M}(N)$. We frequently restrict our attention to the subset $\mathcal{N}_1 \subset \mathcal{N}$ of complete, locally homogeneous three-manifolds with absolute sectional curvature at most 1 and to the related set $\mathcal{M}_1 = \bigcup_{N\in\mathcal{N}_1} \mathcal{M}(N)$. We will study the geometry and topology of surfaces $M \in \mathcal{M}$. For some of our results we will restrict our attention to the case where M has finite topology[2]. For the sake of simplicity, we will assume that both M and N are connected and orientable. We will call M minimal if its mean curvature is zero and will call M a CMC surface if its mean curvature is a positive constant.

The classical examples of complete, embedded, finite topology minimal surfaces are the plane, the helicoid, and the catenoid, which were proven to be minimal by Meusnier [33] in 1776. In 1841 Delaunay [4] gave analytic descriptions of singly-periodic surfaces $M(t)$, $t \in (0,1]$, of revolution in $\mathbb{R}^3$ with constant mean curvature one. The surface $M(1)$ is the cylinder of radius $\frac{1}{2}$ around the x_1-axis and as $t \to 0$, the surfaces $M(t)$ converge to a periodic chain of unit radius spheres with centers on the x_1-axis. By homothetically scaling the surface $M(t)$ by positive constants λ, one obtains surfaces $\lambda M(t)$ of constant mean curvature $\frac{1}{\lambda}$. As $t \to 0$, there exist numbers $\lambda(t) \to \infty$ so that the surfaces $\lambda(t)M(t)$ converge smoothly on compact sets of $\mathbb{R}^3$ to the catenoid obtained by revolving the curve $c(t) = (t, \cosh t, 0)$ around the x_1-axis. We will call the images of the surfaces $\{\lambda M(t) \mid t \in$

*This material is based upon work for the NSF under Award No. DMS - 0405836. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

[1] A Riemannian manifold N is *locally homogeneous* if given any two points $p, q \in N$, there exists an $\varepsilon > 0$ such that the balls $B_N(p,\varepsilon), B_N(q,\varepsilon)$ are isometric.

[2] M has finite topology means that it is homeomorphic to a compact surface $\widehat{M}$ minus a finite number of points.

$(0,1], \lambda \in (0,\infty)\}$ under rigid motions of $\mathbb{R}^3$, Delaunay surfaces. We refer the interested reader to the papers of Hoffman and Meeks [7], Kapouleas [10], Traizet [38, 39] and Weber and Wolf [40] for important methods for constructing finite topology minimal examples in $\mathcal{M}(\mathbb{R}^3)$ and to Kapouleas [9] for the main method for constructing finite topology CMC examples in $\mathcal{M}(\mathbb{R}^3)$.

The first goal of this paper is to extend the present theory for properly embedded, constant mean curvature surfaces in $\mathbb{R}^3$ to the more general complete embedded surfaces in $\mathcal{M}(\mathbb{R}^3)$, which are not necessarily properly embedded; the second goal is to generalize the classical theory for $\mathcal{M}(\mathbb{R}^3)$ to the spaces $\mathcal{M}(N)$, where $N \in \mathcal{N}$. Given Perelman's [35] recent solution of Thurston's Geometrization Conjecture and previous deep applications of minimal surfaces by Gabai [5, 6], Meeks and Scott [27], Meeks and Yau [31, 32] and Scott [37] to classifying certain compact, three-manifolds and to understanding their diffeomorphism groups, it is reasonable to expect that the results presented in this paper might also prove useful in classifying compact three-manifolds and in solving related topological problems.

Our first main theorem describes a fundamental curvature estimate for compact, embedded CMC disks which lie in some $N \in \mathcal{N}_1$. This curvature estimate plays a central role in our theory of CMC surfaces in locally homogeneous three-manifolds. A remarkable property of this estimate is that it does not depend on an upper bound of the mean curvature H_D of the CMC disk D. We recall that the norm of the second fundamental form of a surface $M \subset N$ is $|A_M| = \sqrt{\lambda_1^2+\lambda_2^2}$, where λ_1 and λ_2 are the principal curvatures of M.

Theorem 1.1 (Curvature estimate for CMC disks) *There exists a constant*[3] $\mathbb{K}$ *such that if D is an embedded, compact CMC disk in $N \in \mathcal{N}_1$, then for all $p \in D$,*

$$|A_D|(p) \leq \frac{\mathbb{K}}{\min\{1, H_D)\} \cdot \min\{1, d(p,\partial D)\}},$$

where $d(p,\partial D)$ is the intrinsic distance from p to the boundary of D.

The following definitions are helpful in describing applications of Theorem 1.1.

Definition 1.2 A complete Riemannian surface M is *ε-contractible* if for some $\varepsilon > 0$, every metric ball $B_M(p,\varepsilon)$ is contained in the interior of a simply connected domain in M.

Definition 1.3 An immersed surface $f(\Sigma)$ in a three-manifold X with related immersion $f\colon \Sigma \to X$ is *δ-embedded* if for some $\delta > 0$, f restricted to every metric ball $B_\Sigma(p,\delta)$ is injective.

[3]The universal constant $\mathbb{K}$ must be at least $\sqrt{2}$, as can be seen from consideration of intrinsic disks $D(r)$ of radius $r < 1$ on the unit two sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. We conjecture that Theorem 1.1 holds with $\mathbb{K} = \sqrt{2}$.

The next corollary is an consequence of our curvature estimate for CMC disks.

Corollary 1.4 *If $M \in \mathcal{M}_1$ is a CMC surface which is ε-contractible for some $\varepsilon \in (0,1]$, then:*

$$|A_M| \leq \frac{\mathbb{K}}{\varepsilon \cdot \min\{1, H_M\}}.$$

Furthermore, for any complete, immersed CMC surface $f(\Sigma)$ in a simply connected $N \in \mathcal{N}$, the following statements are equivalent:

1. *$f(\Sigma)$ has bounded second fundamental form.*
2. *For some $\delta > 0$, $f(\Sigma)$ is δ-embedded and Σ has positive injectivity radius.*
3. *For some $\delta > 0$, Σ is δ-contractible and $f(\Sigma)$ is δ-embedded.*

We will prove that the lift of each $M \in \mathcal{M}$ of finite topology to the universal cover of its ambient space is ε-contractible for some $\varepsilon > 0$ (see the proof of Theorem 1.5 in section 5). Thus, Corollary 1.4 implies our first bounded curvature theorem.

Theorem 1.5 (Bounded curvature theorem for finite topology CMC surfaces) *Finite topology CMC surfaces in $\mathcal{M}$ have bounded second fundamental forms.*

We remind the reader that the Gauss equation implies that a constant mean curvature surface in a locally homogeneous three-manifold has bounded second fundamental form (an extrinsic property) if and only if it has bounded Gaussian curvature (an intrinsic property). Hence, the above theorem could be restated as: *Finite topology CMC surfaces in $\mathcal{M}$ have bounded Gaussian curvature.*

A recent theorem of Meeks, Perez and Ros [20] states that the closure of a finite topology minimal surface $M \in \mathcal{M}$ in its ambient space N has the structure of a minimal lamination of N. This lamination closure property is equivalent to the property that M has locally bounded second fundamental form A_M, in the sense that for any ball $B \subset N$, $|A_{M \cap B}|$ is bounded. Theorem 1.5 generalizes this locally bounded curvature property for finite topology minimal $M \in \mathcal{M}$ to a global curvature bound when the surface is CMC. Since there exist noncompact, simply connected minimal surfaces in $\mathcal{M}(\mathbb{H}^3)$ with unbounded second fundamental form, the CMC hypothesis in Theorem 1.5 and in Theorem 1.6 below is a necessary one for obtaining the global curvature bounds described in these theorems; here $\mathbb{H}^3$ is hyperbolic three-space.

Recall that a surface in a three-manifold is incompressible if its inclusion map induces an injective map from the fundamental group of the surface to the fundamental group of the three-manifold. Since a lift of an incompressible surface in a three-manifold to the universal cover of the three-manifold is a simply connected surface, Theorem 1.1 implies our next bounded curvature theorem.

Theorem 1.6 (Bounded curvature theorem for incompressible CMC surfaces) *Let $M \in \mathcal{M}_1$ be a non-spherical, incompressible CMC surface and let $\mathbb{S} \in \mathcal{M}_1$ be a CMC sphere. Then:*

$$|A_M| \leq \frac{\mathbb{K}}{\min\{1, H_M\}} \quad \text{and} \quad |A_{\mathbb{S}}| \leq \frac{\max\{\mathbb{K}, \frac{\mathbb{K}}{r_{\mathbb{S}}}\}}{\min\{1, H_{\mathbb{S}}\}},$$

where $r_{\mathbb{S}}$ is the radius[4] *of $\mathbb{S}$ and $\mathbb{K}$ is the universal constant given in Theorem 1.1.*

An important consequence of the above theorem and the two inequalities $\mathbb{K} \geq \sqrt{2}$ and $\sqrt{2}H_M \leq |A_M|$ is the following: *Every non-spherical, incompressible CMC surface $M \in \mathcal{M}_1$ satisfies $H_M \leq \frac{\mathbb{K}}{\sqrt{2}}$.* This inequality means that the constant mean curvatures of all non-spherical incompressible surfaces in $\mathcal{M}_1$, have a universal upper bound. It turns out that the same proof that leads to the above theorem also shows that if $N \in \mathcal{N}_1$ is simply connected and $M \in \mathcal{M}(N)$ has finite topology and one end, then outside of a compact subset of M, $|A_M| \leq \frac{\mathbb{K}}{\min\{1,H_M\}}$, and so, $H_M \leq \frac{\mathbb{K}}{\sqrt{2}}$ in this case as well. Since the homothety of any surface $M \in \mathcal{M}(\mathbb{R}^3)$ by a small $\lambda > 0$ produces a new surface $\lambda M \in \mathcal{M}(\mathbb{R}^3)$ with the same topology and with the larger mean curvature $H_{\lambda M} = \frac{H_M}{\lambda}$, the next corollary to Theorem 1.6 follows from these observations; this corollary generalizes the classical theorem of Meeks [17] who proved it with the additional hypothesis of properness of the surface.

Corollary 1.7 *If $M \in \mathcal{M}(\mathbb{R}^3)$ has finite topology and one end, then M is a minimal surface. In particular, there do not exist complete, embedded, noncompact, simply connected CMC surfaces in $\mathbb{R}^3$.*

Meeks and Rosenberg [24] have shown that surfaces $M \in \mathcal{M}(\mathbb{R}^3)$ of bounded Gaussian curvature are proper[5] and that properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$ have bounded curvature [26]. The main result of Colding and Minicozzi in [1] states that finite topology minimal surfaces in $\mathcal{M}(\mathbb{R}^3)$ are proper. Thus, these results together with Theorem 1.5 imply surfaces of finite topology in $\mathcal{M}(\mathbb{R}^3)$ have bounded curvature and are proper [26]. This properness and bounded curvature result together with previous theorems contained in papers by Collin [2], Korevaar, Kusner, Solomon [12], Lopez and Ros [14], Meeks [16], Meeks, Perez and Rosenberg [23], Meeks and Rosenberg [26] and Schoen [36] lead to the deep classical result described in the next theorem.

Theorem 1.8 *Suppose $M \in \mathcal{M}(\mathbb{R}^3)$ has finite topology. Then:*

1. *M has bounded Gaussian curvature and is properly embedded in $\mathbb{R}^3$.*

[4]The *radius* r_X of a compact metric space (X, d) is $\min_{p \in X} \max_X d(p, \cdot)$.

[5]In fact, they prove that a surface in $\mathcal{M}(\mathbb{R}^3)$ with bounded second fundamental form has uniformly bounded area in ambient balls of radius 1.

2. *If $H_M = 0$, then each annular end of M is asymptotic to the end of a plane, a catenoid or a helicoid. Furthermore, if M has one end and is simply connected, then it is a plane or helicoid and if M has two ends or genus zero, then it is a catenoid.*

3. *If $H_M > 0$, then each annular end of M is asymptotic to the end of a Delaunay surface. Furthermore, M has at least two ends and if it has exactly two ends, then it is a Delaunay surface.*

In [29], we generalize the properness results of Meeks and Rosenberg [24] for CMC surfaces in $\mathcal{M}(\mathbb{R}^3)$ to CMC surfaces in other constant curvature three-manifolds. In particular, we prove that surfaces in $\mathcal{M}(\mathbb{H}^3)$ with constant mean curvature $H \geq 1$ are proper if they have locally bounded second fundamental form. On the other hand, in [29] we prove that for any $H \in [0, 1)$ there exist nonproper, simply connected surfaces in $\mathcal{M}(\mathbb{H}^3)$ of constant mean curvature H. These examples show that the properness result described in item *1* of Theorem 1.9 below is sharp. The other statements in this theorem are consequences of Theorem 1.5 and previous results contained in the papers of Collin, Hauswirth and Rosenberg [3], Hsiang [8] and Korevaar, Kusner, Meeks and Solomon [11].

Theorem 1.9 *Suppose $M \in \mathcal{M}(\mathbb{H}^3)$ has finite topology. Then:*

1. *M has bounded Gaussian curvature and is properly embedded if $H_M \geq 1$.*

2. *If $H_M = 1$, then each annular end of M is asymptotic to a horosphere. Furthermore, if M has one end, then M is a horosphere and if M has two ends, then it is a catenoid cousin*[6]*.*

3. *If $H_M > 1$, then each annular end of M is asymptotic to the end of a Hsiang surface*[7]*. Furthermore, M has at least two ends and if M has two ends, then it is Hsiang surface.*

Recently, Meeks, Perez and Ros [20] proved the following result in the minimal setting, as well as some related results for CMC surfaces in $\mathcal{M}$. Their theorem answered in the negative the long standing question asking if there exist noncompact minimal $M \in \mathcal{M}(\mathbb{S}^3)$ with finite topology, where $\mathbb{S}^3$ is the unit three-sphere in $\mathbb{R}^4$. This question is partly motivated by work of Lawson [13] who proved that for every nonnegative integer k, there exists a compact, embedded minimal surface of genus k in $\mathbb{S}^3$.

[6]The catenoid cousins in $\mathbb{H}^3$ are surfaces of revolution which arise from the images of the universal covers of catenoids in $\mathbb{R}$ via the Lawson correspondence.

[7]The Hsiang surfaces in $\mathbb{H}^3$ are CMC surfaces of revolution similar in nature to the Delaunay surfaces in $\mathbb{R}^3$ and their definition appears in [8]. The catenoid cousins which are mentioned in the previous item are limits of appropriate choices of Hsiang's examples with mean curvatures converging to one.

Theorem 1.10 *Suppose that $N \in \mathcal{N}$ has scalar curvature S_N and $M \in \mathcal{M}(N)$ satisfies $3H_M^2 \geq -S_N$. If M is not totally geodesic with N flat, then the following hold:*

1. *If M has finite topology, then it is properly embedded in N. In particular, if N is compact and M has finite topology, then M is compact.*

2. *If M is a CMC surface with finite topology, then it has uniformly bounded area in balls of radius one in N.*

3. *If N is compact and M has finite genus with a countable number of ends, then M must be compact. In particular, finite genus surfaces in $\mathcal{M}(\mathbb{S}^3)$ with a countable number of ends are compact.*

By the classification of homogeneous three-manifolds of nonnegative scalar curvature [34], if $N \in \mathcal{N}$ has nonnegative scalar curvature and is not flat, then N is compact or it is finitely covered by the product $\mathbb{S}^2(S_N) \times \mathbb{R}$; here $\mathbb{S}^2(t)$ denotes the sphere of constant Gaussian curvature t. This classification result, Theorem 1.10 and results of Meeks and Rosenberg in [25] imply that if N has nonnegative scalar curvature, is not flat and $M \in \mathcal{M}(N)$ has finite topology, then M has uniformly bounded area in balls of radius one in N.

In section 6, we generalize many of the previously stated results to the larger set of complete, embedded CMC surfaces in general Riemannian three-manifolds. We list several of these results in the next theorem and refer the reader to the section 6 for other related theorems.

Theorem 1.11 *Let X be a Riemannian three-manifold. Then:*

1. *Suppose X is homogeneously regular and M is a complete, embedded CMC surface in X. Then M has bounded second fundamental form if and only if M has positive injectivity radius.*

2. *Let S_X denote the infimum of the scalar curvature of X and let M be a complete embedded CMC surface in X with finite topology. Then:*

   (a) *M has locally bounded second fundamental form in X.*

   (b) *If $3H_M^2 > -S_X$, then M is properly embedded in X. In particular, if $3H_M^2 > -S_X$ and X is compact, then M is compact.*

   (c) *If X is the three-sphere equipped with a metric of nonnegative scalar curvature, then any complete, embedded CMC surface in X of finite genus and a countable number of ends is compact.*

Motivated by our properness results for CMC surfaces of finite topology in $\mathcal{M}(\mathbb{R}^3)$ and the similar recent properness result in the minimal case by Colding and Minicozzi [1], it is natural to ask whether every $M \in \mathcal{M}(\mathbb{R}^3)$ is properly embedded. In the minimal case this classical question is referred to as the Calabi-Yau problem for complete, embedded minimal surfaces in $\mathbb{R}^3$. In regards to this question, we prove in [29] that there exist disconnected, complete, embedded CMC surfaces in $\mathbb{R}^3$ which are proper in an open slab but not proper in the entire space. The first author has conjectured that there exist connected, non-proper minimal surfaces in $\mathcal{M}(\mathbb{R}^3)$ (see Conjecture 15.23 in [18]).

This paper is organized as follows. In section 2, we define the notion of a CMC lamination. We prove here our curvature estimate for CMC Disks (Theorem 1.1) in the special case that the related blow-up points in its proof produce limit surfaces which are properly embedded, minimal planar domains in $\mathbb{R}^3$. In section 4, we resolve the other case where the blow-up points produce limit surfaces which are strongly Alexandrov embedded[8], CMC planar domains in $\mathbb{R}^3$. The proof of this improved curvature estimate depends on our Dynamics Theorem for CMC surfaces in $\mathbb{R}^3$, which is proved in section 3 and which represents a fundamental new result in classical surface theory. In section 5, we prove that the lifts of finite topology surfaces in $\mathcal{M}$ to the universal covers of their ambient manifolds are ε-contractible, which implies Theorem 1.5. In section 6, we prove Theorem 1.11 and the related theorem Theorem 1.10 and obtain some of our deep classification results as applications of these theorems.

We refer the reader to the survey [15] by the first author for an in depth discussion of recent theoretical advances and open problems in the theory of complete, embedded CMC surfaces in locally homogenous three-manifolds. Also, the reader can find in [18] a comprehensive survey of recent advances and open problems in the classical theory of complete, embedded minimal surfaces in $\mathbb{R}^3$.

Acknowledgements: The authors would like to thank Uwe Abresch, Tobias Colding, Bill Minicozzi and Joaquin Perez for their careful reading of a preliminary version of this manuscript and for their numerous and detailed suggestions for improving it.

2 A curvature estimate for CMC disks.

In this section we prove a general result concerning the norm of the second fundamental form of embedded CMC disks in a fixed $N \in \mathcal{N}$. In order to help understand these results we make the following definitions.

Definition 2.1 Let M be a complete, embedded surface in a three-manifold N. A point $p \in N$ is a *limit point* of M if there exist points $\{p_n\}_n \subset M$ which diverge as $n \to \infty$ to infinity on M with respect to the intrinsic Riemannian topology on M but converge in N

[8]See Definition 3.1.

to p as $n \to \infty$. Let $L(M)$ denote the set of all limit points of M in N. In particular, $L(M)$ is a closed subset of N and $\overline{M} - M \subset L(M)$, where $\overline{M}$ denotes the closure of M.

Definition 2.2 A *CMC lamination* $\mathcal{L}$ of a three-manifold N is a collection of immersed surfaces $\{L_\alpha\}_{\alpha \in I}$ of constant positive mean curvature H called *leaves* of $\mathcal{L}$ satisfying the following properties.

1. $\mathcal{L} = \bigcup_{\alpha \in I} \{L_\alpha\}$ is a closed subset of N.
2. For each leaf L_α of $\mathcal{L}$, considered to be the zero section Z_α of its tangent bundle TL_α, there exists a one-sided neighborhood $N(Z_\alpha) \subset TL_\alpha$ of Z_α such that:
   (a) the exponential map $\exp\colon N(Z_\alpha) \to N$ is a submersion;
   (b) with respect to the pull-back metric on $N(Z_\alpha)$, $Z_\alpha \subset \partial N(Z_\alpha)$ is mean convex;
   (c) $\exp^{-1}(\mathcal{L}) \cap Z_\alpha$ is a lamination of $N(Z_\alpha)$.

The reader not familiar with the subject of minimal or CMC laminations should think about a geodesic on a Riemannian surface. If the geodesic is complete and embedded (a one-to-one immersion), then its closure is a geodesic lamination of the surface. When this geodesic has no accumulation points, then it is proper. Otherwise, there pass complete embedded geodesics through the accumulation points forming the leaves of the geodesic lamination of the surface. The similar result is true for a complete, embedded CMC surface of locally bounded second fundamental form (curvature is bounded in compact extrinsic balls) in a Riemannian three-manifold, i.e. the closure of a complete embedded CMC surface of locally bounded second fundamental form has the structure of a CMC lamination. The proof of this elementary fact is straightforward, e.g. see [26] for the proof in the minimal case.

The main goal of this section is to prove the following special case of Theorem 1.1.

Lemma 2.3 *Let $N \in \mathcal{N}$. There exists a constant K depending on N such that if D is an embedded, compact CMC disk in N with $H_D = 1$, then for all $p \in D$,*

$$|A_D|(p) \leq \frac{K}{\min\{1, d(p, \partial D)\}},$$

where $d(p, \partial D)$ is the distance from p to the boundary of D.

Proof. Arguing by contradiction, suppose that Theorem 2.3 fails. In this case, there exists an $N \in \mathcal{N}$, a sequence of compact embedded CMC disks $D(n) \subset N$ with mean curvature 1, and points $q_n \in D(n)$ such that the one has the following estimate:

$$\frac{|A_{D(n)}|(q_n)}{\min\{1, d(q_n, \partial D(n))\}} \geq n.$$

Since the disks $D(n)$ are simply connected, we may lift these disks to the universal cover $\widetilde{N}$ of N, and work in $\widetilde{N}$ instead of N. Hence, we will assume that N is simply connected. Since N is now assumed to be simply connected and has sectional curvature bounded from above, N has positive injectivity radius δ for some $\delta > 0$.

By the local picture theorem on the scale of curvature (Theorem 7.1 and Remark 7.2 in [20]), after replacing by a subsequence, there exist positive numbers $\varepsilon_n \to 0$ and $\lambda_n \to \infty$ such that the component M_n of $M \bigcap B_N(p_n, \varepsilon_n)$ containing p_n satisfies the following properties:

1. After composing by fixed isometry $i_n \colon N \to N$, $p_n = p$ for some $p \in N$.
2. M_n is compact with $\partial M_n \subset \partial B_N(p, \varepsilon_n)$.
3. $\lim_{n\to\infty} \lambda_n \varepsilon_n = \infty$.
4. The scaled surfaces $M(n) = \lambda_n M_n \subset \lambda_n B_N(p, \varepsilon_n)$ have second fundamental forms satisfying:
   (a) $|A_{M(n)}| \leq 1 + \frac{1}{n}$.
   (b) $|A_{M(n)}|(p) = 1$.
5. After picking an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pN, consider $T_pN = \mathbb{R}^3$. Assume that the restriction of the exponential map $\exp \colon T_pN = \mathbb{R}^3 \to N$ to the ball $\mathbb{B}(\varepsilon_n) \subset \mathbb{R}^3$, which maps onto $B_N(p, \varepsilon_n)$, is a diffeomorphism. The surfaces $M(n)$ in the expanding balls $\lambda_n B(p, \varepsilon_n)$, when considered to be balls of radius $\lambda_n \varepsilon_n$ centered at the origin in $\mathbb{R}^3$, converge smoothly with multiplicity one on compact subsets of $\mathbb{R}^3$ to a properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$ passing through the origin $\vec{0}$ and $|A_{M_\infty}| \leq 1$ with $|A_{M_\infty}|(\vec{0}) = 1$.

Since the convergence of the $M(n) \subset \lambda_n N$ to $M_\infty \subset \mathbb{R}^3$ is of multiplicity one, M_∞ is properly embedded and the $M(n)$ are planar domains, then a standard lifting argument of curves on M_∞ to curves on the approximating $M(n)$, implies M_∞ is a planar domain. By Theorem 1 in [22], M_∞ is a catenoid, M_∞ is a properly embedded planar domain with two limit ends or M_∞ is a helicoid. The remainder of the proof will be a case by case study which will show that each of these three possibilities cannot occur.

Assertion 2.4 *M_∞ is not a catenoid.*

Proof. After an appropriate choice of an orthonormal basis $\{e_1, e_2, e_3\}$ of tangent vectors to N at p, we obtain induced coordinates in $\mathbb{R}^3$ such that M_∞ is a vertical catenoid with waist circle α of radius $\frac{1}{\sqrt{2}}$ contained in the (x_1, x_2)-plane P. Now consider P to be the related plane in the tangent space $T_p(\lambda_n N)$ and let P_n be the image of $P \cap \lambda_n \mathbb{B}(4\varepsilon_n)$ in

$\lambda_n N$. Note $P_n \cap M(n) \subset \lambda_n N$ contains a sequence of simple closed curves $\alpha_n \subset M(n)$ which converge smoothly to α in the related coordinates. Let $D_n \subset \lambda_n D(n) \subset \lambda_n N$ denote the compact disks with $\alpha_n = \partial D_n$. The curve α bounds a disk $E \subset P$ and the curves α_n bound nearby disks $E_n \subset P_n \subset \lambda_n N$.

Let S_n be the 2-chain $D_n \cup E_n$, where D_n is oriented by its mean curvature vector and E_n is oriented so that $\partial S_n = 0$. In other words, E_n is oriented so that S_n is an integer 2-cycle. Since $\lambda_n N$ is simply connected, S_n is the boundary of some integer 3-chain $R_n \subset \lambda_n N$, which would be an oriented, connected, piecewise-smooth compact region in $\lambda_n N$ if S_n were an embedded sphere.

Let $e_3 \in \mathbb{R}^3 = T_p(\lambda_n N)$ be the unit tangent vector pointed along the positive x_3-axis in $\mathbb{R}^3$ and let Y_n be a killing field in $\lambda_n N$ with $Y_n(p) = e_3$; choose the Y_n so that they converge to the parallel vector field $\mathbf{E}_3$ on $\mathbb{R}^3$ as $n \to \infty$. From the first variation of area of the flow of Y_n applied to S_n, we have:

$$0 = \delta_{Y_n}(|S_n|) = \int_{S_n} \operatorname{div}(Y_n^T) + \int_{S_n} \operatorname{div}(Y_n^N), \tag{1}$$

where Y_n^T, Y_n^N are the tangential and normal projections of Y_n, respectively.

Applying the divergence theorem to the region R_n, we obtain

$$0 = \delta_{Y_n}(|R_n|) = \int_{R_n} \operatorname{DIV}(Y_n) = \int_{S_n} Y_n \cdot \nu, \tag{2}$$

where ν is the oriented unit normal to ∂R_n and $\cdot$ denotes the Riemannian inner product.

We now calculate that the second term of equation 1 is zero, using equation 2, the fact that Y_n is killing and the fact that the mean curvature $H_{D_n} = \frac{1}{\lambda_n}$; the function H_{E_n} denotes the mean curvature function of the "horizontal" disk E_n.

$$\int_{S_n} \operatorname{div}(Y_n^N) = 2\left(\frac{1}{\lambda_n}\int_{D_n} Y_n \cdot \nu + \int_{E_n} H_{E_n} Y_n \cdot \nu\right) = 2\left(\frac{1}{\lambda_n}\int_{S_n} Y_n \cdot \nu + \int_{E_n} (H_{E_n} - \frac{1}{\lambda_n}) Y_n \cdot \nu\right)$$

$$= 2\int_{E_n} (H_{E_n} - \frac{1}{\lambda_n}) Y_n \cdot \nu$$

Since the function $(H_{E_n} - \frac{1}{\lambda_n})Y \cdot \nu$ is converging to zero on E_n and $\lambda_n \to \infty$ as $n \to \infty$ and the area of E_n is uniformly bounded,

$$\lim_{n\to\infty} \int_{S_n} \operatorname{div}(Y_n^N) = 0. \tag{3}$$

Applying the divergence theorem again, we calculate:

$$\int_{S_n} \operatorname{div}(Y_n^T) = \int_{D_n} \operatorname{div}(Y_n^T) + \int_{E_n} \operatorname{div}(Y_n^T) = \int_{\partial D_n} Y_{D_n}^T \cdot \eta_{\partial D_n} + \int_{\partial E_n} Y_{E_n}^T \cdot \eta_{\partial E_n}.$$

Since Y_n is becoming orthogonal to E_n and of unit length as $n \to \infty$ then $\lim_{n\to\infty} \int_{\partial E_n} Y_{E_n}^T \cdot \eta_{\partial E_n} = 0$. Since portions of D_n near the boundary ∂D_n are converging C^1 to half of the catenoid M_∞ with boundary α, $Y_{D_n}^T$ restricted to ∂D_n is converging to the parallel vector field $\mathbf{E}_3$ on ∂D_n. Hence,

$$\lim_{n\to\infty} \int_{S_n} \mathrm{div}(Y_n^T) = \lim_{n\to\infty} \int_{\partial D_n} Y_{D_n}^T \cdot \eta_{\partial D_n} = \pm\mathrm{length}(\alpha) = \pm\sqrt{2}\pi. \tag{4}$$

Taking the limit as $n \to \infty$ of the equation 1 and plugging in the values from equations 3 and 4, we obtain:

$$0 = \lim_{n\to\infty} \delta_{Y_n}(|S_n|) = \lim_{n\to\infty} \left[\int_{S_n} \mathrm{div}(Y_n^T) + \int_{S_n} \mathrm{div}(Y_n^N) \right] = \pm\sqrt{2}\pi + 0 = \pm\sqrt{2}\pi, \tag{5}$$

which produces a contradiction. This completes the proof of Assertion 2.4. □

Assertion 2.5 *M_∞ does not have two limit ends.*

Proof. All that was really used in the proof of Assertion 2.4 to obtain a contradiction was the existence of a simple closed curve $\alpha \subset M_\infty$ with nonzero vertical flux component and such that α bounds a horizontal disk in $\mathbb{R}^3$. By Theorem**** in [21], after a rotation, such a curve α always exists on a two limit end, genus zero, properly embedded minimal surface $M_\infty \subset \mathbb{R}^3$. Thus, the arguments used to prove Assertion 2.4 also prove that M_∞ cannot have two limit ends. □

To complete the proof of Lemma 2.3, it remains to prove that the possibility M_∞ is a helicoid also does not occur. The proof that this case does not happen is similar to the proofs of the two previous assertions. However, to construct the desired simple closed curves α_n with nonzero flux, we will need to consider certain scales of viewing the disks D_n near p defined in the proof of Assertion 2.4, which are larger that the scale of curvature but still arbitrarily small. □

Assertion 2.6 *M_∞ is not a helicoid*

3 The Dynamics Theorem for CMC surfaces of bounded curvature.

A consequence of the proof of Lemma 2.3 in the previous section is that if D is a compact embedded CMC disk in an $N \in \mathcal{N}_1$, then:

$$|A_D|(p) \leq C \cdot \frac{\max\{1, H_D\}}{\min\{1, H_D)\} \cdot \min\{1, d(p, \partial D)\}},$$

where H_D is the mean curvature of D. The main goal of this section will be to remove the dependence of this curvature estimate on $\max\{1, H_D\}$; see the statement of Theorem 1.1.

In order to prove this better estimate we will need an important dynamics type result for the space $\mathcal{T}(M)$ of certain translational limits of a properly embedded, CMC surface $M \subset \mathbb{R}^3$ of bounded curvature. All of these limit surfaces satisfy the almost-embedded property of Σ described in the next definition.

Definition 3.1 Suppose W is a complete flat three-manifold with boundary $\partial W = \Sigma$ together with an isometric immersion $f\colon W \to \mathbb{R}^3$ such that f restricted to the interior of W is injective. We call the image surface $f(\Sigma)$ a *strongly Alexandrov embedded CMC surface* if $f(\Sigma)$ is a CMC surface and W lies on the mean convex side of Σ.

We note that, by elementary separation properties, any properly embedded CMC surface in $\mathbb{R}^3$ is always strongly Alexandrov embedded. Furthermore, by item *1* of Theorem 3.3 below, any strongly Alexandrov embedded CMC surface in $\mathbb{R}^3$ with bounded Gaussian curvature is properly immersed in $\mathbb{R}^3$.

Definition 3.2 Suppose $M \subset \mathbb{R}^3$ is a strongly Alexandrov embedded CMC surface with bounded second fundamental form.

1. $\mathcal{T}(M)$ is the set of all connected, strongly Alexandrov embedded CMC surfaces $\Sigma \subset \mathbb{R}^3$, which are limits of compact domains $\Delta_n \subset (M - p_n)$ with $\lim_{n\to\infty} |p_n| = \infty$, $\vec{0} \in \Sigma$, and such that the convergence is of class C^2 on compact subsets of $\mathbb{R}^3$. Actually we consider the immersed surfaces in $\mathcal{T}(M)$ to be *pointed* in the sense that if such a surface is not embedded at the origin, then we consider the surface to represent two different surfaces in $\mathcal{T}(M)$ depending on a choice of one of the two preimages of the origin.

2. $\Delta \subset \mathcal{T}(M)$ is called $\mathcal{T}$*-invariant*, if $\Sigma \in \Delta$ implies $\mathcal{T}(\Sigma) \subset \Delta$.

3. A nonempty subset $\Delta \subset \mathcal{T}(M)$ is called a *minimal* $\mathcal{T}$-invariant set, if it is $\mathcal{T}$-invariant and contains no smaller $\mathcal{T}$-invariant subsets; it turns out that a nonempty $\mathcal{T}$-invariant set $\Delta \subset \mathcal{T}(M)$ is a minimal $\mathcal{T}$-invariant set if and only if whenever $\Sigma \in \Delta$, then $\mathcal{T}(\Sigma) = \Delta$.

4. If $\Sigma \in \mathcal{T}(M)$ and Σ lies in a minimal $\mathcal{T}$-invariant subset of $\mathcal{T}(M)$, then Σ is called a *minimal element* of $\mathcal{T}(M)$.

With these definitions in hand, we now state our Dynamics Theorem from [28]; in the statement of this theorem, $\mathbb{B}(R)$ denotes the open ball of radius R centered at the origin in $\mathbb{R}^3$.

Theorem 3.3 (Dynamics Theorem for CMC surfaces in $\mathbb{R}^3$) *Let M be a connected, noncompact, strongly Alexandrov embedded CMC surface with bounded second fundamental form. Then:*

1. *M is properly immersed in $\mathbb{R}^3$. More generally,* $\text{Area}(M \cap \mathbb{B}(R)) \leq cR^3$*, for some constant $c > 0$.*

2. *$\mathcal{T}(M)$ is nonempty.*

3. *$\mathcal{T}(M)$ has a natural metric $d_{\mathcal{T}(M)}$ induced by the Hausdorff distance between compact subsets of $\mathbb{R}^3$. With respect to $d_{\mathcal{T}(M)}$, $\mathcal{T}(M)$ is a compact metric space.*

4. *Every nonempty $\mathcal{T}$-invariant subset of $\mathcal{T}(M)$ contains minimal elements of $\mathcal{T}(M)$. In particular, since $\mathcal{T}(M)$ is itself a nonempty $\mathcal{T}$-invariant set, $\mathcal{T}(M)$ always contains minimal elements.*

5. *A minimal $\mathcal{T}$-invariant set in $\mathcal{T}(M)$ is a compact connected subspace of $\mathcal{T}(M)$.*

6. *If M has finite genus, then every minimal element of $\mathcal{T}(M)$ is a Delaunay surface passing through the origin.*

Proof. Corollary 5.2 in [29] implies that the surfaces $\Sigma \in \mathcal{T}(M)$ have uniform local area estimates, which gives item *1*. The next five items in the above theorem follow from modifications in the proof of the more delicate Dynamics Theorem for properly embedded minimal surfaces in $\mathbb{R}^3$ of Meeks, Perez and Ros [20]; we will now indicate briefly how items *2* and *3* are proved.

The uniform local area and local curvature estimates for M together with standard compactness arguments imply that for any divergent sequence of points $\{p_n\}_n$ in M, a subsequence of the translated surfaces $M - p_n$ converges on compact subsets of $\mathbb{R}^3$ to a strongly Alexandrov embedded CMC surface $\mathbb{M}_\infty$ in $\mathbb{R}^3$. The component M_∞ of $\mathbb{M}_\infty$ passing through the origin is a surface in $\mathcal{T}(M)$, which proves item *2* (if M_∞ is not embedded at the origin, then one needs to make a choice of one of the two pointed components).

Suppose that $\Sigma \in \mathcal{T}(M)$ is embedded at the origin. In this case there exists an $\varepsilon > 0$ depending on the bound of the second fundamental form of M, so that there exists a disk $D_\Sigma \subset (\Sigma \cap \overline{\mathbb{B}}(\varepsilon))$ with $\partial D_\Sigma \subset \partial \mathbb{B}(\varepsilon), \vec{0} \in D_\Sigma$ such that D_Σ is a graph with gradient at most 1 over its projection to the tangent plane $T_{D_\Sigma}(\vec{0}) \subset \mathbb{R}^3$. Given another such $\Sigma' \in \mathcal{T}(M)$, define

$$d_{\mathcal{T}(M)}(\Sigma, \Sigma') = d_{\mathcal{H}}(D_\Sigma, D_{\Sigma'}),$$

where $d_{\mathcal{H}}$ is the Hausdorff distance. If $\vec{0}$ is not a point where Σ is embedded, then consider Σ to represent two different pointed surfaces in $\mathcal{T}(M)$ and one chooses D_Σ to be the disks in $\Sigma \cap \mathbb{B}(\varepsilon)$ containing the chosen point. With this modification, the above metric is well-defined.

Since the surfaces in $\mathcal{T}(M)$ have uniform local area and curvature estimates, standard compactness arguments prove $\mathcal{T}(M)$ is sequentially compact, and so, $\mathcal{T}(M)$ is a compact metric space with respect to the metric $d_{\mathcal{T}(M)}$. This completes the proof of item *3*.

Now we prove item *4* by an application of Zorn's lemma. Suppose $\Delta \subset \mathcal{T}(M)$ is a nonempty $\mathcal{T}$-invariant set and $\Sigma \in \Delta$. Using the definition of $\mathcal{T}$-invariance, it is elementary to prove that $\mathcal{T}(\Sigma)$ is a $\mathcal{T}$-invariant set in Δ which is a closed subset of $\mathcal{T}(M)$; essentially, this is because the set of limit points of a set in a topological space forms a closed set. Consider the set Λ of all nonempty $\mathcal{T}$-invariant subsets of Δ which are closed sets in $\mathcal{T}$, and as we just observed, this collection is nonempty. Observe that Λ has a partial ordering induced by inclusion. We now check that any linearly ordered set in Λ has a lower bound, and then apply Zorn's Lemma to obtain a minimal element of Λ. To do this suppose $\Lambda' \subset \Lambda$ is a nonempty linearly ordered subset and we will prove that the intersection $\bigcap_{\Delta' \in \Lambda'} \Delta'$ is an element of Λ. In our case, this means that we only need to prove that such an intersection is nonempty, because the intersection of closed (resp. $\mathcal{T}$-invariant) sets is closed (resp. $\mathcal{T}$-invariant). This nonempty intersection property follows since each element of Λ is a closed subset of the compact metric space $\mathcal{T}(M)$, and the finite intersection property holds for the collection Λ. By Zorn's lemma, Δ contains a smallest, nonempty, closed $\mathcal{T}$-invariant subset Ω. If Ω' is a nonempty $\mathcal{T}$-invariant subset of Ω, then there exists $\Sigma' \in \Omega'$. By our previous arguments, $\mathcal{T}(\Sigma') \subset \Omega' \subset \Omega$ is a nonempty $\mathcal{T}$-invariant set in Δ which is a closed set in $\mathcal{T}(M)$. Hence, by the minimality property of Ω in Λ, we have $\mathcal{T}(\Sigma') = \Omega' = \Omega$. Thus, Ω is a nonempty, minimal $\mathcal{T}$-invariant subset of Δ, which proves item *4*.

Let $\Delta \subset \mathcal{T}(M)$ be a nonempty, minimal $\mathcal{T}$-invariant set and let $\Sigma \in \Delta$. Since Δ is minimal and $\mathcal{T}(\Sigma)$ is $\mathcal{T}$-invariant, $\mathcal{T}(\Sigma) = \Delta$. Since $\mathcal{T}(\Sigma)$ is closed and $\mathcal{T}(M)$ is compact, then Δ is compact. Note that whenever $W \in \mathcal{T}(M)$, then the path connected set of translates $\text{Trans}(W) = \{W - q \mid q \in W\}$ is a subset of $\mathcal{T}(\Sigma)$. Since Δ is a minimal set, $\Sigma \in \Delta$ implies $\mathcal{T}(\Sigma) = \Delta$, which means $\Sigma \in \mathcal{T}(\Sigma)$. Hence, $\text{Trans}(\Sigma) \subset \mathcal{T}(\Sigma)$ is a path connected subset of $\mathcal{T}(\Sigma)$. By definition of $\mathcal{T}(\Sigma)$ and the metric space structure on $\mathcal{T}(M)$, the closure of $\text{Trans}(\Sigma)$ in $\mathcal{T}(M)$ is precisely $\mathcal{T}(\Sigma)$. Since the closure of a path connected set in a topological space is always connected, we conclude that $\Delta = \mathcal{T}(\Sigma)$ is a connected subspace of $\mathcal{T}(M)$, which completes the proof of item *5*.

Assume now that M has finite genus and we will prove that item *6* of the theorem holds. Since M has finite genus, then the surfaces in $\mathcal{T}(M)$ all have genus zero.

Assertion 3.4 *If $\Sigma \in \mathcal{T}(M)$, then $\mathcal{T}(\Sigma)$ contains an element Σ' with two ends, i.e. an annulus.*

Proof. Since for any sufficiently small $\delta > 0$, the δ-parallel surface to Σ on its mean convex side is properly embedded, we do not lose any generality in the subsequent arguments by assuming Σ is actually properly embedded. Assume now that Σ is properly embedded.

For $R > 0$, let $\Sigma(R)$ be the component of $\Sigma \cap \overline{\mathbb{B}}(R)$ with $\vec{0} \in \Sigma(R)$. We can also assume that $\partial\overline{\mathbb{B}}(R)$ intersects $\Sigma(R)$ transversely for the values of R we are considering. Thus, $\Sigma(R)$ is a smooth, compact subdomain of Σ. We will let $\widehat{\Sigma}(R)$ be the union of $\Sigma(R)$ with the disk components of $\Sigma - \Sigma(R)$. In other words, if a component $\gamma \subset \partial\Sigma(R)$ bounds a disk $D_\gamma \subset \Sigma$, then we glue this disk to $\Sigma(R)$ in the making of $\widehat{\Sigma}(R)$ (see figure 1).

By the definition of the surfaces $\Sigma(R)$ and the fact that Σ is a planar domain, one easily deduces that for $R > R' > 0$, the number of components of $\widehat{\Sigma}(R) - \widehat{\Sigma}(R')$ is equal to the number of boundary curves of $\widehat{\Sigma}(R')$. Furthermore, each of these components has exactly one boundary component in $\partial\widehat{\Sigma}(R')$, at least one boundary curve in $\partial\widehat{\Sigma}(R)$ and is an annulus precisely when it has one boundary component in $\partial\widehat{\Sigma}(R)$. Hence, if no component of $\widehat{\Sigma}(R) - \widehat{\Sigma}(R')$ is an annulus, then the number of boundary components of $\widehat{\Sigma}(R)$ is at least twice the number of boundary components of $\widehat{\Sigma}(R')$.

Claim 3.5 *For each $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $X_k := \widehat{\Sigma}(kn+n) - \widehat{\Sigma}(kn)$ contains a component A_n which is an annulus.*

Proof. Arguing by contradiction, fix an $n \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, X_k does not contain an annulus. Let c_k be the number of components of X_k. By the discussion before the claim, it follows that $c_k \geq 2c_{k-1}$ and so $c_k \geq 2^k$. Notice that there exists an $\varepsilon > 0$ such that for any k, the area of each component of X_k in the ball of radius $kn + n$ is bounded below by ε. Therefore, the area of $X_k \cap \mathbb{B}(kn + n)$ is greater than $2^k\varepsilon$. This contradicts the cubic area growth of Σ given in item *1* of the theorem, which proves the claim. □

For each $n \in \mathbb{N}$, let $k \in \mathbb{N}$ be the integer and A_n be the annulus given in the above claim. Fix a point $p_n \in A_n \cap \partial\mathbb{B}(nk + \frac{1}{2}n)$ and let $E_n = A_n - p_n$. After choosing a subsequence, there exist compact subsets of E_n which converge to a planar domain $E_\infty \in \mathcal{T}(\Sigma)$. We claim that E_∞ is an annulus, which will complete the proof of Assertion 3.4. Arguing by contraction, suppose that E_∞ does not have two ends. By a theorem of Meeks [16], E_∞ is not simply connected[9] and so, E_∞ is a planar domain with at least three ends. In particular, for some R large, $\partial\widehat{E}_\infty(R) = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ where $m \geq 3$.

We will use the following observation which follows from the Alexandrov reflection principle and the height estimate[10] for *CMC* graphs [16]: if D is an embedded compact domain in $\mathbb{R}^3$ with constant mean curvature H and $\partial D \subset \partial\mathbb{B}(r)$, then $D \subset \overline{\mathbb{B}}(r + \frac{2}{H})$. This observation implies that $\widehat{E}_\infty(R) \subset E_\infty(R + \frac{4}{H})$. Without loss of generality, we may

[9]Actually this theorem in [16] is proved for properly embedded *CMC* surfaces but his proof easily generalizes to the case when the surfaces is properly immersed and strongly Alexandrov embedded.

[10]If u is a graph in $\mathbb{R}^3$ with zero boundary values over a domain in $\mathbb{R}^2$ which has nonzero constant mean curvature H, then $\sup |u| \leq \frac{1}{H}$.

assume that there exist components Δ_n of $E_n \cap \mathbb{B}(R + \frac{4}{H})$ which converge smoothly to $E_\infty(R+\frac{4}{H})$ and a collection of curves $\Lambda_n = \{\alpha_1(n), \alpha_2(n), \ldots, \alpha_m(n)\} \subset \Delta_n \cap \partial\mathbb{B}(R)$ such that $\alpha_i(n)$ converges to α_i for $i = 1, \ldots, m$. Notice that for n large, the curves in Λ_n are the boundary of the compact domain $\widehat{E}_n(R) \subset E_n$. Since E_n is an annulus and $m \geq 3$, after possibly reindexing, $\alpha_1(n)$ bounds a disk on E_n, and this disk is contained in $\mathbb{B}(R + \frac{3}{H})$. However, each α_i is part of the boundary of a connected domain $T_i \subset E_\infty(R)$ such that $\partial T_i - \alpha_i$ is nonempty and contained in $\partial\mathbb{B}(R + \frac{4}{H})$. This contradicts the assumption that Δ_n converges smoothly to $E_\infty(R + \frac{4}{H})$. This contradiction proves Assertion 2.5. $\square$

Suppose now that Σ is a minimal element of $\mathcal{T}(M)$ and $\Sigma' \in \mathcal{T}(\Sigma)$ is the annulus given in Assertion 3.4. To complete the proof of Theorem 3.3, we recall that Meeks [16] proved that a properly embedded CMC annulus in $\mathbb{R}^3$ is contained in a fixed size regular neighborhood of some line in $\mathbb{R}^3$; the same proof shows that the strongly Alexandrov embedded surface Σ' which is properly immersed in $\mathbb{R}^3$ must be cylindrically bounded as well. Finally, the results of Korevaar, Kusner and Solomon [12] imply Σ' is a surface of revolution, which means Σ' is a Delaunay surface. In particular, the minimal set $\mathcal{T}(\Sigma)$ contains the Delaunay surface Σ'. Since $\mathcal{T}(\Sigma')$ clearly consists only of certain translates of Σ', then the definition of minimality of $\mathcal{T}(\Sigma)$ implies that $\mathcal{T}(\Sigma') = \mathcal{T}(\Sigma)$ and so every element of $\mathcal{T}(\Sigma)$ is a translate of the Delaunay surface Σ'. In particular, Σ is a Delaunay surface, which completes the proof of Theorem 3.3. $\square$

Remark 3.6 The proof of the Dynamics Theorem for CMC surfaces easily generalizes to show that if a CMC surface $M \in \mathcal{M}(\mathbb{R}^3)$ has bounded curvature and there exist points $p_n \in M$ and numbers $R_n > 0$, such that $R_n \to \infty$ as $n \to \infty$ and the intrinsic balls $B_M(p_n, R_n)$ have uniformly bounded genus, then $\mathcal{T}(M)$ always contains a minimal element which is a Delaunay surface. For example, if the genus of $M \cap \mathbb{B}(R)$ grows sublinearly in R, then $\mathcal{T}(M)$ always contains a Delaunay surface. In [30], we apply these observations to prove that if $M \in \mathcal{M}(\mathbb{R}^3)$ has bounded curvature and, with respect to some point $p \in M$, the genus of $B_M(p, R)$ grows sublinearly in terms of R, then every intrinsic isometry of M extends to an isometry of $\mathbb{R}^3$. In particular, if $M \in \mathcal{M}(\mathbb{R}^3)$ has bounded curvature and finite genus, then its isometry group is induced by ambient isometries.

The first author conjectures that the helicoid is the only surface in $\mathcal{M}(\mathbb{R}^3)$ which admits more that one non-congruent, isometric, constant mean curvature immersion into $\mathbb{R}^3$ with the same constant mean curvature. Since intrinsic isometries of the helicoid extend to ambient isometrics, the second author also makes the following isometry conjecture.

Conjecture 3.7 (Isometry Conjecture) *An intrinsic isometry of a surface in $\mathcal{M}(\mathbb{R}^3)$ extends to an ambient isometry of $\mathbb{R}^3$.*

4 The proof of the improved curvature estimate for CMC disks.

We now complete the proof of Theorem 1.1 which gives our mains curvature estimate for an embedded, compact CMC disk D in a locally homogeneous three-manifold $N \in \mathcal{N}_1$. Since D is simply connected, after lifting D to the universal cover of N, we may assume that N is simply connected. Suppose now that the following curvature estimate fails to hold for such disks in three-manifolds in $\mathcal{N}_1$.

$$|A_D|(p) \leq \frac{\mathbb{K}}{\min\{1, H_D)\} \cdot \min\{1, d(p, \partial D)\}},$$

where $d(p, \partial D)$ is the intrinsic distance from p to the boundary of D and $\mathbb{K}$ is a universal constant.

A similar blow-up argument as the one appearing in the proof of Lemma 2.3 produces a limit surface $M_\infty \subset \mathbb{R}^3$ from arising compact domains on certain embedded CMC disks D_n in three-manifolds in $N_n \in \mathcal{N}$. The surface M_∞ is either a properly immersed embedded minimal planar domain with bounded second fundamental form or a properly, strongly Alexandrov embedded, CMC planar domain in $\mathbb{R}^3$ with bounded second fundamental form. By our previous arguments in the proof of Lemma 2.3, we may assume that M_∞ is a CMC surface. By Theorem 3.3, $\mathcal{T}(M_\infty)$ contains a Delaunay surface, which means that by being more careful in choosing the original blow-up points, we may assume that M_∞ is itself a Delaunay surface. A minimizing geodesic circle α of a Delaunay surface has nonzero CMC flux. This CMC flux arises from the difference of the flux of a parallel unit length Killing field V (pointed along the axis direction of M_∞) across α with $2H_{M_\infty}$ times the area of the planar disk E_α bounded by α. More precisely, if $L(\alpha)$ denotes the length of α, then the CMC flux of M_∞ is:

$$L(\alpha) - 2H_{M_\infty} \cdot \text{Area}(E_\alpha) \neq 0.$$

.

As in the previous case of Lemma 2.3 where M_∞ was minimal (especially see Assertion 2.4), the nonzero property of the flux of M_∞ shows that the related approximating curves $\alpha_n \subset \lambda_n D_n \subset \lambda_n N_n$ bounding related disks $E_{\alpha_n} \subset N_n$ have CMC flux bounded away from zero for n sufficiently large for the related Killing fields Y_n in $\lambda_n N_n$. Since the CMC flux of α_n must be zero, Theorem 1.1 now follows.

5 The proof of the bounded curvature theorem for finite topology CMC surfaces.

In this section we will prove Theorem 1.5, which states that any CMC surface in $\mathcal{M}$ with finite topology has bounded second fundamental form. It suffices to prove that if $M \in \mathcal{M}(N)$ has an annular end F, then the second fundamental form of F is bounded. If the inclusion map $i\colon F \to N$ induces an isomorphism of the fundamental groups, then a lift[11] $\widetilde{F}$ of F to the universal cover $\widetilde{N}$ of N is simply connected and so, as a consequence of our curvature estimate for CMC disks, $\widetilde{F}$, and hence F, has bounded second fundamental form. Thus, we now assume that the induced map $i_*\colon \pi_1(F) = \mathbb{Z} \to \pi_1(N)$ has a kernel. By the loop theorem [31], it follows that $\{e\} = i_*(\pi_1(F)) \subset \pi_1(N)$. Thus, if $\pi\colon \widetilde{N} \to N$ is the universal cover of N, then the lifting lemma implies $i\colon F \to N$ can be factored as $i = \pi \circ \widetilde{i}$. Hence, to complete the proof of the Theorem 1.5, it suffices to prove the next lemma.

Lemma 5.1 *Let $N \in \mathcal{N}$ be simply connected and let $F \subset N$ be a smooth, complete, embedded CMC annulus with boundary a connected compact curve. Then the second fundamental form of F is bounded.*

Proof. As in the proof of Lemma 2.3, there exists a sequence of blow-up points $p_n \in F$ that diverge in F as $n \to \infty$ and that produces a limit M_∞ which is a properly embedded, minimal planar domain in $\mathbb{R}^3$. As in the proof of Lemma 2.3 , M_∞ is a vertical catenoid, a properly embedded, genus-zero minimal surface with two limit ends or a helicoid.

Assertion 5.2 *M_∞ is not a vertical catenoid.*

Proof. Assume that M_∞ is a vertical catenoid and we will obtain a contradiction. We will follow the proofs of Lemma 2.3 and Assertion 2.4 with some minor adaptations.

Our perspective on producing the limit catenoid M_∞ is as follows. Let $p_n, \varepsilon_n, \lambda_n$ be as in the proof of Lemma 2.3. Choose isometries $i_n\colon N \to N$ with $i_n(p_n) = p$, where p is a base point for N. Define $F_n = i_n(F)$ and let $M_n \subset B_N(p, \varepsilon_n) \cap F_n$ be the compact component with $p \in M_n$. As in the Lemma 2.3, after choosing an orthonormal basis for T_pN, we view M_n to lie in the ball $\mathbb{B}(\varepsilon_n) \subset \mathbb{R}^3$ centered at the origin. Then, in this setting the surfaces $M(n) = \lambda_n M_n \subset \lambda_n F_n = F(n) \subset \lambda_n N$ converge in the $\mathbb{R}^3$ coordinates to a vertical catenoid M_∞ in $\mathbb{R}^3$. Note that we consider the surfaces $F(n)$ to lie in the homogeneous three-manifolds $\lambda_n N$. After choosing a subsequence, we may assume that ε_n is decreasing in the index n and that $\lambda_n \geq n\lambda_n$. Thus, for n large, the component $\Sigma(n)$ of $F_n \cap B(i_n(p_{n+1}), \varepsilon_{n+1})$ passing through the point $i_n(p_{n+1})$ is closely approximated by a catenoid of much smaller scale than the almost catenoid $M_n \subset B(p, \varepsilon_n)$. By the

[11]A *lift* $\widetilde{F}$ of F to the universal cover $\pi\colon \widetilde{N} \to N$ is a path component of $\pi^{-1}(F)$ in $\widetilde{N}$.

proof of Lemma 2.3, we may assume that the "waist" circles[12] of these almost catenoids, when considered to lie in $\lambda_n N$, do not bound disks in $F(n)$; hence, these waist circles are homotopically nontrivial on $F(n)$ and bound as annulus $A(n) \subset F(n)$. Let E_n and E_{n+1} denote the corresponding "planar" disks in $\lambda_n N$ bounding the "waist" circles of these almost catenoids and orient them so that the two chain $S_n = A(n) \cup E_n \cup E_{n+1}$ is a 2-cycle bounding an oriented 3-chain R_n in $\lambda_n N$.

By Lemma *** in [19], there exist killing fields Y_n on $\lambda_n N$ satisfying the following properties.

1. $\lim_{n\to\infty} Y_n$ is a parallel killing field on $\mathbb{R}^3$, where we view $\lim_{n\to\infty} \lambda_n N = \mathbb{R}^3$ and p as the origin in $\mathbb{R}^3$.
2. The norm of Y_n on $\lambda_n B_N(p,1) \cup \lambda_n B_N(p_{n+1},1)$ is bounded by 10.
3. The inner product of $Y_n(p)$ with $e_3 \in T_p(\lambda_n N)$ is 1.

The calculations in the proof of Assertion 2.4 imply:

$$\lim_{n\to\infty} \int_{S_n} \mathrm{div}(Y_n^T) = -\lim_{n\to\infty} \int_{S_n} \mathrm{div}(Y_n^N)) = 0.$$

On the other hand, using the fact that the lengths of the "waist" circles of the $\lambda_n \Sigma(n)$ converge to zero as $n \to \infty$, the calculations in the proof of Assertion 2.4 imply:

$$\lim_{n\to\infty} \left| \int_{S_n} \mathrm{div}(Y_n^T) \right| = 2\pi,$$

which gives the desired contradiction. This completes the proof of the assertion. □

So far we have shown that M_∞ is not a catenoid. As in the proof of Assertion 2.4, as modified above, the other possibilities where M_∞ has genus zero and two limit ends or is a helicoid, are also seen to be impossible. This completes the proof of Lemma 5.1. □

Theorem 1.5 now follows from Lemma 5.1.

The arguments in the proof of Theorem 1.5 rather easily generalize to prove the following bounded curvature theorem; we leave the details of the proof to the reader.

Theorem 5.3 *Suppose $N \in \mathcal{N}$ is simply connected and $M \in \mathcal{M}(N)$ is a CMC surface. Suppose M has a finite number of ends and that for some $\varepsilon > 0$, every simple closed geodesic on M with length less than ε separated M. Then M has bounded second fundamental form.*

[12] In order to be more explicit, one can define the "waist" circle to be the nearby simple closed geodesics on the well-formed catenoids.

6 *CMC* surfaces in a Riemannian three-manifold.

Throughout this section, X will denote a Riemannian three-manifold and $\mathcal{M}(X)$ will denote the set of complete, embedded, constant mean curvature surfaces in X. Our first goal is to generalize part of the statement of Corollary 1.4 to this more general setting.

Theorem 6.1 *Suppose X is a simply connected, homogeneously regular three-manifold. For any complete, immersed CMC surface $f(M)$ in X, the following statements are equivalent:*

1. *$f(M)$ has bounded second fundamental form.*

2. *For some $\delta > 0$, $f(M)$ is δ-embedded and has positive injectivity radius.*

Proof. Since X is assumed to be homogeneously regular, then statement *1* implies statement *2*. Assume now that $f(M)$ is δ-embedded and has positive injectivity radius but that $f(M)$ does not have bounded second fundamental form. The proof of Theorem 1.1 and the positive injectivity radius hypothesis imply that, there exist blow-up points $p_n \in M$, which in turn produce a limit surface which is a properly embedded, minimal planar domain M_∞ in $\mathbb{R}^3$. From our analysis of the possibilities for M_n arising in this previous proof, it follows that there exist simple closed geodesics γ_n on M near p_n with lengths converging to zero as $n \to \infty$. Since we are assuming that the injectivity radius of M is positive, we obtain our desired contradiction. □

William H. Meeks, III at bill@math.umass.edu
Mathematics Department, University of Massachusetts, Amherst, MA 01003

Giuseppe Tinaglia gtinagli@nd.edu
Mathematics Department, University of Notre Dame, Notre Dame, IN, 46556-4618

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