ON THE SPECTRUM OF A PARAMETER-DEPENDENT STURM-LIOUVILLE PROBLEM

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Abstract

We study the spectrum of a parameter dependent Sturm-Liouville problem by using the continued fractions, through which necessary and sufficient conditions for eigenvalues are obtained. From these conditions estimates for large eigenvalues depending on the parameter and an asymptotic result for the lowest eigenvalue will follow. Furthermore, the use of the theory of orthogonal polynomials provides upper and lower bounds for the eigenvalues given in terms of the zeros of particular sequences of polynomials.

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1 Introduction

It is well known that the Riemann zeta function (which we will denote by $\zeta = \zeta(s)$) is closely connected to the spectral zeta function of the harmonic oscillator

$$H = -\frac{\partial_x^2}{2} + \frac{x^2}{2}.$$

In fact, if we denote by $\zeta_H = \zeta_H(s)$ the spectral zeta function of H, i.e.

$$\zeta_H(s) = \sum_{n=1}^{+\infty} \frac{1}{\lambda_n^s}, \qquad \lambda_n \text{ eigenvalues of } H, \quad s \in \mathbb{C}, \quad \Re(s) > 1,$$

we have $\zeta_H(s) = (2^s - 1)\zeta(s)$. In this sense we can say that ζ_H is a deformation of ζ.

From here the natural problem of studying another possible deformation of ζ arises, that is to say the spectral ζ-function of the harmonic oscillator, defined on the interval $[-L, L] \subset \mathbb{R}$ with zero Dirichlet conditions, when $L \to +\infty$. The eigenvalue problem of the harmonic oscillator defined on an interval of the real line and with Dirichlet conditions on the boundary has been studied by several authors (see, e.g. [2], [12] and [13]), but it presents relevant difficulties in computations.

The aim of this paper is to study the spectrum of the following operator:

$$P_L : D(P_L) \longrightarrow L^2(-\pi L, \pi L), \quad P_L f = -\frac{1}{2} f'' + V_L f,$$

with

$$V_L(x) = \frac{L^2}{2} \sin^2 \left(\frac{x}{2L} \right), \qquad D(P_L) = H_0^1(-\pi L, \pi L) \cap H^2(-\pi L, \pi L).$$

In particular we will analyse the behaviour of the eigenvalues of P_L when $L \to +\infty$. The study of P_L is related to the aforementioned problems, since V_L tends as $L \to +\infty$ to the harmonic potential, in the sense of tempered distributions.

The spectral zeta is a "spectral observable" which is more regular than the datum of the spectrum itself, for varying L, because it is defined by means of a trace: $\zeta_{P_L}(s) = \operatorname{Tr} P_L^{-s}$, for sufficiently large s. However, the aim of this work is to study in the first place the eigenvalues, in order to control them in as much explicit a fashion as possible. This analysis follows the ideas of [10] (see also [11]).

Notice that the eigenvalue equation of P_L

$$P_L f = \mu f, \qquad 0 \neq f \in D(P_L), \qquad \mu \in \mathbb{C},$$

is similar to the Mathieu equation (see, e.g. [9]) and therefore it presents similar difficulties.

The paper is organized as follows.

In Section 2 we set the spectral problem for P_L on the interval $(-\pi L, \pi L)$. Then we normalize the problem, by removing from $(-\pi L, \pi L)$ the dependence on the parameter L, so that we reduce ourselves to the study of the *semiclassical* problem

$$P(h)f = \mu f, \qquad \mu \in \mathbb{C},$$

with

$$P(h) : H_0^1(-\pi, \pi) \cap H^2(-\pi, \pi) \longrightarrow L^2(-\pi, \pi),$$

$$P(h)f(x) = -\frac{h}{2} f''(x) + \frac{1}{2h} \sin^2 \left(\frac{x}{2} \right) f(x) = \mu f(x), \qquad h := 1/L^2 \to 0^+.$$

Then we expand the eigenfunctions with respect to a carefully chosen orthonormal basis of $L^2(-\pi, \pi)$, getting a three-term recurrence relation for the Fourier coefficients.

Section 3 provides formulas for these Fourier coefficients.

From here we obtain, in Section 4, an equation which involves a particular continued fraction, derived in a natural way from the recurrence relation; this condition characterizes the eigenvalues of $P(h)$ (as zeros of the "determinant" of an infinite size tridiagonal matrix).

In Section 5 we associate, to each given eigenvalue, two sequences converging, one from above and the other from below, to the eigenvalue itself. We obtain some of these results following the ideas used in [9] for studying the eigenvalues of the Mathieu equation; we use in particular the theory of polynomials "with interlaced zeros" (which are essentially orthogonal polynomials).

Moreover, in Section 6, we give estimates for large eigenvalues (large depending on h^{-1}), in other words we study the clustering of the spectrum for high energies.

In the last two sections we study the behaviour, as $h \to 0^+$, of the lowest eigenvalue $\widetilde{\mu}_0(h)$ of the operator defined by $\widetilde{P} := hP$, as a function of h. In Section 7 we show the uniform convergence of the eigenfunction (associated to $\widetilde{\mu}_0(h)$) coefficients, for h in a complex neighbourhood of a fixed h_0.

These results are used, in Section 8, to prove the existence of the limit $\lim\limits_{h\to 0^+} \widetilde{\mu}_0(h)$. This statement could be obtained as a consequence of a general theorem by Helffer and Sjöstrand (see [1], p. 39 and 41), which gives also the asymptotics for low-lying eigenvalues, but it is proved in this simpler setting by following a different approach, using the continued fractions.

2 Necessary conditions for the eigenfunctions

Let $L > 0$ and let P_L be the unbounded operator defined as follows:

$$P_L : D(P_L) \longrightarrow L^2(-\pi L, \pi L), \qquad P_L f = -\frac{1}{2} f'' + V_L f,$$

with

$$V_L(x) = \frac{L^2}{2} \sin^2\left(\frac{x}{2L}\right), \qquad D(P_L) = H_0^1(-\pi L, \pi L) \cap H^2(-\pi L, \pi L).$$

We will study the solutions of the eigenvalue problem related to P_L:

$$P_L f = \mu f, \quad f \in D(P_L), \quad \mu \in \mathbb{C}. \tag{1}$$

In particular we will analyse the eigenvalues' behaviour in the limit $L \to +\infty$.

By a change of variable we remove the parameter L from $(-\pi L, \pi L)$, so that we obtain from (1) a *semiclassical* problem.

Proposition 2.1. *Let $P(h) := P$ be the operator defined by*

$$P : D(P) \longrightarrow L^2(I), \qquad P\psi = -\frac{h}{2}\psi'' + \frac{1}{h} V\psi,$$

with

$$V(t) = \frac{1}{2}\sin^2\left(\frac{t}{2}\right), \qquad D(P) = H_0^1(I) \cap H^2(I), \qquad I := (-\pi, \pi).$$

The eigenvalue problem (1) is equivalent to the following one

$$P\psi = \mu\psi, \qquad \psi \in D(P), \qquad \mu \in \mathbb{C}, \tag{2}$$

upon setting $\psi(t) = \sqrt{L}\, f(Lt)$, $h = 1/L^2$.

We summarize the properties of the spectrum of P in the following (well-known)

Remark 2.2. *The operator P, defined in Proposition 2.1, is selfadjoint and it has a discrete spectrum. In particular the spectrum of P is an unbounded sequence of real numbers $0 < \mu_0 < \mu_1 < \dots$. The eigenvalues of P are all simple.*

We analyse the structure of eigenfunctions of P by using the Fourier series expansion. In particular we want to substitute the Fourier expansion of a generic eigenfunction of P in the eigenvalue equation (2) and then differentiate term by term, getting in this way conditions on the Fourier coefficients of eigenfunctions (as distributions). Notice that, since we are studying a Sturm-Liouville problem, the choice of the Fourier basis in using this procedure is fundamental. In fact, if we chose for instance the classic Fourier basis for $L^2(I)$, i.e. $\{1, \cos(nx), \sin(nx); n \in \mathbb{N}\backslash\{0\}\}$, we would not be able to find all eigenvalues of P (the trouble arises because the eigenfunctions vanish on the boundary of I whereas the $\cos(nx)$ do not).

Since the eigenfunctions of the problem belong to $D(P)$, then a proper basis to be used for their expansion is formed entirely by functions in $D(P)$. We will use, to this purpose, the orthonormal basis

$$B = \left\{ \frac{1}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\, x \right),\ \frac{1}{\sqrt{\pi}} \sin((m+1)x);\quad n, m \in \mathbb{N} \right\}.$$

Each element of this basis is either an odd or an even function.

Thus, since P is self-adjoint with real coefficients, and since it preserves the parity of functions, we can treat separately the odd and the even eigenvalue problems. More precisely we have the following

Remark 2.3. *All eigenfunctions of P are real-valued. Moreover each eigenfunction of P is either an even or an odd fuction.*

We expand the eigenfunctions of P with respect to B. By substituting this expansion in the eigenvalue equation (2), we get a recurrence relation for the Fourier coefficients of the eigenfunctions. In doing this we use the theorem of differentiation term by term, assuming that all the equalities are intended in the distribution sense. In the sequel we will analyse this recurrence relation using the continued fraction theory. This study will provide necessary and sufficient conditions for the eigenvalues of P and at the same time it will show that the Fourier coefficients of the eigenfunctions converge to 0 faster than any negative power of n. This will justify the use of the theorem of differentiation term by term.

Now we state the recurrence relation for the Fourier coefficients of the even eigenfunctions.

Proposition 2.4. *Let $v^+ \in D(P)$ be an even function with Fourier expansion given by*

$$v^+(x) = \sum_{n=0}^{+\infty} \frac{v_n^+}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\, x \right), \qquad v_n^+ = \int_{-\pi}^{\pi} \frac{v^+(x)}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\, x \right) dx. \tag{3}$$

If v^+ is an eigenfunction for P associated to μ, i.e. if

$$P v^+ = \mu v^+, \qquad \text{on } [-\pi, \pi], \qquad v^+(\pm\pi) = 0, \tag{4}$$

then the Fourier coefficients v_n^+ fulfill the following conditions:

$$v_1^+ = \left(h^2 + 1 - 8\mu h\right) v_0^+; \tag{5}$$

$$v_{n+1}^+ = \left((2n+1)^2 h^2 + 2 - 8\mu h\right) v_n^+ - v_{n-1}^+, \qquad n \in \mathbb{N}\backslash\{0\}. \tag{6}$$

Proof. We substitute (3) in (4) and differentiate term by term. □

The analogous conditions for the odd eigenfunctions are given by the following

Proposition 2.5. *Let $v^- \in D(P)$ be an odd function with Fourier series expansion given by*

$$v^- = \sum_{n=0}^{+\infty} \frac{v_n^-}{\sqrt{\pi}} \sin((n+1)x), \qquad v_n^- = \int_{-\pi}^{\pi} \frac{v^-(x)}{\sqrt{\pi}} \sin((n+1)x)dx. \tag{7}$$

If v^- is an eigenfunction for P associated to μ, i.e. if

$$Pv^- = \mu v^-, \quad \text{on } [-\pi,\pi], \quad v^-(\pm\pi) = 0,$$

then the Fourier coefficients v_n^- fulfill the following conditions:

$$v_1^- = \left(4h^2 + 2 - 8\mu h\right) v_0^-; \tag{8}$$

$$v_{n+1}^- = \left(4(n+1)^2 h^2 + 2 - 8\mu h\right) v_n^- - v_{n-1}^-, \qquad n \in \mathbb{N}\backslash\{0\}. \tag{9}$$

By the above propositions, the sequences of the Fourier coefficients of the eigenfunctions, $\{v_n^\pm\}_n$, fulfill recurrence relations of the form

$$g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}. \tag{10}$$

Studying the properties of this type of relation will give information on $\{v_n^\pm\}_n$, and eventually on the eigenvalues of P. To fix notation we state the following

Remark 2.6. *Using the notation of Proposition 2.4 and assuming the same hypotheses let μ be an eigenvalue of P. We set by definition $v_{-1}^+ = 0$ and*

$$\begin{cases} \delta_0^+ = \delta_0^+(\mu) := h^2 + 1 - 8\mu h \\ \delta_n^+ = \delta_n^+(\mu) := (2n+1)^2 h^2 + 2 - 8\mu h, \qquad \forall\, n \in \mathbb{N}\backslash\{0\}. \end{cases} \tag{11}$$

Then the sequence $\{v_n^+\}_{n\geq -1}$ satisfies the following recurrence relation

$$v_{-1}^+ = 0, \qquad v_{n+1}^+ = \delta_n^+ v_n^+ - v_{n-1}^+, \qquad \forall\, n \in \mathbb{N}. \tag{12}$$

Analogously, using the notation fixed in Proposition 2.5 and assuming the same hypotheses, let μ be an eigenvalue of P.

We set by definition $v_{-1}^- = 0$ and

$$\delta_n^- = \delta_n^-(\mu) := 4(n+1)^2 h^2 + 2 - 8\mu h, \qquad \forall\, n \in \mathbb{N}. \tag{13}$$

Then the sequence $\{v_n^-\}_{n\geq -1}$ satisfies the following recurrence relation

$$v_{-1}^- = 0, \qquad v_{n+1}^- = \delta_n^- v_n^- - v_{n-1}^-, \qquad \forall\, n \in \mathbb{N}. \tag{14}$$

3 Formulas for Fourier coefficients

The aim of this section is to provide formulas for solutions of the equation (10), when $\{\vartheta_n\}_n := \{\delta_n^{\pm}\}_n$, $\pm$ respectively.

In the first place we state some general remarks on equation (10), without any conditions on the sequence $\{\vartheta_n\}_n$.

Lemma 3.1. *Let $\{g_n\}_{n\geq -1}$ be different from the 0-sequence, i.e. such that there exists $n_0 \in \mathbb{N}$ with $g_{n_0} \neq 0$. Assume that*

$$g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}. \tag{15}$$

Then the sequence $\{g_n\}_{n\geq -1}$ is not definitely 0 (i.e. there is no n_0 such that $g_n = 0$ for every $n \geq n_0$), in particular $g_n = 0$ implies $g_{n+1} \neq 0$ and $g_{n-1} \neq 0$.

Proof. By contradiction, let $m \in \mathbb{N}$ be such that $g_m \neq 0$ and such that $g_n = 0$ for all $n > m$. From (15), upon setting $n = m + 1$, we get

$$g_{m+2} = \vartheta_{m+1} g_{m+1} - g_m,$$

whence, since $g_{m+2} = g_{m+1} = 0$, we have $g_m = 0$, but this is impossible.

In a similar way it can be proved that if $g_n = 0$ then $g_{n+1} \neq 0$. Indeed, were it not so, we would have, from (15), that $g_{n+2} = g_{n+3} = \cdots = 0$, but this is impossible because we proved that $\{g_n\}_{n\geq -1}$ is not definitely the 0-sequence. Also $g_{n-1} \neq 0$, indeed, were it not so, we would have, from (15), that $g_{n+1} = 0$, which is impossible from what we have just proved. □

Note that Lemma 3.1 can be applied to both recurrence relations (14) and (12). In fact $\{v_n^-\}_n$ or $\{v_n^+\}_n$, being the sequence of Fourier coefficients of an eigenfunction (recall (7), (3)), can never be the 0-sequence.

Hence Lemma 3.1 states that these sequences cannot have two successive terms that are both 0. This implies, in particular, that the eigenfunctions of P cannot be trigonometric polynomials.

Corollary 3.2. *In the hypothesis of Lemma 3.1 if $g_{-1} = 0$ then $g_0 \neq 0$.*

Proof. By contradiction, if $g_0 = 0$ then from (15) we would have $g_n = 0$ for all $n \in \mathbb{N}$, contradicting the hypothesis. □

The following observations, that hold true for generic recurrence equations, are particulary useful for studying the sequences $\{v_n^{\pm}\}_n$. For this reason now we fix the notation with the following

Definition 3.3. *In the sequel the equation*

$$g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N} \tag{16}$$

will denote either equation (14) or equation (12), where we will have

$$\{(g_n, \vartheta_n)\}_n := \{(v_n^{\pm}, \delta_n^{\pm})\}_n, \quad \pm \text{ respectively,}$$

recalling formula (13) for δ_n^- and formula (11) for δ_n^+. We do not assume that μ, appearing in (13) and (11), is an eigenvalue of P. With these assumptions ϑ_n is always a function of the parameter μ.

It is worth recalling that *when μ is an eigenvalue of P* the sequence $\{g_n\}_n$ of Definition 3.3 coincides with the sequence of Fourier coefficients of the eigenfunction associated to μ (by Remark 2.6).

The following remarks define, through $\{g_n\}_n$, another sequence, $\{w_n\}_n$, which fulfills a "normal form" of the recurrence relation. From this relation we can find a formula to determine $\{w_n\}_n$, and consequently $\{g_n\}_n$.

Lemma 3.4. *Let $\{\vartheta_n\}_{n\geq 0}$ be such that $\vartheta_n \neq 0$ for all $n \in \mathbb{N}$. Let $\{g_n\}_{n\geq -1}$ be such that $g_{-1} = 0$. Then $\{g_n\}_{n\geq -1}$ is a solution of (16):*

$$g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}$$

if and only if $\{w_n\}_{n\geq -1}$ is a solution of

$$w_{n+1} = w_n - \alpha_{n-1} w_{n-1}, \qquad n \in \mathbb{N}, \tag{17}$$

with

$$\begin{cases} w_{-1} = 0 \\ w_0 = g_0 \\ w_n = \dfrac{g_n}{\vartheta_0 \dots \vartheta_{n-1}}, \quad n \in \mathbb{N}\backslash\{0\}, \end{cases} \tag{18}$$

and

$$\begin{cases} \alpha_{-1} = 1 \\ \alpha_n = \dfrac{1}{\vartheta_n \vartheta_{n+1}}, \; n \in \mathbb{N}. \end{cases} \tag{19}$$

Proof. The assertion follows immediatly from (18) and (19), upon dividing (16) by $\vartheta_0 \dots \vartheta_n$. □

In other words, Lemma 3.4 states that we can relate the solutions of equations (16) and (17) if the coefficients ϑ_n are all different from 0. In this hypotesis we can obtain $\{g_n\}_n$ from the values of $\{w_n\}_n$. We will see that this can be done also if $\vartheta_n(\mu) = 0$ for some $n \in \mathbb{N}$.

It is now convenient to assume that all sequences we will consider from now on take values in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 3.5. *Given $\{\alpha_N\}_{N\in\mathbb{N}} \subseteq \widehat{\mathbb{C}}$, we denote by $\{[\alpha_0, \dots, \alpha_j]\}_{j\in\mathbb{N}}$ the sequence defined by recurrence as*

$$\begin{cases} [\alpha_0] = 1 - \alpha_0 \\ [\alpha_0, \dots, \alpha_n] = 1 - \dfrac{\alpha_n}{[\alpha_0, \dots, \alpha_{n-1}]}, \qquad \forall\; n \in \mathbb{N}\backslash\{0\}, \end{cases}$$

where we set, by convention, $1/0 = \infty$ and $1/\infty = 0$.

The following Proposition provides a formula that gives the w_n depending on the coefficients α_n in (17) (for the detailed proof see [10], p. 570).

Proposition 3.6. *Let $\{w_n\}_{n\geq -1}$ and $\{\alpha_n\}_{n\geq -1}$ be two sequences such that $w_{-1} = 0$ and $\alpha_{-1} = 1$. We assume that $\{w_n\}_n$ fulfills the recurrence equation*

$$w_{n+1} = w_n - \alpha_{n-1} w_{n-1}, \qquad n \in \mathbb{N}.$$

Moreover, put $z_n = [\alpha_0, \dots, \alpha_n]$ for every $n \in \mathbb{N}$ and let $\{z_n^*\}_{n\geq 0}$ be defined by

$$z_j^* = \begin{cases} z_j & \text{if} \quad z_j \neq 0, \infty \\ -\alpha_{j+1} & \text{if} \quad z_j = 0 \\ 1 & \text{if} \quad z_j = \infty. \end{cases} \tag{20}$$

Then we have

$$w_0 = w_1, \qquad w_N = \begin{cases} z_0^* z_1^* \dots z_{N-2}^* w_0 & \text{if} \quad z_{N-2} \neq 0, \\ 0 & \text{if} \quad z_{N-2} = 0, \end{cases} \qquad N \geq 2. \tag{21}$$

Notice that $z_{N-2} = 0$ implies that $w_N = 0$, so that, for notational simplicity, we will write (21) as

$$w_0 = w_1, \qquad w_N = z_0^* z_1^* \dots z_{N-2}^* w_0, \qquad N \geq 2, \tag{22}$$

using (20) for the z_j^* when $0 \leq j < N-2$, and with the convention of setting, in (20), $z_{N-2}^* = 0$ if $z_{N-2} = 0$, for the last index $j = N-2$.

Proof (sketch). We consider the w_n as determinants of proper tridiagonal matrices (depending on coefficients α_n). By triangularizing these matrices we obtain essentially z_n as diagonal elements. □

From Proposition 3.6 we get a formula for coefficients g_n. In particular we have the following

Lemma 3.7. *Following the notation fixed in Definition 3.3, Lemma 3.4 and Proposition 3.6 assume that $\vartheta_n \neq 0$, for all $n \in \mathbb{N}$. Then the solution $\{g_n\}_{n\geq -1}$ of equation (16):*

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}$$

satisfies

$$\begin{cases} g_1 = \vartheta_0 g_0 \\ g_n = \vartheta_0 \dots \vartheta_{n-1} z_0^* \dots z_{n-2}^* \; g_0, \qquad \forall \; n \geq 2, \end{cases} \tag{23}$$

with the convention, given in Proposition 3.6, that if $z_{n-2} = 0$ then $g_n = 0$.

Proof. Equation (16) fulfills the hypothesis of Lemma 3.4 and therefore can be related to equation (17) through the relations (18). From Proposition 3.6 we prove the assertion just by substituting (18) in (22). □

Using Lemma 3.7 we will be able to study the behaviour of the Fourier coefficients, g_n, of the eigenfunctions as $n \to +\infty$. Moreover we will obtain as a consequence a necessary and sufficient condition for the eigenvalues. Before doing this, we give results analogous to Lemma 3.7 also in the case there exists $n_0 \in \mathbb{N}$ such that $\vartheta_{n_0} = 0$. Since ϑ_n actually depends linearly on μ we will have $\vartheta_{n_0}(\mu) = 0$ for particular values of μ. More precisely, by recalling Definition 3.3, (11) and (13) we have the following

Remark 3.8. *Let the sequence $\{\vartheta_n\}_n = \{\vartheta_n(\mu)\}_n$ be defined either by*

$$\vartheta_n := \delta_n^- = 4(n+1)^2 h^2 + 2 - 8\mu h, \qquad \forall \; n \in \mathbb{N},$$

or by

$$\vartheta_n := \delta_n^+ = \begin{cases} h^2 + 1 - 8\mu h, & \text{if } n = 0 \\ (2n+1)^2 h^2 + 2 - 8\mu h, & \forall\, n \in \mathbb{N}\backslash\{0\}. \end{cases}$$

We have that ϑ_n depends linearly on μ and therefore, if there exists μ such that $\vartheta_{n_0}(\mu) = 0$ for some n_0, then $\vartheta_n(\mu) \neq 0$ for all $n \neq n_0$.

Assuming that there exists μ such that $\vartheta_{n_0}(\mu) = 0$, for some n_0, we will obtain for g_n a formula similar to (23). In particular, for the first $n_0 + 1$ terms of the sequence we get the following

Lemma 3.9. *Let $\{g_n\}_{n\geq -1}$ be a solution of (16) and suppose there exists $n_0 \in \mathbb{N}$ and $\mu \in \mathbb{R}$ such that $\vartheta_{n_0}(\mu) = 0$. Then, using the notation as in Lemma 3.4, we have*

$$w_{-1} = 0, \qquad w_{n+1} = w_n - \alpha_{n-1} w_{n-1}, \qquad n = 0, 1, \ldots, n_0. \tag{24}$$

Proof. From Remark 3.8 if $n \leq n_0 - 1$ we have $\vartheta_n \neq 0$. To prove (24) it suffices to follow the procedure used in Lemma 3.4. □

From the proof of Proposition 3.6 (see [10], p. 570) it follows that formula (21) can be used as well for a finite number of terms of the sequence. From this we will get at once a formula to compute g_n, with $n = 0, 1, \ldots, n_0$. In particular we get the following

Remark 3.10. *In the hypothesis of Lemma 3.9, from Proposition 3.6 it follows that*

$$w_N = z_0^* \ldots z_{N-2}^* w_0, \qquad N = 0, 1, \ldots, n_0, \tag{25}$$

with the convention that if $z_{N-2} = 0$ then $w_N = 0$. From here, by (18), we get

$$g_N = \vartheta_0 \ldots \vartheta_{N-1} z_0^* \ldots z_{N-2}^*\ g_0, \qquad N = 0, 1, \ldots, n_0, \tag{26}$$

with the convention that if $z_{N-2} = 0$ then $g_N = 0$.

Remark 3.10 gives a formula for the first $n_0 + 1$ terms of $\{g_n\}_n$, in case $\vartheta_{n_0} = 0$. We will show that we can obtain an analogous formula for the remaining terms of the sequence. This will be done by showing that $\{g_n\}_n$ satisfies, from a certain index onward, the hypothesis of Lemma 3.4 and by applying then Lemma 3.7. We will treat separately the cases $n_0 = 0$ and $n_0 \geq 1$.

Proposition 3.11. *Let $\{g_n\}_{n\geq -1}$ be a solution of (16):*

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}$$

and let $\mu \in \mathbb{R}$ such that $\vartheta_0 = \vartheta_0(\mu) = 0$. Then, upon setting

$$\begin{cases} d_{-1} = 0 \\ d_0 = -g_0 \\ d_k = g_{k+2}, \ \forall\, k \in \mathbb{N}\backslash\{0\}, \end{cases} \qquad \eta_k = \vartheta_{n+2}, \quad \forall\, n \in \mathbb{N}, \tag{27}$$

we get

$$d_{n+1} = \eta_n d_n - d_{n-1}, \qquad n \in \mathbb{N}. \tag{28}$$

Proof. By substituting (27) in (16), and by recalling that $\vartheta_0 = 0$, we obtain (28). □

When $n_0 \neq 0$ we have the following

Proposition 3.12. *Let $\{g_n\}_{n\geq -1}$ be a solution of (16):*

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N}.$$

If $\mu \in \mathbb{R}$ and $n_0 \in \mathbb{N}\backslash\{0\}$ are such that $\vartheta_{n_0} = \vartheta_{n_0}(\mu) = 0$, then

a) If $g_{n_0} \neq 0$ and $g_{n_0-1} \neq 0$ we have

$$f_{n+1} = \gamma_n f_n - f_{n-1}, \qquad n \in \mathbb{N}, \tag{29}$$

where

$$\begin{cases} f_{-1} = 0 \\ f_0 = g_{n_0} \\ f_1 = -g_{n_0-1} \\ f_k = g_{n_0+k}, & \forall\ k \geq 2, \end{cases} \qquad \begin{cases} \gamma_0 = -\dfrac{g_{n_0-1}}{g_{n_0}} \\ \gamma_k = \vartheta_{n_0+k}, & \forall\ k \in \mathbb{N}\backslash\{0\}. \end{cases}$$

b) If $g_{n_0} \neq 0$ and $g_{n_0-1} = 0$ we have

$$p_{n+1} = \nu_n p_n - p_{n-1}, \qquad n \in \mathbb{N}, \tag{30}$$

where

$$\begin{cases} p_{-1} = 0 \\ p_k = g_{n_0+2+k}, & \forall\ k \in \mathbb{N}, \end{cases} \qquad \nu_k = \vartheta_{n_0+2+k}, \quad \forall\ k \in \mathbb{N}.$$

c) If $g_{n_0} = 0$ we have

$$q_{n+1} = \rho_n q_n - q_{n-1}, \qquad n \in \mathbb{N}, \tag{31}$$

where

$$\begin{cases} q_{-1} = 0 \\ q_0 = -g_{n_0-1} \\ q_k = g_{n_0+1+k}, & \forall\ k \in \mathbb{N}\backslash\{0\}, \end{cases} \qquad \rho_k = \vartheta_{n_0+1+k}, \quad \forall\ k \in \mathbb{N}.$$

Moreover we have

$$\begin{cases} \gamma_k \neq 0 \\ \nu_k \neq 0 \\ \rho_k \neq 0 \end{cases}, \quad \forall\ k \in \mathbb{N}. \tag{32}$$

Proof. Notice that if $\vartheta_{n_0} = 0$ then, from Remark 3.8, $\vartheta_k \neq 0$ for all $k \neq n_0$, whence (32) is proved, on recalling the definitions of $\{\gamma_n\}_n$, $\{\nu_n\}_n$, $\{\rho_n\}_n$.

a) By substituting

$$\begin{cases} f_n = g_{n_0+n}, \\ \gamma_n = \vartheta_{n_0+n}, \end{cases} \quad \forall\ n \in \mathbb{N},$$

in (16) we easily obtain (29) for $n \geq 1$. If $n = 0$, (29) becomes

$$f_1 = \gamma_0 f_0 - f_{-1}. \tag{33}$$

But since $f_{-1} = 0$, $\gamma_0 = -\frac{g_{n_0-1}}{g_{n_0}}$, $f_0 = g_{n_0}$ and $f_1 = g_{n_0+1}$, substituting these values in (33) gives $g_{n_0+1} = -g_{n_0-1}$. This last equation is satisfied by hypothesis, by substituting $\vartheta_{n_0} = 0$ in

$$g_{n_0+1} = \vartheta_{n_0} g_{n_0} - g_{n_0-1}.$$

We can obtain b) and c) in a similar way. □

To summarize, in case $\mu \in \mathbb{R}$ is such that there exists $n_0 \in \mathbb{N}$ with $\vartheta_{n_0}(\mu) = 0$ we can use Remark 3.10 to compute the first $n_0 + 1$ terms of the sequence $\{g_n\}_n$. Moreover, from Propositions 3.11 and 3.12, we can get a formula for the remaining terms of $\{g_n\}_n$. In fact we can apply Lemma 3.7 to relations (28), (29), (30), (31) obtaining from (23) the desired formula.

4 Continued fractions and charachterization of the eigenvalues

We will now use the theory of continued fractions in order to study the convergence of the coefficients of the eigenfunctions of P. To this purpose we recall several definitions and a classical result on 1-periodic continued fractions (see [7] pp. 7, 8, 9, 59, 103, 150). We recall that the sequences used in these arguments take value in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 4.1. *A* ***continued fraction*** *is an ordered pair*

$$((\{a_n\}_n, \{b_n\}_n), \{f_n\}_n),$$

where the sequences $\{a_n\}_n$, $\{b_n\}_n \subseteq \mathbb{C}$ *and* $\{f_n\}_n \subseteq \widehat{\mathbb{C}}$ *is given by*

$$f_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\ddots + \cfrac{a_n}{b_n}}}.$$

We will call f_n *the* ***n-th approximant*** *of the continued fraction. Besides, we say that two continued fractions are* ***equivalent*** *if they have the same sequence of approximants.*

Moreover, setting $f_n = \frac{A_n}{B_n}$, *we call* A_n *and* B_n *the* ***n-th canonical numerator*** *and* ***denominator****, respectively.*

If

$$\lim_{n\to+\infty} f_n = f \in \widehat{\mathbb{C}}$$

we will write

$$f = b_0 + K_{n=1}^{+\infty}(a_n/b_n).$$

We state some properties on the sequences $\{A_n\}_n$ and $\{B_n\}_n$ which will be useful in the sequel.

Remark 4.2. *Let* $f = b_0 + K_{n=1}^{+\infty}(a_n/b_n)$ *be a continued fraction and let* $\{A_n\}_n$ *and* $\{B_n\}_n$ *be the sequences of canonical numerators and denominators, respectively. If we set* $A_{-1} = 1,\ \ A_0 = b_0,\ \ B_{-1} = 0,\ \ B_0 = 1$, *then we have:*

$$\begin{cases} A_{n+1} = b_{n+1}A_n + a_nA_{n-1}, \\ B_{n+1} = b_{n+1}B_n + a_nB_{n-1}, \end{cases} \qquad n \in \mathbb{N}, \tag{34}$$

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}\prod_{k=1}^{n} a_k, \qquad n > 1. \tag{35}$$

Notice that equations (34) provide a natural correspondence between a continued fraction and the recurrence relation satisfied by the sequence of its canonical numerators and denominators.

Following this idea we will apply the following definitions to study the recurrence relations of Fourier coefficients of eigenfunctions of P, found in the previous section. In particular, as we will see in detail, by showing that $\{z_n\}_n$ is a tail sequence for the continued fraction $K_{n=1}^{+\infty}(-\alpha_n/-1)$, (recall the notation fixed in Lemma 3.4 and in Proposition 3.6) we will obtain an equation, involving this continued fraction, which is a necessary and sufficient condition for μ to be an eigenvalue of P.

Definition 4.3. *We say that a sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \widehat{\mathbb{C}}$ is a **tail sequence** for the continued fraction $b_0 + K_{n=1}^{+\infty}(a_n/b_n)$ if*

$$t_{n-1} = \frac{a_n}{b_n + t_n}, \qquad n = 1, 2, 3, \ldots.$$

We can associate to each term of a tail sequence a Möbius transformation, in a natural way, so obtaining a sequence of Möbius transformations. Studying the limit transformation of this sequence will give us particular properties of the continued fraction. An important result is obtained if this limit Möbius transformation is loxodromic.

Definition 4.4. *Let*

$$t : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}, \quad w \longmapsto t(w) = \frac{aw + b}{cw + d},$$

*with $ad - bc \neq 0$, be a Möbius transformation. Let x and y be two fixed points for t, that is $\lim_{n\to+\infty} t^n(x) = x$ and $\lim_{n\to+\infty} t^n(y) = y$. Then t is said to be **loxodromic** if $x \neq y$ and*

$$\begin{cases} |cx + d| > |cy + d|, & \text{if } c \neq 0, \\ |a| \neq |d|, & \text{if } c = 0. \end{cases}$$

Definition 4.5. *Let $K_{n=1}^{+\infty}(a_n/b_n)$ be a limit 1-periodic continued fraction, i.e. such that the following limits exist*

$$\lim_{n\to+\infty} a_n = a^*, \qquad \lim_{n\to+\infty} b_n = b^*,$$

*with $a^*, b^* \in \widehat{\mathbb{C}}$. Then $K_{n=1}^{+\infty}(a_n/b_n)$ is said to be **of loxodromic type** if $a^* \in \mathbb{C}$, $b^* \in \mathbb{C}$ and if the following implications hold:*

a) if $a^* \neq 0$ then $T(w) := \frac{a^*}{b^*+w}$ is loxodromic as a Möbius transformation;

*b) if $a^* = 0$ then $b^* \neq 0$. In this last case T is a singular transformation, with $T(w) = 0$ for all $w \neq b^*$. We say that $x = 0$ is the **attractive fixed point** of T and $y = -b^*$ is the **repulsive fixed point** of T.*

We state a very important property of tail sequences of limit 1-periodic continued fractions of loxodromic type (see [7], p. 151).

Theorem 4.6. *Let $K_{n=1}^{+\infty}(a_n/b_n)$ be a limit 1-periodic continued fraction of loxodromic type, where T has attractive fixed point x and repulsive fixed point* *y. Then* $K_{n=1}^{+\infty}(a_n/b_n)$ *converges to a value* $f \in \widehat{\mathbb{C}}$*. Moreover, for every tail sequence* $\{z_n\}_n$*, we have*

$$\lim_{n\to+\infty} z_n = \begin{cases} x & \text{if } z_0 = f \\ y & \text{if } z_0 \neq f. \end{cases} \tag{36}$$

These results will now be used to analyse the convergence of the coefficients of eigenfunctions of P and, moreover, this will provide the necessary and sufficient condition on eigenvalues of P.

Lemma 4.7. *Using the notation of Proposition 3.6 the sequence* $\{z_n\}_n$ *is a tail sequence for the continued fraction* $K_{n=1}^{+\infty}(-\alpha_n/-1)$.

Proof. By definition (see Definition 3.5) we have

$$z_n = 1 - \frac{\alpha_n}{z_{n-1}}, \qquad \forall\, n \in \mathbb{N}\backslash\{0\},$$

which, recalling Definition 4.3, proves the claim. □

Thus, since $K_{n=1}^{+\infty}(-\alpha_n/-1)$ *is limit 1-periodic of loxodromic type*, we can use Theorem 4.6 to have information on $\lim\limits_{n\to+\infty} z_n$.

Proposition 4.8. *Using the notation fixed in Lemma 3.4, let* $z_n = [\alpha_0, \dots, \alpha_n]$, $n \in \mathbb{N}$, *(see Definition 3.5) then*

$$\lim_{n\to+\infty} z_n = \begin{cases} 0 & \text{if} \quad z_0 = f = K_{n=1}^{+\infty}(-\alpha_n/-1) \\ 1 & \text{if} \quad z_0 \neq f. \end{cases}$$

Proof. By Definition 3.3 and recalling (19) we have that $K_{n=1}^{+\infty}(-\alpha_n/-1)$ is limit 1-periodic of loxodromic type. In fact, for every fixed μ we have

$$\lim_{n\to+\infty} \alpha_n = \lim_{n\to+\infty} \frac{1}{\delta_n^- \delta_{n+1}^-} = \lim_{n\to+\infty} \frac{1}{\delta_n^+ \delta_{n+1}^+} = 0.$$

Besides, following the notation of Definition 4.4, we have, in this case $b^* = -1 \neq 0$. Moreover, by Lemma 4.7, z_n is a tail sequence for $K_{n=1}^{+\infty}(-\alpha_n/-1)$. From Theorem 4.6 we obtain the assertion. □

Now we want to prove that all values of μ such that $\lim\limits_{n\to+\infty} z_n = 0$, and only those values, are related, through the recurrence relations, to the Fourier coefficients of the eigenfunctions associated to μ. In the first place we suppose that $\vartheta_n = \vartheta_n(\mu) \neq 0$ for every $n \in \mathbb{N}$, analysing the case $\vartheta_m = 0$ for some m afterwards.

Theorem 4.9. *Using the notation of Proposition 3.6 and of Lemma 3.4, assume that* $\vartheta_n = \vartheta_n(\mu) \neq 0$ *for all* $n \in \mathbb{N}$*. Then*

a) if μ *is such that* $z_0 = K_{n=1}^{+\infty}(-\alpha_n/-1)$ *i.e.*

$$1 - \frac{1}{\vartheta_0\vartheta_1} = K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\vartheta_n\vartheta_{n+1}}}{-1}\right) \tag{37}$$

then μ *is an eigenvalue of* P *and the* g_n *in the recurrence relation (16)*

$$g_{-1} = 0, \qquad g_{n+1} = \vartheta_n g_n - g_{n-1}, \qquad n \in \mathbb{N},$$

are the Fourier coefficients of an eigenfunction associated to μ; *moreover* $g_n \to 0$, *as* $n \to +\infty$, *faster than any negative power of* n;

b) if μ is such that $z_0 \neq K_{n=1}^{+\infty}(-\alpha_n/-1)$ then the coefficients in (16) do not converge and the function series associated to them does not represent an eigenfunction of P; moreover $|g_n| \to +\infty$, as $n \to +\infty$, faster than any power of n.

Proof. a) If μ is such that $z_0 = K_{n=1}^{+\infty}(-\alpha_n/-1)$ then from Proposition (4.8) we have that $\lim_{n\to+\infty} z_n = 0$. In these hypotheses we will prove that if $\{g_n\}_{n\geq -1}$ is a solution of (16) then $g_n \to 0$, as $n \to +\infty$, faster than any negative power of n. From this, recalling Definition 3.3, we obtain that the series given by

$$v^+ := \sum_{n=0}^{+\infty} v_n^+ \frac{1}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}\, x\right) \tag{38}$$

or by

$$v^- := \sum_{n=1}^{+\infty} v_n^- \frac{1}{\sqrt{\pi}} \sin\left((n+1)\,x\right), \tag{39}$$

converges uniformly to the eigenfunction v^+ or to the eigenfunction v^-. We show now the convergence of the coefficients. Recalling Lemma 3.7 and Proposition 3.6 we have:

$$g_n = \vartheta_0 \ldots \vartheta_{n-1} z_0^* \ldots z_{n-2}^*\, g_0, \qquad n \geq 2,$$

with

$$z_j^* = \begin{cases} z_j & \text{if} \quad z_j \neq 0, \infty, \\ -\alpha_{j+1} & \text{if} \quad z_j = 0, \\ 1 & \text{if} \quad z_j = \infty, \end{cases} \qquad \text{if } j \neq n-2, \tag{40}$$

and with

$$z_{n-2}^* = \begin{cases} z_{n-2} & \text{if} \quad z_{n-2} \neq 0,\ \infty, \\ 0 & \text{if} \quad z_{n-2} = 0, \\ 1 & \text{if} \quad z_{n-2} = \infty. \end{cases} \tag{41}$$

As already noticed $\lim_{n\to+\infty} z_n = 0$. Thus, upon fixing $0 < \varepsilon < \frac{1}{2}$,

$$\exists\ n_0 \in \mathbb{N} \text{ such that } |z_n| \leq \varepsilon,\ \forall\ n \geq n_0. \tag{42}$$

We have, for every $n \geq n_0$, $|1 - z_{n+1}| \geq 1 - \varepsilon$. Since $\{z_n\}_n$ is a tail sequence for $K_{n=1}^{+\infty}(-\alpha_n/-1)$ (see Definition 4.3) we have

$$|z_n| = \frac{\left|\frac{1}{\vartheta_{n+1}\vartheta_{n+2}}\right|}{|1 - z_{n+1}|} \leq \frac{\left|\frac{1}{\vartheta_{n+1}\vartheta_{n+2}}\right|}{1-\varepsilon}. \tag{43}$$

Moreover, for all $n \geq n_0$ we get $z_n \neq 0, \infty$. Indeed, were z_{n_0} to vanish for some n_0 we would have $z_{n_0+1} = 1 - \frac{\alpha_{n_0+1}}{z_{n_0}} = \infty$, which is impossible because of (42). Relation (42) implies also that $z_n \neq \infty$ for every $n \in \mathbb{N}$.

Hence, recalling (40) and (41), if $n \geq n_0$ we have $z_n = z_n^*$ and therefore

$$|g_N| = |g_0|\ \left|\vartheta_0 \ldots \vartheta_{n_0+1} z_0^* \ldots z_{n_0}^* \vartheta_{n_0+2} \ldots \vartheta_{N-1} z_{n_0+1} \ldots z_{N-2}\right|, \quad N \geq 2. \tag{44}$$

By (23) we have

$$|g_{n_0+2}| = |g_0|\ \left|\vartheta_0 \ldots \vartheta_{n_0+1} z_0^* \ldots z_{n_0}^*\right|. \tag{45}$$

From (43), (44) and (45) it follows, by simplifying the common terms, that

$$|g_N| \le |g_{n_0+2}| \, \frac{1}{(1-\varepsilon)^{N-n_0-2} \, |\vartheta_{n_0+3} \ldots \vartheta_N|}. \tag{46}$$

Recall that, by Definition 3.3, $\{\vartheta_n\}_n$ denotes either $\{\delta_n^+\}_n$ or $\{\delta_n^-\}_n$, defined respectively by (11) or (13). Suppose, to fix ideas, that $\{\vartheta_n\}_n := \{\delta_n^+\}_n$ (for $\{\delta_n^-\}_n$ the proof is similar.) We will show that the right-hand side of (46) tends to zero as $N \to +\infty$. Notice that, by hypothesis, we have $\vartheta_n = \delta_n^+ = \delta_n^+(\mu) \neq 0$ for all $n \in \mathbb{N}$, so that (46) makes sense. We write δ_n^+ in the form

$$\delta_n^+ = (2n+1)^2 h^2 \left(1 - \frac{8\mu h - 2}{h^2(2n+1)^2}\right), \quad n = n_0+3, \ldots, N. \tag{47}$$

By substituting (47) in (46) we get, by noting that

$$(2n_0+7)(2n_0+9) \, \ldots \, (2N+1) < 2^{N-n_0-2}(n_0+3)(n_0+4) \, \ldots \, N,$$

$$|g_N| \le \frac{\left((n_0+2)!\right)^2 |g_{n_0+2}|}{\left((1-\varepsilon)h^2 4\right)^{N-n_0-2} \, (N!)^2 \prod_{k=n_0+3}^N \left|1 - \frac{8\mu h-2}{h^2(2k+1)^2}\right|}. \tag{48}$$

Consider the term $\prod_{k=n_0+3}^N \left|1 - \frac{8\mu h-2}{h^2(2k+1)^2}\right|$. By assuming that n_0 is such that

$$\left|\frac{8\mu h - 2}{h^2(2k+1)^2}\right| < 1, \qquad \forall\, k \in \mathbb{N}, \quad k \ge n_0+3, \tag{49}$$

we have

$$\prod_{k=n_0+3}^{N} \left|1 - \frac{8\mu h - 2}{h^2(2k+1)^2}\right| \ge \prod_{k=n_0+3}^{N} \left(1 - \frac{|\,8\mu h - 2\,|}{h^2(2k+1)^2}\right) := D_N. \tag{50}$$

Whence, recalling (48), we get

$$|g_N| \le \frac{\left((n_0+2)!\right)^2 |g_{n_0+2}|}{\left((1-\varepsilon)h^2 4\right)^{N-n_0-2} \, (N!)^2 \prod_{k=n_0+3}^N \left(1 - \frac{|8\mu h-2|}{h^2(2k+1)^2}\right)}. \tag{51}$$

By (49) we have

$$0 < D_N = \exp\left[\sum_{k=n_0+3}^{N} \log\left(1 - \frac{|8\mu h - 2|}{h^2(2k+1)^2}\right)\right]. \tag{52}$$

Therefore taking the limit in (52) gives

$$\lim_{N\to+\infty} D_N = \exp\left[\sum_{k=n_0+3}^{+\infty} \log\left(1 - \frac{|8\mu h - 2|}{h^2(2k+1)^2}\right)\right] = a \in \mathbb{R}_+. \tag{53}$$

In order to use Stirling's formula (see e.g. [6], p.423), we multiply and divide (51) by $2\pi N^{2N+1}e^{-2N}$. Then, upon setting

$$C_N = \frac{2\pi N^{2N+1} e^{-2N}}{(N!)^2}$$

we have, from (51),

$$|g_N| \leq \frac{|g_{n_0+2}|\,((n_0+2)!)^2\,C_N}{\left((1-\varepsilon)h^2 4\frac{N^2}{e^2}\right)^{N-n_0-2}\;D_N 2\pi N}\left(\frac{e^2}{N^2}\right)^{n_0+2}. \tag{54}$$

By recalling (53), and since by Stirling's formula $C_n \to 1$, we have that the right-hand side of (54), for ε fixed and for $N \to +\infty$, approaches to zero faster than every negative power of N. As already remarked, an analogous result can be proved also when $\{\vartheta_n\}_n = \{\delta_n^-\}_n$ (see (13)). From this we get that the series given either by (38) or by (39) converges uniformly on $[-\pi, \pi]$, with all its derivatives and therefore it represents a function of the space $D(P)$ and an eigenfunction associated with μ. Moreover, from what just stated, the eigenfunction obtained in this way is C^∞ on the interval $[-\pi, \pi]$.

b) Conversely, let μ be such that $z_0 \neq K_{n=1}^{+\infty}(-\alpha_n/-1)$ and let $\vartheta_n \neq 0$ for every $n \in \mathbb{N}$. Then from Proposition 4.8 we have that $\lim\limits_{n\to+\infty} z_n = 1$. Let $\{g_n\}_n$ be a solution of (16). We will show that $|g_n| \to +\infty$ as $n \to +\infty$. This implies that the series given by either the expansion (38) or the expansion (39) does not converge to a function of $D(P)$. Since $\lim\limits_{n\to+\infty} z_n = 1$ we have that, for a fixed $\varepsilon > 0$, exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $|z_n - 1| < \varepsilon$. Let be $\varepsilon < \frac{1}{2}$.

We have that $1-\varepsilon \leq z_n \leq 1+\varepsilon$ for every $n \geq n_0$ so that, in particular, we have $z_n \neq 0,\ \infty$. Whence for all $n \geq n_0$ holds the equality $z_n = z_n^*$ (see (40) and (41)). Thus, by reasoning as in the proof of a), we get

$$|g_N| \geq |g_{n_0+2}|\ \ |\vartheta_{n_0+2}\ldots\vartheta_{N-1}|\,(1-\varepsilon)^{N-2-n_0}. \tag{55}$$

We show the divergence of g_n only for $\{\vartheta_n\}_n := \{\delta_n^+\}_n$ (the proof for $\{\delta_n^-\}_n$ is similar). We use here, as in a), formula (47).

By substituting (47) in (55), and by using

$$(2n_0+5)(2n_0+7)\ldots(2N-1) \leq 2^{N-2-n_0}(n_0+2)(n_0+3)\ldots(N-1),$$

we have

$$|g_N| \geq |g_{n_0+2}|\left[(1-\varepsilon)4h^2\right]^{N-2-n_0}\left[\frac{(N-1)!}{(n_0+1)!}\right]^2\prod_{k=n_0+2}^{N-1}\left|1-\frac{8\mu h-2}{h^2(2k+1)^2}\right|. \tag{56}$$

Suppose that n_0 is such that we have

$$\left|\frac{8\mu h-2}{h^2(2k+1)^2}\right| < 1, \qquad \forall\ k \geq n_0+2.$$

Thus, from (56), as in the proof of a), we obtain

$$|g_N| \geq |g_{n_0+2}|\left[(1-\varepsilon)4h^2\right]^{N-2-n_0}\left[\frac{(N-1)!}{(n_0+1)!}\right]^2\prod_{k=n_0+2}^{N-1}\left[1-\frac{|8\mu h-2|}{h^2(2k+1)^2}\right]. \tag{57}$$

We prove that the right-hand side of (57) goes to infinity as $N \to +\infty$. In a way similar to that of case a) we obtain

$$\lim_{N\to+\infty}\prod_{k=n_0+2}^{N-1}\left(1-\frac{|8\mu h-2|}{h^2(2k+1)^2}\right) = a \in \mathbb{R}_+.$$

Now set

$$C_N = \prod_{k=n_0+2}^{N-1} \left(1 - \frac{|8\mu h - 2|}{h^2(2k+1)^2}\right), \quad B_N = \frac{((N-1)!)^2}{2\pi(N-1)(N-1)^{2(N-1)}\ e^{-2(N-1)}}.$$

Note that, by Stirling's formula, $\lim_{N\to+\infty} B_N = 1$. Multiplying and dividing by

$$2\pi(N-1)(N-1)^{2(N-1)}\ e^{-2(N-1)}$$

the right-hand side of (57) we have

$$|g_N| \geq \frac{|g_{n_0+2}|B_N}{[(n_0+1)!]^2} \left[(1-\varepsilon)4h^2 \left(\frac{N-1}{e}\right)^2\right]^{N-2-n_0} 2\pi(N-1) \left(\frac{N-1}{e}\right)^{2n_0+2} C_N.$$

Taking the limit as $N \to +\infty$ gives $\lim_{N\to+\infty} |g_N| = +\infty$. Therefore the series (38) and (39) in this case do not converge to functions of $L^2(I)$ and thus they cannot represent any function in $D(P)$. □

By recalling the remarks made at the end of the previous Section we can use the same procedure of Theorem 4.9 to prove analogous characterization of the eigenvalues of P, also in case μ is such that $\vartheta_{n_0}(\mu) = 0$, for a certain $n_0 \in \mathbb{N}$. In fact, from Propositions 3.11, 3.12 and Lemma 3.7 we obtain, from a certain index onward, a formula similar to (44) for the g_n and then we can conclude the proof as in Theorem 4.9. Thus we state the following

Theorem 4.10. *Let μ be such that $\vartheta_{n_0}(\mu) = 0$ for a certain $n_0 \in \mathbb{N}$.*

a) Using the notation of Proposition 3.11 if $n_0 = 0$, necessary and sufficient condition for μ to be an eigenvalue of P is that

$$1 - \frac{1}{\eta_0\eta_1} = K_{n=1}^{+\infty} \left(\frac{-\frac{1}{\eta_n\eta_{n+1}}}{-1}\right).$$

b) Using the notation of Proposition 3.12 if $n_0 \neq 0$, $g_{n_0} \neq 0$, $g_{n_0-1} \neq 0$, necessary and sufficient condition for μ to be an eigenvalue of P is that

$$1 - \frac{1}{\gamma_0\gamma_1} = K_{n=1}^{+\infty} \left(\frac{-\frac{1}{\gamma_n\gamma_{n+1}}}{-1}\right).$$

c) Using the notation of Proposition 3.12 if $n_0 \neq 0$, $g_{n_0} \neq 0$, $g_{n_0-1} = 0$, necessary and sufficient condition for μ to be an eigenvalue of P is that

$$1 - \frac{1}{\nu_0\nu_1} = K_{n=1}^{+\infty} \left(\frac{-\frac{1}{\nu_n\nu_{n+1}}}{-1}\right).$$

d) Using the notation of Proposition 3.12 if $n_0 \neq 0$, $g_{n_0} = 0$, necessary and sufficient condition for μ to be an eigenvalue of P is that

$$1 - \frac{1}{\rho_0\rho_1} = K_{n=1}^{+\infty} \left(\frac{-\frac{1}{\rho_n\rho_{n+1}}}{-1}\right).$$

We can state condition (37) of Theorem 4.9 in terms of an equivalent continued fraction (see Definition 4.1) in the following form:

$$f = \vartheta_0 - \cfrac{1}{\vartheta_1 - \cfrac{1}{\vartheta_2 - \ddots}} = 0. \tag{58}$$

From Theorems 4.9, 4.10 we get further information on the Fourier coefficients of the eigenfunctions. For instance we can prove that for every $N \in \mathbb{N}$ there exists $n_0 > N$ such that g_{n_0}, g_{n_0+1}, $g_{n_0+2} \neq 0$, where g_n represent, as usual, Fourier coefficients of eigenfunctions. This will be proved for a general solution $\{g_n\}_n$ of the recurrence relation (16) of Definition 3.3, even if g_n does not represent an eigenfunction's Fourier coefficient (i.e. if μ is not an eigenvalue of P). Besides, when μ is an eigenvalue of P, we will obtain in addition that there exist $m \in \mathbb{N}$ such that the Fourier coefficients g_n do not vanish for all $n \geq m$.

Remark 4.11. *Let $\mu \in \mathbb{R}$ and let $\{g_n\}_{n\geq -1}$ be the solution, different from the 0-sequence, of the recurrence relation*

$$g_{-1} = 0, \quad g_{n+1} = \vartheta_n(\mu) g_n - g_{n-1}, \qquad \forall\, n \in \mathbb{N}, \tag{59}$$

where we use the notation fixed in Definition 3.3. Then, for every $N \in \mathbb{N}$ there exists $n_0 > N$ such that g_{n_0}, g_{n_0+1}, $g_{n_0+2} \neq 0$.

Proof. By Theorems 4.9, 4.10, we have either $\lim_{n\to+\infty} |g_n| = 0$ or $\lim_{n\to+\infty} |g_n| = +\infty$. In the second case, that is when μ is not an eigenvalue of P, the assertion follows immediately.

Let μ be an eigenvalue of P and assume that $g_n = 0$ for infinitely many values of n (otherwise the assertion follows immediatly).

In the first place we prove that there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$ we have $(g_n, g_{n+2}) \neq (0, 0)$. Set $n_1 \in \mathbb{N}$ such that $\vartheta_n(\mu) \neq 0$ for all $n > n_1$ (the existence of such an n_1 follows from Remark 3.8). By contradiction let $g_n = g_{n+2} = 0$ for $n > n_1$. Then, by (59), we have

$$0 = g_{n+2} = \vartheta_{n+1}(\mu) g_{n+1} - g_n = \vartheta_{n+1}(\mu) g_{n+1}.$$

As $\vartheta_{n+1}(\mu) \neq 0$ this implies that $g_{n+1} = 0$. Since $g_n = 0$ this is a contradiction, by Lemma 3.1.

Up to now we have shown that, for all $n > n_1$, $g_n = 0$ implies that both g_{n+1} and g_{n+2} are different from 0. We reason again by contradiction to conclude the proof. Suppose there exists $N \in \mathbb{N}$ such that, for all $n > N$ if g_n, $g_{n+1} \neq 0$ then $g_{n+2} = 0$. Fix $n_0 > \max\{N, n_1\}$, such that $g_{n_0} = 0$ (recall that we are in the hypothesis that $g_n = 0$ for infinitely many values of n). Then g_{n_0+1}, $g_{n_0+2} \neq 0$. Thus $g_{n_0+3} = 0$ and this implies g_{n_0+4}, $g_{n_0+5} \neq 0$ and so on. Substituting these values in (59) gives

$$\begin{aligned} &g_{n_0} = 0, && g_{n_0+1} = -g_{n_0-1} \neq 0, && g_{n_0+2} = -\vartheta_{n_0+1} g_{n_0-1} \neq 0, \\ &g_{n_0+3} = 0, && g_{n_0+4} = \vartheta_{n_0+1} g_{n_0-1} \neq 0, && g_{n_0+5} = \vartheta_{n_0+4}\vartheta_{n_0+1} g_{n_0-1} \neq 0, \dots . \end{aligned}$$

By induction, since $|\vartheta_n(\mu)| \to +\infty$, as $n \to +\infty$, we see that

$$\lim_{j\to+\infty} |g_{n_0+m_j}| = +\infty, \quad \text{when} \quad m_j \in \mathbb{N}, \quad m_j \equiv 1 \bmod 3,$$

and $|g_{n_0+m_k}| = 0$ for all $m_k \in \mathbb{N}$ such that $m_j \equiv 0 \bmod 3$. Thus the sequence $\{|g_n|\}_n$ does not have a limit, but this is a condradiction because, as μ is an eigenvalue of P, by Theorems 4.9, 4.10 the sequence $\{|g_n|\}_n$ must converge to 0. □

Looking at the proof of Theorem 4.9 (see (44), (45)) we recall that we have, for a sufficiently large n_0

$$g_N = g_{n_0+2}\vartheta_{n_0+2}\dots\vartheta_{N-1}z_{n_0+1}\dots z_{N-2}, \qquad N > n_0+2, \tag{60}$$

where $\{z_n\}_n$ is recursively defined by

$$z_0 = 1 - \frac{1}{\vartheta_0\vartheta_1}; \qquad z_n = 1 - \frac{\frac{1}{\vartheta_n\vartheta_{n+1}}}{z_{n-1}}, \qquad n > 0. \tag{61}$$

Besides we have, if μ is an eigenvalue of P, that

$$z_0 = 1 - \frac{1}{\vartheta_0\vartheta_1} = K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\vartheta_n\vartheta_{n+1}}}{-1}\right). \tag{62}$$

In other words z_0 can be written as a continued fraction. From (61) and (62) we find out that we can write all z_n in (61) as continued fractions which are the tails of the continued fraction in (62). To prove this we recall the following statement about tail sequences (see [7], p. 60).

Remark 4.12. *Let $\{t_n\}_n$, $\{\tilde{t}_n\}_n$ be two tail sequences for $b_0 + K(a_n/b_n)$, with $t_k = \tilde{t}_k$ for one index k. Then $t_n = \tilde{t}_n$ for all $n \in \mathbb{N}$.*

Proposition 4.13. *Let μ be an eigenvalue of P. Using the notation of Theorem 4.9 we have that*

$$z_n = K_{j=n}^{+\infty}\left(\frac{-\frac{1}{\vartheta_j\vartheta_{j+1}}}{-1}\right), \qquad n \in \mathbb{N}. \tag{63}$$

Proof. By Lemma 4.7 $\{z_n\}_n$ is a tail sequence for $K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\vartheta_n\vartheta_{n+1}}}{-1}\right)$. The right-hand side of (63), for $n = 1, 2, \ldots$, is obviously a tail sequence for the same continued fraction (see Definition 4.3). The assertion follows from Remark 4.12 and Theorem 4.9, as the two tail sequences have the first term in common. □

Using this proposition will give estimates on coefficients z_n, appearing in (60). To this purpose we recall the following theorem about continued fractions (for the proof see [7], p. 35).

Theorem 4.14 (Worpitzky). *If $\{a_n\}_n \subseteq \mathbb{C}$ is such that $|a_n| \leq 1/4$, for all $n \in \mathbb{N}\backslash\{0\}$, then $K_{n=1}^{+\infty}(a_n/1)$ converges. Moreover all approximants f_n verify $|f_n| < \frac{1}{2}$ and we have*

$$|f| = \left|K_{n=1}^{+\infty}(a_n/1)\right| \leq \frac{1}{2}.$$

Applying this theorem to (63) gives the following

Corollary 4.15. *Using the notation of Definition 3.3, let μ be a real number. There exists $n_0 \in \mathbb{N}$ such that*

$$\left| \frac{1}{\vartheta_n(\mu)\vartheta_{n+1}(\mu)} \right| < \frac{1}{4}, \qquad \forall\, n > n_0,$$

so that we have, for z_n defined by (63), $|z_n| \leq 1/2$, for all $n > n_0$.

We consider once again equation (60), shown in the proof of Theorem 4.9. Notice that the recurrence relation (59) gives an unique expression for g_{n_0+2} in both cases $\vartheta_n(\mu) \neq 0$ for all n and $\vartheta_n(\mu) = 0$ for some n. Moreover we can compute also coefficients z_n, appearing in (60), with a procedure independent to whether or not $\vartheta_n(\mu)$ vanishes. This will be done by computing z_{n_0}, for n_0 large enough, independently to z_0, z_1,..., z_{n_0-1}. Following these ideas we shall find out the following general form of the Fourier coefficients, g_n, of the eigenfunctions of P, which is fulfilled in both cases $\vartheta_n(\mu) = 0$ or $\vartheta_n(\mu) \neq 0$. Besides, this analysis will provide a general necessary and sufficient condition for the eigenvalues of P that unifies the notation of the two cases considered in Theorems 4.9, 4.10.

Proposition 4.16. *Using the notation of Remark 4.11 let $\mu \in \mathbb{R}$. Let n_0 be such that $|\vartheta_n(\mu)| > 2$ for all $n \geq n_0$ and such that g_{n_0}, g_{n_0+1}, $g_{n_0+2} \neq 0$. Then μ is an eigenvalue of P if and only if*

$$1 - \frac{1}{\dfrac{g_{n_0+1}}{g_{n_0}}\vartheta_{n_0+1}} = \frac{\dfrac{1}{\vartheta_{n_0+1}\vartheta_{n_0+2}}}{1 - \dfrac{\dfrac{1}{\vartheta_{n_0+2}\vartheta_{n_0+3}}}{1 - \ddots}}. \tag{64}$$

Furthermore, if μ is an eigenvalue of P, we have

$$g_{n_0+1+m} = \vartheta_{n_0+1} \ldots \vartheta_{n_0+m} z_{n_0} \ldots z_{n_0+m-1} g_{n_0+1}, \qquad \forall\, m > 0, \tag{65}$$

$$z_{n_0+m} = K_{j=m}^{+\infty} \left(\frac{-\frac{1}{\vartheta_{n_0+j+1}\vartheta_{n_0+j+2}}}{-1} \right), \qquad \forall\, m \in \mathbb{N}. \tag{66}$$

Notice that $z_{n_0+m} \neq 0$ for all $m \in \mathbb{N}$ and thus $g_{n_0+1+m} \neq 0$ for all $m > 0$.

Proof. The existence of n_0 such that $|\vartheta_n(\mu)| > 2$ for all $n \geq n_0$ and such that g_{n_0}, g_{n_0+1}, $g_{n_0+2} \neq 0$ is a consequence of Remark 4.11 and of Definition 3.3. We write the recurrence relation as

$$g_{n_0+1} = \left(\frac{g_{n_0+1}}{g_{n_0}} \right) g_{n_0}, \tag{67}$$

$$g_{n_0+m+2} = \vartheta_{n_0+m+1} g_{n_0+m+1} - g_{n_0+m}, \qquad m \in \mathbb{N}. \tag{68}$$

Thus we apply Lemma 3.4, Proposition 3.6 and Proposition 4.8 to this recurrence relation as in the proof of Theorem 4.9; notice that the analogous of the sequence $\{z_n\}_n$ in this case is

$$\tilde{z}_0 = 1 - \frac{1}{\dfrac{g_{n_0+1}}{g_{n_0}}\vartheta_{n_0+1}}; \qquad \tilde{z}_n = 1 - \frac{\dfrac{1}{\vartheta_n\vartheta_{n+1}}}{\tilde{z}_{n-1}}, \quad n > 0. \tag{69}$$

So we have that (64) is a necessary and sufficient condition for μ to be an eigenvalue of P and furthermore we have

$$g_{n_0+1+m} = \vartheta_{n_0+1} \dots \vartheta_{n_0+m} \tilde{z}_0^* \dots \tilde{z}_{m-1}^* g_{m_0+1}, \tag{70}$$

where $\tilde{z}_j^*$ are defined in analogy with z_j^* in (40). We reason as in the proof of Proposition 4.13. From (64) and Remark 4.12, as $\{\tilde{z}_n\}_n$ is a tail sequence for $K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\vartheta_{n_0+n+1}\vartheta_{n_0+n+2}}}{-1}\right)$ we get

$$\tilde{z}_n = \cfrac{\cfrac{1}{\vartheta_{n_0+n+1}\vartheta_{n_0+n+2}}}{1 - \cfrac{\cfrac{1}{\vartheta_{n_0+n+2}\vartheta_{n_0+n+3}}}{1 - \ddots}}, \qquad \forall\, n \in \mathbb{N}. \tag{71}$$

Notice that from $g_{n_0+2} \neq 0$, by using (68), it follows

$$\tilde{z}_0 = 1 - \frac{1}{\frac{g_{n_0+1}}{g_{n_0}}\vartheta_{n_0+1}} \neq 0.$$

Moreover, as $|\vartheta_n(\mu)| > 2$ for every $n \geq n_0$, we have, from (71) and Corollary 4.15, $|\tilde{z}_n| \leq \frac{1}{2}$. From here and by (69) we have $\tilde{z}_n \neq 0, \infty$ for all $n \in \mathbb{N}$. Thus $\tilde{z}_n^* = \tilde{z}_n$ for all n, so (70) and (71) imply (65) and (66). □

5 Upper and lower bounds for eigenvalues

In this section we will provide for each eigenvalue two sequences; one converging to the eigenvalue from above and the other converging to the eigenvalue from below. The following results can be found in [9] and we just give the statements in a form tailored to our particular situation.

We recall again the recurrence relations fulfilled by the coefficients of eigenfunctions $\{v_n^{\pm}\}_n$. We have

$$\begin{cases} v_{n+1}^+ = \delta_n^+ v_n^+ - v_{n-1}^+, & \forall\, n \in \mathbb{N} \\ v_{n+1}^- = \delta_n^- v_n^- - v_{n-1}^-, & \forall\, n \in \mathbb{N}, \end{cases} \tag{72}$$

with

$$\begin{cases} \delta_0^+ = h^2 + 1 - 8\mu h \\ \delta_n^+ = (2n+1)^2 h^2 + 2 - 8\mu h, & \forall\, n \in \mathbb{N}\backslash\{0\} \end{cases} \tag{73}$$

and

$$\delta_n^- = 4(n+1)^2 h^2 + 2 - 8\mu h, \qquad \forall\, n \in \mathbb{N}. \tag{74}$$

Following the notation fixed in Definition 3.3 we will consider $\{g_n\}_{n\geq -1} = \{g_n(\mu)\}_{n\geq -1}$ as a particular sequence of polynomials in μ. We will see that the eigenvalues of P are the limits of zeros of these polynomials.

To start this analysis it is useful to give the following definition.

Definition 5.1. *Let $\{\Pi_n\}_n$ be a sequence of polynomials in the variable μ, with real coefficients. Denote by $r_{n,1} \leq r_{n,2} \leq \dots \leq r_{n,k}$ the real zeros (in case they exist) of Π_n and put, by definition, $r_{n,0} = -\infty$ and $r_{n,k+1} = +\infty$. We shall say that $\{\Pi_n\}_{n\geq 0}$ is a* ***sequence of polynomials with interlaced zeros*** *if* *(i)* Π_0 *is not the zero polynomial, it has degree* $d \geq 0$ *and all its zeros are real with multeplicity* 1.

(ii) Π_1 *has degree* $d+1$, *all its zeros are real-valued with multeplicity* 1 *and each zero of* Π_1 *is located between two consecutive zeros of* Π_0, *i.e.*

$$r_{0,i-1} < r_{1,i} < r_{0,i}, \quad i = 1, 2, \dots, d+1.$$

(iii) There exists a sequence $\{\beta_n\}_n$ *of polynomials (in* μ*) of degree* 1 *such that*

$$\Pi_{n+1} = \beta_n \Pi_n + \Pi_{n-1}, \qquad n = 1, 2, \dots \ . \tag{75}$$

(iv) $\lim_{n\to+\infty} \Pi_n(\mu) := \Pi_n(+\infty)$ *and* $\lim_{n\to+\infty} \Pi_{n+2}(\mu) := \Pi_{n+2}(+\infty)$ *have opposite signs for all* $n \in \mathbb{N}$.

If Π_0 *has degree* 0 *we say that* $\{\Pi_n\}_n$ *is a* ***sequence of polynomials with interlaced zeros*** *if* $\{\Pi_n\}_n$ *fulfills (i), (iii), (iv).*

Now we change the sequences of coefficients of eigenfunctions $\{v_n^{\pm}\}_n$, so that they satisfy Definition 5.1.

Lemma 5.2. *We put, by definition*

$$\begin{cases} c_{2n}^- = (-1)^n v_{2n}^-, \\ c_{2n+1}^- = (-1)^n v_{2n+1}^-, \end{cases} \quad n \in \mathbb{N}, \qquad \begin{cases} c_{2n}^+ = (-1)^n v_{2n}^+, \\ c_{2n+1}^+ = (-1)^n v_{2n+1}^+, \end{cases} \quad n \in \mathbb{N}. \tag{76}$$

Moreover let $\{\chi_n^-\}_n$, $\{\chi_n^+\}_n$ *be such that*

$$\chi_n^- = (-1)^n \delta_n^-, \qquad n \in \mathbb{N}; \qquad \chi_n^+ = (-1)^n \delta_n^+, \qquad n \in \mathbb{N}. \tag{77}$$

Then $\{c_n^-\}_n$, $\{c_n^+\}_n$ *satisfy the following recurrence relations:*

$$c_{n+1}^- = \chi_n^- c_n^- + c_{n-1}^-, \quad n \in \mathbb{N}; \qquad c_{n+1}^+ = \chi_n^+ c_n^+ + c_{n-1}^+, \quad n \in \mathbb{N}. \tag{78}$$

In particular $\{c_n^-\}_n$, $\{c_n^+\}_n$ *are sequences of polynomials in* μ *with interlaced zeros.*

Proof. It follows immediatly from relations (76) and (77), recalling (72), (73) and (74). □

We next state an important property of sequences of polynomials with interlaced zeros.

Theorem 5.3. *If* $\{\Pi_n\}_n$ *is a sequence of polynomials with interlaced zeros, then* Π_n *has all real and distinct zeros, for every* $n \in \mathbb{N}$. *Moreover, for all* $n \geq 1$, *each zero of* Π_{n-1} *is located between two consecutive zeros of* Π_n; *in other words*

$$r_{n-1,i-1} < r_{n,i} < r_{n-1,i}, \qquad i = 1, 2, \dots, d+n, \qquad \forall\ n \geq 1.$$

From here we obtain, in particular, that the sequences $\{r_{n,i}\}_n$, *for fixed* i, *are monotone decreasing.*

The following result defines the polynomials Θ_n and Ψ_n and gives a property for their zeros; we will see that particular sequences defined using these zeros converge to eigenvalues of P.

Theorem 5.4. *Let $\{\Pi_n\}_n$ be a sequence of polynomials with interlaced zeros and define $\Theta_n = \Pi_n - \Pi_{n-1}$ and $\Psi_n = \Pi_n + \Pi_{n-1}$. Then all the zeros of $\Theta_n,\ \Psi_n$ are real and distinct. We denote by $\rho_{n,1} < \rho_{n,2} < \cdots < \rho_{n,d+n}$, the zeros of Θ_n, and we denote by $\rho'_{n,1} < \rho'_{n,2} < \cdots < \rho'_{n,d+n}$ those of Ψ_n. We set, by definition $\rho^-_{n,i} = \min\{\rho_{n,i},\ \rho'_{n,i}\}$. Then, for every $n \in \mathbb{N}$ and for every $i = 1, 2, \ldots, d+n$, we have $\rho^-_{n,i} \in (r_{n-1,i-1},\ r_{n,i})$.*

We will use the following definition to prove the monotonicity of $\{\rho^-_{n,i}\}$, for $n \geq n_0$, for some $n_0 \in \mathbb{N}$.

Definition 5.5. *Using the notation of Definition 5.1 and writing β_n as $\beta_n(\mu) = \xi_n(\mu - B_n)$, for $\xi_n \in \mathbb{R}$, we say that a sequence of polynomials with interlaced zeros is* ***admissible*** *if there exist $\xi > 0$ and $N_0 \in \mathbb{N}$ such that $|\xi_n| \geq \xi$ for all $n \in \mathbb{N}$ and $B_{n+1} - B_n > \frac{2}{\xi}$ for all $n \geq N_0$.*

We show that the sequences $\{\chi^{\pm}_n\}_n$, defined in Lemma 5.2 satisfy Definition 5.5.

Lemma 5.6. *The sequences $\{\chi^-_n\}_n,\ \{\chi^+_n\}_n$, defined by (77) are admissible.*

Proof. Upon recalling relations (77) and (73) we have

$$\chi^+_n(\mu) = (-1)^{n+1} 8h \Bigg(\underbrace{-\frac{(2n+1)^2 h}{8} - \frac{1}{4h}}_{B_n} + \mu \Bigg),$$

with $n \in \mathbb{N}\backslash\{0\}$. Using the notation fixed in Definition 5.5, we set $\xi = 8h$, $\xi_n = (-1)^{n+1} 8h$. In order to verify the definition it suffices to prove that there exists $N_0 \in \mathbb{N}$ such that

$$\frac{(2n+3)^2 h}{8} - \frac{(2n+1)^2 h}{8} \geq \frac{1}{4h} \tag{79}$$

for all $n \geq N_0$. From (79) we get $N_0 \geq \frac{1}{4h^2} - 1$. In a similar way, recalling (77) and (74), we obtain that $\{\chi^-_n\}_n$ fulfills the admissibility hypothesis, for $\xi = 8h$, $\xi_n = (-1)^{n+1} 8h$ and $N_0 \geq \frac{1}{4h^2} - \frac{3}{2}$. □

The following theorem ensures the monotonicity of $\{\rho^-_{n,i}\}$, for sufficiently large n, in the case the sequence of polynomials is admissible.

Theorem 5.7. *Let $\{\Pi_n\}_n$ be an admissible sequence of polynomials with interlaced zeros and let $r_{n,i}$ be the zeros of Π_n (using the notation fixed in Definition 5.1). Fix $i \in \mathbb{N}$. By Definition 5.5 there exists $n_i \in \mathbb{N}$ such that $|\beta_{n+1}(\mu)| > 2$ for all $\mu < r_{n,i}$ and for all $n \geq n_i$. Then, for $n \geq n_i$ we have*

$$r_{n+1,i} \in \left[\rho^-_{n,i},\ r_{n,i}\right), \qquad \rho^-_{n+1,i} > \rho^-_{n,i}.$$

As a consequence for every i the sequence $\{r_{n,i}\}_n$ converges (recall that $\{r_{n,i}\}_n$ is monotone, by Theorem 5.3) and $\rho^-_{n,i}$, with $n \geq n_i$, are lower bounds for $\lim\limits_{n\to+\infty} r_{n,i}$.

The following result shows that, for some values of the variable, the limit of the absolute value of admissible sequences of polynomials with interlaced zeros is 0. This will give us information about eigenvalues of P, since the sequence of Fourier coefficients of eigenfunctions converges to zero (see Theorems 4.9, 4.10), and by Lemma 5.2 the same sequence is an admissible sequence of polynomials, with interlaced zeros.

Lemma 5.8. *Using the notation of Theorem 5.4, let* $\{\Pi_n\}_n$ *be an admissible sequence of polynomials with interlaced zeros and let* $r_i = \lim_{n\to+\infty} r_{n,i}$ *and* $l_i = \lim_{n\to+\infty} \rho^-_{n,i}$. *Then, for every* $i \in \mathbb{N}$, *we have*

$$a \in [l_i,\ r_i] \Rightarrow \lim_{n\to+\infty} |\Pi_n(a)| = 0.$$

We now fix the notation we will use hereafter.

Definition 5.9. *From now on we will denote with* $\{\Pi_n\}_n$ *one of the two sequences* $\{c^-_n\}_n,\ \{c^+_n\}_n$ *defined by (76). Furthermore we will use the notation fixed in Definition 5.1 and in Theorem 5.4, recalling that either* $\{\Pi_n\}_n := \{c^-_n\}_n$ *or* $\{\Pi_n\}_n := \{c^+_n\}_n$.

Lemma 5.8 implies, recalling Theorems 4.9 and 4.10 and their proofs, that $r_i = l_i$ and that these values are exactly the eigenvalues of P. In particular we have the following

Corollary 5.10. *Using the notation of Definition 5.9 and of Lemma 5.8 we have, by Lemma 5.6, that* $\{\Pi_n\}_n = \{\Pi_n(\mu)\}_n$ *is an admissible sequence of polynomials in* μ *with interlaced zeros and*

$$\lim_{n\to+\infty} r_{n,i} = r_i = l_i = \lim_{n\to+\infty} \rho^-_{n,i} \qquad \forall\ i \in \mathbb{N}.$$

Furthermore the set $\{r_i,\ i \in \mathbb{N}\backslash\{0\}\}$ *coincides with the set of eigenvalues of* P.

Proof. By Theorems 4.9 and 4.10 we have, by recalling (76), that $|\Pi_n(\mu)| \to 0$ if and only if μ is an eigenvalue of P. From Lemma 5.8 we have that $|\Pi_n(a)| \to 0$ for all $a \in [l_i, r_i]$. By Remark 2.3 P has discrete spectrum, therefore $l_i = r_i$ and r_i is an eigenvalue of P. □

We next show that Corollary 5.10 implies that all Fourier coefficients of the eigenfunction associated with the lowest eigenvalue of P can not vanish.

Corollary 5.11. *Let* μ_0 *be the lowest eigenvalue of* P. *If*

$$v^+ = \sum_{n=0}^{+\infty} v^+_n \frac{1}{\sqrt{\pi}} \cos\left(\frac{2n+1}{2}x\right) \tag{80}$$

is the eigenfunction associated with μ_0 *then we have* $v^+_n \neq 0$ *for all* $n \in \mathbb{N}$.

Proof. Notice that v^+ is an even eigenfunction which does not vanish in the interior of I (see e.g. [3], Theorem 4.1 p. 337). This justify the expansion (80). From Corollary 5.10 we have that μ_0 is the limit of the sequence $\{r_{n,1}\}_n$, where $r_{n,1}$ denotes the lowest zero of $v^+_n = v^+_n(\mu)$, considered as a polynomial in μ.

Notice that, by Lemma 5.2 and Theorem 5.3, the sequence $\{r_{n,1}\}_n$ is monotone decreasing. Therefore, since

$$\mu_0 = \lim_{n\to+\infty} r_{n,1}$$

it follows immediatly that μ_0 can not be a zero of any of the v_n^+. □

We now state an important result about the continued fraction

$$f = f(\mu) = \vartheta_0(\mu) + K_{n=1}^{+\infty}\left(-1/\vartheta_n(\mu)\right),$$

where $\{\vartheta_n\}_n$ represents, as established in Definition 3.3, one of the sequences $\{\delta_n^+\}$, $\{\delta_n^-\}$. This function appears in the necessary and sufficient condition for the eigenvalues of P (see (58)).

In particular this function is meromorphic in μ (for the proof see [9]).

Notice that the continued fraction $K_{n=1}^{+\infty}\left(-1/\vartheta_n(\mu)\right)$ is equivalent to (see Definition 4.1) $K_{n=1}^{+\infty}\left(1/(-1)^n\vartheta_n(\mu)\right)$. For this reason, recalling (77), we give the following

Definition 5.12. *Define the function*

$$f = f(\mu) = \beta_0(\mu) + K_{n=1}^{+\infty}\left(1/\beta_n(\mu)\right), \tag{81}$$

with $\{\beta_n\}_n := \{\chi_n^{\pm}\}_n$, $\pm$ *respectively (see (77)).*

We write the approximants of f (see Definition 4.1) as $f_n = \frac{P_n}{Q_n}$. Thus, from Remark 4.2, upon setting $P_{-1} = 1$, $P_0 = \beta_0$, $Q_{-1} = 0$, $Q_0 = 1$, we have

$$P_{n+1} = \beta_{n+1}P_n + P_{n-1}, \qquad n \in \mathbb{N}, \tag{82}$$

$$Q_{n+1} = \beta_{n+1}Q_n + Q_{n-1}, \qquad n \in \mathbb{N}. \tag{83}$$

From (82) we have that, when $\{\beta_n\}_n := \{\chi_n^-\}_n$, the sequence $\{P_n\}_n$ coincides with the sequence of coefficients of eigenfunctions $\{c_n^-\}_n$ (see (76) and (77)), and we have $\{P_n\}_n = \{c_n^+\}_n$ when $\{\beta_n\}_n := \{\chi_n^+\}_n$ (see (76) and (77)). Thus, by Corollary 5.10 *the eigenvalues of P are the limits of zeros of the canonical numerators of f*. Furthermore f is meromorphic and all *its poles are limits of the zeros of the canonical denominators Q_n*. (For the proof see [9].)

Proposition 5.13. *The function f, defined by (81), is meromorphic on $\mathbb{C}$, and it has a pole in z if* $\lim_{n\to+\infty}|Q_n(z)| = 0$.

If we treat $\{Q_n\}_n$ as a sequence of polynomials with interlaced zeros then, from Theorem 5.4, we get two sequences converging, one from above, the other from below, to the poles of f.

6 Estimates for large eigenvalues

In this section we will study the behaviour of the eigenvalues μ such that $\mu > C = C(h)$, for fixed h. In particular we will provide upper and lower bounds for these eigenvalues. To prove these results we will use Worpitzky's theorem (Theorem 4.14) applied to the continued fraction appearing in the necessary and sufficient condition (37) for the eigenvalues. As usual we will analyse in the first place the eigenvalues associated to even eigenfunctions and afterwards those associated to odd eigenfunctions.

In order to apply Worpitzky's Theorem to $K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\delta_n^+\delta_{n+1}^+}}{-1}\right)$ we will study the values of $\left|\delta_n^+\delta_{n+1}^+\right| = \left|\delta_n^+(\mu)\delta_{n+1}^+(\mu)\right|$ for varying μ. For this reason is useful to recall the definition of δ_n^+ :

$$\begin{cases} \delta_0^+ = \delta_0^+(\mu) = h^2 + 1 - 8\mu h \\ \delta_n^+ = \delta_n^+(\mu) = (2n+1)^2h^2 + 2 - 8\mu h, \qquad \forall\, n \in \mathbb{N}\backslash\{0\}. \end{cases} \tag{84}$$

To have a better understanding of the problem, it helps using a geometric approach. More precisely we can think of the functions $\delta_n^+(\mu)\delta_{n+1}^+(\mu)$, for every n, as parabolas in the variable μ. In this way we get a sequence of polynomials of degree 2, $\{\delta_n^+(\mu)\delta_{n+1}^+(\mu)\}_n$, with the property that the maximum zero of $\delta_n^+(\mu)\delta_{n+1}^+(\mu)$ is the minimum zero of $\delta_{n+1}^+(\mu)\delta_{n+2}^+(\mu)$, for every $n \in \mathbb{N}$. Furthermore the sequence of the vertices of the parabolas defined by these polynomials, for $n \geq 1$, is monotone decreasing. These properties are straightforward consequences of (84). In the following results we get sufficient conditions for these polynomials to have absolute value greater than or equal to 4. We will see that if μ is such that this last condition is fulfilled then μ can not be an eigenvalue of P.

Lemma 6.1. *Fix $n \in \mathbb{N}$ and let μ be such that $\delta_n^+(\mu) < 0$ and $\delta_{n+1}^+(\mu) > 0$, i.e. such that*

$$\frac{(2n+1)^2}{8}h + \frac{1}{4h} < \mu < \frac{(2n+3)^2}{8}h + \frac{1}{4h}. \tag{85}$$

Then we have

$$\left|\delta_n^+(\mu)\delta_{n+1}^+(\mu)\right| < \left|\delta_{n+1}^+(\mu)\delta_{n+2}^+(\mu)\right|. \tag{86}$$

Proof. We prove that $|\delta_n^+(\mu)| < \left|\delta_{n+2}^+(\mu)\right|$, from which (86) follows immediatly.

By (84) and by (85) we have $\delta_{n+2}^+(\mu) > \delta_{n+1}^+(\mu) > 0$. Thus, to obtain (86), it suffices to show that $-\delta_{n+2}^+(\mu) < \delta_n^+(\mu) < \delta_{n+2}^+(\mu)$. It is straightforward that $\delta_n^+(\mu) < \delta_{n+2}^+(\mu)$, for $\delta_{n+2}^+(\mu) > 0$ and $\delta_n^+(\mu) < 0$. We now prove that $-\delta_{n+2}^+(\mu) < \delta_n^+(\mu)$. From (85) it follows that $-8\mu h > -(2n+3)^2h^2 - 2$, so that we have

$$\delta_n^+(\mu) = (2n+1)^2h^2 + 2 - 8\mu h > (-8n-8)h^2. \tag{87}$$

In addition, from (85) it also follows that

$$-\delta_{n+2}^+(\mu) = -(2n+5)^2h^2 - 2 + 8\mu h < (-8n-16)h^2. \tag{88}$$

From (87) and (88), being $(-8n-8)h^2 > (-8n-16)h^2$, we get $\delta_n^+(\mu) > -\delta_{n+2}^+(\mu)$ and hence (86). □

In the hypotheses of Lemma 6.1, we now study the values of the function defined by

$$\mu \;\longmapsto\; R_n(\mu) := \min\{\left|\delta_n^+(\mu)\delta_{n+1}^+(\mu)\right|, \left|\delta_n^+(\mu)\delta_{n-1}^+(\mu)\right|\}. \tag{89}$$

Proposition 6.2. *Fix $n \in \mathbb{N}$. In the hypotheses of Lemma 6.1 we have that*

1) if $\mu < \frac{1}{4h} + \frac{[(2n+1)^2+4]h}{8}$ then $\left|\delta_n^+(\mu)\delta_{n-1}^+(\mu)\right| < \left|\delta_n^+(\mu)\delta_{n+1}^+(\mu)\right|$;

2) if $\mu > \dfrac{1}{4h} + \dfrac{[(2n+1)^2+4]h}{8}$ *then* $\left|\delta_n^+(\mu)\delta_{n-1}^+(\mu)\right| > \left|\delta_n^+(\mu)\delta_{n+1}^+(\mu)\right|$;

3) if $\mu = \dfrac{1}{4h} + \dfrac{[(2n+1)^2+4]h}{8}$ *then* $\left|\delta_n^+(\mu)\delta_{n-1}^+(\mu)\right| = \left|\delta_n^+(\mu)\delta_{n+1}^+(\mu)\right| = 16h^4(2n+1)$.

Proof. Points 1) and 2) follow from (84) and (85) by analysing the inequality $\left|\delta_{n-1}^+(\mu)\right| < \left|\delta_{n+1}^+(\mu)\right|$. To obtain 3), we simply replace the value of μ in $\left|\delta_{n-1}^+(\mu)\delta_n^+(\mu)\right|$ (see (84)). □

We will show in the following Propositions that $\left|\delta_m^+(\mu)\delta_{m+1}^+(\mu)\right| > R_n(\mu)$ for every $m \in \mathbb{N}$, with $m \neq n-1,\ n$, and for μ fulfilling the hypothesis of Proposition 6.2. In this way, by (89), conditions on values of $\left|\delta_n^+(\mu)\delta_{n-1}^+(\mu)\right|$ and $\left|\delta_n^+(\mu)\delta_{n+1}^+(\mu)\right|$ give rise to conditions on all the other terms $\left|\delta_m^+(\mu)\delta_{m+1}^+(\mu)\right|$. Hereafter we occasionally denote $\delta_n^+(\mu)$ for short simply by δ_n^+.

Using (84), with straightforward computations we can prove the following

Proposition 6.3. *Fix n in $\mathbb{N}$. Let μ be such that it verifies equation (85). Then we have $\left|\delta_m^+\delta_{m+1}^+\right| > \left|\delta_n^+\delta_{n+1}^+\right|$, for all $m > n$. Furthermore*

a) if $\mu < \dfrac{1}{4h} + \dfrac{[(2n+1)^2+4]h}{8}$ *we have*

$$\left|\delta_m^+(\mu)\delta_{m+1}^+(\mu)\right| > \left|\delta_n^+(\mu)\delta_{n-1}^+(\mu)\right| \quad \textit{for every } m = 0, 1, \ldots, n-2;$$

b) if $\mu > \dfrac{1}{4h} + \dfrac{[(2n+1)^2+4]h}{8}$ *we have*

$$\left|\delta_m^+(\mu)\delta_{m+1}^+(\mu)\right| > \left|\delta_n^+(\mu)\delta_{n+1}^+(\mu)\right| \quad \textit{for every } m = 0, 1, \ldots, n-2.$$

From Proposition 6.3, and by recalling (89), it follows the following

Corollary 6.4. *Let $n \in \mathbb{N}$. If μ satisfies equation (85) then*

$$\left|\delta_m^+(\mu)\delta_{m+1}^+(\mu)\right| > R_n(\mu), \qquad \forall\ m \neq n, n-1.$$

By Theorem 4.9 we have that μ is an eigenvalue of P if and only if it fulfills

$$1 - \frac{1}{\delta_0^+\delta_1^+} = K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\delta_n^+\delta_{n+1}^+}}{-1}\right), \tag{90}$$

in case $\delta_n^+ \neq 0$ for every $n \in \mathbb{N}$. Notice that this last condition, $\delta_n^+(\mu) \neq 0$, is immediatly fulfilled when μ satisfies the hypotheses of Corollary 6.4 (see equations (85) and (84)).

Now we apply Worpitzky's Theorem (Theorem 4.14) to the continued fraction appearing in (90) to obtain estimates for the eigenvalues.

Theorem 6.5. *Fix n in $\mathbb{N}$. If μ is such that*

$$\frac{(2n+1)^2}{8}h + \frac{1}{4h} < \mu < \frac{(2n+3)^2}{8}h + \frac{1}{4h}$$

and, at the same time, $R_n(\mu) \geq 4$ (recall (89)) then μ is not an eigenvalue for P.

Proof. By Corollary 6.4 and since $R_n(\mu) \geq 4$ we have $\left|\delta_n^+ \delta_{n+1}^+\right| \geq 4$ for every $n \in \mathbb{N}$. Then the continued fraction $K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\delta_n^+\delta_{n+1}^+}}{-1}\right)$ verifies the hypothesis of Worpitzky's Theorem and thus

$$\left|K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\delta_n^+\delta_{n+1}^+}}{-1}\right)\right| \leq \frac{1}{2}. \tag{91}$$

Notice that $\delta_n^+ \neq 0$ for every n, because $\left|\delta_n^+ \delta_{n+1}^+\right| > 4$ for every n, hence the hypothesis of Theorem 4.9 are verified. Therefore in case μ is an eigenvalue for P, equation (90) is satisfied by μ and, recalling (91), we get

$$\frac{1}{2} \geq \left|K_{n=1}^{+\infty}\left(\frac{-\frac{1}{\delta_n^+\delta_{n+1}^+}}{-1}\right)\right| = \left|1 - \frac{1}{\delta_0^+\delta_1^+}\right| \geq \left|1 - \left|\frac{1}{\delta_0^+\delta_1^+}\right|\right| \geq \frac{3}{4}$$

which is a contradiction. ☐

From Theorem 6.5 we obtain two different estimates for the eigenvalues, depending on the value of $R_n(\mu) = \min\{|\delta_n^+(\mu)\delta_{n-1}^+(\mu)|,\ |\delta_n^+(\mu)\delta_{n+1}^+(\mu)|\}$. Proposition 6.2 establishes that if

$$\frac{(2n+1)^2}{8}h + \frac{1}{4h} < \mu < \frac{[(2n+1)^2+4]h}{8} + \frac{1}{4h} \tag{92}$$

then $R_n(\mu) = |\delta_n^+(\mu)\delta_{n-1}^+(\mu)|$, and if

$$\frac{[(2n+1)^2+4]h}{8} + \frac{1}{4h} < \mu < \frac{(2n+3)^2 h}{8} + \frac{1}{4h}$$

then $R_n(\mu) = |\delta_n^+(\mu)\delta_{n+1}^+(\mu)|$. In addition, by recalling 3) of Proposition 6.2, we obtain from Theorem 6.5 two different situations according to whether $4 > 16h^4(2n+1)$ or $4 < 16h^4(2n+1)$. In particular we have the following

Theorem 6.6. *Let n be a natural number such that $n \geq \frac{1}{2h^2} - 1$. Then we have the following.*

1) If $4 \geq 16h^4(2n+1)$, let μ be such that

$$\begin{cases} \mu \geq \dfrac{4(n+1)^2+1}{8}h + \dfrac{1}{4h} - \dfrac{\sqrt{4(n+1)^2h^4-1}}{4h} \\ \mu \leq \dfrac{4(n+1)^2+1}{8}h + \dfrac{1}{4h} + \dfrac{\sqrt{4(n+1)^2h^4-1}}{4h}. \end{cases} \tag{93}$$

Then μ cannot be an eigenvalue for P, associated to an even eigenfunction.

2) If $4 < 16h^4(2n+1)$, let μ be such that

$$\begin{cases} \mu \geq \dfrac{(4n^2+1)}{8}h + \dfrac{1}{4h} + \dfrac{\sqrt{4n^2h^4+1}}{4h} \\ \mu \leq \dfrac{4(n+1)^2+1}{8}h + \dfrac{1}{4h} + \dfrac{\sqrt{4(n+1)^2h^4-1}}{4h}. \end{cases} \tag{94}$$

Then μ cannot be an eigenvalue for P, associated to an even eigenfunction.

Proof. Note that the hypoteses of 1) and 2) imply that μ fulfills (92). Notice, furthermore, that $n \geq \frac{1}{2h^2} - 1$ is a necessary condition for the estimates (93) and (94) to make sense. In fact this condition assures that the radicand which appears in these expressions is greater than or equal to 0.

1) If $4 \geq 16h^4(2n+1)$ then the condition of Theorem 6.5, $R_n(\mu) \geq 4$, is equivalent to $|\delta_n^+ \delta_{n+1}^+| > 4$, by Proposition 6.2. This last relation can be written as $-\delta_n^+ \delta_{n+1}^+ > 4$, because $\delta_n^+ < 0$ and $\delta_{n+1}^+ > 0$. Thus, by (84) we obtain

$$(8\mu h)^2 - 8\mu h\big[(8(n+1)^2+2)h^2+4\big] + \big[4(n+1)^2-1\big]^2 h^4 + \big[16(n+1)^2+4\big]h^2 + 8 < 0$$

and from here 1).

2) Similarly to 1) by Proposition 6.2 we have that if $4 < 16h^4(2n+1)$ then the condition of Theorem 6.5, $R_n(\mu) \geq 4$, is equivalent to

$$a \leq \mu \leq b, \tag{95}$$

where a is the maximal solution of the equation $|\delta_n^+ \delta_{n-1}^+| = 4$ and b is the maximal solution of the equation $|\delta_n^+ \delta_{n+1}^+| = 4$. This follows immediatly from Proposition 6.2 and from (84). We compute a and b. By (84) we have $|\delta_n^+ \delta_{n-1}^+| = 4$ if

$$(8\mu h)^2 - 8\mu h\big[(8n^2+2)h^2+4\big] + \left(4n^2-1\right)^2 h^4 + \left(16n^2+4\right)h^2 = 0,$$

thus

$$a = \frac{(4n^2+1)}{8}h + \frac{1}{4h} + \frac{\sqrt{4n^2h^4+1}}{4h}.$$

The computation of b has been already done in the proof of 1). Replacing the values of a and b in (95) we get the assertion. □

The approach of this section applies, in a similar way, to eigenvalues associated to odd eigenfunctions. In this way we get estimates similar to those of Theorem 6.6. We just state an analogous theorem, this time about odd eigenfunctions.

Theorem 6.7. *Let n be a natural number such that $n \geq \frac{1}{2h^2} - \frac{1}{2}$.*

1) If $4 \geq 32nh^4$, let μ be such that it satisfies

$$\begin{cases} \mu \geq \dfrac{n^2+(n+1)^2}{4}h + \dfrac{1}{4h} - \dfrac{\sqrt{(2n+1)^2h^4-1}}{4h} \\ \mu \leq \dfrac{n^2+(n+1)^2}{4}h + \dfrac{1}{4h} + \dfrac{\sqrt{(2n+1)^2h^4-1}}{4h}. \end{cases} \tag{96}$$

Then μ cannot be an eigenvalue of P, associated to an odd eigenfunction.

2) If $4 < 32nh^4$, let μ be such that it satisfies

$$\begin{cases} \mu \geq \dfrac{n^2+(n-1)^2}{4}h + \dfrac{1}{4h} + \dfrac{\sqrt{(2n-1)^2h^4+1}}{4h} \\ \mu \leq \dfrac{n^2+(n+1)^2}{4}h + \dfrac{1}{4h} + \dfrac{\sqrt{(2n+1)^2h^4-1}}{4h}. \end{cases} \tag{97}$$

Then μ cannot be an eigenvalue of P, associated to an odd eigenfunction.

From Theorems 6.6 and 6.7 it follows that the eigenvalues of P belong to the union of an infinite number of intervals. In particular we have the following

Corollary 6.8. *Let $n_0 \in \mathbb{N}$ be such that $n_0 \geq \frac{1}{2h^2} - 1$, and let $\mathrm{Spec}_+(P)$ denote the set of all eigenvalues of P associated with even eigenfunctions.*

Then, upon setting

$$C_n = \frac{4(n+1)^2+1}{8}h + \frac{1}{4h} - \frac{\sqrt{4(n+1)^2h^4-1}}{4h},$$

$$D_n = \frac{4(n+1)^2+1}{8}h + \frac{1}{4h} + \frac{\sqrt{4(n+1)^2h^4-1}}{4h},$$

$$E_n = \frac{(4n^2+1)}{8}h + \frac{1}{4h} + \frac{\sqrt{4n^2h^4+1}}{4h},$$

$$F_n = \frac{4(n+1)^2+1}{8}h + \frac{1}{4h} + \frac{\sqrt{4(n+1)^2h^4-1}}{4h},$$

there exists $n_1 \in \mathbb{N}$, with $n_1 \geq \max\left\{n_0+1, \frac{1}{8h^4} - \frac{1}{2}\right\}$, such that

$$\mathrm{Spec}_+(P) \cap [C_{n_0}, +\infty) \subset \left(\bigcup_{n=n_0}^{n_1-1} (D_n,\ C_{n+1})\right) \cup \left(\bigcup_{n=n_1}^{+\infty} (F_n,\ E_{n+1})\right). \quad (98)$$

Furthermore, all the intervals appearing in (98) are pairwise disjoint and the length $(E_{n+1} - F_n) \to 0$ as $n \to +\infty$. By writing (F_n, E_{n+1}) in terms of their center, denoted by T_n, we get $(F_n, E_{n+1}) = (T_n - U_n, T_n + U_n)$, with

$$T_n = \frac{4(n+1)^2+1}{8}h + \frac{1}{4h} + \frac{\sqrt{4(n+1)^2h^4+1} + \sqrt{4(n+1)^2h^4-1}}{8h},$$

$$U_n = \frac{\sqrt{4(n+1)^2h^4+1} - \sqrt{4(n+1)^2h^4-1}}{8h}, \qquad \lim_{n\to+\infty} U_n = 0.$$

Proof. It follows immediatly from Theorem 6.6. □

An analogous result holds for eigenvalues associated to odd eigenfunctions.

Corollary 6.9. *Let $n_0 \in \mathbb{N}$ be such that $n_0 \geq \frac{1}{2h^2} - \frac{1}{2}$ and let $\mathrm{Spec}_-(P)$ denote the set of all the eigenvalues of P associated to odd eigenfunctions. Then, putting*

$$G_n = \frac{n^2+(n+1)^2}{4}h + \frac{1}{4h} - \frac{\sqrt{(2n+1)^2h^4-1}}{4h},$$

$$H_n = \frac{n^2+(n+1)^2}{4}h + \frac{1}{4h} + \frac{\sqrt{(2n+1)^2h^4-1}}{4h},$$

$$L_n = \frac{n^2+(n-1)^2}{4}h + \frac{1}{4h} + \frac{\sqrt{(2n-1)^2h^4+1}}{4h},$$

$$M_n = \frac{n^2+(n+1)^2}{4}h + \frac{1}{4h} + \frac{\sqrt{(2n+1)^2h^4-1}}{4h},$$

there exists $n_1 \in \mathbb{N}$, with $n_1 \geq \max\left\{n_0+1, \frac{1}{8h^4}\right\}$, such that

$$\mathrm{Spec}_-(P) \cap [G_{n_0}, +\infty) \subset \left(\bigcup_{n=n_0}^{n_1-1} (H_n,\ G_{n+1})\right) \cup \left(\bigcup_{n=n_1}^{+\infty} (M_n,\ L_{n+1})\right). \quad (99)$$

Furthermore, all the intervals appearing in (99) are pairwise disjoint and the length $(L_{n+1} - M_n) \to 0$ as $n \to +\infty$. By writing (M_n, L_{n+1}) in terms of their center, denoted by V_n, we get $(M_n, L_{n+1}) = (V_n - W_n, V_n + W_n)$, with

$$V_n = \frac{n^2 + (n+1)^2}{4} h + \frac{1}{4h} + \frac{\sqrt{(2n+1)^2 h^4 - 1} + \sqrt{(2n+1)^2 h^4 + 1}}{8h},$$

$$W_n = \frac{\sqrt{(2n+1)^2 h^4 + 1} - \sqrt{(2n+1)^2 h^4 - 1}}{8h}, \qquad \lim_{n \to +\infty} W_n = 0.$$

It is easy to see that there exists $n_2 \in \mathbb{N}$ such that $(F_n,\ E_{n+1})$, appearing in (98), do not intersect $(M_n,\ L_{n+1})$, appearing in (99), for all $n \geq n_2$.

It is worth to recall that there are classical asymptotic estimates of large eigenvalues of general Sturm-Liouville problems, (see e.g. [4], pp. 270-273) which can be applied to our case. Although our approach is not general, since it is tied to a particular Sturm-Liouville problem, it makes an interesting use of the continued fractions. Besides, the estimates proved using the continued fractions approach give a more precise result than the classical asymptotics (notice that asymptotics given in [4] depend on the eigenfunction, whereas the bounds C_n to F_n, G_n to M_n are explicit).

7 Uniform convergence of the eigenfunction coefficients

It is known that some classes of parameter-dependent operators admit asymptotic expansions (in the same parameter) for their eigenvalues. We recall a result about these expansions (see [1], pp. 39 and 41) which can be applied to the operator $\widetilde{P}$, defined by $\widetilde{P} = 2hP$, $D(\widetilde{P}) = D(P)$.

Recall the definition of P :

$$P\,(h) := P : D(P) \longrightarrow L^2(I), \qquad Pf = -\frac{h}{2} f'' + \frac{1}{h} V f,$$

where

$$V(x) = \frac{1}{2} \sin^2\left(\frac{x}{2}\right), \qquad D(P) = H_0^1(I) \cap H^2(I) \subset L^2(I), \qquad I = (-\pi, \pi).$$

Then if we multiply the eigenvalue equation for P by $2h$, we get

$$\widetilde{P} f = -h^2 f'' + 2Vf = 2\mu h f, \quad f \in D(P), \quad \mu \in \mathbb{C}. \tag{100}$$

Therefore, μ is an eigenvalue of P if and only if $2\mu h$ is an eigenvalue of $\widetilde{P}$.

To fix notation we give the following

Definition 7.1. *We put $\widetilde{P} = \widetilde{P}(h) = 2hP(h)$, $D(\widetilde{P}) = D(P)$, and we set $\widetilde{V}(x) = \sin^2\left(\frac{x}{2}\right)$.*

Now we recall the theorem, by Helffer and Sjöstrand, that gives the asymptotic expansion of eigenvalues of a general class of operators, which contains $\widetilde{P}$ (we just state this result in our particular case, for the general case see [1], pp. 39 and 41).

Theorem 7.2. *Let $\widetilde{P}_0$ be the harmonic oscillator, in $L^2(\mathbb{R})$,*

$$\widetilde{P}_0(h)f = -h^2 f'' + \frac{1}{4}x^2 f$$

and let

$$\{E_n\}_{n\in\mathbb{N}} := \left\{\frac{2n+1}{2}\right\}_{n\in\mathbb{N}}$$

be the sequence of eigenvalues of $\widetilde{P}_0(1)$. Fix $0 < C_0 \notin \{E_0, E_1, \dots\}$ and let $N_0 \in \mathbb{N}$ be such that $E_{N_0-1} < C_0 < E_{N_0}$.

Then there exists $h_0 > 0$ such that for $0 < h \leq h_0$, $\widetilde{P}$ has precisely N_0 eigenvalues $0 < \widetilde{\lambda}_0(h) \leq \dots \leq \widetilde{\lambda}_{N_0-1}(h)$ in $[0, C_0 h]$. Moreover, $\widetilde{\lambda}_n$ has an asymptotic expansion

$$\widetilde{\lambda}_n(h) \sim h(E_n + a_1 h + a_2 h^2 + \dots), \qquad a_n \in \mathbb{R}, \quad h \to 0^+. \tag{101}$$

Notice that $\frac{1}{4}x^2$ represents the first non-zero term in the Taylor expansion of $\widetilde{V}(x)$ at $x = 0$.

It is interesting to see if this general theorem can be proved in our particular, one-dimensional case, using simpler techniques. In what follows we give a partial answer to this question, by analysing the case of the lowest eigenvalue of $\widetilde{P}$, which we will denote by $\widetilde{\mu}_0$. The main tool used to this aim is once again the continued fractions theory. In particular we will prove the monotonicity of $\widetilde{\mu}_0(h)$ with respect to h, from which it will follow the existence of $\lim_{h\to 0^+} \widetilde{\mu}_0(h)$. We want to notice here that the limit $h \to 0$ is not a perturbative limit for the problem, so that we cannot use directly the classic results of Perturbation Theory.

For the sake of completeness we state the asymptotic expansion for $\widetilde{\mu}_0(h)$ as follows from (101) of Theorem 7.2:

$$\widetilde{\mu}_0(h) \sim h\left(\frac{1}{2} + a_1 h + a_2 h^2 + \dots\right), \qquad a_n \in \mathbb{R}.$$

Recalling Definition 7.1, as $\widetilde{\mu}_0(h) = 2\mu_0(h)h$, where μ_0 is the lowest eigenvalue of P, we have

$$\mu_0(h) \sim \frac{1}{4} + \frac{a_1}{2}h + \frac{a_2}{2}h^2 + \dots, \qquad a_n \in \mathbb{R}.$$

Now we fix $h_0 \in \mathbb{R}$, with $h_0 > 0$. We will show that $\frac{d}{dh}\widetilde{\mu}_0(h)|_{h=h_0}$ is positive for every $h_0 > 0$, from this the monotonicity of $\widetilde{\mu}_0(h)$, with respect to h, will follow (and from here the existence of $\lim_{h\to 0^+} \widetilde{\mu}_0(h)$). Since we will analyse $\frac{d}{dh}\widetilde{\mu}_0(h)|_{h=h_0}$ we assume that $|h - h_0|$ is small, so that we can use the Perturbation Theory (for a reference see [5]). Through this approach we will prove uniform estimates on coefficients of the eigenfunction associated with $\widetilde{\mu}_0(h)$, for h in a *complex* neighbourhood of h_0. Then, using an integral equation which relates $\widetilde{\mu}_0(h)$ and its associated eigenfunction, we will get information on $\widetilde{\mu}_0(h)$.

Now we recall two classical results of Perturbation Theory which will be used for this aim (see [5], p. 377, 392).

Theorem 7.3. *Let $T^{(0)}$ be a closable operator from X to Y, with $D(T^{(0)}) = D$. Let $T^{(n)}$, $n = 1, 2, \ldots$, be operators from X to Y with domains containing D, and let there be constants a, b, $c \geq 0$ such that*

$$\left\| T^{(n)} u \right\| \leq c^{n-1}(a\|u\| + b\|T^{(0)}u\|), \qquad u \in D,\ n = 1, 2, \ldots \ . \tag{102}$$

Then the series

$$T(\xi)u = T^{(0)}u + \xi T^{(1)}u + \xi^2 T^{(2)}u + \ldots, \qquad u \in D$$

defines an operator $T(\xi)$ with domain D for $|\xi| < \frac{1}{c}$. If $|\xi| < (b+c)^{-1}$ then $T(\xi)$ is closable and the closures for such ξ form a holomorphic family of type (A).

Theorem 7.4. *Let $T(\xi)$ a selfadjoint holomorphic family of type (A), defined in a neighborhood of an interval I_0 of the real axis. Furthermore, let $T(\xi)$ have compact resolvent for every ξ. Then all eigenvalues of $T(\xi)$ can be represented by functions which are holomorphic on I_0. More precisely, there is a sequence of scalar-valued functions $\mu_n(\xi)$ and a sequence of vector-valued functions $\varphi_n(\xi)$, all holomorphic on I_0, such that for every $\xi \in I_0$, the $\mu_n(\xi)$ represent all the repeated eigenvalues of $T(\xi)$ and the $\varphi_n(\xi)$ form a complete orthonormal family of the associated eigenvectors of $T(\xi)$.*

Consider now $-\frac{d^2}{dx^2}$ in $L^2(I)$ with domain $D(P)$. We can write

$$\widetilde{P} = \widetilde{P}(h) = -h^2 \frac{d^2}{dx^2} + \widetilde{V} = \widetilde{P}(h_0) + (h^2 - h_0^2)\left(-\frac{d^2}{dx^2}\right). \tag{103}$$

From (103) we can use Theorem 7.3 to show that the $h-$dependent family of operators $\widetilde{P} = \widetilde{P}(h)$ forms an holomorphic family of type (A) in the parameter $(h^2 - h_0^2)$.

Proposition 7.5. *The family of operators $\widetilde{P} = \widetilde{P}(h)$ (see Definition 7.1) is a selfadjoint holomorphic family of type (A) in the perturbative parameter $(h^2 - h_0^2)$.*

Proof. By (103) it suffices to prove that there exist $a, b \geq 0$ such that $\| -f''\| \leq a\|f\| + b\|\widetilde{P}(h_0)f\|$. We have

$$\| -f''\| = \frac{1}{h_0^2}\| -h_0^2 f'' + \widetilde{V}f - \widetilde{V}f\| \leq \frac{1}{h_0^2}(\|P(h_0)f\| + \|f\|), \tag{104}$$

where the last inequality follows from $\max_{|x|\leq\pi} \widetilde{V}(x) = \max_{|x|\leq\pi}\left[\sin^2\left(\frac{x}{2}\right)\right] = 1$. Thus, by the same theorem, $\widetilde{P} = \widetilde{P}(h)$ forms an holomorphic family of type (A) in $(h^2 - h_0^2)$, for $|h^2 - h_0^2| < h_0^2$. Moreover we have that $\widetilde{P}(h)$ is selfadjoint (see [5], p. 385). □

By Proposition 7.5 and Theorem 7.4 we can expand all eigenfunctions and eigenvalue of $\widetilde{P}$ in power series of the perturbative parameter $(h^2 - h_0^2)$. Notice that these series are defined for complex values of the perturbative parameter, thus we will consider, from now on, h as a *complex* parameter, varying in a neighbourhood of the real parameter h_0. From these expansions uniform estimates on coefficients of the eigenfunction in the *same* complex neighbourhood of h_0 will follow.

We give the expansion for the lowest eigenvalue $\widetilde{\mu}_0$ and its associated eigenfunction, $\widetilde{\psi}$.

Proposition 7.6. *Let $\widetilde{\mu}_0$ be the lowest eigenvalue of $\widetilde{P}$ (see Definition 7.1). Then, for every $h \in \mathbb{C}$ such that $|h^2 - h_0^2| < h_0^2$, $\widetilde{\mu}_0 = \widetilde{\mu}_0(h)$ admits the following power series expansion*

$$\widetilde{\mu}_0 = \sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \widetilde{\mu}_{0n}. \tag{105}$$

Let $\widetilde{\psi} = \widetilde{\psi}(h)$ be the eigenfunction associated with $\widetilde{\mu}_0$. We have, for every $h \in \mathbb{C}$ such that $|h^2 - h_0^2| < h_0^2$, that $\widetilde{\psi}$ admits the following expansion

$$\widetilde{\psi} = \sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \widetilde{\psi}_n, \qquad \widetilde{\psi}_n \in L^2(I).$$

We prove next some technical results which give estimates for coefficients of the expansions of $\widetilde{\psi}$ and $\widetilde{\mu}_0$. Later on we will write $\widetilde{\mu}_0$ in terms of its associated eigenfunction $\widetilde{\psi}$ and we will use these estimates to obtain information about the monotonicity of $\widetilde{\mu}_0(h)$. Now we show the convergence to 0, as $n \to +\infty$, of the coefficients $\widetilde{\mu}_{0n}$ is uniform on the set $\left\{h \in \mathbb{C}; |h^2 - h_0^2| \leq \alpha^2\right\}$, for some $\alpha > 0$. For future reference we give the following

Definition 7.7. *Let $\alpha > 0$ such that $\alpha < h_0$. We denote by $S_\alpha(h_0)$ the following set*

$$S_\alpha(h_0) = \left\{h \in \mathbb{C};\ |h^2 - h_0^2| \leq \alpha^2\right\}.$$

Lemma 7.8. *Fix $h_0 > 0$. Let $\widetilde{\mu}_0(h)$ be the lowest eigenvalue of $\widetilde{P}(h)$. Then there exist C, α, $\alpha_1 > 0$, with $0 < \alpha < \alpha_1 < h_0$, such that*

$$|\widetilde{\mu}_0(h)| \leq \frac{C\alpha_1^2}{\alpha_1^2 - \alpha^2}, \qquad \forall\ h \in S_\alpha(h_0).$$

Proof. By equation (105) of Proposition 7.6, if we fix $\alpha_1 > 0$ such that $0 < \alpha_1^2 < h_0^2 \leq \rho^2$ there exists $C > 0$ such that

$$|\widetilde{\mu}_{0n}| \leq C \left|\frac{1}{\alpha_1^2}\right|^n, \qquad \forall\ n \in \mathbb{N}$$

(see e. g. [6], p. 56). Therefore equation (105) gives

$$|\widetilde{\mu}_0(h)| \leq \sum_{n=0}^{+\infty} |h^2 - h_0^2|^n\ C \left|\frac{1}{\alpha_1^2}\right|^n. \tag{106}$$

Now we fix $\alpha > 0$ such that $0 < \alpha < \alpha_1$. Therefore, for all h such that $|h^2 - h_0^2| \leq \alpha^2$, the right-hand side of (106) can be dominated by a geometric series. Computing the value of this series gives the assertion. □

As already noticed the eigenfunction $\widetilde{\psi}$, associated to the lowest eigenvalue $\widetilde{\mu}_0$, is even and does not vanish on the interior of I. Since $\widetilde{\psi}$ is analytic in $h^2 - h_0^2$, (see Proposition 7.6), we can give the following expansion for $\widetilde{\psi}$

$$\widetilde{\psi}(h, x) = \sum_{m=0}^{+\infty} \left(\sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \widetilde{\psi}_{mn}\right) \frac{1}{\sqrt{\pi}} \cos\left(\frac{2m+1}{2} x\right). \tag{107}$$

Recalling the notation fixed in Section 2 (see equation (3)) we will write

$$\sum_{n=0}^{+\infty}(h^2-h_0^2)^n\widetilde{\psi}_{mn}=v_m^+=v_m^+(h). \tag{108}$$

Furthermore, as $\widetilde{\psi}$ is an eigenfunction of $\widetilde{P}$ associated to $\widetilde{\mu}_0$, then $\widetilde{\psi}$ is an eigenfunction also of P, associated to the eigenvalue $\widetilde{\mu}_0/2h$, which is the lowest eigenvalue of P. Thus, by Proposition 2.4 and Remark 2.6, using the notation fixed by (108), we have that

$$v_{-1}^+:=0 \qquad v_{n+1}^+=\delta_n^+v_n^+-v_{n-1}^+, \qquad n\in\mathbb{N}, \tag{109}$$

where

$$\begin{cases} \delta_0^+=\delta_0^+\left(\dfrac{\widetilde{\mu}_0}{2h}\right)=h^2+1-4\widetilde{\mu}_0 \\ \delta_n^+=\delta_n^+\left(\dfrac{\widetilde{\mu}_0}{2h}\right)=(2n+1)^2h^2+2-4\widetilde{\mu}_0, \qquad \forall\, n\in\mathbb{N}\backslash\{0\}. \end{cases} \tag{110}$$

From Lemma 7.8 follows an estimate on $\delta_n^+\left(\widetilde{\mu}_0/2h\right)$, which we will use, exploiting relation (109), to estimate the Fourier coefficients $v_m^+(h)$ (see (108)).

Lemma 7.9. *Fix $h_0>0$. Let $\widetilde{\mu}_0=\widetilde{\mu}_0(h)$ be the lowest eigenvalue of $\widetilde{P}(h)$. Then there exist α, with $0<\alpha<h_0$ and $n_0\in\mathbb{N}$ such that (recall (110))*

$$\left|\delta_n^+\left(\frac{\widetilde{\mu}_0}{2h}\right)\right|\geq 2, \qquad \forall\, n\geq n_0, \qquad \forall\, h\in S_\alpha(h_0). \tag{111}$$

Proof. If we prove $(2n+1)^2|h|^2-|4\widetilde{\mu}_0(h)-2|\geq 2$, by (110), we obtain as a consequence (111). From here, by the triangle inequality, if we prove that $(2n+1)^2|h|^2\geq 4+4|\widetilde{\mu}_0(h)|$ we get (111). By Lemma 7.8 it suffices to show that

$$(2n+1)^2|h|^2\geq 4+\frac{4C\alpha_1^2}{\alpha_1^2-\alpha^2}. \tag{112}$$

Notice that

$$\frac{1}{|h|^2}\leq\frac{1}{h_0^2-\alpha^2}, \qquad \forall\, h\in S_\alpha(h_0).$$

Therefore, since there exists $n_0\in\mathbb{N}$ such that

$$(2n+1)^2\geq\frac{1}{(h_0^2-\alpha^2)}\left[4+\frac{4C\alpha_1^2}{(\alpha_1^2-\alpha^2)}\right], \qquad \forall\, n\geq n_0, \tag{113}$$

the assertion follows. □

We will show the convergence to 0 of the coefficients $v_m^+=\sum_{n=0}^{+\infty}(h^2-h_0^2)^n\widetilde{\psi}_{mn}$ in (108), as $m\to+\infty$, uniformly for $|h^2-h_0^2|$ in the same neighborhood of 0. To this purpose we will reason as in the proof of Proposition 4.16. Recall that, as the eigenfunction associated to $\widetilde{\mu}_0$ is even, we have $\{\vartheta_n\}_n:=\{\delta_n^+\}_n$ (see Definition 3.3).

In the sequel we will consider the Fourier coefficients v_m^+ in (108) as *complex* functions in the parameter $(h^2-h_0^2)$, $h\in S_\alpha(h_0)$, for a fixed $\alpha>0$. To do this we just substitute a complex value of h in (110) and we compute the value for v_m^+ using the recurrence relation (109). Then, again using (109), we will show, as in the proof of Proposition 4.16, the following formula for v_m^+ :

$$v^+_{n_0+1+m} = \delta^+_{n_0+1} \dots \delta^+_{n_0+m} z_{n_0} \dots z_{n_0+m-1} v^+_{n_0+1}, \qquad \forall\, m > 0, \tag{114}$$

and for all *complex* h such that $|h^2 - h_0^2| \leq \alpha^2$, with

$$z_{n_0+m} = \cfrac{\cfrac{1}{\delta^+_{n_0+m+1}\delta^+_{n_0+m+2}}}{1 - \cfrac{\cfrac{1}{\delta^+_{n_0+m+2}\delta^+_{n_0+m+3}}}{1 - \ddots}}, \qquad \forall\, m \in \mathbb{N}, \quad \forall\, h \in S_\alpha(h_0). \tag{115}$$

We will prove in the first place that the functions $\delta^+_{n_0+m}\left(\frac{\widetilde{\mu}_0}{2h}\right) z_{n_0+m-1}\left(\frac{\widetilde{\mu}_0}{2h}\right)$ are holomorphic in $h^2 - h_0^2$. Afterwards, from relation (114), we will obtain an estimate on coefficients v_m^+, uniform with respect to $h^2 - h_0^2$.

Now we prove that equations (114) and (115) hold.

Proposition 7.10. *Let $\widetilde{\mu}_0$ be the lowest eigenvalue of $\widetilde{P}$ and let $\widetilde{\psi}$ be the associated eigenfunction given by (107):*

$$\widetilde{\psi}(h,x) = \sum_{m=0}^{+\infty} v_m^+(h) \frac{1}{\sqrt{\pi}} \cos\left(\frac{2m+1}{2}x\right), \qquad v_m^+(h) = \sum_{n=0}^{+\infty} (h^2 - h_0^2)^n \widetilde{\psi}_{mn}.$$

Then there exists $\beta \in \mathbb{R}$, $0 < \beta < h_0$, and $n_0 \in \mathbb{N}$ such that

$$v^+_{n_0+1+m} = \delta^+_{n_0+1} \dots \delta^+_{n_0+m} z_{n_0} \dots z_{n_0+m-1} v^+_{n_0+1}, \qquad \forall\, m > 0, \tag{116}$$

with

$$z_{n_0+m} = \cfrac{\cfrac{1}{\delta^+_{n_0+m+1}\delta^+_{n_0+m+2}}}{1 - \cfrac{\cfrac{1}{\delta^+_{n_0+m+2}\delta^+_{n_0+m+3}}}{1 - \ddots}}, \qquad \forall\, m \in \mathbb{N}, \quad h \in S_\beta(h_0). \tag{117}$$

Furthermore the functions $\delta^+_{n_0+m+1}(\widetilde{\mu}_0/2h)\, z_{n_0+m}(\widetilde{\mu}_0/2h)$ are holomorphic on $S_\beta(h_0)$, for all $m \in \mathbb{N}$.

Proof. We follow the proof of Proposition 4.16. Notice that, by Lemma 7.9 there exist α, with $0 < \alpha < h_0$, and $n_1 \in \mathbb{N}$ such that

$$\left|\delta_n^+\left(\frac{\widetilde{\mu}_0}{h^2}\right)\right| \geq 2, \qquad \forall\, n \geq n_1, \qquad \forall\, h \in S_\beta(h_0). \tag{118}$$

Recall that, by the recurrence relation (109), all v_n^+ are holomorphic in the parameter $h^2 - h_0^2$. By Corollary 5.11, as $\widetilde{\mu}_0(h_0)$ is the lowest eigenvalue of $\widetilde{P}$, we have that $v_m^+(h_0) \neq 0$ for all $m \in \mathbb{N}$. As $v_m^+(h)$ are holomorphic in $h^2 - h_0^2$ then

$$\lim_{h^2 \to h_0^2} v_m^+(h) = v_m^+(h_0) \neq 0, \qquad \forall\, m \in \mathbb{N}.$$

Thus there exist $n_0 > n_1$ and $0 < \beta < \alpha < h_0$ such that $v^+_{n_0}(h)$, $v^+_{n_0+1}(h)$, $v^+_{n_0+2}(h) \neq 0$, for all $h \in S_\beta(h_0)$. Therefore, since we chose $n_0 > n_1$ (so that $\delta^+_{n_0+1} \neq 0$), we have that the function appearing in Proposition 4.16, this time considered as complex valued,

$$1 - \frac{1}{\frac{v^+_{n_0+1}}{v^+_{n_0}} \delta^+_{n_0+1}}, \tag{119}$$

is holomorphic on $S_\beta(h_0)$. We recall equality (64), which holds for *real* h :

$$1 - \frac{1}{\frac{v^+_{n_0+1}}{v^+_{n_0}} \delta^+_{n_0+1}} = \frac{\frac{1}{\delta^+_{n_0+1}\delta^+_{n_0+2}}}{1 - \frac{\frac{1}{\delta^+_{n_0+2}\delta^+_{n_0+3}}}{1 - \ddots}}. \tag{120}$$

From what we have just proved the left-hand side of (120) makes sense also for complex value of h. The right-hand side of (120) makes sense too, in the same neighborhood of 0 in which the function (119) is holomorphic, that is for all h belonging to $S_\beta(h_0)$. In fact, as $n_0 > n_1$ we have

$$\left| \frac{1}{\delta^+_{n_0+m+1}\delta^+_{n_0+m+2}} \right| < \frac{1}{4}, \qquad \forall\, m \in \mathbb{N}, \quad \forall\, h \in S_\beta(h_0).$$

Thus, by Worpitzky's theorem (Theorem 4.14), we have that the continued fraction in the right-hand side of (120) converges for all h in $S_\beta(h_0)$. Moreover we will show that this function is analytic on $S_\beta(h_0)$. Therefore, on recalling that the left-hand side of (120) is analytic too, in the same neighbourhood of 0, and as (120) holds for real h, equality (120) will follow for all h in $S_\beta(h_0)$. We put

$$z_n = K_{i=n}^{+\infty} \left(\frac{-\frac{1}{\delta^+_{i+1}\delta^+_{i+2}}}{-1} \right). \tag{121}$$

By proving the analyticity of the functions z_n we will obtain also (116) and (117). In fact from Proposition 4.16 we know that (116) is true for real values of h. Furthermore, as already noticed, $v^+_{n_0+1}$ is holomorphic on $S_\beta(h_0)$ and $v^+_{n_0+1+m}$ is holomorphic in the same neighborhood, by the recurrence relation (109). Thus we can obtain (116) for all *complex* h, in $S_\beta(h_0)$, if we show that all $z_n(h)$ are holomorphic in the same set, for $n \geq n_0$. We prove this by showing that $\delta^+_{n+1} z_n$ are holomorphic for all $n \geq n_0$.

By (121) we have, writing an equivalent continued fraction (see Defintion 4.1)

$$\delta^+_{m+1} z_m = \frac{1}{\delta^+_{m+2} - \frac{1}{\delta^+_{m+3} - \ddots}}. \tag{122}$$

Let $f_n = \widetilde{A}_n/\widetilde{B}_n$ be the *n-th* approximant of the continued fraction in (122). As $\widetilde{A}_n$, $\widetilde{B}_n$ represent the *n-th* numerator and denominator for this continued fraction, from Remark 4.2, upon setting $\widetilde{A}_{-1} = 1$, $\widetilde{A}_0 = 0$, $\widetilde{B}_{-1} = 0$, $\widetilde{B}_0 = 1$, we have

$$\begin{cases} \widetilde{A}_n = \delta^+_{m+1+n}\widetilde{A}_{n-1} - \widetilde{A}_{n-2}, & n \geq 1, \\ \widetilde{B}_n = \delta^+_{m+1+n}\widetilde{B}_{n-1} - \widetilde{B}_{n-2}, & n \geq 1. \end{cases} \tag{123}$$

Through these relations, on recalling (110) and (105), we obtain that $\widetilde{A}_n$, $\widetilde{B}_n$ are holomorphic functions of $h^2 - h_0^2$. We prove that $\widetilde{B}_n$ is never 0 on $S_\beta(h_0)$ and that $\left\{\frac{\widetilde{A}_n}{\widetilde{B}_n}\right\}_n$ converges to $\delta^+_{m+1}z_m$, for all $m \geq n_0$, uniformly on every compact subset of $S_\beta(h_0)$. From here, it will follow that $\delta^+_{m+1}z_m$ are holomorphic on $S_\beta(h_0)$, for all $m \in \mathbb{N}$ (see e.g. [6], p. 156).

We recall that, as $n_0 > n_1$, for all $m \geq n_0$ and for all h in $S_\beta(h_0)$, we have $\left|\delta^+_m\left(\frac{\widetilde{\mu}_0}{2h}\right)\right| > 2$ (see equation (118)).

Notice that $|\widetilde{B}_1| = |\delta^+_{m+2}\widetilde{B}_0| = |\delta^+_{m+2}| > 2 > 1 = |\widetilde{B}_0|$. As $|\widetilde{B}_1| > |\widetilde{B}_0|$ and as $|\delta^+_m| > 2$, for all $m \geq n_0$, from the recurrence relation (123) we get $|\widetilde{B}_2| \geq 2|\widetilde{B}_1| - |\widetilde{B}_0| \geq |\widetilde{B}_1|$. We can use the same procedure inductively, as $|\delta^+_m| > 2$ for all $m \geq n_0$; so we find that $|\widetilde{B}_{n+1}| > |\widetilde{B}_n|$ for all n. Again from the recurrence relation we get

$$|\widetilde{B}_n| \geq |\delta^+_{m+1+n}||\widetilde{B}_{n-1}| - |\widetilde{B}_{n-2}| \geq 2|\widetilde{B}_{n-1}| - |\widetilde{B}_{n-2}|.$$

This implies that $|\widetilde{B}_n| - |\widetilde{B}_{n-1}| \geq 1$ for all n and thus $|\widetilde{B}_n| \geq n$. We prove the uniform convergence of $f_n = f_n(h^2 - h_0^2)$ (i.e. as functions of $h^2 - h_0^2$). If $n > j$ we have

$$|f_n - f_j| = \left|\sum_{k=1}^{n}(f_k - f_{k-1}) - \sum_{k=1}^{j}(f_k - f_{k-1})\right| \leq \sum_{k=j+1}^{n} |f_k - f_{k-1}|. \tag{124}$$

Notice that by relation (35) we have

$$f_k - f_{k-1} = \frac{A_k}{B_k} - \frac{A_{k-1}}{B_{k-1}} = \frac{(-1)^{k-1}}{B_k B_{k-1}} \prod_{j=1}^{k} a_j.$$

From here and from (124) it follows

$$|f_n(h^2 - h_0^2) - f_j(h^2 - h_0^2)| \leq \sum_{k=j+1}^{n} \frac{1}{|B_k||B_{k-1}|} \leq \sum_{k=j+1}^{n} \frac{1}{k(k-1)},$$

for all $n > j$ and for every h in $S_\beta(h_0)$.

Therefore we have the uniform convergence on all compact set of $S_\beta(h_0)$. From here the assertion follows. □

Now, by using (116) and (117), we obtain uniform estimates for v^+_n.

Proposition 7.11. *Let $\widetilde{\mu}_0$ be the lowest eigenvalue of $\widetilde{P}$ and let $\widetilde{\psi}$ be the associated eigenfunction. Using the notation of Proposition 7.10, there exists $\alpha \in \mathbb{R}$, $0 < \alpha < h_0$, such that the coefficients $v^+_m(h)$ tend to zero, as $m \to +\infty$, faster than any negative power of m and uniformly in $h \in S_\alpha(h_0)$.*

Proof. On recalling Proposition 7.10 and its proof we have that there exists $\beta \in \mathbb{R}$, $0 < \beta < h_0$, and $n_0 \in \mathbb{N}$ such that (116) and (117) hold true for all h in $S_\beta(h_0)$. Furthermore, for the same values of h, we have that $|\delta^+_{n_0+m}| > 2$ for all $m \in \mathbb{N}$.

Thus $\{z_m\}_m$ fulfills the hypothesis of Worpitzky's theorem (Theorem 4.14) and therefore we get $|z_m| \leq 1/2$, for every $m \geq n_0$ and for every h in $S_\beta(h_0)$. Then, from $|1 - z_{m+1}| \geq 1/2$, by recalling (116) and (117), we get

$$|v^+_{n_0+1+m}| \leq |v^+_{n_0+1}| \frac{2^m}{|\delta^+_{n_0+2} \cdots \delta^+_{n_0+m+1}|}. \tag{125}$$

As in the proof of Theorem 4.9, we write δ^+_m as

$$\delta^+_m = (2m+1)^2 h^2 \left(1 - \frac{4\widetilde{\mu}_0 - 2}{h^2(2m+1)^2}\right). \tag{126}$$

Plugging (126) into (125) gives

$$|v^+_{n_0+1+m}| \leq \frac{\left((n_0+2)!\right)^2 |v^+_{n_0+1}|}{|2h^2|^m \left((n_0+m+2)!\right)^2 \prod_{k=n_0+2}^{n_0+m+1} \left|1 - \frac{4\widetilde{\mu}_0 - 2}{h^2(2k+1)^2}\right|} \tag{127}$$

We have (upon possibly increasing n_0)

$$\prod_{k=n_0+2}^{n_0+m+1} \left|1 - \frac{4\widetilde{\mu}_0 - 2}{h^2(2k+1)^2}\right| \geq \prod_{k=n_0+2}^{n_0+m+1} \left(1 - \frac{|4\widetilde{\mu}_0 - 2|}{|h^2|(2k+1)^2}\right). \tag{128}$$

From Lemma 7.8, by assuming, without loss of generality, that $\alpha < \alpha_1 < \beta$, we have

$$\frac{1}{|h^2|}|4\widetilde{\mu}_0 - 2| \leq \frac{1}{|h^2|}(|4\widetilde{\mu}_0| + 2) \leq \frac{1}{h_0^2 - \alpha^2}\left(\frac{4C\alpha_1^2}{\alpha_1^2 - \alpha^2} + 2\right) \tag{129}$$

From (128) and (129), upon possibly enlarge n_0, we get

$$\prod_{k=n_0+2}^{n_0+m+1} \left|1 - \frac{4\widetilde{\mu}_0 - 2}{h^2(2k+1)^2}\right| \geq \prod_{k=n_0+2}^{n_0+m+1} \left[1 - \frac{\frac{4C\alpha_1^2}{\alpha_1^2 - \alpha^2} + 2}{(h_0^2 - \alpha^2)(2k+1)^2}\right] > 0. \tag{130}$$

Thus, upon setting

$$\widetilde{D}_m := \prod_{k=n_0+2}^{n_0+m+1} \left[1 - \frac{\frac{4C\alpha_1^2}{\alpha_1^2 - \alpha^2} + 2}{(h_0^2 - \alpha^2)(2k+1)^2}\right], \widetilde{C}_m := \frac{2\pi(m+1)^{2(m+1)+1} e^{-2(m+1)}}{\left((m+1)!\right)^2}$$

we use the same procedure of the proof of Theorem 4.9. In this way, by (130) and (127), we get

$$|v^+_{n_0+1+m}| \leq \frac{|v^+_{n_0+1}| \ \left((n_0+2)!\right)^2 \widetilde{C}_m e^2}{\left(2\left|\frac{h\,(m+1)}{e}\right|^2\right)^m \widetilde{D}_m \, 2\,\pi\,(m+1)^3}. \tag{131}$$

As we assumed that $|h^2 - h_0^2| \leq \alpha^2$, from (131) it follows that

$$|v_{n_0+1+m}^+| \leq \frac{|v_{n_0+1}^+| \; ((n_0+2)!)^2 \; \widetilde{C}_m e^2}{\left(2(h_0^2-\alpha^2)\left(\dfrac{m+1}{e}\right)^2\right)^m \widetilde{D}_m \, 2\,\pi\,(m+1)^3}. \tag{132}$$

Notice that, since v_m^+ verifies the recurrence relation

$$v_{m+1}^+ = \delta_m^+ v_m^+ - v_{m-1}^+, \qquad \forall\, m \in \mathbb{N},$$

we have that $v_{n_0+1}^+ = v_{n_0+1}^+(h)$ represents an analytic function in $(h^2 - h_0^2)$. Thus, upon possibly shrinking α, the value of $|v_{n_0+1}^+|$ is bounded for all h in $S_\alpha(h_0)$. Thus, by (132), if we notice that α and h_0 are fixed the assertion follows. □

We re-write inequality (132) in a simpler form.

Corollary 7.12. *In the hypotheses of Proposition 7.11 there exist D, $\alpha > 0$ such that*

$$|v_m^+| = |v_m^+(h)| = \left|\sum_{n=0}^{+\infty}(h^2-h_0^2)^n \widetilde{\psi}_{mn}\right| \leq \frac{D}{m^m}, \;\; \forall\, h \in S_\alpha(h_0), \;\; \forall\, m \in \mathbb{N}. \tag{133}$$

Notice that the constant D is independent of m.

8 Monotonicity of $\widetilde{\mu}_0(h)$

Using (133) we will prove some other needed estimates on coefficients of the eigenfunction $\widetilde{\psi}$ and its derivative with respect to h. These estimates, together with the Picone identity (see e.g. [4], p.226), which links $\widetilde{\mu}_0(h)$ and its associated eigenfunction, will be used later on to show the monotonicity of $\widetilde{\mu}_0(h)$, with respect to h. To this purpose we recall a result on analytic functions (see [8], p. 6).

Proposition 8.1. *Let f be an holomorphic function in U and let $|f(z)| \leq M$ for every $z \in U$. Then for any compact set $K \subset U$ and any α we have*

$$|D^\alpha f(z)| \leq M\alpha!\delta^{-|\alpha|} \qquad \forall\, z \in K,$$

where δ is the distance of K from the boundary of U.

Now we set $\zeta = h^2 - h_0^2$ and consequently write $v_m^+ = v_m^+(\zeta)$.

Using Corollary 7.12 and Proposition 8.1 we can immediatly prove an estimate on $\frac{d}{d\zeta}v_m^+(\zeta)$.

Proposition 8.2. *Using the notation of Corollary 7.12 let γ be such that $0 < \gamma < \alpha$. Then we have*

$$\left|\frac{d}{d\zeta}v_m^+(\zeta)\right| = \left|\frac{d}{d\zeta}\sum_{n=0}^{+\infty}\zeta^n\widetilde{\psi}_{mn}\right| \leq \frac{D}{(\alpha^2-\gamma^2)m^m}, \qquad \forall\, \zeta \in \mathbb{C}, \quad |\zeta| \leq \gamma^2.$$

From here, by the mean value theorem, it follows an estimate for $|v_m^+(\zeta) - v_m^+(0)|$.

From now on we will consider again the parameter h (and thus ζ) as a *real* number.

Lemma 8.3. *Using the notation of Corollary 7.12 let $0 < \gamma < \alpha$. We have, when ζ is real,*

$$|v_m^+(\zeta) - v_m^+(0)| \leq \frac{|\zeta|\ D}{(\alpha^2 - \gamma^2)\, m^m} = \frac{D\ |h^2 - h_0^2|}{(\alpha^2 - \gamma^2)\, m^m}$$

Now we use the Picone identity (see e.g. [4], p.226), to get an expression of $\widetilde{\mu}_0(h)$ which we will use in computing $\frac{d}{dh}\widetilde{\mu}_0(h)|_{h=h_0}$.

Remark 8.4. *Let $\widetilde{\mu}_0(h)$ and $\widetilde{\mu}_0(h_0)$ represent the lowest eigenvalue of the operators $\widetilde{P}(h)$ and $\widetilde{P}(h_0)$, respectively, and let $\widetilde{\psi}(h)$, $\widetilde{\psi}(h_0)$ be the associated eigenfunctions. Assume also that these eigenfunctions are normalized, so that*

$$\|\widetilde{\psi}(h)\|^2 = \|\widetilde{\psi}(h_0)\|^2 = 1. \tag{134}$$

Then we have (with $\widetilde{\psi}' = \frac{d\widetilde{\psi}}{dx}$)

$$\widetilde{\mu}_0(h) - \widetilde{\mu}_0(h_0) = (h^2 - h_0^2) \int_{-\pi}^{\pi} (\widetilde{\psi}'(h))^2 dx +$$

$$+h_0^2 \int_{-\pi}^{\pi} (\widetilde{\psi}'(h) - \widetilde{\psi}'(h_0))^2 dx + h_0^2 \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) \right]^2 dx +$$

$$+2h_0^2 \int_{-\pi}^{\pi} (\widetilde{\psi}'(h) - \widetilde{\psi}'(h_0)) \widetilde{\psi}'(h_0) \left(\frac{\widetilde{\psi}(h_0) - \widetilde{\psi}(h)}{\widetilde{\psi}(h_0)} \right) dx. \tag{135}$$

Proof. By hypothesis we have that $\widetilde{\psi}(h_0)$ is a solution of the problem

$$\begin{cases} \widetilde{P}(h)y = -h^2 y'' + V(x)y = \widetilde{\mu}_0(h)y \\ y(\pm\pi) = 0 \end{cases} \tag{136}$$

and $\widetilde{\psi}(h_0)$ is a solution of the problem

$$\begin{cases} \widetilde{P}(h_0)z = -h_0^2 z'' + V(x)z = \widetilde{\mu}_0(h_0)z \\ z(\pm\pi) = 0. \end{cases} \tag{137}$$

Then, by Picone's identity

$$0 = \int_{-\pi}^{\pi} (\widetilde{\mu}_0(h_0) - \widetilde{\mu}_0(h)) \widetilde{\psi}(h)^2 dx + \int_{-\pi}^{\pi} (h^2 - h_0^2)(\widetilde{\psi}'(h))^2 dx +$$

$$+ \int_{-\pi}^{\pi} h_0^2 \left[\widetilde{\psi}'(h) - \frac{\widetilde{\psi}(h)\widetilde{\psi}'(h_0)}{\widetilde{\psi}(h_0)} \right]^2 dx.$$

Thus, recalling (134)

$$\widetilde{\mu}_0(h) - \widetilde{\mu}_0(h_0) = \int_{-\pi}^{\pi} (h^2 - h_0^2)(\widetilde{\psi}'(h))^2 dx + \int_{-\pi}^{\pi} h_0^2 \left[\widetilde{\psi}'(h) - \frac{\widetilde{\psi}(h)\widetilde{\psi}'(h_0)}{\widetilde{\psi}(h_0)} \right]^2 dx.$$

Adding and subtracting $\widetilde{\psi}'(h_0)$ in the second intergral gives the assertion. □

We will analyse

$$\frac{d}{dh}\widetilde{\mu}_0(h)\Big|_{h=h_0} = \lim_{h\to h_0} \frac{\widetilde{\mu}_0(h)-\widetilde{\mu}_0(h_0)}{h-h_0}, \quad h_0>0.$$

By (135) we have

$$\lim_{h\to h_0} \frac{\widetilde{\mu}_0(h)-\widetilde{\mu}_0(h_0)}{h-h_0} = \lim_{h\to h_0}(h+h_0)\int_{-\pi}^{\pi}(\widetilde{\psi}'(h))^2dx+$$

$$+\lim_{h\to h_0}\frac{h_0^2}{h-h_0}\int_{-\pi}^{\pi}(\widetilde{\psi}'(h)-\widetilde{\psi}'(h_0))^2dx+$$

$$+\lim_{h\to h_0}\frac{h_0^2}{h-h_0}\int_{-\pi}^{\pi}\left[\widetilde{\psi}'(h_0)\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{\widetilde{\psi}(h_0)}\right)\right]^2dx+$$

$$+\lim_{h\to h_0}\frac{2h_0^2}{h-h_0}\int_{-\pi}^{\pi}(\widetilde{\psi}'(h)-\widetilde{\psi}'(h_0))\widetilde{\psi}'(h_0)\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{\widetilde{\psi}(h_0)}\right)dx. \tag{138}$$

We will prove that the left-hand side of (138) is greater than 0, which will show the monotonicity of $\widetilde{\mu}_0(h)$.

Recalling the notation used up to now, by (107) and (108) we have

$$\widetilde{\psi}(h,x) = \sum_{m=0}^{+\infty}\frac{v_m^+}{\sqrt{\pi}}\cos\left(\frac{2m+1}{2}x\right), \quad v_m^+ = v_m^+(h) = \sum_{n=0}^{+\infty}(h^2-h_0^2)^n\widetilde{\psi}_{mn}; \tag{139}$$

and we have

$$\widetilde{\psi}(h_0,x) = \sum_{m=0}^{+\infty}\widetilde{\psi}_{m0}\frac{1}{\sqrt{\pi}}\cos\left(\frac{2m+1}{2}x\right). \tag{140}$$

In order to compute the derivatives of $\widetilde{\psi}$, appearing in (138), notice at first that from Proposition 7.11 and Corollary 7.12 we have $v_m^+\to 0$, as $m\to+\infty$, faster than any negative power of m and uniformly with respect to h. Thus we can differentiate term by term the series for $\widetilde{\psi}$ in (139). Moreover, by Theorem 4.9, we can differentiate term by term equation (140), since $\widetilde{\psi}_{m0}\to 0$, as $m\to+\infty$, faster than any negative power of m. In particular we have the following

Remark 8.5. *Let $\widetilde{\mu}_0(h)$ and $\widetilde{\mu}_0(h_0)$ represent the lowest eigenvalue of $\widetilde{P}(h)$ and $\widetilde{P}(h_0)$ respectively. Let $\widetilde{\psi}(h)$, $\widetilde{\psi}(h_0)$ be the associated eigenfunctions given by (139) and (140). We have*

$$\widetilde{\psi}'(h) = \sum_{m=0}^{+\infty}v_m^+\left(-\frac{2m+1}{2\sqrt{\pi}}\right)\sin\left(\frac{2m+1}{2}\,x\right), \tag{141}$$

$$\widetilde{\psi}'(h_0) = \sum_{m=0}^{+\infty}\widetilde{\psi}_{m0}\left(-\frac{2m+1}{2\sqrt{\pi}}\right)\sin\left(\frac{2m+1}{2}\,x\right). \tag{142}$$

Using the estimates proved up to now we show that $\widetilde{\mu}_0(h)$ is monotone increasing with respect to h.

Theorem 8.6. *The eigenvalue $\widetilde{\mu}_0 = \widetilde{\mu}_0(h)$ is monotone increasing as a function of h (for $h > 0$); as a consequence there exists $\lim_{h\to 0^+} \widetilde{\mu}_0(h)$.*

Proof. We have to show that

$$\frac{d}{dh}\widetilde{\mu}_0(h)\Big|_{h=h_0} = \lim_{h\to h_0} \frac{\widetilde{\mu}_0(h) - \widetilde{\mu}_0(h_0)}{h - h_0} > 0, \qquad \forall\ h_0 > 0.$$

To do this we compute each term in equation (138). Consider the first term in the right-hand side of (138). From (141) of Remark 8.5 we get

$$\lim_{h\to h_0} (h + h_0) \int_{-\pi}^{\pi} (\widetilde{\psi}'(h))^2 dx = 2h_0 \lim_{h\to h_0} \int_{-\pi}^{\pi} (\widetilde{\psi}'(h))^2 dx =$$

$$= 2h_0 \lim_{h\to h_0} \int_{-\pi}^{\pi} \left[\sum_{m=0}^{+\infty} -\frac{2m+1}{2\sqrt{\pi}} \sin\left(\frac{2m+1}{2}x\right) v_m^+(h) \right]^2 dx. \tag{143}$$

By Corollary 7.12 and since the $v_m^+(h)$ are analytic in h we can exchange in (143) the limit with the integral and then with the sum, thus obtaining

$$\lim_{h\to h_0} (h + h_0) \int_{-\pi}^{\pi} (\widetilde{\psi}'(h))^2 dx =$$

$$= 2h_0 \int_{-\pi}^{\pi} \left[\sum_{m=0}^{+\infty} -\frac{2m+1}{2\sqrt{\pi}} \sin\left(\frac{2m+1}{2}x\right) \widetilde{\psi}_{m0} \right]^2 dx = 2h_0 \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0)\right]^2 dx > 0,$$

where the last equality follows from (142).

We will next see that all the other terms in the right-hand side of (138) vanish, thus concluding the proof.

We consider the second term in (138). By (141) and (142) we have

$$\lim_{h\to h_0} \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left(\widetilde{\psi}'(h) - \widetilde{\psi}'(h_0)\right)^2 dx =$$

$$= \lim_{h\to h_0} \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left[\sum_{m=0}^{+\infty} \left(-\frac{2m+1}{2\sqrt{\pi}}\right) \sin\left(\frac{2m+1}{2}\ x\right) \left(v_m^+(h) - \widetilde{\psi}_{m0}\right) \right]^2 dx.$$

As $\widetilde{\psi}_{m0} = v_m^+(h_0)$, by Lemma 8.3 there exist D, α, $\gamma > 0$ such that

$$\lim_{h\to h_0} \left| \frac{h_0^2}{h - h_0} \int_{-\pi}^{\pi} \left(\widetilde{\psi}'(h) - \widetilde{\psi}'(h_0)\right)^2 dx \right| \le$$

$$\le \lim_{h\to h_0} \frac{h_0^2}{|h - h_0|} \int_{-\pi}^{\pi} \left[\sum_{m=0}^{+\infty} \left| -\frac{2m+1}{2\sqrt{\pi}} \sin\left(\frac{2m+1}{2}\ x\right) \right| \frac{|h^2 - h_0^2|\ D}{(\alpha^2 - \gamma^2)\ m^m} \right]^2 dx =$$

$$= \lim_{h\to h_0} \frac{h_0^2 |h - h_0| |h + h_0|^2 D^2}{(\alpha^2 - \gamma^2)^2} \int_{-\pi}^{\pi} \left[\sum_{m=0}^{+\infty} \frac{2m+1}{2\sqrt{\pi}} \left|\sin\left(\frac{2m+1}{2}\ x\right)\right| \frac{1}{m^m} \right]^2 dx,$$

and this is 0.

Now we compute the third term in the right-hand side of (138).

As already noticed the eigenfunctions $\widetilde{\psi}(h)$ and $\widetilde{\psi}(h_0)$ are even functions, without any zeros on $(-\pi, \pi)$. So we have

$$\lim_{h\to h_0} \frac{h_0^2}{h-h_0} \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0)\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{\widetilde{\psi}(h_0)}\right)\right]^2 dx =$$

$$= 2 \lim_{h\to h_0} \frac{h_0^2}{h-h_0} \int_{0}^{\pi} \left[\widetilde{\psi}'(h_0)\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{\widetilde{\psi}(h_0)}\right)\right]^2 dx. \tag{144}$$

Multiplying and dividing the right-hand side of (144) by $(x-\pi)$ gives

$$\lim_{h\to h_0} \frac{h_0^2}{h-h_0} \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0)\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{\widetilde{\psi}(h_0)}\right)\right]^2 dx =$$

$$= 2 \lim_{h\to h_0} \frac{h_0^2}{h-h_0} \int_{0}^{\pi} \left[\widetilde{\psi}'(h_0)\left(\frac{\dfrac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{x-\pi}}{\dfrac{\widetilde{\psi}(h_0)}{x-\pi}}\right)\right]^2 dx. \tag{145}$$

The function $\frac{\widetilde{\psi}(h_0,x)}{x-\pi}$ does not vanish on the interval $[0,\pi)$. Moreover, by De L'Hospital's theorem, we have

$$\lim_{x\to\pi^-} \frac{\widetilde{\psi}(h_0,x)}{x-\pi} = \lim_{x\to\pi^-} \widetilde{\psi}'(h_0,x) = \widetilde{\psi}'(h_0,\pi).$$

The right-hand side of this equation is obviously different from zero, because it cannot be $\widetilde{\psi}'(h_0,\pi) = \widetilde{\psi}(h_0,\pi) = 0$, as $\widetilde{\psi}$ is a non-trivial solution of a second order Sturm-Liouville problem. Thus we have proved that there exists $R > 0$ such that

$$\left|\frac{\widetilde{\psi}(h_0,x)}{x-\pi}\right| > R, \qquad \forall\, x \in [0,\pi]. \tag{146}$$

From (146) and (145) it follows

$$\left|\lim_{h\to h_0} \frac{h_0^2}{h-h_0} \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0)\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{\widetilde{\psi}(h_0)}\right)\right]^2 dx\right| \le$$

$$\le 2 \lim_{h\to h_0} \frac{h_0^2}{|h-h_0|} \int_{0}^{\pi} \left[\frac{\widetilde{\psi}'(h_0)}{R}\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{x-\pi}\right)\right]^2 dx. \tag{147}$$

We set

$$S_m(x) = \frac{(-1)^{m+1}\sin\left(\frac{2m+1}{2}(x-\pi)\right)}{x-\pi}.$$

Thus, by (147), (139) and (140) and recalling Lemma 8.3 we have

$$\left|\lim_{h\to h_0} \frac{h_0^2}{h-h_0} \int_{-\pi}^{\pi} \left[\widetilde{\psi}'(h_0)\left(\frac{\widetilde{\psi}(h_0)-\widetilde{\psi}(h)}{\widetilde{\psi}(h_0)}\right)\right]^2 dx\right| \le$$

$$\leq 2 \lim_{h\to h_0} \frac{h_0^2}{|h-h_0|} \int_0^\pi \left[\frac{\widetilde{\psi}'(h_0)}{R} \left(\sum_{m=0}^{+\infty} (v_m^+(h) - v_m^+(h_0)) \frac{\cos\left(\frac{2m+1}{2}x\right)}{x-\pi} \right) \right]^2 dx \leq$$

$$\leq 2 \lim_{h\to h_0} \frac{h_0^2}{|h-h_0|} \int_0^\pi \left[\frac{\widetilde{\psi}'(h_0)}{R} \left(\sum_{m=0}^{+\infty} \frac{|h^2-h_0^2|\ D\ S_m(x)}{(\alpha^2-\gamma^2)m^m} \right) \right]^2 dx =$$

$$= \lim_{h\to h_0} \frac{2h_0^2|h-h_0||h+h_0|^2 D^2}{(\alpha^2-\gamma^2)^2} \int_0^\pi \left[\frac{\widetilde{\psi}'(h_0)}{R} \sum_{m=0}^{+\infty} \frac{S_m(x)}{m^m} \right]^2 dx = 0.$$

With analogous procedures one can prove that also the limit of the last term in the right-hand side of (138) is 0, concluding the proof. □

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