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Some Remarks in Differential and Integral Geometry of Carnot Groups

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Introduction

Over the last few years the project of developing the methods of Geometric Measure Theory in very general metric spaces has been carried out along the lines originally suggested in Federer's book [33]. In many respects, deep contributions to this task have been inspired and carried out by the works of Ambrosio & Kirchheim [3, 4], Cheeger [18], David & Semmes [25], De Giorgi [27, 28, 29, 30], Gromov [49, 50], Montgomery [77], Pansu [81, 82], Preiss & Tisěr [84], just to mention some examples.

Many of advances are somehow connected with a contemporary development of a foundational theory of Sobolev spaces in abstract metric settings, culminated in the paper [52].

Geometries associated with a family of vector fields and Carnot-Carathéodory spaces are, of course, the main models of this research. On this subject, there is a wide literature and we shall refer the reader to [9], [14], [23], [36], [37], [38], [39, 40], [45], [57], [69], [73], [77], [80], [82], [90], [93], [94]. Clearly, this list is far from being complete, but illustrates fairly well some of the directions followed by the contemporary research.

The closeness of Analysis and Geometry is here particularly stressed by the fact that, initially, these questions had arisen in the field of hypoelliptic differential equations. In this respect, we mention the important paper by Rothschild and Stein, [85]. We have also to emphasize the special importance of the related studies on nilpotent Lie groups; as references we would cite the papers of Folland and Stein [34], [35], [89] and Goodman [47] as regards the analytical aspects, and, for instance, those of Pansu [81, 82] and Korányi & Reimann [61] to better appreciate the geometrical features involved in this kind of problems. See also [53], [77] and [78] for useful comments and more detailed references.

Finally, we would stress that the mathematical interest for these largely non-euclidean geometries, at least from É. Cartan's work (see, for example, [15]), seems motivated by the fact that they constitute a model for the so-called *non-holonomic* physical systems, i.e. non-integrable in the sense of Frobenious theorem. See, for instance, the very interesting survey by Vershik & Gershkovich in [95], but also [9] and [49].

The geometric setting of this PhD thesis is that of *Carnot groups*, also known in the current literature as *non-Abelian vector spaces* or *subriemannian groups* [9], [49], [73], [77].

They constitute an important class of examples of subriemannian geometries, and they have become the subject of many papers of geometric analysis. See, for instance, [14], [45], [41, 42, 43], [49], [66, 67], [73], [79], [82], [94]).

Roughly speaking, Carnot groups are nilpotent stratified Lie groups endowed with a one-parameter family of dilations adapted to the Lie algebra stratification. They are naturally equipped with an m_1*-planes distribution*, constructed by left translation of the first m_1-dimensional step H of the Lie algebra stratification. This m_1-planes distribution, still denoted by H, is a subbundle of the tangent bundle of the group whose elements are called *horizontal vectors*. A subriemannian structure on them is defined whenever the fibres of this bundle are endowed with an inner product.

The crucial role played by Carnot groups in the theory of Carnot-Carathéodory geometries, comes from a deep theorem of Mitchell which states that the tangent cone -*in the sense of Gromov-Hausdorff* - of any Carnot-Carathéodory space is a suitable Carnot group, [73]. See also [77] for many clarifying discussions about this point.

Since Carnot groups are *homogenous groups*, according to a definition given by Stein, [89], harmonic analysis and P.D.E.'s on them have been an extensive subject of research. Furthermore, many classical tools of Calculus of Variations have been generalized to this context and, in particular, the theory of *bounded* H*-variation functions* and that of H*-Caccioppoli sets*, [39, 41, 42], [45], [67], [79]. Notice that, in both these notions, H means horizontal, i.e. *they are notions related only to the horizontal subbundle of the Carnot group.* For a specific survey of these results and for more detailed bibliographic references, we shall refer the reader to [5], [14], [24], [40, 41, 42, 43], [44], [45], [65], [66, 67], [79], [83].

In the present PhD thesis we shall try to give some contributions to the study of both integral and differential-geometric properties of submanifolds of Carnot groups.

The thesis is subdivided into 7 sections, the first of which is foundational and introduces many of the notions useful for the sequel.

Section 2, 3 and 4 are devoted to illustrate some new results about the Integral Geometry of Carnot groups.

All the results given in this part of the thesis are contained in the paper *Some relations among volume, intrinsic perimeter and one-dimensional restrictions of BV functions in Carnot groups*, [75].

Section 5, 6 and 7 are mainly concerned with a differential-geometric study of "suitably regular" hypersurfaces, particularly in the case of 2-step Carnot groups. This part contains the results of an unpublished preprint *Some remarks about the geometry of non-characteristic hypersurfaces in Carnot groups*, [76].

More precisely, in Section 2, our starting point will be a *Fubini type theorem* for codimension one *H-regular submanifolds* (see Definition 1.38). It can be stated as follows:

Theorem 0.1. *Let $\mathbb{G}$ be a* k*-step n-dimensional Carnot group and let $S \subset \mathbb{G}$ be a H-regular hypersurface. By the Implicit Function Theorem 1.39 (see [42]), without loss of generality, we may assume that $S = \partial E$ globally, where $E \subset \mathbb{G}$ is an open H-Caccioppoli set with locally $\mathbf{C}^1_H$ boundary. Let $X \in H$, $|X|_H = 1$, be a unit horizontal left invariant vector field which is transverse to S. Let γ_y be the horizontal X-line starting from $y \in S$ and let us suppose that $\gamma_y(\mathbb{R}) \cap S = \{y\}$ for every $y \in S$. Finally, let $D \subseteq \mathcal{R}^X_S$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from S. Then we have*

(i) $D_y := \gamma_y(\mathbb{R}) \cap D$ *is $\mathcal{H}^1_c$-measurable for $|\partial E|_H$-a.e. $y \in S$;*

(ii) the mapping $S \ni y \longmapsto \mathcal{H}^1_c(D_y)$ is $|\partial E|_H$-measurable on S and

$$\mathcal{L}^n(D) = \int_{pr^X_S(D)} \mathcal{H}^1_c(D_y)\, |\langle X, \nu_E \rangle_{H_y}|\, d\,|\partial E|(y) = \int_{pr^X_S(D)} \mathcal{H}^1_c(D_y)\, d\,|\partial_X E|(y),$$

where $|\partial_X E|$ denotes the partial X-perimeter of E (see Section 1.4).

The proof of the theorem follows by an approximation argument for H-regular hypersurfaces, after using Proposition 2.3 which proves the result for $\mathbf{C}^1$-smooth hypersurfaces. We would emphasize the fact that the main problem to obtain these formulae is that of a good choice of projection maps. Here we use, for a great number of integral formulae, the projections along the integral curves of horizontal vectors of Lie algebra $\mathfrak{g}$, that we shall call *horizontal projections*.

In Section 3.1, we use this kind of results to work with *one-dimensional slices* of functions. We then apply this procedure to state a characterization of the space *HBV* (see Theorem 3.7). A similar characterization was proved in [96] for Sobolev spaces in Carnot groups.

More precisely, this will be done by linking, through a *horizontal slicing*, the total H-variation of a function with that of its one-dimensional restrictions. We refer the reader to Section 3.1 for related definitions and precise statement of the results.

In Section 3.2, we prove some integral-geometric formulae and one in particular, which characterizes the intrinsic perimeter measure. We can write the result as follows (see Proposition 3.14):

Proposition 0.2. *[Integral geometric H-perimeter] Let $U \subseteq \mathbb{G}$ be open and fix $z \in \mathbb{G}$. If $D \subset \mathbb{G}$ is a H-Caccioppoli set, we have then*

$$|\partial D|_H(U) = \frac{1}{2\kappa_{m_1-1}} \int_{\mathbb{S}^{m_1-1}} d\,\sigma_{m_1-1}(X) \int_{pr^X_{\mathcal{I}_z(X)}(D\cap U)} var^1_X[\mathbf{1}_{D^X_y}](U^X_y)\, d\,\mathcal{H}^{n-1}(y),$$

where κ_{m_1-1} is the m_1-1-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{m_1-1}$. Here, $\mathcal{I}_z(X)$ denotes the generic "vertical hyperplane" through $z \in \mathbb{G}$ and $var^1_X[\cdot](\cdot)$ the one-dimensional variation (see Section 3.2).

Afterwards, in Section 3.2, we introduce a notion of horizontal convexity, called *H-convexity*, explaining some of its main features.

This notion turns out to be analogous to that recently given in [24] and in [65]. We then prove that H-convex sets verify an integral *Cauchy type formula* for the H-perimeter (see Theorem 3.21), and then a related inequality, showing, in a sense, this kind of convex sets to minimize the intrinsic H-perimeter.

Section 4 is devoted to state and prove a *horizontal Santaló type formula* and some of its possible applications (see Theorem 4.5). We stress that our result generalizes to arbitrary Carnot groups a result already proved in Pansu's thesis, [81]. This formula is strictly connected with the introduction of a measure on the so-called unit horizontal bundle of $\mathbb{G}$ and with its invariance under a suitable restriction of the Riemannian geodesic flow. We refer to Section 4 for a detailed introduction.

We then apply Theorem 4.5 to show a geometric inequality among volume, H-perimeter and diameter of smooth bounded domains.

As an application to Analysis in Carnot groups, we perform some explicit computations to find two lower bounds for the first eigenvalue of the Dirichlet problem for the Carnot sub-Laplacian on smooth domains. This will be done by adapting some methods of Riemannian geometry inspired by the Crooke's article, [21]; see also [17] and [26] for a classical setting.

From now on, we shall illustrate the results of the second part of the thesis, from Sections 5 until Section 7.

Here, the main task we try to carry out, is that of a better understanding of how the study of hypersurfaces in Carnot groups may be approached. We shall tract, with more emphasis, the case of 2-step Carnot groups, since in this case a *Rectifiability Theory*, has been developed, due to Franchi, Serapioni and Serra Cassano [41, 42, 43]; see also [1], [7], [66, 67].

The point of view developed in this thesis seems to be slightly different from that of the current literature. Indeed, all the geometric structures that we consider are supposed to be smooth, as is usually assumed in Riemannian geometry. In fact, since Carnot groups can be regarded as Riemannian manifolds, we shall make use of some basic tools of differential geometry such as connections, differential forms and moving frames which allow us to describe the local geometric properties of a suitable regular hypersurface. For an introduction to these methods and for the definitions used, we refer the reader to Section 1.1 and 1.2, while a detailed study of these topics will begin from Section 5.

In this way, we are able to define some local geometric invariants associated with the H-perimeter measure as, for instance, a suitable notion of *horizontal mean curvature* of a hypersurface. We stress that this notion gives the same scalar invariant used in some recent papers, as that of N.Garofalo and S.Pauls, [44].

Since we will restrict ourselves to consider the case of *regular, non-characteristic hypersurfaces* (see Section 5), we shall define a smooth differential form $\sigma_{_H}$ on such hypersurfaces, that is the *horizontal perimeter form*, that will play the role of the H-perimeter measure. More precisely, we give the following:

Definition 0.3. *[H-perimeter form $\sigma_{_H}$] Let $S \subset \mathbb{G}$ be a smooth, non-characteristic hypersurface with unit horizontal normal $\nu_{_H}$. Then the H*-**perimeter form** $\sigma_{_H}$ *on S is the differential $n-1$-form on S given by contraction with $\nu_{_H}$ of the volume form Ω^n, i.e.*

$$\sigma_{_H}|_S := (\nu_{_H} \rfloor \Omega^n)|_S. \tag{1}$$

For the case of 2-step Carnot groups, in Section 6.1, we shall develop a basic differential-geometric formalism using suitable moving frames that turns out to be *adapted* to the horizontal tangent space HTS of a non-characteristic hypersurface. This will be done because we try to study the H-perimeter form $\sigma_{_H}$ instead of the Riemannian area form. We refer the reader to Section 6 for precise statements, and, in particular, to Definition 6.1, for the notion of *H-adapted moving frame.*

We stress that our choice to define a such H-adapted frame is motivated by the fact that we cannot use the usual Riemannian approach (see [63], [16], [87], [88]) in proving variational formulas concerning the H-perimeter form $\sigma_{_H}$ as, for instance, divergence-type theorems on hypersurfaces or the 1st and the 2nd variation of $\sigma_{_H}$.

In the same perspective, we introduce some affine connections, that turn out to be naturally associated with a suitable decomposition of the tangent space. One of them is the so-called *horizontal connection* as defined in [60] (see Section 1.2, for these definitions). For this *restricted H-connection* we show a generalized version of the classical *Gauss formulae* (see Proposition 6.17).

Then, the formalism of differential forms on Lie groups and the methods that we have previously developed, allow us to deduce basic results such as *integration by parts* on regular hypersurfaces. This is the subject of Section 6.2 where we shall prove *Gauss-Green type formulas* on regular hypersurfaces and consequently, *Green's type identities*; see Section 6.2.

One of the main consequences of our approach is indeed the following:

Theorem 0.4. *[Divergence type theorems on regular hypersurfaces] Let $\mathbb{G}$ be a 2-step Carnot group with Lie algebra $\mathfrak{g} = H \oplus Z$. Let $S \subset \mathbb{G}$ be a smooth immersed non-characteristic hypersurface with unit normal vector along S denoted by N. Let $\mathcal{U} \subset S$ be compact and suppose that the boundary $\partial\mathcal{U}$ is a smooth $n-2$-dimensional Riemannian submanifold with outward pointing unit normal η. Then*

(i) For every smooth vector field $X \in \mathbf{C}^{\infty}(\mathbb{G}, VS)$ we have

$$\int_{\mathcal{U}} \mathrm{div}_{VS} X \, \sigma_{H} + \int_{\mathcal{U}} \Big\{ \Big\langle \sum_{\beta \in I_2} n_{\beta}\, \mathcal{C}^{\beta} \nu_{H}, \mathcal{P}_H(X) \Big\rangle_H + \mathcal{H}^{sc}_{H} \langle \mathcal{P}_Z(N), \mathcal{P}_Z(X) \rangle \Big\} \, \sigma^{n-1} = \int_{\partial\mathcal{U}} \langle X, \eta \rangle \, |\mathcal{P}_H(N)|_H \, \sigma^{n-2};$$

(ii) For every smooth vector field $X \in \mathbf{C}^{\infty}(S, HTS)$ we have

$$\int_{\mathcal{U}} \mathrm{div}_{HTS} X \, \sigma_{H} \quad + \quad \int_{\mathcal{U}} \Big\langle \sum_{\beta \in I_2} n_{\beta}\, \mathcal{C}^{\beta} \nu_{H}, X \Big\rangle_H \sigma^{n-1} = \int_{\partial\mathcal{U}} \langle X, \eta \rangle \, |\mathcal{P}_H(N)|_H \, \sigma^{n-2};$$

(iii) For every smooth vector field $X \in \mathfrak{X}(S)$ we have

$$\int_{\mathcal{U}} \Big\{ \mathrm{div}_{VS} X - \mathcal{H}^{sc}_{H} \langle X, \nu_{H} \rangle \Big\} \, \sigma_{H} + \int_{\mathcal{U}} \Big\langle \sum_{\beta \in I_2} n_{\beta}\, \mathcal{C}^{\beta} \nu_{H}, \mathcal{P}_H(X) \Big\rangle_H \sigma^{n-1} = \int_{\partial\mathcal{U}} \langle X, \eta \rangle \, |\mathcal{P}_H(N)|_H \, \sigma^{n-2}.$$

Here, VS denotes the vertical bundle over S, HTS the horizontal tangent bundle over S, $\{\mathcal{C}^{\beta}\}_{\beta\in I_2}$ *denotes a family of linear operators depending on the structural constants of the Lie algebra. Moreover* div_{VS} *and* div_{HTS} *denote, respectively, the divergence operators on the vector bundles VS and HTS. Finally,* $\mathcal{H}^{sc}_{H}$ **is the scalar mean horizontal curvature along** $\mathcal{U}$*; see Section 5 and Section 6.1.*

In Section 7, we shall compute, by using the method of H-adapted moving frames, *the 1st and the 2nd variation* of the H-perimeter form σ_H in 2-step Carnot groups. The theorem about the 1st variation is the following (see Section 7.2):

Theorem 0.5. *Let* $\mathbb{G}$ *be a* 2*-step Carnot group and let* $\imath:\mathcal{U}\longrightarrow\mathbb{G}$ *be the inclusion into* $\mathbb{G}$ *of a smooth non-characteristic hypersurface* $\mathcal{U}$ *with boundary* $\partial\mathcal{U}$*. Moreover, let* $\vartheta:(-\epsilon,\epsilon)\times\mathcal{U}\longrightarrow\mathbb{G}$ *be be a smooth variation of* $\imath$*, with variation vector field* W*, and assume that* $\mathcal{U}_t=\vartheta_t(\mathcal{U})$ *is non-characteristic for* $t\in(-\epsilon,\epsilon)$*. Let* $\Gamma(t)=\vartheta_t^{*}\sigma_{H,t}$ *be a* $\mathbf{C}^{\infty}$ 1*-parameter family of* $n-1$*-forms on* $\mathcal{U}$*. If* $I_{\mathcal{U}}(\sigma_H):=\frac{d}{dt}\int_{\mathcal{U}}\Gamma(t)\big|_{t=0}$*, then*

$$I_{\mathcal{U}}(\sigma_H)=-\int_{\mathcal{U}}\mathcal{H}^{sc}_{H}\,\langle\mathcal{P}_H(W),\nu_H\rangle_H\,\sigma_H-\int_{\mathcal{U}}\mathcal{H}^{sc}_{H}\,\langle\mathcal{P}_Z(W),\mathcal{P}_Z(N)\rangle\,\sigma^{n-1}+\int_{\partial\mathcal{U}}\langle W,\eta\rangle\,|\mathcal{P}_H(N)|_H\,\sigma^{n-2}.$$

For 2-step Carnot groups we then prove a formula for the 2nd variation of σ_H, without boundary terms, since in this case we will make use of compactly supported vector fields. The calculation itself is quite difficult and also the result has a quite complicated expression, at least in the general case of *normal variations* (see Theorem 7.8).

The other interesting theorem of Section 7.3 gives the *2nd horizontal normal variation of* σ_H. Also this formula is stated without boundary terms. The result reads as follows:

Theorem 0.6. *Let* $\mathbb{G}$ *be a* 2*-step Carnot group and* $\imath:\mathcal{U}\longrightarrow\mathbb{G}$ *be the inclusion into* $\mathbb{G}$ *of a smooth non-characteristic hypersurface* $\mathcal{U}$ *with boundary* $\partial\mathcal{U}$*. Moreover, let* $\vartheta:(-\epsilon,\epsilon)\times\mathcal{U}\longrightarrow\mathbb{G}$ *be a* **smooth normal** H**-variation of** $\imath$*, with variation* *vector field* $W = w\nu_{_H} \in \mathbf{C}_0^\infty(\mathbb{G}, H)$ *such that* $\mathrm{spt}(W) \cap \mathcal{U} \Subset \mathcal{U}$*. Assume that* $\mathcal{U}_t = \vartheta_t(\mathcal{U})$ *is non-characteristic for* $t \in (-\epsilon, \epsilon)$ *and let* $\Gamma(t) = \vartheta_t^* \sigma_{_{H,t}}$ *be a* $\mathbf{C}^\infty$ *1-parameter family of* $n-1$*-forms on* $\mathcal{U}$*. If* $II_{\mathcal{U}}^{int}(\sigma_{_H}) := \frac{d^2}{dt^2} \int_{\mathcal{U}} \Gamma(t)\big|_{t=0}$*, then*

$$\begin{aligned} II_{\mathcal{U}}^{int}(\sigma_{_H}) &= \int_{\mathcal{U}} \Big\{ - W(w)\mathcal{H}_{_H}^{sc} + w^2\Big((\mathcal{H}_{_H}^{sc})^2 - \|\mathtt{b}_H\|^2_{Gram}\Big) - w\,\Delta_{_{HTS}} w \\ &\quad -w^2 \sum_{\alpha\in I_2} \langle \nabla_{\tau_\alpha^S} \nu_{_H}, \mathcal{C}^\alpha \nu_{_H}\rangle + \frac{w^2}{2} \sum_{\alpha\in I_2} \|\mathcal{C}^\alpha \nu_{_H}\|^2_{Gram} - w\,\mathrm{div}_{_{HTS}}(w\mathcal{C}\nu_{_H})\Big\}\sigma_{_H}; \end{aligned}$$

see Section 7.1. Here $\tau_\alpha^S = X_\alpha - \langle X_\alpha, N\rangle N$ $(\alpha \in I_2 = \{m_1+1, ..., n\})$.

The Addendum of Section 7 is then devoted to prove, by using a different method, integration by parts theorems on regular non-characteristic hypersurfaces and the 1st variation formula for $\sigma_{_H}$ in the general case of k-step Carnot groups.

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1 Preliminaries

1.1 The geometry of Carnot groups

In this section we recall the main notions about Carnot groups, introducing also the main geometric structures as connections and differential forms that we will use throughout this thesis. As a basic references for the topics developed in this section we cite [9], [20], [53], [54], [59], [49], [64], [71], [77], [92], [95].

Let $\mathbb{G}$ be a n-dimensional, connected and simply connected Lie group over $\mathbb{R}$ with group law denoted by $\bullet$. Let $\mathfrak{X}(\mathbb{G})$ be the set of all smooth sections of $T\mathbb{G}$, i.e. $\mathfrak{X}(\mathbb{G}) := \mathbf{C}^{\infty}(\mathbb{G}, T\mathbb{G})$. As usual, any $x \in \mathbb{G}$ defines smooth maps $L_x, R_x : \mathbb{G} \longrightarrow \mathbb{G}$, called left translation and right translation, respectively, by $L_x(y) := x \bullet y,\ R_x(y) := y \bullet x$, $y \in \mathbb{G}$. Let $\mathfrak{g}$ denote the *Lie algebra of* $\mathbb{G}$, i.e. the linear subspace of $\mathfrak{X}(\mathbb{G})$ of all left invariant vector fields of $\mathbb{G}$, endowed with the bracket operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$. This algebra is canonically isomorphic to $T_e\mathbb{G}$, *i.e. the tangent space at the identity* e *of* $\mathbb{G}$, via the identification of any left invariant vector field X of $\mathbb{G}$ with its value X_e at e. Here the isomorphism is explicitly given by $L_{x*} : T_e\mathbb{G} \longrightarrow T_x\mathbb{G}$, *i.e. the differential of the left translation by* x *at* e, so that *hereafter we will think of* $\mathfrak{g}$ *either as the vector space* $T_e\mathbb{G}$ *with the rule composition* $[\cdot,\cdot]$, *as well as the Lie algebra of vector fields of* $\mathbb{G}$.

Remark 1.1. *Since the tangent space at any point $x \in \mathbb{G}$ is completely determined by the structure of the tangent space at the identity $e \in \mathbb{G}$, we shall use the following notation: if K is a vector subspace of $\mathfrak{g} = T_e\mathbb{G}$ we denote by K_x its corresponding image through L_{x*} in $T_x\mathbb{G}$ and by W the smooth subbundle of $T\mathbb{G}$ whose fibre at the point $x \in \mathbb{G}$ is K_x.*

For any $X \in \mathfrak{g}$ we shall denote by $\gamma_X : \mathbb{R} \longrightarrow \mathbb{G}$ the one-parameter subgroup of $\mathbb{G}$ generated by X, or equivalently, the integral curve of X starting at the identity e of $\mathbb{G}$.

Now let *exp* $: \mathfrak{g} \longrightarrow \mathbb{G}$ denote the Lie group *exponential map* defined by $exp\,(X) := \gamma_X(1)$, for $X \in \mathfrak{g}$. It is well known that, in general, *exp* is a diffeomorphism of an open neighborhood O_0 of 0 in $\mathfrak{g}$ onto an open neighborhood O_e of e

in $\mathbb{G}$, but since $\mathbb{G}$ is connected and simply connected, we have that *exp* is a global diffeomorphism of $\mathfrak{g}$ onto $\mathbb{G}$. Therefore we denote by *log* : $\mathbb{G} \longrightarrow \mathfrak{g}$ its inverse. If X is a left invariant vector field of $\mathbb{G}$ and $x \in \mathbb{G}$, then γ_{X_x} denotes the integral curve of X starting from x. We remark that γ_{X_x} is given by right translation of x by $exp\,(tX)$, $t \in \mathbb{R}$. Now if $\{\mathrm{e}_1, ..., \mathrm{e}_n\}$ is a linear basis of $\mathfrak{g} = T_e\mathbb{G}$, then the corresponding coordinates in $\mathbb{G}$, given by the inverse *log* of the exponential map, will be called a system of *exponential coordinates in* $\mathbb{G}$ or also, *canonical coordinates of the first kind.* The differential of the exponential map can be described, in general, as follows [54]:

$$d\exp_X = d\Big(L_{\exp X}\Big)_e \circ \frac{1 - e^{-\mathrm{ad}X}}{\mathrm{ad}\,X} \qquad (X \in \mathfrak{g}). \tag{2}$$

Here $1-e^{-A}/A$ stands for $\sum_{h=0}^{\infty}(-A)^h/(h+1)!$. Moreover $\mathrm{ad}\,X$ denotes the linear transformation of $\mathfrak{g}$ given by $\mathrm{ad}\,X(Y) = [X, Y]$.

Henceforth we assume that $\mathbb{G}$ be endowed with a left invariant Riemannian metric. We stress that, fixing some basis $\{\mathrm{e}_1, ..., \mathrm{e}_n\}$ of the vector space $\mathfrak{g}$, there is only one left invariant Riemannian metric on $\mathbb{G}$ such that the corresponding left invariant vector fields $(X_1, ..., X_n)$, (where $(X_j)_x = (L_x)_*\mathrm{e}_j,\ j = 1, ..., n)$ are everywhere orthonormal; see [71]. More precisely, we will fix on $\mathbb{G}$ the left invariant Riemannian metric obtained by left translation of the Euclidean metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ that makes $\mathrm{e}_1, ..., \mathrm{e}_n$ an orthonormal basis, i.e.

$$\langle X, Y\rangle := \langle X_e, Y_e\rangle, \quad \forall\ X,\ Y \in \mathfrak{X}(\mathbb{G}). \tag{3}$$

For each positive integer i, we set $\mathfrak{g}^i := [\mathfrak{g}, \mathfrak{g}^{i-1}]$, where $\mathfrak{g}^1 := \mathfrak{g}$. We say that $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}^i = \{0\}$ for some positive integer i; in this case the center of $\mathfrak{g}$ contains $\mathfrak{g}^{i-1}$. The Lie algebra $\mathfrak{g}$ is k-step if $\mathfrak{g}^{\mathrm{k}+1} = \{0\}$ and $\mathfrak{g}^{\mathrm{k}} \neq \{0\}$; in this case we say that the Lie group $\mathbb{G}$ is k-*step.*

For a k-step nilpotent Lie group $\mathbb{G}$ the group law $\bullet$ is completely determined, by the *Campbell-Hausdorff formula*, from the structure of $\mathfrak{g}$, [20, 92]. Indeed we have that

$$exp\,(X) \bullet exp\,(Y) = exp\,(\widetilde{\mathcal{P}}(X, Y)) \qquad \forall\ X,\ Y \in \mathfrak{g},$$

where $\widetilde{\mathcal{P}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ is defined by the following identity

$$\widetilde{\mathcal{P}}(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \mathcal{R}(X,Y).$$

Here above $\mathcal{R}(X,Y)$ denotes a formal series of brackets of length at least 3 and at most $\mathrm{k}-1$.

Remark 1.2. *In exponential coordinates, the group law* $\bullet$ *of* $\mathbb{G}$ *turns out to be a polynomial function. Indeed, let* $x, y \in \mathbb{G}$ *and* $X = \sum_{i=1}^n x_i \, \mathrm{e}_i$, $Y = \sum_{i=1}^n y_i \, \mathrm{e}_i \in \mathfrak{g}$ *be such that* $x = \exp(X)$ *and* $y = \exp(Y)$. *Then* $z = x \bullet y$ *if, and only if, there exists* $Z = \sum_{i=1}^n z_i \, \mathrm{e}_i \in \mathfrak{g}$ *such that* $z = \exp(Z)$ *and* $Z = \widetilde{\mathcal{P}}(X,Y)$. *Now, setting* $\mathcal{P}(\cdot,\cdot) := \exp(\widetilde{\mathcal{P}}(\log(\cdot), \log(\cdot)))$, *we get* $\mathcal{P}(x,y) = x \bullet y$. *Note also that, in exponential coordinates, the identity* e *of* $\mathbb{G}$ *is given by* $e = (0,...,0)$ *and if* $x = (x_1,...,x_n) \in \mathbb{G}$, *then* $x^{-1} = (-x_1,...,-x_n)$. *Hereafter, we shall set*

$$\mathcal{P}(x,y) := x + y + \mathcal{Q}(x,y),$$

where $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_n)$ *and* $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_n)$ *are* $\mathbb{G}$*-valued polynomial functions on* $\mathbb{G}$, *written in exponential coordinates, [20], [78].*

A k-step nilpotent Lie group $\mathbb{G}$ is *stratified* if its Lie algebra $\mathfrak{g}$ admits a k-*step stratification*, i.e. there exist linear subspaces $V_1,...,V_{\mathrm{k}}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_{\mathrm{k}}, \quad [V_1, V_{i-1}] = V_i \text{ for } i = 2,...,\mathrm{k} \text{ and } V_{\mathrm{k}+1} = \{0\}. \tag{4}$$

In this case we set $H := V_1$ to denote the *horizontal layer* of the stratification of $\mathfrak{g}$. Note that, by iterated brackets, H generates the whole Lie algebra $\mathfrak{g}$.

Definition 1.3. *[Carnot groups] We say that a finite-dimensional, connected, simply connected nilpotent Lie group is a Carnot group if its Lie algebra is stratified.*

Warning. *Throughout the thesis, unless otherwise mentioned,* $\mathbb{G}$ *will denote a* n*-dimensional Carnot group with Lie algebra* $\mathfrak{g}$, *and the number* k *its step.*

Any Carnot group can be naturally endowed with a family of *Carnot dilations* $\{\delta_\lambda(x_1,\ldots,x_n) = (\lambda^{\alpha_1}x_1,\ldots,\lambda^{\alpha_n}x_n)\}_{t>0}$. To construct these dilations we first consider the family of linear operators $\widetilde{\delta_t} : \mathfrak{g} \longrightarrow \mathfrak{g}$, $t \in \mathbb{R}_+$, which act by scalar multiplication by t^i on V_i for $i = 1, ..., \mathrm{k}$; then we extend these operators to group automorphisms by setting

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n) := exp \circ \widetilde{\delta_t} \circ log : \mathbb{G} \longrightarrow \mathbb{G}.$$

Hereafter, we identify $\mathfrak{g} = T_e\mathbb{G}$ with $\mathbb{R}^n$ and we choose as a basis, the standard one of $\mathbb{R}^n$, denoted by $\{\mathrm{e}_1, \dots, \mathrm{e}_n\}$. This basis can be adapted to the stratification of $\mathfrak{g}$ as follows. First, we set $m_i := \dim V_i$ and $h_i := m_1 + \dots + m_i$, for $i = 1, \dots, \mathrm{k}$, where $h_0 := 0$ and $h_\mathrm{k} := n$; then we assume that $\mathrm{e}_{h_{j-1}+1}, \dots, \mathrm{e}_{h_j}$ is a basis of V_j for each $j = 1, \dots, \mathrm{k}$.

Remark 1.4. *If $x = exp\,(X)\,(X = \sum_{i=1}^n x_i\, \mathrm{e}_i)$, we get*

$$\delta_t x = \delta_t(x_1, ..., x_n) = (t^{\alpha_1} x_1, ..., t^{\alpha_n} x_n) \quad \forall\, x \in \mathbb{G},\ t > 0, \tag{5}$$

where $\alpha_i \in \mathbb{N}$ is called the **homogeneity** *of the variable x_i, and we have that $\alpha_j := i$ whenever $h_{i-1} + 1 \leq j \leq h_i$. Hence*

$$1 = \alpha_1 = ... = \alpha_{m_1} < \alpha_{m_1+1} = 2 \leq ... \leq \alpha_n = \mathrm{k}.$$

Following [89], we note that $\mathbb{G}$ is a **homogeneous group** *with respect to Carnot dilations. Thus, $Q := \sum_{i=1}^k i \dim\,(V_i)$ denotes its* **homogeneous dimension**. *We have that $\mathcal{P}_i$ and $\mathcal{Q}_i$ are homogeneous polynomials of degree α_i with respect to $\{\delta_t\}_{t>0}$, i.e. $\mathcal{P}_i(\delta_t x, \delta_t y) = t^{\alpha_i}\mathcal{P}_i(x, y)$, $\mathcal{Q}_i(\delta_t x, \delta_t y) = t^{\alpha_i}\mathcal{Q}_i(x, y)$ for any $x, y \in \mathbb{G}$. Moreover the following items hold, [42, 43], [79]:*

(i) $\mathcal{Q}_1(x, y) = ... = \mathcal{Q}_{m_1}(x, y) = 0$;

(ii) $\mathcal{Q}_j(x, 0) = \mathcal{Q}_j(0, y) = 0$ *and* $\mathcal{Q}_j(x, x) = \mathcal{Q}_j(x, -x) = 0$ *for* $m_1 < j \leq n$;

(iii) $\mathcal{Q}_j(x, y) = \mathcal{Q}_j(x_1, \dots, x_{h_{i-1}}, y_1, \dots, y_{h_{i-1}})$ *if* $1 < i \leq k$ *and* $j \leq h_i$;

(iv) *$\mathcal{Q}_j(x, y)$ is a sum of terms each of which contains a factor $(x_i y_l - x_l y_i)$ for some $1 \leq i, l < j$, whenever $j > m_1$.*

Starting from the basis $\{e_1, ..., e_n\}$ of $\mathfrak{g}$, a *smooth global frame* $(X_1, ..., X_n)$ *for* $\mathbb{G}$, is defined by

$$(X_j)_x := L_{x*}e_j, \quad \forall\, x \in \mathbb{G},\ j = 1, ..., n. \tag{6}$$

Note that, putting $A(x) := [A_{ij}(x)]_{\{i,j=1,...,n\}}$ $(x \in \mathbb{G})$, where

$$A_{ij}(x) := \frac{\partial \mathcal{P}_i(x, 0)}{\partial y_j},$$

we get that $A(x)$ is the $n \times n$-matrix representing, in exponential coordinates, the pushforward associated with L_x.

Remark 1.5. *Each left invariant section of the frame $(X_1, ..., X_n)$ have polynomial coefficients and it can be written as follows, [43], [79]:*

$$X_j(x) = e_j + \sum_{i>h_l}^{n} A_{ij}(x)\, e_i \qquad \forall\, j \leq h_l,\ (j = 1, \ldots, n). \tag{7}$$

If $j \leq h_l$, we get that

$$A_{ij}(x) = A_{ij}(x_1, ..., x_{h_{l-1}}),\ A_{ij}(e) = 0, \qquad A_{ij}(\delta_t(x)) = t^{\alpha_i - \alpha_j} A_{ij}(x).$$

Moreover, X_j turns out to be homogeneous of degree α_j with respect to Carnot dilations, i.e.

$$X_j(\psi \circ \delta_t)(x) = t^{\alpha_j} X_j(\psi)(\delta_t x) \quad \forall\, \psi \in \mathbf{C}^\infty(\mathbb{G}),\ \forall\, x \in \mathbb{G},\ t > 0.$$

According to Remark 1.1 we shall denote by H the *horizontal bundle* for $\mathbb{G}$, that is, *the smooth subbundle of the tangent bundle $T\mathbb{G}$ given by $H := \coprod_{x\in\mathbb{G}} H_x$*, where H_x denotes the horizontal space at the point $x \in \mathbb{G}$, i.e. $H_x = L_{x*}H$. Here the bundle projection map $\pi_{H} : H \longrightarrow \mathbb{G}$ is just the restriction to H of the natural projection map π of $T\mathbb{G}$. A subriemannian structure on $\mathbb{G}$ is given by endowing each fibre of H with an inner product $\langle \cdot, \cdot \rangle_H : H \times H \longrightarrow \mathbb{R}$. In this thesis we shall assume that $\langle \cdot, \cdot \rangle_H := \langle \cdot, \cdot \rangle|_H$ and we denote by $|\cdot|_{H}$ its associated norm.

A curve $\gamma \subset \mathbb{G}$ is *horizontal* if its tangent vector $\dot{\gamma}$ is everywhere tangent to H.

Definition 1.6. *[9] The* **Carnot-Carathéodory distance** *of any two points x, $y \in \mathbb{G}$ is given by*

$$d_{\rm c}(x,y) := \inf \int_0^1 |\dot{\gamma}(t)|_H \, dt,$$

where the infimum is taken over all horizontal curves $\gamma : [0,1] \longrightarrow \mathbb{G}$, joining x to y, i.e. $\gamma(0) = x$, $\gamma(1) = y$.

Since the rank of the Lie algebra of vector fields generated by $(X_1, \ldots, X_{m_1})$ is n, *Chow's Theorem*, [9, 49], implies that the set of all horizontal curves joining two different points is not empty and hence $d_{\rm c}$ *is a metric on* $\mathbb{G}$. Moreover, d_c *induces on* $\mathbb{G}$ *the same topology as the Riemannian one and is well behaved with respect to left translations and group dilations*, i.e. $d_{\rm c}(z \bullet x, z \bullet y) = d_{\rm c}(x,y)$ and $d_{\rm c}(\delta_t(x), \delta_t(y)) = t\, d_{\rm c}(x,y)$ for all x, y, $z \in \mathbb{G}$, $t > 0$.

Remark 1.7. *Because the special importance of the first layer of the Lie algebra $H (= V_1 \cong \mathbb{R}^{m_1})$, in the light of what we have said here above, we say that any fixed orthonormal frame $(X_1, ..., X_{m_1})$ for H,* **is a generating family of** $\mathbb{G}$

The following Remark 1.8 says that the metric space $(\mathbb{G}, d_{\rm c})$ can be suitably approximated by means of a family of Riemannian metrics on $\mathbb{G}$ preserving H, as proved by Pansu, [81].

Remark 1.8. *If $t \in \mathbb{R}_+$, let us set*

$$g_t := \frac{1}{t} \delta_\lambda(x_1, \ldots, x_n) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n)^* \langle \cdot, \cdot \rangle^{\frac{1}{2}},$$

where $\delta_\lambda(x_1, \ldots, x_n) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n)^* \langle \cdot, \cdot \rangle^{\frac{1}{2}}$ denotes the pull-back by Carnot dilations of the metric $\langle \cdot, \cdot \rangle^{\frac{1}{2}}$ on $\mathbb{G}$. Note that $\{g_t\}_{t>0}$ defines a family of left-invariant Riemannian metrics on $\mathbb{G}$. We may think of $\{(\mathbb{G}, g_t)\}_{t>0}$ as a family of metric spaces. This family $\{(\mathbb{G}, g_t)\}_{t>0}$ converges, in the sense of Gromov-Hausdorff convergence (see [6], [49], [73], [77], [81]), as $t \longrightarrow +\infty$, to the metric space $(\mathbb{G}, d_{\rm c})$.

Theorem 1.9. *Let $\zeta : [0, T] \longrightarrow \mathbb{G}$ be an horizontal curve with tangent vector at each point given, in canonical coordinates, by $a = (a_1, ..., a_{m_1}, 0..., 0)$. Then* *there exists the* **metric derivative** *of* ζ *for* $\mathcal{L}^1$ *a.e.* $t \in [0,T]$ *and it is equal to* $|a| = \sqrt{a_1^2 + ... + a_{m_1}^2}$, *i.e.*

$$|\dot{\zeta}|(t) := \lim_{\epsilon \to 0} \frac{d_c(\zeta(t+\epsilon), \zeta(t))}{|\epsilon|} = |a(t)| \quad \textit{for } \mathcal{L}^1 - \textit{a.e. } t \in [0,T].$$

Moreover, if $\mathrm{Var}(\zeta)$ *denotes the* **total variation** *of* ζ *with respect to the cc-distance* d_c*, then*

$$\mathrm{Var}(\zeta) = \int_0^T |a(t)|\, dt \geq \mathcal{H}_c^1(\zeta([0,T]))$$

and the equality holds if and only if ζ *is injective.*

Proof. The theorem follows by Theorem 4.4.1 of [6] and Theorem 1.3.5 of [78]. □

Now let γ be an integral curve of a fixed horizontal left invariant vector $X \in H$ and let $a = (a_1, ..., a_{m_1}) \in \mathbb{R}^{m_1} (\cong V_1 = H)$ denote the vector of coordinates of X, i.e.

$$X_x = \sum_{i=1}^{m_1} L_{x*} a_i \mathrm{e}_i;$$

since $|a|$ is constant, for all $K \subset \gamma$ compact we have

$$\mathcal{H}_c^1(K) = \int_{\gamma^{-1}(K)} |a(t)|\, dt = |a| \cdot \mathcal{L}^1(\gamma^{-1}(K)), \tag{8}$$

where $\mathcal{H}_c^1$ denotes the 1-*dimensional Hausdorff measure* with respect to the cc-distance d_c.

1.2 Connections and Curvatures on Carnot groups

Here below we introduce the notion of *connection* and that of *covariant derivative*. As a general reference for connections we refer the reader to [16], [54], [59], [88].

Definition 1.10. *[54] An affine connection on a* $\mathbf{C}^\infty$ *manifold* M *is a rule* ∇ *which assigns to each* $X \in \mathfrak{X}(M)$ *an* $\mathbb{R}$*-linear map* $\nabla_X : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ *called covariant differentiation with respect to* X*, such that for all* X, Y, $Z \in \mathfrak{X}(M)$ *and all* f, $g \in \mathbf{C}^\infty(M)$ *we have:*

(1) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$;

(2) $\nabla_X fY = f\nabla_X Y + (Xf)Y$.

If M is a Riemannian manifold with metric denoted by $\langle\cdot,\cdot\rangle$, then ∇ is called the Levi-Civita connection on M if for every X, Y, $Z \in \mathfrak{X}(M)$, satisfies the following further conditions [1]

(3) $X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$;

(4) $\nabla_X Y - \nabla_Y X = [X, Y]$.

Whenever we shall work with a Carnot group $\mathbb{G}$ endowed with the metric $\langle\cdot,\cdot\rangle$ defined before, we shall denote by ∇ its left invariant Levi-Civita connection. We explicitly mention that (see, for instance, [54]), if X_e, $Y_e \in \mathfrak{g}(= T_e\mathbb{G})$, we have

$$(\nabla_{X_x} Y)_x = L_{x*}(\nabla_{X_e} Y)_e.$$

The next proposition describes the Levi-Civita connection for general nilpotent Lie groups equipped with a left invariant Riemannian metric, so that it applies as well to the case of Carnot groups; see, for instance, [71] and [31].

Proposition 1.11. *Let $\mathbb{G}$ denote any nilpotent Lie group endowed with a left invariant Riemannian metric denoted by $\langle\cdot,\cdot\rangle$. Then the Levi-Civita connection ∇ of $\mathbb{G}$ satisfies the following formula*

$$\langle\nabla_X Y, Z\rangle = \frac{1}{2}\big(\langle[X, Y], Z\rangle - \langle[Y, Z], X\rangle + \langle[Z, X], Y\rangle\big) \tag{9}$$

for all left invariant vector fields X, Y, $Z \in \mathfrak{X}(\mathbb{G})$. Moreover, let $\{\mathrm{e}_1, ..., \mathrm{e}_n\}$ be an orthonormal basis of $\mathfrak{g}\,(= T_e\mathbb{G})$, let $(X_1, ..., X_n)$ be the associated frame on $\mathbb{G}$, and let $c_{ij}^k := \langle[\mathrm{e}_i, \mathrm{e}_j], \mathrm{e}_k\rangle$ denote the **structural constants** *of $\mathfrak{g}$. See also Section 1.3, equations* (15) *and* (16). *Then*

$$\langle\nabla_{X_i} X_j, X_k\rangle = \frac{1}{2}\big(c_{ij}^k - c_{jk}^i + c_{ki}^j\big) \qquad \forall\ i,\ j,\ k = 1, ..., n. \tag{10}$$

[1] We stress that, on any Riemannian manifold, there exists one and only one affine connection ∇ satisfying (1), (2), (3) and (4) (see [54, 59, 88]).

Notice that the above equation (10) can be rewritten as follows

$$\nabla_{X_i} X_j = \frac{1}{2}\Big(c_{ij}^k - c_{jk}^i + c_{ki}^j\Big) X_k \qquad \forall\ i,\ j = 1, ..., n. \tag{11}$$

If $\mathbb{G}$ denote a nilpotent Lie group with Levi-Civita connection ∇, we remind that, in general, the *torsion* T and the *curvature* R of ∇ are defined as follows:
for $x \in \mathbb{G}$, *let*

$$\mathsf{T} : T_x\mathbb{G} \times T_x\mathbb{G} \longrightarrow T_x\mathbb{G}, \qquad \mathsf{R} : T_x\mathbb{G} \times T_x\mathbb{G} \times T_x\mathbb{G} \longrightarrow T_x\mathbb{G},$$

denote the multilinear maps defined by

$$\mathsf{T}(\xi, \eta) := \nabla_\eta X - \nabla_\xi Y - [Y, X]_{|x}, \tag{12}$$

$$\mathsf{R}(\xi, \eta)\zeta := \nabla_\eta \nabla_X Z - \nabla_\xi \nabla_Y Z - \nabla_{[Y,X]_{|x}} Z, \tag{13}$$

where ξ, η, $\zeta \in T_x\mathbb{G}$ *and* X, Y, Z *are extensions of* ξ, η, ζ, *respectively, to vector fields on a neighborhood of* x. Obviously, since ∇ is the Levi-Civita connection of $\mathbb{G}$, by item (4) of Definition 1.10 it follows that T is identically 0. Moreover the curvature tensor R can be explicitly computed in terms of structural constants of $\mathfrak{g}$. This can be done using (11) and the definition of R, as in [71]. This method allows us to compute also the *sectional curvature* $\mathsf{K}(\xi, \eta)$ of two orthonormal vectors ξ, $\eta \in T_x\mathbb{G}$. For instance, for a nilpotent Lie group $\mathbb{G}$, according to [71], by using the previous notation of Proposition 1.11, for every i, $j = 1, ..., n$, the sectionals curvatures are given by

$$\mathsf{K}((X_i)_x, (X_j)_x) := \langle \mathsf{R}((X_i)_x, (X_j)_x)(X_i)_x, (X_j)_x \rangle$$

$$= \sum_{k=1}^{n} \Big[-\frac{1}{2} c_{ij}^k \Big(-c_{ij}^k + c_{jk}^i + c_{ki}^j \Big) + \frac{1}{4}\Big(c_{ij}^k - c_{jk}^i + c_{ki}^j \Big)\Big(c_{ij}^k + c_{jk}^i - c_{ki}^j \Big) + c_{ki}^i c_{kj}^j \Big].$$

We now introduce a general definition of *restricted connection*, as suggested in [60]. We only mention that such connection was originally defined and used by É. Cartan in his works on non-holonomic geometries; see [15].

Definition 1.12. *Let M be a Riemannian manifold and let (E, π_E, M), (F, π_F, M) be smooth subbundles of TM. An* E**-connection** $\nabla^{(E,F)}$ *on F is a rule which assigns to each vector field $X \in \mathbf{C}^\infty(M, E)$ an $\mathbb{R}$-linear transformation*

$$\nabla_X^{(E,F)} : \mathbf{C}^\infty(M, F) \longrightarrow \mathbf{C}^\infty(M, F)$$

such that for all $X, Y \in \mathbf{C}^\infty(M, E)$, for all $Z \in \mathbf{C}^\infty(M, F)$ and all $f, g \in \mathbf{C}^\infty(M)$ we have

(1) $\nabla_{fX+gY}^{(E,F)} Z = f\nabla_X^{(E,F)} Z + g\nabla_Y^{(E,F)} Z$;

(2) $\nabla_X^{(E,F)} fY = f\nabla_X^{(E,F)} Y + (Xf)\, Y$.

If $E = F$ we shall set $\nabla^E := \nabla^{(E,E)}$, while if $E = TM$ we set $D^F := \nabla^{(TM,F)}$ and we call D^F a **full connection** *on F.*

Remark 1.13. *The above definition enables us to work with many connections. We emphasize that, if $E = TM$, then the definition of full connection D^F on F recaptures the usual notion of connection on a vector bundle (see [72]), with the further hypothesis that this vector bundle is a vector subbundle of TM. In facts, the difference between these definitions is that in the latter we may covariantly differentiate along every curve of M, while in the first one, we may consider only curves that are tangent to the subbundle E.*

Remark 1.14. *Note that if (F, π_F, M) is a subbundle of TM, then from any (full) connection ∇ on TM we may get a full connection D^F on F as follows: denoting by $\mathcal{P}_F$ the projection operator on F, we set*

$$D_X^F Y := \mathcal{P}_F(\nabla_X \mathcal{P}_F(Y)) \quad \forall\, X, Y \in \mathfrak{X}(M).$$

Clearly, if we have a decomposition of the tangent space given by $TM = E \oplus F$, the previous construction holds as well for both the layers E and F; see [49], [60]. In the sequel we will use some of these notions to get computations in the setting of Carnot groups. Now we may give the following:

Definition 1.15. *Let $\mathbb{G}$ denote a Carnot group and let H its horizontal subbundle. Then, using the notation of Definition 1.12, we will denote by ∇^H the H-connection on H and by D^H the full connection on H. Moreover if $\psi \in \mathbf{C}^\infty(\mathbb{G})$, using the subriemannian metric on H, $\langle\cdot,\cdot\rangle_H$, we define the* **horizontal gradient of** ψ, *also denoted by the symbol $\nabla^H\psi$, as the (unique) left invariant horizontal vector field such that $\langle\nabla^H\psi, X\rangle_H = d\psi(X) = X\psi$ $(\forall\, X \in H)$. Finally, we define the* **horizontal divergence of** $X \in H$, *denoted by $\mathrm{div}_{_H} X$, to be the function given at each point $x \in \mathbb{G}$ by $\mathrm{div}_{_H} X := \mathrm{Trace}\big(Y \longrightarrow \nabla^H_Y X\big)$ $(Y \in H_x)$.*

Note that, with respect to any orthonormal frame $(Y_1, ..., Y_{m_1})$ for H, we have

$$\mathrm{div}_{_H} X = \sum_{i=1}^{m_1} \langle\nabla^H_{Y_i} X, Y_i\rangle_H.$$

In particular, with respect to the frame $(X_1, ..., X_{m_1})$, by using equation (10) and the stratification hypothesis on $\mathfrak{g}$ (see also Remark 1.22 below), we get that

$$\mathrm{div}_{_H} X = \sum_{i=1}^{m_1} X_i(x_i), \qquad \Big(X = \sum_{i=1}^{m_1} x_i\, X_i\Big).$$

Finally, we remind some elementary definitions and results about calculus in Carnot groups.

We say that a map $T : \mathbb{G} \longrightarrow \mathbb{R}$ is *H-linear* if is a group homomorphism of $(\mathbb{G}, \bullet)$ onto $(\mathbb{R}, +)$ and if it is positively homogeneous of degree 1 with respect to the positive dilations of $\mathbb{G}$, i.e. $T(\delta_\lambda x) = \lambda T(x)$ for every $\lambda > 0$ and $x \in \mathbb{G}$. The $\mathbb{R}$-linear set of *H-linear real valued functionals* is denoted by $\mathcal{L}_H$; it is endowed with the norm $\|T\|_{\mathcal{L}_H} := \sup\{|T(x)| :\ d_c(x, 0) \leq 1\}$. For a fixed left invariant frame $(X_1, \ldots, X_n)$ of $\mathbb{G}$, every H-linear map can be represented as follows, [43].

Proposition 1.16. *A function $T : \mathbb{G} \longrightarrow \mathbb{R}$ turns out to be H-linear if, and only if, there exists $a = (a_1, \ldots, a_{m_1}) \in \mathbb{R}^{m_1}$ such that, whenever $y = (y_1, \ldots, y_n) \in \mathbb{G}$, one has $T(y) = \sum_{j=1}^{m_1} a_j y_j$.*

Definition 1.17. *Let $U \subseteq \mathbb{G}$ be open and $x_0 \in U$. We say that $f : U \longrightarrow \mathbb{R}$ is* **Pansu-differentiable at** x_0 *if there exists an H-linear map T such that*

$$\lim_{\lambda\to 0^+} \frac{f(L_{x_0}(\delta_\lambda y)) - f(x_0)}{\lambda} = T(y)$$

uniformly with respect to y belonging to a compact set in $\mathbb{G}$. In particular, T is unique and we shall write, in the sequel, $d_H f(x_0)(y) := T(y)$.

Remark 1.18. *This definition depends only on $\mathbb{G}$ and not on the particular choice of the canonical generating vector fields. If $U \subseteq \mathbb{G}$ is open we denote by $\mathbf{C}^1_H(U)$ the set of all continuous real functions in U such that the map $d_H f : U \longrightarrow \mathcal{L}_H$ is continuous in U and by $\mathbf{C}^1_H(U, H)$ the set of all sections ψ of H whose canonical coordinates ψ_j belongs to $\mathbf{C}^1_H(U)$ $(j = 1, \dots, m_1)$. We remark that*

$$\mathbf{C}^1(U) \subsetneqq \mathbf{C}^1_H(U).$$

In general, **the inclusion is strict**. *We say that f* **is differentiable along** X_j **(**$j = 1, \dots, m_1$**) at** x_0 *if the map $\lambda \mapsto f(L_{x_0}(\delta_\lambda \mathrm{e}_j))$ is differentiable at $\lambda = 0$ where e_j is the j-th vector of the standard basis of $\mathfrak{g}\,(= T_e\mathbb{G} \cong \mathbb{R}^n)$.*

Notation 1.19. *For a fixed $x_0 \in \mathbb{G}$ we set*

$$\Pi_{x_0}(y) := \sum_{j=1}^{m_1} y_j\, X_j(x_0)$$

for $y = (y_1, \dots, y_n) \in \mathbb{G}$. The map $y \longrightarrow \Pi_{x_0}(y)$ is a smooth section of H.

Proposition 1.20. *[79] If f is Pansu-differentiable at x_0, then f is differentiable along X_j at x_0 $(j = 1, \dots, m_1)$ and*

$$d_H f(x_0)(y) = \langle \nabla^H f, \Pi_{x_0}(y) \rangle_{H_{x_0}} \quad \forall\, y \in \mathbb{G}. \tag{14}$$

1.3 Differential forms and Structure equations

We now introduce the main features of differential forms on Lie groups which can be found, for instance, in Helgason's book, [54]; see also [16, 33, 59, 64]. All that we will state in the sequel for general or nilpotent Lie groups applies as well to the case of Carnot groups. From now on let $\Lambda^k(\mathbb{G})$ denote the bundle of alternating covariant k-tensors on $\mathbb{G}$, i.e.

$$\Lambda^k(\mathbb{G}) = \coprod_{x \in \mathbb{G}} \Lambda^k(T_x\mathbb{G}).$$

The sections of $\Lambda^k(\mathbb{G})$ are called differential k-forms. We say that a *differential form* ω *on* $\mathbb{G}$ *is left invariant* if $L_x^*\omega = \omega$ for all $x \in \mathbb{G}$. Here the map

$$L_x^* : T_y^*\mathbb{G} \longmapsto T_{L_x(y)}^*\mathbb{G}, \qquad (y \in \mathbb{G})$$

denotes the pullback associated with the left translation L_x. The *right invariant differential forms* are defined analogously. Moreover, a differential form is called bi-invariant if it is both left and right invariant.

A *smooth global coframe* $(\omega_1, ..., \omega_n)$ for $\mathbb{G}$ is determined by the condition $\omega_i(X_j) = \delta_i^j$ for $i, j = 1, ..., n$, where $(X_1, ..., X_n)$ is the smooth global frame for $\mathbb{G}$ defined before, and δ_i^j denotes the Kronecker delta. By the previous definitions, it follows that this coframe $(\omega_1, ..., \omega_n)$ for $\mathbb{G}$, automatically, turns out to be adapted to the stratification of $\mathfrak{g}$. Now let $\mathfrak{g}^*$ denote the dual space of the Lie algebra $\mathfrak{g}$ and let $\{\mathrm{e}_1^*, ..., \mathrm{e}_n^*\}$ denote its basis. Obviously, $\mathfrak{g}^* = T_e^*\mathbb{G} = \mathrm{span}\{\mathrm{e}_1^*, ..., \mathrm{e}_n^*\}$ and using Cartesian coordinates $(x_1, ..., x_n)$ of $\mathfrak{g}$ with respect to the basis $\mathrm{e}_1, ..., \mathrm{e}_n$, we may notice that $\mathrm{e}_i^* = dx_i \, (i = 1, ..., n)$. Moreover, let $c_{ij}^k \, (i,\, j,\, k = 1, ..., n)$ denote the *structural constants* of the Lie algebra $\mathfrak{g}$, defined by

$$[\mathrm{e}_i, \mathrm{e}_j] := \sum_{k=1}^{n} c_{ij}^k \mathrm{e}_k. \tag{15}$$

Notice that the structural constants c_{ij}^k satisfy the relations:

$$(i)\;\; c_{ij}^k + c_{ji}^k = 0; \qquad\qquad (ii)\;\; \sum_{j=1}^{n} c_{jl}^i c_{km}^j + c_{jm}^i c_{lk}^j + c_{jk}^i c_{ml}^j = 0. \tag{16}$$

The following proposition introduces the so-called *Maurer-Cartan equations* (see [59, 54]).

Proposition 1.21. *Let* $\omega_1, ..., \omega_n$ *be the global coframe for a Lie group* $\mathbb{G}$ *uniquely determined by requiring that* $\omega_i(X_j) = \delta_i^j$ $(i, j = 1, ..., n)$*, where* $X_1, ..., X_n$ *is the global frame for* $\mathbb{G}$*. Then*

$$d\omega_k = -\frac{1}{2} \sum_{i,j=1}^{n} c_{ij}^k \, \omega_i \wedge \omega_j. \tag{17}$$

Remark 1.22. **[The case of Carnot groups]** *In the case of Carnot groups, the stratification hypothesis on the Lie algebra $\mathfrak{g}$ implies that, if $\mathrm{e}_i \in V_r$ and $\mathrm{e}_j \in V_s$, then*

$$[\mathrm{e}_i, \mathrm{e}_j] \in V_{r+s}.$$

Therefore

$$c_{ij}^k \neq 0 \Longrightarrow h_{r+s-1} < k < h_{r+s+1} \quad \forall\ i,\ j,\ k = 1, ..., n.$$

In particular, $c_{ij}^i = c_{ij}^j = 0 \quad \forall\ i, j = 1, ..., n$. Moreover let k be such that

$$h_{l-1} < k < h_{l+1}.$$

Then $c_{ij}^k \neq 0$ only if, for any i, j such that $h_{r-1} < i < h_{r+1}$ and $h_{s-1} < j < h_{s+1}$, we have that $l = r+s$. This means that, in the above formula (17), we may rewrite the summation as

$$d\omega_k = -\frac{1}{2} \sum_{1 \leq i,j \leq h_{l-1}} c_{ij}^k\, \omega_i \wedge \omega_j \quad \text{whenever} \quad h_{l-1} < k < h_{l+1}. \tag{18}$$

Remark 1.23. *The 1-forms ω_i can explicitly be determined in terms of structural constants on any Lie group, [54]. More precisely, let $(x_1, ..., x_n)$ be the Cartesian coordinates of $\mathfrak{g}$ with respect to the basis $\{\mathrm{e}_1, ..., \mathrm{e}_n\}$ and let $X = \sum_{i=1}^n x_i \mathrm{e}_i$ (note that $(x_1, ..., x_n)$ is the n-tuple of the exponential coordinates of $x = \exp(X) \in \mathbb{G}$). Then, there exist functions*

$$B_{ih}(x) = B_{ih}(x_1, ..., x_n) \in \mathbf{C}^\infty(\mathbb{R}^n) \quad (i, h = 1, ..., n)$$

such that $(\omega_i)_x = \sum_{h=1}^n B_{ih}(x) dx_h$ and

$$\begin{aligned} B_{ih}(x) &= (\omega_i)_x(d\exp_X(\mathrm{e}_h)) = (\omega_i)_e\Big(\frac{1 - e^{-\mathrm{ad}X}}{\mathrm{ad}\,X}(\mathrm{e}_h)\Big) \\ &= dx_i\Big(\frac{1 - e^{-\mathrm{ad}\,X}}{\mathrm{ad}\,X}(\mathrm{e}_h)\Big). \end{aligned}$$

For nilpotent Lie groups, and thus for Carnot groups, the formal series of linear operators $1 - e^{-\mathrm{ad}X}/\mathrm{ad}X$ have only a finite number of non 0 terms, so that it can be explicitly computed. This will be done for the case of 2-step Carnot groups in Section 1.3.

Notation 1.24. *Throughout the thesis, if* $h \geq 0$*, then* $\mathcal{H}^h$ *will denote the* h*-dimensional Hausdorff measure associated with the Riemannian metric of* $\mathbb{G}$*, while* $\mathcal{H}^h_c$ *and* $\mathcal{S}^h_c$ *will denote, respectively, theh-dimensional Hausdorff measure and the* h*-dimensional spherical Hausdorff measure, obtained from the cc-distance* d_c *using Carathédory's construction, [33]. We also denote by* $\mathcal{L}^p$ $(p \in \mathbb{N})$ *the standard* p*-dimensional Lebesgue measure on* $\mathbb{R}^p$*.*

The following remark explain what is the canonical volume measure on Carnot groups. This construction holds as well for nilpotent Lie groups.

Remark 1.25. *[Volume measure on groups] Since* $\mathbb{G}$ *is a* k*-step nilpotent Lie group equipped with exponential coordinates, it follows that, if* $x \in \mathbb{G}$*, left translations* L_x *and right translations* R_x *are maps whose jacobian determinants are identically equal to* 1*. Moreover, the exponential map* $\exp : \mathfrak{g} \longrightarrow \mathbb{G}$ *takes Lebesgue measure on* $\mathfrak{g}$ *to a left invariant (Haar) measure* dV^n *on* $\mathbb{G}$*. This measure turns out to be also right invariant (see [20, 64]). Therefore, Carnot groups equipped with the measure* dV^n *are unimodular. Since* $\mathbb{G}$ *is naturally identified with* $\mathbb{R}^n$*, throughout the thesis we shall use indifferently the symbol* dV^n *or that* $\mathcal{L}^n$ *to denote the volume measure on* $\mathbb{G}$*.*

We may restated the previous remark in terms of left invariant differential forms. Indeed, the left invariant volume element of $\mathbb{G}$ is just the differential n-form defined by

$$\Omega^n := \omega_1 \wedge ... \wedge \omega_n \in \Lambda^n(\mathbb{G}), \tag{19}$$

so that Ω^n turns out to be a bi-invariant n-form, usually called the Haar volume form on $\mathbb{G}$. We explicitly remark that Ω^n is the *Riemannian volume element* with respect to the chosen Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{G}$. From now on we shall either use the notation dV^n or Ω^n to denote the volume form of $\mathbb{G}$. Note also that a deep theorem of Mitchell [73] states that the Hausdorff dimension of a Carnot group with respect to the cc-distance d_c equals its homogeneous dimension Q.

Remark 1.26. *We stress that since* $\mathcal{S}^Q_c$ *- i.e. the* Q*-dimensional spherical Hausdorff measure of* $\mathbb{G}$*- is a Haar measure of* $\mathbb{G}$ *and since, up to scale, there is only* *one Haar measure on locally compact Lie groups, we must have*

$$dV^n \llcorner B = k_Q \cdot \mathcal{S}_c^Q \llcorner B \qquad \forall\, B \in \mathcal{B}(\mathbb{G}), \tag{20}$$

where k_Q is an absolute constant. Hereafter $\mathcal{B}(\mathbb{G})$ will denote the family of Borel subsets of $\mathbb{G}$.

We finally introduce a specific notation for the class of hyperplanes which are, in a sense, orthogonal to the horizontal distribution, the so-called *vertical hyperpanes*. These hyperplanes will be very useful in some mean integral formulae stated in the sequel.

Notation 1.27. *If $z \in \mathbb{G}$ and $X \in H$, we set*

$$\mathcal{I}_z(X) := L_z(\exp(X_e^{\perp})) = L_z\{y \in \mathbb{G} : \langle \Pi_e(y), X_e\rangle_{H_e} = 0\}, \tag{21}$$

where $X_e^{\perp}$ is the orthogonal complement of X_e in $\mathfrak{g}$. Explicitly, if $X_e = \sum_{j=1}^{m_1} a_j \mathrm{e}_j$,

$$\mathcal{I}_z(X) = \Big\{ y \in \mathbb{G} : \sum_{j=1}^{m_1} [y_j - x_j]\, a_j = 0 \Big\}.$$

We call $\mathcal{I}_z(X)$ **vertical hyperplane through** x **and orthogonal to** X *and we denote by $\mathcal{V}_z$ the family of all vertical hyperplanes through x, i.e.*

$$\mathcal{V}_z := \big\{ \mathcal{I}_z(X) : X \in H \big\}.$$

1.4 HBV and H-Caccioppoli sets

For the classical theory of BV functions and Caccioppoli sets we shall refer the reader to [2], [32] and [97], while many generalizations to metric spaces as Carnot-Carathéodory ones or Carnot groups we may cite [1], [3], [4], [14], [39, 41, 42, 43], [45], [73], [78], [79]. We shall make now a quick overview of main definitions and properties that will be used later on.

Definition 1.28. *Let $U \subseteq \mathbb{G}$ be open and $f \in L^1(U)$. Then, f has bounded H-variation in U if*

$$|\nabla^H f|(U) := \sup \left\{ \int_U f \operatorname{div}_H(\psi)\, d\mathcal{L}^n : \psi \in \mathbf{C}_0^1(U, H),\ |\psi| \leq 1 \right\} < \infty, \tag{22}$$

where $|\nabla^H f|(U)$ is called H-variation of f in U. We denote by $HBV(U)$ the vector space of functions of bounded H-variation in U and by $HBV_{loc}(U)$ the set of functions belonging to $HBV(U)$ for each open set $U \Subset U$.

Theorem 1.29. *[Structure of HBV functions] If $f \in HBV(U)$ then $|\nabla^H f|$ is a Radon measure in U and there exists a $|\nabla^H f|$-measurable horizontal section $\sigma_f : U \to H$ such that $|\sigma_f| = 1$ for $|\nabla^H f|$-a.e. $x \in U$ and*

$$\int_U f \operatorname{div}_H(\psi)\, d\mathcal{L}^n = \int_U \langle \psi, \sigma_f \rangle_H \, d\,|\nabla^H f| \quad \forall\, \psi \in \mathbf{C}_0^1(U, H). \tag{23}$$

Moreover ∇^H can be extended as a vector valued measure to functions in HBV setting

$$\nabla^H f := -\sigma_f \llcorner |\nabla^H f| = \Big(-(\sigma_f)_1 \llcorner |\nabla^H f|, \dots, -(\sigma_f)_{m_1} \llcorner |\nabla^H f| \Big), \tag{24}$$

where $(\sigma_f)_j$ $(j = 1, \dots, m_1)$ is j-th component of the vector valued measure σ_f, with respect to the horizontal frame.

The next results hold for general Carnot-Carathéodory geometries associated with vector fields as proved in [39], [45].

Theorem 1.30. *[Lower semicontinuity] Let $f, f_k \in L^1(U)$, $k \in \mathbb{N}$, be such that $f_k \to f$ in $L^1(U)$; then*

$$|\nabla^H f|(U) \leq \liminf_{k\to\infty} |\nabla^H f_k|(U). \tag{25}$$

Theorem 1.31. *[Compactness] $HBV_{loc}(\mathbb{G})$ is compactly embedded in $L^p_{loc}(\mathbb{G})$ for $1 \leq p < \frac{Q}{Q-1}$, where Q denotes the homogeneous dimension of $\mathbb{G}$.*

Definition 1.32. *Let U be an open subset of $\mathbb{G}$; then a measurable set $E \subset \mathbb{G}$ has finite H-perimeter in U, or is a H-Caccioppoli set in U, if its characteristic function $\mathbf{1}_E$ belongs to $HBV_{loc}(U)$. In this case we call H-perimeter of E in U the (Radon) measure given by*

$$|\partial E|_H := |\nabla^H \mathbf{1}_E| \tag{26}$$

and we call generalized inward $\mathbb{G}$-normal along ∂E in U the vector valued measure

$$\nu_E := -\sigma_{\mathbf{1}_E}. \tag{27}$$

Remark 1.33. *The H-perimeter measure is invariant under group translations, i.e.*

$$|\partial E|_H(B) = |\partial(L_x E)|_H(L_x B) \qquad \forall\, x \in \mathbb{G} \quad \forall\, B \in \mathcal{B}(\mathbb{G}); \tag{28}$$

indeed $\mathrm{div}_{_H}$ *is invariant under group translations and the Jacobian determinant of L_x is* 1. *Moreover the H-perimeter is $(Q-1)$-homogeneous with respect to the intrinsic dilations, i.e.*

$$|\partial(\delta_t E)|_H(\delta_t B) = t^{Q-1}|\partial E|_H(B) \qquad \forall\, B \in \mathcal{B}(\mathbb{G}). \tag{29}$$

This fact can be easily proved by a change of variables in formula (22).

Proposition 1.34 ([14]). *Let E be a H-Caccioppoli set in U having $\mathbf{C}^1$-smooth boundary. Then*

$$|\partial E|_H(U) = \int_{\partial E \cap U} \sqrt{\langle X_1, N\rangle^2 + ... + \langle X_{m_1}, N\rangle^2}\, d\,\mathcal{H}^{n-1}, \tag{30}$$

where N is the unit outward normal along ∂E. In this case we have

$$(\nu_E)_x = \frac{(\langle X_{1x}, N_x\rangle, ..., \langle X_{m_1 x}, N_x\rangle)}{\sqrt{\langle X_{1x}, N_x\rangle^2 + ... + \langle X_{m_1 x}, N_x\rangle^2}} \qquad \forall\, x \in \partial E \cap U.$$

The regularization technique of convolution with mollifiers enables us to obtain approximation results for both Sobolev and HBV functions in Carnot groups as well as in more general contexts; see [39], [45]. To this end we introduce a family of spherically symmetric mollifiers J_ϵ ($\epsilon > 0$) by $J_\epsilon(x) := \epsilon^{-n}J(\epsilon^{-1}x)$, where $J \in \mathbf{C}_0^\infty(\mathbb{R}^n)$, $J \geq 0$, $\mathrm{spt}\,(J) \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\int_{\mathbb{G}} J\, d\mathcal{L}^n = 1$.

Lemma 1.35. *Let $U \subseteq \mathbb{G}$ be open and $f \in HBV(U)$. If $\widetilde{U} \Subset U$ is open and $|\nabla^{H} f|(\partial\widetilde{U}) = 0$, then*

$$\lim_{\epsilon \to 0} |\nabla^{H}(J_\epsilon * f)|(\widetilde{U}) = |\nabla^{H} f|(\widetilde{U}). \tag{31}$$

Theorem 1.36. *[Density for HBV functions] Let $f \in HBV(U)$; then there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset \mathbf{C}^\infty(U) \cap HBV(U)$ such that*

$$\lim_{j\to\infty} \|f_j - f\|_{L^1(U)} = 0 \quad \textit{and} \quad \lim_{j\to\infty} |\nabla^{H} f_j|(U) = |\nabla^{H} f|(U). \tag{32}$$

The following coarea formula for HBV functions is a key tool to understand the interplay between HBV functions and H-Caccioppoli sets. For a proof see [45], [39, 41], [67], [79].

Theorem 1.37. *Let $f \in HBV(U)$ and set $E_t := \{x \in U : f(x) > t\}$ for $t \in \mathbb{R}$. Then*

(i) E_t has finite H-perimeter in U for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$;

(ii) $|\nabla^H f|(U) = \int_{-\infty}^{+\infty} |\partial E_t|(U)\, dt$.

(iii) Conversely, if $f \in L^1(U)$ and $\int_{-\infty}^{+\infty} |\partial E_t|(U)\, dt < \infty$, then $f \in HBV(U)$ and (ii) holds.

As in $\mathbb{R}^n$ a $\mathbf{C}^1$-smooth hypersurface can be regarded as the zero set of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with non-vanishing gradient, in Carnot groups we must follow the same approach in defining the so-called *H-regular hypersurfaces*; see [41, 42, 43]. This choice is motivated by the fact that it is not possible to follows Federer's approach to rectifiability (see [4], [41, 42, 43], [67]).

Definition 1.38. *[42] $S \subset \mathbb{G}$ is a H-**regular hypersurface** if for every $x \in S$ there exist a neighborhood U of x and a function $f \in \mathbf{C}^1_H(U)$ such that*

(i) $S \cap U = \{y \in U : f(y) = 0\}$;

(ii) $\nabla^H f(y) \neq 0 \qquad \forall\, y \in U$.

The following important Implicit Function Theorem was proved in [42].

Theorem 1.39. *[Implicit Function Theorem] Let $\Omega \subseteq \mathbb{G}$ be open such that $0 \in \Omega$; let $f \in \mathbf{C}^1_H(\Omega)$ be such that $f(0) = 0$ and $X_1 f(0) > 0$. Put*

$$E := \{x \in U : f(x) < 0\}, \;\; S := \{x \in \Omega : f(x) = 0\}$$

and for $h,\, \delta > 0$ set $J_h := [-h, h]$ and

$$I_\delta := \{\xi = (\xi_2, \dots, \xi_n) \in \mathbb{R}^{n-1} : |\xi_j| \leq \delta,\, j = 2, \dots, n\}.$$

If $\xi \in \mathbb{R}^{n-1}$ *and* $t \in J_h$ *we denote by* $\gamma^1_{(0,\xi)}(t)$ *the integral curve of the horizontal left invariant vector field* $X_1 \in H$ *at the time* t *issued from*

$$(0,\xi) \in \{(0,\eta) \in \mathbb{G} : \eta \in \mathbb{R}^{n-1}\},$$

i.e. $\gamma^1_{(0,\xi)}(t) = \exp[tX_1](0,\xi)$. *Then there exist* δ, $h > 0$ *such that*

$$\mathbb{R} \times \mathbb{R}^{n-1} \ni (t,\xi) \longmapsto \gamma^1_{(0,\xi)}(t)$$

is a diffeomorphism of a neighborhood of $J_h \times I_\delta$ *onto an open subset of* $\mathbb{R}^n$ *and denoting by* $U \Subset \Omega$ *the image of* $\mathrm{Int}\{J_h \times I_\delta\}$ *under this mapping the following statements hold:*

(i) E *has finite* H*-perimeter in* U;

(ii) $\partial E \cap \Omega = S \cap U$;

(iii) if ν_E *is the generalized inner unit normal of* E *then*

$$\nu_E(x) = -\frac{\nabla^H f(x)}{|\nabla^H f(x)|_H} \quad \forall\, x \in S\cap\Omega, \qquad |\nu_E|_{H_x} = 1 \ \text{ for } |\partial E|_H\text{-}a.e.\, x \in U.$$

Moreover there exists a unique continuous function $\phi = \phi(\xi) : I_\delta \longrightarrow J_h$ *such that, setting* $\Phi(\xi) = \gamma^1_{(0,\xi)}(\phi(\xi))$ *for* $\xi \in I_\delta$, *we have*

(iv) $S \cap U = \{x \in U : x = \Phi(\xi),\, \xi \in I_\delta\}$;

(v) the H*-perimeter has the following integral representation:*

$$|\partial E|_H(U) = \int_{I_\delta} \frac{\sqrt{\sum_{j=1}^m |X_j f(\Phi(\xi))|^2}}{X_1 f(\Phi(\xi))}\, d\xi.$$

We end this subsection with the definition of *partial perimeter along an horizontal direction*, while in the next Lemma 1.41 we explicitly characterize it.

Definition 1.40. *Let* Ω *be open and let* $X \in H$. *Let* E *be a Lebesgue measurable subset of* $\mathbb{G}$ *such that* $\mathcal{L}^n(E \cap \Omega) < \infty$. *Then we say that* E *has finite* X*-perimeter in* Ω *if*

$$|\partial_X E|(\Omega) := \sup\left\{ \int_\Omega \mathbf{1}_E\, X\varphi\, d\mathcal{L}^n : \varphi \in \mathbf{C}^1_0(\Omega),\ |\varphi| \le 1 \right\} < \infty \tag{33}$$

and we call this quantity the X**-perimeter of** E **in** Ω *; see also [39], [74].*

We will see in Section 3.1 that this notion agrees with that more general of X-variation of a L^1 function; see, for instance, Definition 3.1 and Remark 3.2 below.

Lemma 1.41. *Let Ω be open and let $X \in H$. If E is a H-Caccioppoli set in Ω, then*

$$|\partial_X E|(\Omega) = \int_\Omega |\langle X, \nu_E\rangle_H| \, d\,|\partial E|_H.$$

Proof. Firstly, putting $\Phi := \varphi X \in H$, where $\varphi \in \mathbf{C}_0^1(\Omega)$, $|\varphi| \leq 1$, we get

$$\begin{aligned}\int_\Omega \mathbf{1}_E X \varphi \, d\mathcal{L}^n &= \int_\Omega \mathbf{1}_E \langle \nabla^H \varphi, X\rangle_H \, d\mathcal{L}^n = \int_\Omega \mathbf{1}_E \operatorname{div}_H \Phi \, d\mathcal{L}^n \\ &= -\int_\Omega \langle \Phi, \nu_E\rangle_H \, d\,|\partial E|_H = -\int_\Omega \varphi \langle X, \nu_E\rangle_H \, d\,|\partial E|_H.\end{aligned}$$

Since for every $x \in \Omega$ we have $\varphi \langle X, \nu_E\rangle_H \leq |\langle X, \nu_E\rangle_H|$, from Definition 1.40 it follows that

$$|\partial_X E|(\Omega) \leq \int_\Omega |\langle X, \nu_E\rangle_H| \, d\,|\partial E|_H.$$

Now we shall prove the reverse inequality. Let $\epsilon > 0$ and set

$$\Omega_\epsilon := \left\{ x \in \Omega : |x| < \frac{1}{\epsilon},\ \operatorname{dist}(x, \partial\Omega) > \epsilon \right\}, \quad \zeta_\epsilon := \frac{J_\epsilon * \Big(\mathbf{1}_{\Omega_\epsilon} \operatorname{sign}\langle X, \nu_E\rangle_H\Big)}{\sqrt{\epsilon^2 + \Big(J_\epsilon * \mathbf{1}_{\Omega_\epsilon} \operatorname{sign}\langle X, \nu_E\rangle_H\Big)^2}},$$

where, as above, J_ϵ is a Friedrichs' mollifier. Using standard properties of mollifiers we get that $\zeta_\epsilon \in \mathbf{C}_0^\infty(\Omega)$, $|\zeta_\epsilon| < 1$, and $\zeta_\epsilon \longrightarrow \mathbf{1}_\Omega \operatorname{sign}\langle X, \nu_E\rangle_H$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{G}$, as $\epsilon \to 0$. Finally, from Definition 1.40 together with the previous computations and Fatou's Lemma we get

$$\begin{aligned}|\partial_X E|(\Omega) &\geq \liminf_{\epsilon \to 0} \int_\Omega \zeta_\epsilon \langle X, \nu_E\rangle_H \, d\,|\partial E|_H \\ &\geq \int_\Omega \liminf_{\epsilon \to 0} \zeta_\epsilon \langle X, \nu_E\rangle_H \, d\,|\partial E|_H = \int_\Omega |\langle X, \nu_E\rangle_H| \, d\,|\partial E|_H.\end{aligned}$$

□

Remark 1.42. *From Lemma 1.41 and the regularity of the measures $|\partial E|_H$ and $|\partial_X E|_H$ one gets equality of measures, i.e.*

$$|\partial_X E| \llcorner B = |\langle X, \nu_E\rangle_H| \cdot |\partial E|_H \llcorner B \quad \forall\, B \in \mathcal{B}(\mathbb{G}).$$

2 Integral geometry in Carnot groups

2.1 A Fubini type Theorem in Carnot groups

Let $S \subset \mathbb{G}$ be a fixed $\mathbf{C}^1$-smooth hypersurface. By the classical Implicit Function Theorem we may assume that $S = \partial E$ where $E \subset \mathbb{G}$ is an open H-Caccioppoli set. Moreover let us choose a horizontal left invariant direction $X \in H$ which is *globally transverse to* S, i.e.

$$\langle X_y, N_y \rangle \neq 0 \qquad \forall\, y \in S, \tag{34}$$

where N is the euclidean unit inward normal along S. We explicitly notice that if $X \in H$ is a horizontal left invariant vector field and $S \subset \mathbb{G}$ is a $\mathbf{C}^1$-smooth hypersurface we have that

$$\langle X, \nu_E \rangle_{H_y} \neq 0 \Longleftrightarrow \langle X_y, N_y \rangle \neq 0 \qquad \forall\, y \in S.$$

Indeed by Proposition 1.34 the inward unit H-normal along $S = \partial E$ is given by

$$(\nu_E)_y = \frac{\sum_{j=1}^{m_1} \langle (X_j)_y, N_y \rangle (X_j)_y}{\sqrt{\sum_{j=1}^{m_1} \langle (X_j)_y, N_y \rangle^2}} \qquad \forall\, y \in S$$

and if $X = \sum_{i=1}^{m_1} a_i X_i$ we get

$$\langle X, \nu_E \rangle_H = \frac{\sum_{j=1}^{m_1} \langle X_j, N \rangle\, a_j}{\sqrt{\sum_{j=1}^{m_1} \langle X_j, N \rangle^2}} = \frac{\langle X, N \rangle}{\sqrt{\sum_{j=1}^{m_1} \langle X_j, N \rangle^2}}.$$

Condition (34) is therefore equivalent to require that $X_y \in H\mathbb{G}_y \setminus T_y S$ for $y \in S$.

Consider now the following Cauchy problem

$$\begin{cases} \dot{\gamma}(t) = X(\gamma(t)), \\ \gamma(0) = y \in S. \end{cases}$$

There exists a unique smooth solution of this problem which is defined on all of $\mathbb{R}$ and, throughout this section, we shall write $\gamma_{X_y}(t) = exp\,[tX](y)$ for $t \in \mathbb{R}$ and $y \in S$. If $X \in H$ is fixed, we shall remove the apex just writing γ_y. Notice that $\gamma_{X_y}(t) = y \bullet exp\,(tX) = \mathcal{P}(y, exp\,(tX))$. Following [69], we call such a trajectory a *horizontal X-line,* or simply *horizontal line*. Now let us consider the family of horizontal X-lines starting from S.

Notation 2.1. *We shall denote by* $\mathcal{R}_S^X$ *the subset of* $\mathbb{G}$ **reachable from** S **by means of horizontal** X**-lines**, *i.e.*

$$\mathcal{R}_S^X := \Big\{x \in \mathbb{G} : \exists\ y \in S,\ \exists\ t \in \mathbb{R}\ \ s.t.\ x = exp\,[tX](y)\Big\}.$$

From now on we assume that S enjoys the following further property:

$$\gamma_y(\mathbb{R}) \cap S = y \qquad \forall\ \{y\} \in S. \tag{35}$$

Since X is transverse to S, from the uniqueness of the solutions of the Cauchy problem and the hypothesis (35) it follows that any subset D of $\mathcal{R}_S^X$ has a *natural projection* on S along the horizontal direction X. More precisely we may define a mapping $pr_S^X : D \subseteq \mathcal{R}_S^X \longmapsto S$ as follows; for $x \in D$ and $y \in S$ we set $y = pr_S^X(x)$ if, and only if, there exists $t \in \mathbb{R}$ such that $x = exp\,[tX](y)$. Using this projection every subset D of $\mathcal{R}_S^X$ can be foliated with one-dimensional leaves that are horizontal X-lines. In fact, setting $D_y := \gamma_y(\mathbb{R}) \cap D$, one has:

$$D = \coprod_{y \,\in\, pr_S^X(D)} D_y \qquad \text{and} \quad y_1 \neq y_2 \Longrightarrow D_{y_1} \cap D_{y_2} = \emptyset \quad \forall\ y_1,\, y_2 \in pr_S^X(D).$$

Remark 2.2. *We remark that if* S *is a* $\mathbf{C}^1$*-smooth closed hypersurface without boundary and globally transverse to* $X \in H$ *one can prove, by applying the Tubular Neighborhood Theorem (see [56]), that any integral curve of* X *cut* S *in at most one point and hence (35) is automatically verified.*

In many subsequent integration formulae we shall adopt the so-called *vertical hyperplanes* (see Notation 1.27). We emphasize that every subset of $\mathbb{G}$ is reachable from any vertical hyperplane. We would also stress that, although this projection turns out to be useful in many integral formulas, it is not Lipschitz with respect to the Carnot-Carathéodory distance d_c and so one cannot to assimilate it to an euclidean orthogonal projection. For more details, see [61].

We may state our first result of this section:

Proposition 2.3. *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^1$ smooth hypersurface and $X \in H$, $|X|_H = 1$, be a unit horizontal left invariant vector field which is transverse to S, i.e.*

$$\langle X, \nu_E \rangle_{H_y} \neq 0 \qquad \forall\, y \in S.$$

Let γ_y be the horizontal X-line starting from $y \in S$, i.e.

$$\gamma_y : \mathbb{R} \longmapsto \mathbb{G}, \quad \gamma_y(t) = exp\,[tX](y) \ \textit{ for } \ y \in S.$$

Moreover we assume that

$$\gamma_y(\mathbb{R}) \cap S = \{y\} \qquad \forall\, y \in S.$$

Let $D \subseteq \mathcal{R}^X_S$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from S by means of horizontal X-lines. Since locally $S = \partial E$, for a suitable open set $E \subset \mathbb{G}$, without loss of generality we may assume that $S = \partial E$ globally, where E has locally finite H-perimeter. Then we have

(i) $D_y := \gamma_y(\mathbb{R}) \cap D$ *is $\mathcal{H}^1_c$-measurable for $|\partial E|_H$-a.e. $y \in S$;*

(ii) the mapping $S \ni y \longmapsto \mathcal{H}^1_c(D_y)$ is $|\partial E|_H$-measurable on S and

$$\begin{aligned} \mathcal{L}^n(D) &= \int_{pr^X_S(D)} \mathcal{H}^1_c(D_y)\, |\langle X, \nu_E \rangle_{H_y}|\, d\,|\partial E|_H(y) \\ &= \int_{pr^X_S(D)} \mathcal{H}^1_c(D_y)\, d\,|\partial_X E|, \end{aligned}$$

where $pr^X_S(D) \subseteq S$ is the horizontal X-projection of D on S.

This proposition may be generalized to H-regular hypersurfaces and, more precisely, we can state our main theorem as follows:

Theorem 2.4. *Let $S \subset \mathbb{G}$ be a H-regular hypersurface. By Theorem 1.39, without loss of generality, we may assume that $S = \partial E$ globally, where $E \subset \mathbb{G}$ is an open H-Caccioppoli set with locally $\mathbf{C}^1_H$ boundary. Let $X \in H$, $|X|_H = 1$, be a unit horizontal left invariant vector field which is transverse to S. Let γ_y be the horizontal X-line starting from $y \in S$ and let us suppose that $\gamma_y(\mathbb{R}) \cap S = \{y\}$ for every $y \in S$. Let $D \subseteq \mathcal{R}^X_S$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from S. Then we have:*

(i) $D_y := \gamma_y(\mathbb{R}) \cap D$ *is* $\mathcal{H}^1_c$*-measurable for* $|\partial E|_H$*-a.e.* $y \in S$.

(ii) the mapping $S \ni y \longmapsto \mathcal{H}^1_c(D_y)$ *is* $|\partial E|_H$*-measurable on* S *and*

$$\begin{aligned}\mathcal{L}^n(D) &= \int_{pr_S^X(D)} \mathcal{H}^1_c(D_y)\,|\langle X, \nu_E\rangle_{H_y}|\,d\,|\partial E|_H(y)\\ &= \int_{pr_S^X(D)} \mathcal{H}^1_c(D_y)\,d\,|\partial_X E|(y).\end{aligned}$$

The proof of this results will be given in the next subsection. Nevertheless we can state a first useful consequence.

Corollary 2.5. *Let* $S \subset \mathbb{G}$ *be a* H*-regular hypersurface and assume that* $S = \partial E$ *globally, where* $E \subset \mathbb{G}$ *is a suitable open* H*-Caccioppoli set. Let* $X \in H$, $|X|_H = 1$, *be a unit horizontal left invariant vector field which is transverse to* S *and denote by* γ_y *the horizontal* X*-line starting from* $y \in S$. *We assume that* $\gamma_y(\mathbb{R}) \cap S = \{y\}$ *for every* $y \in S$. *Finally let* $D \subseteq \mathcal{R}^X_S$ *be a Lebesgue measurable subset of* $\mathbb{G}$ *that is reachable from* S *by means of* X*-lines. Then, for every function* $\psi \in L^1(D)$ *the following statements hold:*

(i) Let $\psi_{|D_y}$ *denote the restriction of* ψ *to* $D_y := \gamma_y(\mathbb{R}) \cap D$ *and let us define the mapping*

$$\psi_y : \gamma_y^{-1}(D_y) \subseteq \mathbb{R} \longmapsto \mathbb{R}, \qquad \psi_y(t) = (\psi \circ \gamma_y)(t).$$

Then ψ_y *is* $\mathcal{L}^1$*-measurable for* $|\partial E|_H$*-a.e.* $y \in S$. *Equivalently, we have that the restriction* $\psi_{|D_y}$ *is* $\mathcal{H}^1_c$*-measurable for* $|\partial E|_H$*-a.e.* $y \in S$.

(ii) The mapping defined by

$$S \ni y \longmapsto \int_{D_y} \psi\,d\mathcal{H}^1_c = \int_{\gamma_y^{-1}(D_y)} \psi_y(t)\,dt$$

is $|\partial E|_H$*-measurable on* S *and the following formula holds:*

$$\begin{aligned}\int_D \psi\,d\,\mathcal{L}^n &= \int_{pr_S^X(D)} \int_{D_y} \psi\,d\mathcal{H}^1_c\ d\,|\partial_X E|(y)\\ &= \int_{pr_S^X(D)} \int_{\gamma_y^{-1}(D_y)} \psi_y(t)\,dt\ |\langle X, \nu_E\rangle_{H_y}|\,d\,|\partial E|_H(y).\end{aligned}$$

Proof. Having at our disposal Theorem 2.4, is enough to use a standard argument of measure theory to approximate the function ψ with a finite linear combination of characteristic functions, as for instance in Theorem 3.2.5 of [33]. $\square$

Till now we have used the intrinsic H-perimeter as a measure for hypersurfaces in $\mathbb{G}$ but also different measures can be considered. In fact, the comparison of different surface measures is a one of the main problems of the Geometric Measure Theory in Carnot groups and in general Carnot-Carathéodory spaces. In particular an interesting problem for Carnot groups is that to compare the H-perimeter with the the $(Q-1)$-dimensional Hausdorff measure associated with either the cc-distance d_c or with some suitable homogeneous distance on $\mathbb{G}$, in the case of euclidean smooth hypersurfaces (see [7], [42, 43], [67]).
The following result for general Carnot groups is proved in [67].

Remark 2.6. *Let S be a $\mathbf{C}^1$-smooth hypersurface and let assume that S is locally the boundary of an open set E. Then*

$$|\partial E|_H \llcorner B = k_{Q-1}(\nu_E)\, \mathcal{H}_c^{Q-1} \llcorner \Big(S \cap B\Big) \quad \forall\, B \in \mathcal{B}(\mathbb{G}) \tag{36}$$

where the measure $\mathcal{H}_c^{Q-1}$ is the spherical [2] *$(Q-1)$-dimensional Hausdorff measure associated with the cc-distance d_c and k_{Q-1} is a function depending on ν_E, called metric factor (see Definition 2.17 of [67]).*

By means of this result, we may reformulate Proposition 2.3 using Hausdorff measures with respect to the cc-distance. We have, more precisely, the following:

Corollary 2.7. *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^1$-smooth hypersurface and let $X \in H$, $|X|_H = 1$, be a unit horizontal left invariant vector field which is transverse to S. Let γ_y be the horizontal X-line starting from $y \in S$ and assume that $\gamma_y(\mathbb{R}) \cap S = \{y\}$ for every*

[2] Notice that $\mathcal{H}_c^{Q-1}(S) = \lim_{\delta\to 0^+} \mathcal{H}_{c,\delta}^{Q-1}(S)$ where, up to a constant multiple,

$$\mathcal{H}_{c,\delta}^{Q-1}(S) = \inf\Big\{ \sum_i \Big(\mathrm{diam}_c(\mathcal{B}_i)\Big)^{Q-1} \,:\, S \subset \bigcup_i \mathcal{B}_i;\ \mathrm{diam}_c(\mathcal{B}_i) < \delta \Big\}$$

and the infimum is taken with respect to closed d_c-balls $\mathcal{B}_i$.

$y \in S$. Finally, let $D \subseteq \mathcal{R}_S^X$ be a $\mathcal{H}_c^Q$- measurable subset of $\mathbb{G}$ that is reachable from S by means of horizontal X-lines. Then

(i) $D_y := \gamma_y(\mathbb{R}) \cap D$ is $\mathcal{H}_c^1$-measurable for $\mathcal{H}_c^{Q-1}$-a.e. $y \in S$.

(ii) The mapping $S \ni y \longmapsto \mathcal{H}_c^1(D_y)$ is $\mathcal{H}_c^{Q-1}$-measurable on S and

$$\mathcal{H}_c^Q(D) = \int_{pr_S^X(D)} \mathcal{H}_c^1(D_y)\, |\langle X, \nu_E \rangle_{H_y}| \, \frac{k_{Q-1}(\nu_E)}{k_Q} \, d\,\mathcal{H}_c^{Q-1}(y)$$

where k_Q is the constant defined in Remark 1.26. Moreover

$$\int_D \psi \, d\,\mathcal{H}_c^Q = \int_{pr_S^X(D)} \int_{D_y} \psi \, d\mathcal{H}_c^1 \, |\langle X, \nu_E \rangle_{H_y}| \, \frac{k_{Q-1}(\nu_E)}{k_Q} \, d\,\mathcal{H}_c^{Q-1}(y).$$

Proof. We have already observed in Remark 1.26 that Lebesgue measure $\mathcal{L}^n$ and Q-dimensional spherical Hausdorff measure $\mathcal{H}_c^Q$ coincide up to the constant k_Q. Thus, using Proposition 2.3, Corollary 2.5 and the identity of measures stated in (36) the thesis follows. □

2.2 Proofs of Proposition 2.3 and Theorem 2.4

This subsection is entirely devoted to prove Proposition 2.3 and Theorem 2.4. The proof of Proposition 2.3 relies mainly on the next Lemma 2.9 and on the classical change of variables formula with some non trivial computations. The proof of Theorem 2.4 follows from Proposition 2.3 using an approximation argument inspired by a recent work of Franchi, Serapioni and Serra Cassano about an implicit function theorem in Carnot groups (see Theorem 1.39 or [42]).

We begin by stating two technical lemmas. For the notation used in the sequel we refer the reader to Section 1.1. We just recall here that the group law $\bullet$ on $\mathbb{G}$ is also denoted by $\mathcal{P}(x, y) = x + y + \mathcal{Q}(x, y)$ for $x, y \in \mathbb{G}$, where $\mathcal{P}_j(x, y) = x_j + y_j$ for $1 \le j \le m_1 (= \dim V_1)$ and $\mathcal{P}_j(x, y) = x_j + y_j + \mathcal{Q}_j(x, y)$ for $j > m_1$.

Lemma 2.8. *If $X \in V_1$ and $j > m_1$, then*

$$\mathcal{Q}_j(y, \exp((t_1 + t_2)X)) = \mathcal{Q}_j(y, \exp(t_1 X)) + \mathcal{Q}_j(\mathcal{P}(y, \exp(t_1 X)), \exp(t_2 X)) \quad (37)$$

whenever $y \in \mathbb{G}$ and $t_1, t_2 \in \mathbb{R}$.

Proof. First, by Remark 1.5 we get that if $X \in V_1$

$$\mathcal{P}(exp\,(t_1X), exp\,(t_2X)) = exp\,(t_1X) + exp\,(t_2X) \quad \forall\ t_1, t_2 \in \mathbb{R}. \tag{38}$$

Now, starting from the associativity property of the group law and using (38), it follows that

$$\mathcal{P}(\mathcal{P}(y, exp\,(t_1X)), exp\,(t_2X)) = \mathcal{P}(y, \mathcal{P}(exp\,(t_1X), exp\,(t_2X)))$$

and so

$$\mathcal{P}_j(\mathcal{P}(y, exp\,(t_1X)), exp\,(t_2X)) = \mathcal{P}_j(y, \mathcal{P}(exp\,(t_1X), exp\,(t_2X))). \tag{39}$$

Moreover the following identities hold

$$\mathcal{P}_j(\mathcal{P}(y, exp\,(t_1X)), exp\,(t_2X)) = \mathcal{P}_j(y, exp\,(t_1X)) + \mathcal{Q}_j(\mathcal{P}(y, exp\,(t_1X)), exp\,(t_2X))$$

$$= y_j + \mathcal{Q}_j(y, exp\,(t_1X)) + \mathcal{Q}_j(\mathcal{P}(y, exp\,(t_1X)), exp\,(t_2X)); \tag{40}$$

$$\mathcal{P}_j(y, \mathcal{P}(exp\,(t_1X), exp\,(t_2X))) = y_j + \mathcal{Q}_j(y, \mathcal{P}(exp\,(t_1X), exp\,(t_2X)))$$

$$= y_j + \mathcal{Q}_j(y, exp\,((t_1 + t_2)X)). \tag{41}$$

Thus the claim easily follows by substituting (40) and (41) in (39). $\square$

Lemma 2.9. *If $X \in V_1$ we have that*

$$\frac{\partial}{\partial\, t}\mathcal{P}(y, \exp\,(tX)) = \left[\frac{\partial}{\partial\, y}\mathcal{P}(y, \exp\,(tX))\right] X(y) \quad \forall\ t \in \mathbb{R}\ \forall\ y \in \mathbb{G}. \tag{42}$$

Notation 2.10. *In some of the following formulae we shall write*

$$\mathcal{J}_y\mathcal{P}(y, z) := \frac{\partial}{\partial\, y}\mathcal{P}(y, z) \qquad (\textit{for }\ y, z \in \mathbb{G}).$$

Proof. We prove this lemma by components. First, we assume that $X = \sum_{j=1}^{m_1} a_j \mathrm{e}_j$ so that

$$exp\,(tX) = (ta_1, ..., ta_{m_1}, 0, ..., 0).$$

If $1 \leq j \leq m_1$ we have that $\mathcal{P}_j(y, \exp(tX)) = y_j + ta_j$ and since we may easily prove that

$$\langle \Big[\mathcal{J}_y \mathcal{P}(y, \exp(tX))\Big] X(y), \mathrm{e}_j\rangle = a_j,$$

in this case the thesis follows. Now if $j > m_1$, we have to show that

$$\frac{\partial}{\partial t}\mathcal{P}_j(y, \exp(tX)) = \langle \nabla_y \mathcal{P}_j(y, \exp(tX)), X(y)\rangle.$$

Since $(\exp(tX))_j = 0$, we have that $\mathcal{P}_j(y, \exp(tX)) = y_j + \mathcal{Q}_j(y, \exp(tX))$. Now, note that the following identities hold

$$\frac{\partial}{\partial t}\mathcal{P}_j(y, \exp(tX)) = \frac{\partial}{\partial t}\mathcal{Q}_j(y, \exp(tX)); \tag{43}$$

$$\langle \nabla_y \mathcal{P}_j(y, \exp(tX)), X(y)\rangle = (X(y))_j + \langle \nabla_y \mathcal{Q}_j(y, \exp(tX)), X(y)\rangle. \tag{44}$$

Therefore, by (43) and (44) we have to prove that

$$\frac{\partial}{\partial t}\mathcal{Q}_j(y, \exp(tX)) = (X(y))_j + \langle \nabla_y \mathcal{Q}_j(y, \exp(tX)), X(y)\rangle \quad \forall\, t \in \mathbb{R}\ \forall\, y \in \mathbb{G}. \tag{45}$$

Now, by differentiating both sides of (37) of the previous Lemma 2.8 with respect to t_1 at the time $t_1 = 0$ and putting $t_2 = t$, we get that

$$\begin{aligned}
&\frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{Q}_j(y, \exp((t_1+t)X)) \\
&= \frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{Q}_j(y, \exp(t_1X)) + \frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{Q}_j(\mathcal{P}(y, \exp(t_1X)), \exp(tX)) \\
&= \frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{P}_j(y, \exp(t_1X)) + \Big\langle \nabla_y \mathcal{Q}_j(\mathcal{P}(y,0), \exp(tX)), \Big[\frac{\partial}{\partial t_1}\Big|_{t_1=0}\mathcal{P}(y, \exp(t_1X))\Big]\Big\rangle \\
&= (X(y))_j + \langle \nabla_y \mathcal{Q}_j(y, \exp(tX)), X(y)\rangle
\end{aligned}$$

that is nothing but (45). $\square$

Proof of Proposition 2.3. Let S_α be an open neighborhood of $pr_S^X(D)$ on S. Of course, with no loss of generality, we may think S_α to be globally parameterized through a smooth map Φ_α, where $\Phi_\alpha : U_\alpha \subseteq \mathbb{R}^{n-1} \longmapsto S_\alpha$ and $\Phi_\alpha \in \mathbf{C}^1(U_\alpha, \mathbb{G})$. In the general case we shall use a partition of unity related to an atlas $\{(S_\alpha, \Psi_\alpha)\}_{\alpha\in\mathcal{A}}$ of S, where $\Psi_\alpha := \Phi_\alpha^{-1}$ for $\alpha \in \mathcal{A}$ and (S_α, Ψ_α) is a coordinate chart on S. However, for sake of simplicity, we omit the index α from U_α, Φ_α and S_α just writing U, Φ and S. Let us consider the map $S \times \mathbb{R} \ni (y,t) \longmapsto \gamma_y(t) \in \mathbb{G}$ given by $\gamma_y(t) = exp\,[tX](y)$. The last one enables us to carry out the parametrization of D we were looking for. Indeed, more precisely, starting from the parametrization of S, we may put

$$\gamma_{\Phi(\xi)}(t) = exp\,[tX](\Phi(\xi))$$

whenever $\xi \in U$ and $t \in \mathbb{R}$. For simplicity, we shall drop the dependence on the variables and we denote this mapping just by γ_Φ. This one enjoys an important property that we summarize in the next lemma.

Lemma 2.11. *The Jacobian matrix of the mapping γ_Φ with respect to $(\xi,t) \in U\times\mathbb{R}$ satisfies the following identity*

$$\left|\det\left[\mathcal{J}_{(\xi,t)}\gamma_\Phi\right]\right| = \left|\left\langle X, \nu_E\right\rangle_{H_\Phi}\right| \left(\sum_{j=1}^{m_1}\langle X_j(\Phi), N(\Phi)\rangle^2\right)^{\frac{1}{2}} |\Phi_{\xi_1}\wedge\ldots\wedge\Phi_{\xi_{n-1}}|, \quad (46)$$

where we have set

$$\Phi_{\xi_h} := \frac{\partial\,\Phi}{\partial\,\xi_h} \qquad \textit{for } h \in \{1,...,n-1\}.$$

Proof of Lemma 2.11. We have to compute the expression of the Jacobian matrix of γ_Φ, i.e.

$$\mathcal{J}_{(\xi,t)}\gamma_\Phi = \left[\frac{\partial\,\gamma_\Phi}{\partial\,\xi}, \frac{\partial\,\gamma_\Phi}{\partial\,t}\right] = \left[\frac{\partial\,\gamma_\Phi}{\partial\,\xi_1}, ..., \frac{\partial\,\gamma_\Phi}{\partial\,\xi_{n-1}}, \frac{\partial\,\gamma_\Phi}{\partial\,t}\right].$$

By definition we have that $\gamma_{\Phi(\xi)}(t) = \mathcal{P}(\Phi(\xi), exp\,(tX))$ and so we get

$$\frac{\partial\,\gamma_\Phi}{\partial\,\xi} = \left[\frac{\partial}{\partial\,y}\Big|_{y=\Phi(\xi)}\mathcal{P}(y, \exp\,(tX))\right]\frac{\partial\,\Phi}{\partial\,\xi}.$$

We have then

$$\mathcal{J}_{(\xi,t)}\gamma_\Phi = \left[\left[\frac{\partial}{\partial\,y}\Big|_{y=\Phi(\xi)}\mathcal{P}(y, \exp\,(tX))\right]\frac{\partial\,\Phi}{\partial\,\xi}\,,\,\frac{\partial}{\partial\,t}\mathcal{P}(\Phi(\xi), \exp\,(tX))\right] \quad (47)$$

and, for sake of simplicity, we will set

$$
\begin{aligned}
\mathbf{A} &:= \left[\frac{\partial}{\partial y}\Big|_{y=\Phi(\xi)} \mathcal{P}(y, \exp(tX))\right], \\
\mathbf{b} &:= \frac{\partial}{\partial t}\mathcal{P}(\Phi(\xi), \exp(tX))
\end{aligned}
$$

So we get

$$
\begin{aligned}
\Big|\det\Big[\mathcal{J}_{(\xi,t)}\gamma_{\Phi}\Big]\Big| &= \Big|\det\Big[\mathbf{A}\,\frac{\partial\,\Phi}{\partial\,\xi}\,,\ \mathbf{b}\Big]\Big| \\
&= \Big|\det\Big[\mathbf{A}\,\frac{\partial\,\Phi}{\partial\,\xi}\,,\ \mathbf{A}\mathbf{A}^{-1}\,\mathbf{b}\Big]\Big|.
\end{aligned}
$$

Now we may notice that $|\det\mathbf{A}\,| = 1$. Indeed, in general, one has

$$
\frac{\partial}{\partial\,y}\mathcal{P}(y,z) = \mathbf{I}_n + \frac{\partial}{\partial\,y}\mathcal{Q}(y,z)
$$

whenever $y, z \in \mathbb{G}$, where $\mathbf{I}_n$ is the $n\times n$ identity matrix and $\frac{\partial}{\partial y}\mathcal{Q}$ is a $n\times n$ nilpotent matrix, because it is lower triangular with the entries in the main diagonal all equal to 1. Furthermore, by Lemma 2.9 we infer that

$$
X(y) = \left[\frac{\partial}{\partial\,y}\mathcal{P}(y, \exp(tX))\right]^{-1}\frac{\partial}{\partial\,t}\mathcal{P}(y, \exp(tX))
$$

whenever $y \in \mathbb{G}$ and $t \in \mathbb{R}$ and so, in particular, we get that $\mathbf{A}^{-1}\,\mathbf{b} = X(\Phi(\xi))$. Therefore

$$
\begin{aligned}
\Big|\det\Big[\mathcal{J}_{(\xi,t)}\gamma_{\Phi}\Big]\Big| &= |\det\mathbf{A}\,|\cdot\Big|\det\Big[\frac{\partial\,\Phi}{\partial\,\xi}\,,\ \mathbf{A}^{-1}\,\mathbf{b}\Big]\Big| \\
&= \Big|\det\Big[\frac{\partial\,\Phi}{\partial\,\xi}\,,\ X(\Phi(\xi))\Big]\Big| \\
&= \Big|\det\Big[\frac{\partial\,\Phi}{\partial\,\xi_1}, ..., \frac{\partial\,\Phi}{\partial\,\xi_{n-1}}\,,\ X(\Phi(\xi))\Big]\Big| \\
&= \Big|\langle\frac{\partial\,\Phi}{\partial\,\xi_1}\wedge ...\wedge\frac{\partial\,\Phi}{\partial\,\xi_{n-1}}, X(\Phi(\xi))\rangle\Big| \\
&= \Big|\langle N(\Phi(\xi)), X(\Phi(\xi))\rangle\Big|\cdot|\Phi_{\xi_1}\wedge\ldots\wedge\Phi_{\xi_{n-1}}|.
\end{aligned}
$$

Here above we have used two standard properties of Linear Algebra and, more precisely, the following identity

$$
\det\big[\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_{n-1}, \mathbf{b}\big] = \langle\mathbf{a}_1\wedge\mathbf{a}_2\wedge ...\wedge\mathbf{a}_{n-1}, \mathbf{b}\rangle \qquad \forall\ \mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_{n-1}, \mathbf{b}\in\mathbb{R}^n,
$$

and the fact that

$$\det\Big[\mathbf{A}\mathbf{b}_1, \mathbf{A}\mathbf{b}_2, ..., \mathbf{A}\mathbf{b}_n\Big] = \det\mathbf{A}\cdot\det\big[\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\big]$$

for any invertible $n\times n$ matrix $\mathbf{A}$. Notice also that in the last line, we have used the explicit expression of the euclidean unit inward normal along a parametric hypersurface.

Now, keeping in mind that, whenever $S = \partial E$ is smooth, we have

$$\nu_E(y) = \frac{\Big(\langle X_{1y}, N_y\rangle, \dots, \langle X_{m_1 y}, N_y\rangle\Big)}{\Big(\sum_{j=1}^{m_1}\langle X_{j_y}, N_y\rangle^2\Big)^{\frac{1}{2}}}$$

for every $y\in S$, we get the thesis observing that

$$\Big|\Big\langle N_y, X_y\Big\rangle\Big| = \Big|\langle \nu_{Ey}, X_y\rangle_{H_y}\Big|\cdot\Big(\sum_{j=1}^{m_1}\langle X_{j_y}, N_y\rangle^2\Big)^{\frac{1}{2}}.$$

$\square$

Starting from this lemma we carry out the proof of Proposition 2.3 by means of a partition of unity $\{(\mathcal{W}_\alpha, \sigma_\alpha)\}_{\alpha\in\mathcal{A}}$ related to the atlas $\{(S_\alpha, \Psi_\alpha)\}_{\alpha\in\mathcal{A}}$ for S, where $\mathcal{W}_\alpha = \mathrm{spt}\{\sigma_\alpha\}\Subset S_\alpha$. Indeed, by the classical change of variables formula

$$\mathcal{L}^n(D)$$

$$= \sum_{\alpha\in\mathcal{A}}\int_{\Psi_\alpha(pr_S^X(D)\cap S_\alpha)}(\sigma_\alpha\circ\Phi_\alpha)(\xi)\Big[\int_{\gamma^{-1}_{\Phi_\alpha(\xi)}(D_{\Phi_\alpha(\xi)})}\Big|\det\Big[\mathcal{J}_{(\xi,t)}\gamma_{\Phi_\alpha(\xi)}(t)\Big]\Big|\,dt\Big]\,d\xi, \tag{48}$$

where

$$D_{\Phi_\alpha(\xi)} := \gamma_{\Phi_\alpha(\xi)}(\mathbb{R})\cap D$$

and

$$\gamma^{-1}_{\Phi_\alpha(\xi)}(D_{\Phi_\alpha(\xi)}) = \Big\{t\in\mathbb{R} : \gamma_{\Phi_\alpha(\xi)}(t)\cap D\neq\emptyset\Big\}.$$

Then, by (46) the right-hand side of (48) is equal to

$$\sum_{\alpha\in\mathcal{A}}\int_{\Psi_\alpha(pr_S^X(D)\cap S_\alpha)}(\sigma_\alpha\circ\Phi_\alpha)(\xi)\times$$

$$\times \int_{\gamma^{-1}_{\Phi_\alpha(\xi)}(D_{\Phi_\alpha(\xi)})} |\langle X, \nu_E\rangle_{H_{\Phi_\alpha(\xi)}}| \Big[\sum_{j=1}^m \langle X_j(\Phi_\alpha(\xi)), N(\Phi_\alpha(\xi))\rangle^2\Big]^{\frac{1}{2}} |(\Phi_\alpha)_{\xi_1} \wedge \ldots \wedge (\Phi_\alpha)_{\xi_{n-1}}| \, dt \, d\xi$$

$$= \int_{pr^X_S(D)} \Big[\int_{\mathbb{R}} \mathbf{1}_{D_y}(t)\, dt\Big] \Big|\langle X, \nu_E\rangle_{H_y}\Big| \Big[\sum_{j=1}^m \langle X_j(y), N(y)\rangle^2\Big]^{\frac{1}{2}} d\mathcal{H}^{n-1}(y)$$

$$= \int_{pr^X_S(D)} \mathcal{H}^1_c(D_y)\, |\langle X, \nu_E\rangle_{H_y}|\, d\,|\partial E|_H(y)$$

$$= \int_{pr^X_S(D)} \mathcal{H}^1_c(D_y)\; d\,|\partial_X E|(y),$$

where we have used Theorem 1.9, Proposition 1.34 and Remark 1.42. □

Before the beginning of the proof of Theorem 2.4 we recall the basic statements of Implicit Function Theorem 1.39. We assume, by hypothesis, that S is a H-regular hypersurface and so for every $\widetilde{x} \in S$ there exist an open neighborhood U of $\widetilde{x}$ and a real valued function $f \in \mathbf{C}^1_H(U)$ such that $S \cap U = \{x \in U : f(x) = 0\}$ and $\nabla^H f(x) \neq 0$ for all $x \in U$. Thus S is locally the boundary of $E = \{x \in U : f(x) < 0\}$ and without loss of generality we assume that $X_1 f(x) > 0$ for $x \in U$. Let now h, $\delta > 0$ and set

$$J_h := [-h, h], \quad I_\delta := \{\xi = (\xi_2, \ldots, \xi_n) \in \mathbb{R}^{n-1} : |\xi_j| \leq \delta,\, j = 2, \ldots, n\}.$$

If $\xi \in \mathbb{R}^{n-1}$ and $t \in J_h$ we denote by $\gamma^1_{(0,\xi)}(t)$ the integral curve of the left invariant horizontal vector field $X_1 \in H$ at the time t issued from $(0, \xi) \in \{(0, \eta) \in \mathbb{G} : \eta \in \mathbb{R}^{n-1}\}$. Then Theorem 1.39 states that there exist δ, $h > 0$ such that the mapping

$$\mathbb{R} \times \mathbb{R}^{n-1} \ni (t, \xi) \longmapsto \gamma^1_{(0,\xi)}(t)$$

is a diffeomorphism of a neighborhood of $J_h \times I_\delta$ onto an open subset of $\mathbb{G}$. In what follows we denote by U the image of $\mathrm{Int}\{J_h \times I_\delta\}$ through this mapping. The set E has finite H-perimeter in U and if ν_E is the generalized inward unit normal of E we have

$$\nu_E(x) = -\frac{\nabla^H f(x)}{|\nabla^H f(x)|_H} \quad \forall x \in S \cap U.$$

Furthermore there exists a unique continuous function $\phi = \phi(\xi) : I_\delta \longrightarrow J_h$ such that, setting $\Phi(\xi) = \gamma^1_{(0,\xi)}(\phi(\xi))$ for $\xi \in I_\delta$, we have

$$S \cap U = \{x \in U : x = \Phi(\xi),\, \xi \in I_\delta\}$$

and the H-perimeter has the following integral representation

$$|\partial E|_H(U) = \int_{I_\delta} \frac{\sqrt{\sum_{j=1}^m |X_j f(\Phi(\xi))|^2}}{X_1 f(\Phi(\xi))}\, d\xi. \tag{49}$$

Let now J_ϵ be a Friedrichs' mollifier; putting $f_\epsilon = f * J_\epsilon$ by the continuity of f we have that $f_\epsilon \longrightarrow f$ as $\epsilon \to 0$ uniformly in $\mathcal{U}$ and analogously $(X_j f) * J_\epsilon \longrightarrow X_j f$ as $\epsilon \to 0$ uniformly in $\mathcal{U}$ (for $j = 1, \ldots, m$). Arguing as in [40], p. 90, we obtain

$$X_j f_\epsilon = (X_j f) * J_\epsilon - ((X_j f) * J_\epsilon - X_j f_\epsilon) \quad \text{for } j \in \{1, \ldots, m\}$$

and also

$$(X_j f) * J_\epsilon - X_j f_\epsilon \longrightarrow 0$$

uniformly in $\mathcal{U}$ as $\epsilon \to 0$. We note that starting from the regularization of f by the classical Implicit Function Theorem we get the existence of a smooth function $\phi_\epsilon : I_\delta \longrightarrow J_h$ such that $\phi_\epsilon \longrightarrow \phi$ as $\epsilon \to 0$ uniformly in I_δ. Thus we may construct a family $\{S_\epsilon\}_{\epsilon>0}$ of smooth hypersurfaces which uniformly converges in U to $S \cap U$ as $\epsilon \to 0$. Moreover every hypersurface S_ϵ is the boundary of a smooth open set E_ϵ which also converges in U to $E \cap U$ as $\epsilon \to 0$. Here an explicit parametrization of S_ϵ is given by the mapping $\Phi_\epsilon : I_\delta \longrightarrow \mathbb{G}$, $\Phi_\epsilon(\xi) := \gamma^1_{(0,\xi)}(\phi_\epsilon)$ for $\xi \in I_\delta$. Finally, we have that $\Phi_\epsilon \longrightarrow \Phi$ uniformly for $\xi \in I_\delta$ as $\epsilon \to 0$. To see this, notice that

$$\begin{aligned} |\Phi_\epsilon(\xi) - \Phi(\xi)| &= |\gamma^1_{(0,\xi)}(\phi_\epsilon(\xi)) - \gamma^1_{(0,\xi)}(\phi(\xi))| \\ &\le \Big| \int_{\phi_\epsilon(\xi)}^{\phi(\xi)} |X_1(exp\,[tX_1](0,\xi))|\, dt \Big| \end{aligned}$$

and that $d_c(exp\,[tX_1](0,\xi), (0,\xi)) \le |t| \le h$. So if K is a compact subset of I_δ

$$exp\,[tX_1]((0,\xi)) \in K_h := \{z \in \mathbb{G} : d_c(z, \{0 \times K\}) \le h\},$$

and, keeping in mind that $\phi_\epsilon \longrightarrow \phi$ as $\epsilon \to 0$ uniformly in I_δ, we get the claim observing that

$$|\Phi_\epsilon(\xi) - \Phi(\xi)| \le |\phi_\epsilon(\xi) - \phi(\xi)| \cdot \max_{z \in K_h} |X_1(z)|.$$

Proof of Theorem 2.4. The proof will be divided in some claims and we shall use notation and statements of Theorem 1.39. From now on we assume that the hypersurface S is globally parameterized by a unique map Φ as above and, more precisely, we may suppose that there exists $\delta > 0$ such that S is the image of $\Phi : \mathrm{Int}\{I_\delta\} \longmapsto \mathbb{G}$, where $\Phi(\xi) = \gamma^1_{(0,\xi)}(\phi(\xi))$ and $I_\delta = \{\xi \in \mathbb{R}^{n-1} : |\xi|_\infty \le \delta\}$. So we have

$$S = \{y \in \mathbb{G} : y = \Phi(\xi),\ \xi \in I_\delta\} = \{y \in \mathbb{G} : f(y) = 0\}$$

where $f \in \mathbf{C}^1_H(\mathbb{G})$ is an *implicit function* which defines S such that $X_1 f > 0$ near S.

Claim 1. Let $\alpha \in L^\infty(\mathbb{G}) \cap \mathbf{C}^\infty(\mathbb{G})$ be such that $\alpha \ge 0$. Then we have

$$\int_{\mathbb{G}} \alpha \, d\mathcal{L}^n = \int_S \Big[\int_{\mathbb{R}} (\alpha \circ \gamma_y)(t)\, dt \Big]\, d\,|\partial_X E|(y).$$

Proof. More explicitly, we note that the right-hand side is equal to

$$\int_S \int_{\mathbb{R}} (\alpha \circ \gamma_y)(t)\, |\langle X, \nu_E \rangle_{H_y}|\, dt\, d\,|\partial E|_H(y).$$

To prove this claim, we first set

$$I := \int_S \Big[\int_{\mathbb{R}} (\alpha \circ \gamma_y)(t)\, dt \Big]\, d\,|\partial_X E|(y).$$

By (iii) and (v) of Theorem 1.39 we get

$$I = \int_{I_\delta} \int_{\mathbb{R}} \frac{|\langle X(\Phi(\xi)), \nabla^H f(\Phi(\xi)) \rangle_{H_{\Phi(\xi)}}|\, (\alpha \circ \gamma_{\Phi(\xi)})(t)}{X_1 f(\Phi(\xi))}\, dt\, d\xi.$$

Now we shall prove that

$$I = \lim_{\epsilon \to 0} \int_{I_\delta} \int_{\mathbb{R}} \frac{|\langle X(\Phi_\epsilon(\xi)), \nabla^H f_\epsilon(\Phi_\epsilon(\xi)) \rangle_{H_{\Phi_\epsilon(\xi)}}|\, (\alpha \circ \gamma_{\Phi_\epsilon(\xi)})(t)}{X_1 f(\Phi_\epsilon(\xi))}\, dt\, d\xi. \tag{50}$$

Indeed, if (50) holds we get the claim observing that

$$I = \lim_{\epsilon \to 0} \int_{S_\epsilon} \int_{\mathbb{R}} (\alpha \circ \gamma_y)(t) \, |\langle X, \nu_{E_\epsilon} \rangle_{H_y}| \, dt \, d\, |\partial E_\epsilon|_H(y)$$

and that from Corollary 2.5 we have

$$\int_{\mathbb{G}} \alpha \, d\mathcal{L}^n = \int_{S_\epsilon} \int_{\mathbb{R}} (\alpha \circ \gamma_y)(t) \, |\langle X, \nu_{E_\epsilon} \rangle_{H_y}| \, dt \, d\, |\partial E_\epsilon|_H(y).$$

To prove (50) notice that, as we have seen above, $\Phi(\xi) \longrightarrow \Phi_\epsilon(\xi)$ uniformly in I_δ as $\epsilon \to 0$ and so, keeping in mind that $\nabla^H f_\epsilon \longrightarrow \nabla^H f$ uniformly on compact sets, we get

$$\nabla^H f_\epsilon(\Phi_\epsilon(\xi)) \longrightarrow \nabla^H f(\Phi(\xi)) \tag{51}$$

as $\epsilon \to 0$ for $\xi \in I_\delta$. Thus, by (51) and by the continuous dependence of the Cauchy problem on the initial data, the integrand in (50) tends to the integrand of I. On the other hand $\Phi_\epsilon(\xi)$ lies in a fixed compact neighborhood of $\Phi(I_\delta)$ so that, by Weierstrass Theorem and our assumptions on α, the integrand in (50) is bounded by a constant for $(\xi, t) \in I_\delta \times \mathbb{R}$ and (50) follows by Dominate Convergence Theorem. □

Claim 2. Let $Q \subset \mathcal{R}_S^X$ be a compact, rectangular n-box. Then

$$\int_S \mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap Q) \, d\, |\partial_X E| \le \mathcal{L}^n(Q).$$

Proof. To prove this we may consider a sequence of functions $\{\alpha_h\}_{h \in \mathbb{N}}$ such that

$$\lim_{h \to \infty} \alpha_h(x) = \mathbf{1}_Q(x) \quad \forall \, x \in \mathbb{G}.$$

For $y \in S$ we set $\gamma_y^{-1}(Q) := \Big\{ t \in \mathbb{R} : \, \gamma_y(t) \in Q \Big\}$. So $\alpha_h(\gamma_y(t)) \longrightarrow \mathbf{1}_{\gamma_y^{-1}(Q)}(t)$ for all $(y, t) \in S \times \mathbb{R}$ as $h \to \infty$. Therefore we get the claim observing that

$$\begin{aligned}
\int_S \mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap Q) \, d\, |\partial_X E|(y) \quad &= \quad \int_S \int_{\mathbb{R}} \mathbf{1}_{\gamma_y^{-1}(Q)}(t) \, d\,t \, d\, |\partial_X E|(y) \\
= \int_S \int_{\mathbb{R}} \lim_{h \to \infty} \alpha_h(\gamma_y(t)) \, d\,t \, d\, |\partial_X E|(y) \quad &\le \quad \liminf_{h \to \infty} \int_S \int_{\mathbb{R}} \alpha_h(\gamma_y(t)) \, d\,t \, d\, |\partial_X E|(y) \\
&= \quad \lim_{h \to \infty} \int_{\mathbb{G}} \alpha_h(x) \, d\mathcal{L}^n(x) = \mathcal{L}^n(Q).
\end{aligned}$$

□

Claim 3. *Let $F \subset \mathcal{R}_S^X$ be a measurable subset of $\mathbb{G}$ such that $\mathcal{L}^n(F) = 0$. Setting*

$$S_0 := \Big\{ y \in S : \mathcal{H}_c^1(\gamma_y^1(\mathbb{R}) \cap F) > 0 \Big\},$$

we have that $|\partial E|_H(S_0) = 0$.

Proof. Indeed let $\epsilon > 0$ and $\{Q_j\}_{j\in\mathbb{N}}$ be a countable family of compact, rectangular, n-box such that

$$F \subseteq \bigcup_{j=1}^{\infty} Q_j, \qquad \sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) < \epsilon.$$

We have then

$$\begin{aligned}
\int_S \mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap F)\, d\,|\partial_X E|(y) &\le \int_S \sum_{j=1}^{\infty} \mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap Q_j)\, d\,|\partial_X E|(y) \\
&= \int_S \lim_{k\to\infty} \sum_{j=1}^{k} \mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap Q_j)\, d\,|\partial_X E|(y) \\
&\le \lim_{k\to\infty} \sum_{j=1}^{k} \int_S \mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap Q_j)\, d\,|\partial_X E|(y) \\
&\le \sum_{j=1}^{\infty} \int_S \mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap Q_j)\, d\,|\partial_X E|(y) \\
&\le \sum_{j=1}^{\infty} \mathcal{L}^n(Q_j) < \epsilon.
\end{aligned}$$

Therefore

$$\int_S \mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap F) \cdot |\langle X, \nu_E \rangle_{H_y}|\, d\,|\partial E|_H(y) = 0$$

and since $\langle X, \nu_E \rangle_{H_y} \neq 0$ for any $y \in S$, we get the claim observing that

$$\mathcal{H}_c^1(\gamma_y(\mathbb{R}) \cap F) = 0 \quad \text{for } |\partial E|_H - \text{a. e. } y \in S.$$

$\square$

At this point we can achieve the proof of the theorem in the following way. Let J_ϵ be a Friedrichs' mollifier and put $\alpha_\epsilon := \mathbf{1}_D * J_\epsilon$. Since $\alpha_\epsilon \in L^\infty(\mathbb{G}) \cap \mathbf{C}^\infty(\mathbb{G})$ and $\alpha_\epsilon \longrightarrow \mathbf{1}_D$ in L^1_{loc}, up to a subsequence, we may assume that

$$\lim_{\epsilon\to 0} \alpha_\epsilon = \mathbf{1}_D \quad \text{for} \quad \mathcal{L}^n - \text{a.e. } x \in \mathbb{G}.$$

Now setting

$$F := D \setminus \Big\{ x \in D : \lim_{\epsilon \to 0} \alpha_h(x) = \mathbf{1}_D(x) \Big\}$$

and

$$S_0 := \Big\{ y \in S : \mathcal{H}^1_c(\gamma_y(\mathbb{R}) \cap F) > 0 \Big\}$$

by Claim 3 we get $|\partial E|_H(S_0) = 0$. Moreover by Claim 1 we obtain

$$\begin{aligned}
\int_{\mathbb{G}} \alpha_\epsilon \, d\,\mathcal{L}^n &= \int_S \int_{\mathbb{R}} (\alpha_\epsilon \circ \gamma_y)(t)\, dt\, d\,|\partial_X E|(y) \\
&= \int_{S \setminus S_0} \int_{\mathbb{R}} (\alpha_\epsilon \circ \gamma_y)(t)\, dt\, d\,|\partial_X E|(y).
\end{aligned}$$

Therefore $\alpha_\epsilon(\gamma_y(t)) \longrightarrow \mathbf{1}_{\gamma_y^{-1}(Q)}(t)$ for $\mathcal{L}^1 -$ a.e. $t \in \mathbb{R}$ and $|\partial E| -$ a.e. $y \in S$ as $\epsilon \to 0$. Thus we get the thesis by letting $\epsilon \to 0$ in (52). $\square$

3 Slicing of HBV functions and H-perimeter

3.1 One-dimensional restrictions of HBV functions

We introduce the concept of *variation along a horizontal direction* of a locally summable function in a Carnot group $\mathbb{G}$ and we summarize its main properties. Afterwards, we define the notion of *X-variation along a horizontal line* and we consider the space of functions of bounded variation along a fixed horizontal line. Then, in Theorem 3.7, we establish a link between the notion of variation of a function along a horizontal direction and that of variation of the restrictions of such a function to a family of horizontal lines. Finally, we generalize to Carnot groups a well-known characterization of the usual space BV by means of one-dimensional restrictions of its elements. These topics in the classical setting can be found in [2] or in [32], while, for many other results about function of bounded variation in Carnot-Carathéodory spaces, one can see [1], [5], [14], [39, 40], [45], [79] [78], [96].

Definition 3.1. *Let $U \subseteq \mathbb{G}$ be open and let $X \in H$ be a horizontal left invariant vector field. We say that $f \in L^1(U)$ has bounded X-variation in U if*

$$|Xf|(U) = \sup\Big\{ \int_U f X\varphi \, d\mathcal{L}^n : \varphi \in \mathbf{C}^1_0(U),\ |\varphi| \leq 1 \Big\} < \infty;$$

we refer to the quantity $|Xf|(U)$ as the X-variation of f in U and we denote by $BV_X(U)$ the vector space of bounded X-variation functions in U.

In the next remark we summarize some well-known properties of the variation:

Remark 3.2. *Let $U \subseteq \mathbb{G}$ be open and let $X \in H$. Then the following items hold:*

(i) let $f, f_k \in L^1(U)$ for $k \in \mathbb{N}$ be such that $f_k \longrightarrow f$ in $L^1(U)$ as $k \to \infty$. Then

$$|Xf|(U) \leq \liminf_{k \longrightarrow \infty} |Xf_k|(U);$$

(ii) if $f \in BV_X(U)$ then $|Xf|$ is a Radon measure in U and

$$\int_U f\, X\varphi \, d\,\mathcal{L}^n = -\int_U \varphi \, d\,|Xf| \quad \forall\, \varphi \in \mathbf{C}^\infty_0(U);$$

(iii)

$$|Xf|(U) = \int_U |Xf|\, d\mathcal{L}^n \qquad \forall\ f \in \mathbf{C}^1(U);$$

(iv) *if* $f \in BV_X(U)$ *then there exists a sequence* $\{f_j\}_{j\in\mathbb{N}} \subset \mathbf{C}^\infty(U) \cap BV_X(U)$ *such that*

$$\lim_{j\to\infty} \|f_j - f\|_{L^1(U)} = 0 \quad \textit{and} \quad \lim_{j\to\infty} |Xf_j|(U) = |Xf|(U).$$

From now on, let U denote an open subset of $\mathbb{G}$ and let $f : U \longrightarrow \mathbb{R}$. Moreover let us fix a horizontal direction $X \in H$ and let us denote by $\gamma : \mathbb{R} \longrightarrow U$ the corresponding horizontal X-line. Theorem 1.9 implies that, if $a = (a_1, ..., a_{m_1})$ is the vector of the canonical coordinates of γ, then for all compact set $K \subset \gamma$ one has

$$\mathcal{H}^1_c(K) = \int_{\gamma^{-1}(K)} |a|\, dt,$$

where $|a|$ is constant ($|a| = |X|_H$). Therefore, if $f \circ \gamma \in L^1(\gamma^{-1}(K))$, putting $|X|_H = 1$, we get that the integral of f along the horizontal X-line γ is

$$\int_K f\, d\mathcal{H}^1_c = \int_{\gamma^{-1}(K)} (f \circ \gamma)(t)\, dt \tag{52}$$

for every compact $K \subset \gamma$. In the sequel, if $U \subset \gamma$ is an open subset of γ, we shall denote by $L^1(U, d\mathcal{H}^1_c \llcorner \gamma)$ the space of all $\mathcal{H}^1_c$-summable functions defined on U.

Proposition 3.3. *Let* $X \in H$, $|X|_H = 1$, *and let* γ *be a horizontal* X*-line starting from* $x \in \mathbb{G}$, *i.e.* $\gamma(t) = exp\,[tX](x)$ *for* $t \in \mathbb{R}$. *If* U *is an open subset of* γ *and* $f \in L^1(U, d\mathcal{H}^1_c \llcorner \gamma)$ *the following two statement are equivalent:*

(i) $f \circ \gamma \in BV(\gamma^{-1}(U))$;

(ii)

$$|D(f \circ \gamma)|(\gamma^{-1}(U)) = \sup \left\{ \int_\gamma f\, d\psi,\ \psi \in \mathbf{C}^1_0(U), |\psi| \le 1 \right\} < \infty.$$

Moreover, setting

$$var^1_X[f](U) := \sup \left\{ \int_U f\, X\varphi\, d\mathcal{H}^1_c : \varphi \in \mathbf{C}^1_0(B),\ |\varphi| \le 1,\ \ B \subset \mathbb{G}\ \textit{open s.t.}\ \gamma \cap B = U \right\},$$

we get that $var^1_X[f](U) = |D(f \circ \gamma)|(\gamma^{-1}(U))$.

Remark 3.4. *Here above we have used the usual definition (see [2], [32]) of total variation for real functions of one variable. We remind that, whenever*

$$h : I \subset \mathbb{R} \longrightarrow \mathbb{R}, \ \ h \in L^1(I),$$

the total variation $|Dh|(I)$ *of* h *in* I *is given by*

$$|Dh|(I) := \sup\Big\{ \int_I h \,\frac{d\phi}{dt}\, dt : \phi \in \mathbf{C}^1_0(I), \ |\phi| \le 1| \Big\}.$$

Also, $BV(I)$ *denotes the space of functions belonging to* $L^1(I)$ *and of finite total variation in* I*.*

Proof of Proposition 3.3. Since

$$\int_\gamma f \, d\psi = \int_{\mathbb{R}} (f \circ \gamma) \frac{d}{dt}(\psi \circ \gamma)\, dt$$

it follows that (i) is equivalent to (ii) because if $\psi \in \mathbf{C}^1_0(U), \ |\psi| \le 1$, we may put

$$\phi = (\phi \circ \gamma^{-1}) \circ \gamma = \psi \circ \gamma,$$

where $\phi \in \mathbf{C}^1_0(\mathbb{R}), \ \mathrm{spt}(\phi) \subset \gamma^{-1}(U), \ |\phi| \le 1$. To prove the last statement we notice that, for any $\psi \in \mathbf{C}^1_0(U), \ |\psi| \le 1$, we may find $\varphi \in \mathbf{C}^1_0(\mathbb{R}^n)$ such that $\psi = \varphi_{|\gamma}$, $\mathrm{spt}(\varphi) \cap \gamma = \mathrm{spt}(\psi)$ and $|\varphi| \le 1$. Thus the following chain of equalities holds:

$$\begin{aligned} &\sup\left\{ \int_\gamma f \, d\psi : \psi \in \mathbf{C}^1_0(U), |\psi| \le 1 \right\} \\ &= \sup\left\{ \int_\gamma f \, d\varphi : \varphi \in \mathbf{C}^1_0(\mathbb{R}^n), \mathrm{spt}(\varphi) \cap \gamma \subset U, \ |\varphi| \le 1 \right\} \\ &= \sup\left\{ \int_{\mathbb{R}} (f \circ \gamma) \frac{d}{dt}(\varphi \circ \gamma)\, dt : \varphi \in \mathbf{C}^1_0(\mathbb{R}^n), \mathrm{spt}(\varphi) \cap \gamma \subset U, |\varphi| \le 1 \right\} \\ &= \sup\left\{ \int_{\mathbb{R}} (f \circ \gamma) \langle \dot\gamma(t), \nabla\varphi(\gamma(t)) \rangle \, dt : \varphi \in \mathbf{C}^1_0(\mathbb{R}^n), \mathrm{spt}(\varphi) \cap \gamma \subset U, |\varphi| \le 1 \right\} \\ &= \sup\left\{ \int_U f \, X\varphi \, d\mathcal{H}^1_c : \varphi \in \mathbf{C}^1_0(U), \ |\varphi| \le 1 \right\}, \quad (53) \end{aligned}$$

where (53) follows by tacking an open set $B \subset \mathbb{G}$ such that $\gamma \cap B = U$. ☐

Definition 3.5. *Let $X \in H$, $|X|_H = 1$, and let γ be a horizontal X-line. If U is an open subset of γ and $f \in L^1(U, d\mathcal{H}_c^1 \llcorner \gamma)$ we call $var_X^1[f](U)$ the X-variation of f along γ and we define $BV_X^1(U)$ as the space of functions of finite X-variation in $U \subset \gamma$.*

Proposition 3.6. *Let $X \in H$, $|X|_H = 1$; let γ be a horizontal X-line. Then for every $\mathcal{H}_c^1$-measurable set $\mathcal{E} \subset \gamma$ one has*

$$\begin{aligned} var_X^1[\mathbf{1}_{\mathcal{E}}](\gamma) &= |D\mathbf{1}_{\gamma^{-1}(\mathcal{E})}|(\mathbb{R}) \\ &= var_X^1[\mathbf{1}_{L_y\mathcal{E}}](L_y\gamma) \quad \forall\, y \in \mathbb{G} \end{aligned} \tag{54}$$

where $\gamma^{-1}(\mathcal{E}) = \Big\{ t \in \mathbb{R} : \gamma(t) \in \mathcal{E} \Big\}$; moreover

$$var_X^1[\mathbf{1}_{\mathcal{E}}](\gamma) \geq 2 \tag{55}$$

and equality holds if and only if $\gamma^{-1}(\mathcal{E})$ is a bounded interval of $\mathbb{R}$.

Proof. Equalities (54) follow from Definition 3.5. Moreover, using the first identity of (54) we get that $var_X^1[\mathbf{1}_{\mathcal{E}}](\gamma)$ is equal to the euclidean one-dimensional perimeter of $\gamma^{-1}(\mathcal{E})$ in $\mathbb{R}$. Thus, using the one-dimensional isoperimetric inequality of [91], page 103, section 3.6, we get (55). $\square$

It seems interesting to find some results that reduce the study of HBV functions to that one of their one-dimensional restrictions, being this one a very useful approach of Calculus of Variations (see [2], [46]). Here below we state a theorem modeled on an analogous euclidean result (see [2], [32]). A similar theorem has been proved in [96] for Sobolev functions in Carnot groups and in [19] in the case of vertical planes in "rototranslation groups".

Theorem 3.7. *Let $S \subset \mathbb{G}$ be a H-regular hypersurface and assume that $S = \partial E$ globally, where $E \subset \mathbb{G}$ is a suitable open H-Caccioppoli set. Let $X \in H$, $|X|_H = 1$, be a unit horizontal left invariant vector field which is transverse to S and denote by γ_y the horizontal X-line starting from $y \in S$. We assume that $\gamma_y(\mathbb{R}) \cap S = \{y\}$*

for every $y \in S$. Finally let $U \subset \mathcal{R}_S^X$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from S by means of X-lines. Then

$$|Xf|(U) = \int_{pr_S^X(U)} var_X^1[f_{\gamma_y}](U_y)\, d\,|\partial_X E|(y) \tag{56}$$

where $f_{\gamma_y} := f \circ \gamma_y$ and $U_y := \gamma_y \cap U$.

Proof. Using (ii) of Corollary 2.5 we get

$$\begin{aligned}\int_U fX\varphi\, d\,\mathcal{L}^n &= \int_{pr_S^X(U)} \int_{\gamma_y^{-1}(U_y)} (f\circ\gamma_y)\frac{d}{d\,t}(\varphi\circ\gamma_y)\, dt\, d\,|\partial_X E|(y)\\ &\le \int_{pr_S^X(U)} var_X^1[f_{\gamma_y}](U_y)\, d\,|\partial_X E|(y),\end{aligned}$$

whenever $\varphi \in \mathbf{C}_0^1(\Omega)$. In a similar way we obtain the equality if $f \in \mathbf{C}^1(U)$. Now let us set

$$U^h := \left\{x \in U : |x| < \frac{1}{h},\ \operatorname{dist}(x, \partial U) > h\right\}$$

and choose $h > 0$ such that $|Xf|(\partial U^h) = 0$. Notice that this can be done for $\mathcal{L}^1$-a.e. $h > 0$, as for instance in [2], Example 1.63. Therefore, using Lemma 1.35, we get that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{U^h} |(f * J_\epsilon) - f|\, d\,\mathcal{L}^n\\ =\ &\lim_{\epsilon\to 0}\int_{pr_S^X(U^h)} \|(f*J_\epsilon)_y - f_y\|_{L^1(\gamma_y{}^{-1}((U^h)_y))}\, d\,|\partial_X E|(y) = 0,\end{aligned}$$

and so we may choose a sequence $\{\epsilon_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty}\int_{\gamma_y{}^{-1}((U^h)_y)} |(f*J_{\epsilon_j})_y - f_y|\, dt = 0 \qquad \text{for } |\partial_X E| - \text{a.e. } y \in pr_S^X(U^h).$$

By the lower semicontinuity of the X-variation (see (i) of Remark 3.2) we get

$$\begin{aligned}&\int_{pr_S^X(U^h)} var_X^1[f_{\gamma_y}]((U^h)_y)\, d\,|\partial_X E|(y)\\ \le\ &\int_{pr_S^X(U^h)} \liminf_{j\to\infty} var_X^1[(f*J_{\epsilon_j})_{\gamma_y}]((U^h)_y)\, d\,|\partial_X E|(y)\\ \le\ &\liminf_{j\to\infty}\int_{pr_S^X(U^h)} var_X^1[(f*J_{\epsilon_j})_y]((U^h)_y)\, d\,|\partial_X E|(y)\\ =\ &\lim_{j\to\infty} |X(f*J_{\epsilon_j})|(U^h)\\ =\ &|Xf|(U^h) \le |Xf|(U)\end{aligned}$$

and the claim follows letting $h \to 0$. $\square$

We would stress that for any $j = 1, ..., m_1$, the following inequalities hold

$$|X_j f|(U) \leq |\nabla^H f|(U) \leq \sum_{j=1}^{m_1} |X_j f|(U)$$

for $f \in HBV(U)$, where $\{X_1, ..., X_{m_1}\}$ is the fixed generating family vector fields of the group, i.e. a fixed orthonormal frame for H. This easily follows from Definition 1.28 and Definition 3.1 and, using Theorem 3.7, it allows to state the following.

Corollary 3.8. *[HBV functions and 1-dimensional slicing] Let $\{X_1, \ldots, X_{m_1}\}$ be any generating family of vector fields for $\mathbb{G}$. Let $S_j \subset \mathbb{G}$ ($j = 1, ..., m_1$) be a H-regular hypersurface such that $S_j = \partial E_j$ globally, where $E_j \subset \mathbb{G}$ is a suitable open H-Caccioppoli set, and suppose that $X_j \pitchfork S_j$ (i.e. X_j is transverse to S_j). Denoting by γ^j_y the horizontal X_j-line starting from $y \in S_j$, we assume that*

$$\gamma^j_y(\mathbb{R}) \cap S_j = \{y\}$$

for every $y \in S_j$. Finally let $U \subset \mathcal{R}^j_{S_j}$ be a Lebesgue measurable subset of $\mathbb{G}$ that is reachable from each S_j by means of X_j-lines. Then, $f \in HBV(U)$ **if, and only if,** $f_{\gamma^j_y} \in BV^1_{X_j}(U^j_y)$ **for** $|\partial E_{X_j}|$**-a.e.** $y \in pr^j_{S_j}(U)$ **and**

$$\int_{pr^j_{S_j}(U)} var^1_{X_j}[f_{\gamma^j_y}](U^j_y)\, d\,|\partial_{X_j} E|(y) < \infty \qquad \forall\, j = 1, ..., m_1.$$

[Here: $pr^j_{S_j}(U) := pr^{X_j}_{S_j}(U)$, $U^j_y := U^{X_j}_y$]

Remark 3.9. *Let $\mathcal{I}_e(X_j)$ the vertical hyperplane through the identity $e \in \mathbb{G}$ and orthogonal to X_j (see Notation 1.27), we may assume that $S_j = \mathcal{I}_e(X_j)$ for $j = 1, ..., m_1$. For such hypersurfaces the hypotheses of Corollary 3.8 are automatically verified, since* **each subset of $\mathbb{G}$ is reachable from any vertical hyperplane**. *Thus, every $U \subset \mathbb{G}$ can be U foliated by a family of horizontal X_j-lines ($j = 1, ..., m_1$) starting from $\mathcal{I}_e(X_j)$, and hence* **the previous characterization of $HBV(U)$ can be reformulated by means of vertical hyperplanes.**

3.2 Integral geometric measures, H-normal sets and H-convexity

In this subsection we give some applications of the previous results. To this end, we introduce a measure on UH, i.e. *unit horizontal bundle on* $\mathbb{G}$ (see also Sections 4 for further results), that we need to state some integral geometric formulae for volume and H-perimeter. Afterwards, we give a definition of *H-normality* with respect to a vertical hyperplane that generalizes the euclidean one ([27], [91]). Then we formulate an intrinsic definition of convexity, named *H-convexity* (see Definition 3.16), that seems to be natural from a geometric point of view. Indeed, by this definition, we state a Cauchy-type formula and a related inequality which says that, in some sense, among all sets containing a fixed H-convex set, this one minimizes the H-perimeter. See Theorem 3.21 and Corollary 3.22 below and also [17] and [86] for the classical results. We would emphasize that equivalent definitions of convexity in Carnot groups has been introduced recently in [24] and in [65]; see also [8] and [51] for some further developments.

Definition 3.10. *[Unit horizontal bundle] Let us set $\overset{\circ}{H} := H \setminus \{0_H\}$, where 0_H is the zero section of H. Denoting by UH the quotient of $\overset{\circ}{H}$ by the positive dilations we obtain a bundle structure on $\mathbb{G}$, called unit horizontal bundle on $\mathbb{G}$, whose projection map on the base space $\mathbb{G}$, $\pi_{UH} : UH \longrightarrow \mathbb{G}$, is given by $\pi_{UH}(z; Z) = z$ for $(z; Z) \in H$. Notice that each fiber UH_z of π_{UH} can be identified with the unit sphere $\mathbb{S}^{m_1-1}$ of $\mathbb{R}^{m_1}$. Roughly speaking, UH_z is the subset of H_z of all unit vectors with respect to the norm $|\cdot|_H$ on the fiber.*

We define the *volume form on UH* to be the differential $n + m_1 - 1$-form

$$\Omega^n \wedge \sigma_{m_1-1} \in \Lambda^*(UH),$$

where $\Omega^n = \omega_1 \wedge ... \wedge \omega_n$ is the bi-invariant volume form on $\mathbb{G}$ defined in Section 1.3 and σ_{m_1-1} is the canonical volume form on the unit sphere $\mathbb{S}^{m_1-1}$ of $\mathbb{R}^{m_1}$ identified with the generic fiber of UH. We denote by μ_0 the measure on UH obtained by integration of $\Omega^n \wedge \sigma_{m_1-1}$ and if $x \in \mathbb{G}$ we denote by $d\mu_{0x}$ the measure on UH_x

obtained by integration of σ_{m_1-1}. Explicitly, for $f \in L^1(UH)$, one has

$$\int_{UH} f(x;X)\, d\,\mu_0(x;X) := \int_{\mathbb{G}} d\,\mathcal{L}^n(x) \int_{UH_x} f(x;X)\, d\,\mu_{0x}(X).$$

Notation 3.11. *If D is a subset of $\mathbb{G}$, then UHD will denote the restriction to D of the bundle structure UH i.e.*

$$UHD := \Big\{X \in UH : \pi_{UH}(X) \in D\Big\}.$$

We also remind that, if $z \in \mathbb{G}$ and $X \in UH$, then $\mathcal{I}_z(X)$ denotes the vertical hyperplane through z and orthogonal to X while $\mathcal{V}_z$ the family of all vertical hyperplanes through z. Finally, γ_{X_y} denotes the horizontal X-line starting from $y \in \mathcal{I}_z(X)$ (i.e. $\gamma_{X_y}(t) = exp\,[tX](y),\ t \in \mathbb{R}$) and if $D \subset \mathbb{G}$ we set

$$D_y^X := \gamma_{X_y}(\mathbb{R}) \cap D.$$

Notice that, if $X_e = \sum_{j=1}^{m_1} a_j \mathrm{e}_j$, then $\mathcal{I}_z(X)$ can be regarded as the boundary of the half-space

$$\mathcal{I}_z^-(X) := \Big\{y \in \mathbb{G} : \sum_{j=1}^{m_1} [y_j - z_j]\, a_j \leq 0\Big\}$$

and so

$$\nu_{\mathcal{I}_z^-(X)}(y) = (a_1, ..., a_{m_1}) \in \mathbb{R}^{m_1} \cong H \qquad \forall\ y \in \mathcal{I}_z(X).$$

So we get that the H-perimeter of $\mathcal{I}_z^-(X)$ is just the $n-1$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on the vertical hyperplane $\mathcal{I}_z(X)$ and from Proposition 2.3 we deduce the following:

Corollary 3.12. *Let D be a Lebesgue measurable subset of $\mathbb{G}$ and fix $z \in \mathbb{G}$. Then*

$$\mu_0(UHD) = \int_{UH_z} d\,\mu_{0z}(X) \int_{pr^X_{\mathcal{I}_z(X)}(D)} \mathcal{H}^1_c(D_y^X)\, d\,\mathcal{H}^{n-1}(y)$$

or, equivalently,

$$\mathcal{L}^n(D) = \frac{1}{O_{m_1-1}} \int_{UH_z} d\,\mu_{0z}(X) \int_{pr^X_{\mathcal{I}_z(X)}(D)} \mathcal{H}^1_c(D_y^X)\, d\,\mathcal{H}^{n-1}(y),$$

where O_{m_1-1} denotes the (m_1-1)-dimensional surface measure of the sphere $\mathbb{S}^{m_1-1}$ of $\mathbb{R}^{m_1}$.

Proof. From Proposition 2.3 we have that

$$\mathcal{L}^n(D) = \int_{pr^X_{\mathcal{I}_z(X)}(D)} \mathcal{H}^1_c(D^X_y)\, d\,\mathcal{H}^{n-1}(y) \quad \forall\, X \in UH. \tag{57}$$

Then we get the claim by integrating both sides of (57) over $X \in UH_z$. $\square$

Corollary 3.13. *Let $U \subseteq \mathbb{G}$ be open and $X \in UH$. Assume that $D \subset \mathbb{G}$ is a H-Caccioppoli set, then*

$$|\partial_X D|(U) = \int_{pr^X_{\mathcal{I}_z(X)}(D\cap U)} var^1_X[\mathbf{1}_{D^X_y}](U^X_y)\, d\,\mathcal{H}^{n-1}(y). \tag{58}$$

Proof. This follows using Lemma 1.41 and Theorem 3.7 and observing that, for the half-space $\mathcal{I}^-_z(X)$, we have

$$|\partial_X\, \mathcal{I}^-_z(X)|(B) = \mathcal{H}^{n-1}(B \cap \mathcal{I}_z(X)) \quad \forall\, B \in \mathcal{B}(\mathbb{G}).$$

$\square$

As application of the last corollary we may establish the following:

Proposition 3.14. *[Integral geometric H-perimeter] Let $U \subseteq \mathbb{G}$ be open and fix $z \in \mathbb{G}$. If $D \subset \mathbb{G}$ is a H-Caccioppoli set, we have then*

$$|\partial D|_H(U) = \frac{1}{2\kappa_{m_1-1}} \int_{UH_z} d\,\mu_{0z}(X) \int_{pr^X_{\mathcal{I}_z(X)}(D\cap U)} var^1_X[\mathbf{1}_{D^X_y}](U^X_y)\, d\,\mathcal{H}^{n-1}(y), \tag{59}$$

where κ_{m_1-1} is the $m_1 - 1$-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{m_1-1}$.

Proof. Starting from Corollary 3.13, we integrate both sides of (58) over $X \in UH_z$. Thus

$$\begin{aligned}
&\int_{UH_z} d\,\mu_{0z}(X) \int_{pr^X_{\mathcal{I}_z(X)}(D\cap U)} var^1_X[\mathbf{1}_{D^X_y}](U^X_y)\, d\,\mathcal{H}^{n-1}(y) \\
=\ & \int_{UH_z} |\partial D|_H(U)\, d\,\mu_{0x_0}(X) \\
=\ & \int_{UH_z} d\,\mu_{0z}(X) \int_{D\cap U} |\langle X, \nu_D\rangle_H|\, d\,|\partial D|_H \\
=\ & \int_{D\cap U} d\,|\partial D|_H \int_{UH_z} |\langle X, \nu_D\rangle_H|\, d\,\sigma_{m_1-1}(X) = 2\kappa_{m_1-1}|\partial D|_H(U),
\end{aligned}$$

where we have used Fubini's Theorem and spherical coordinates to compute the integrating of the last line. $\square$

We now give the notion of H-normality with respect to any vertical hyperplane.

Definition 3.15. *[H-normality in Carnot groups] If $z \in \mathbb{G}$ and $X \in H$, let $\mathcal{I}_z(X)$ denote the vertical hyperplane through z and orthogonal to X. We say that $D \subseteq \mathbb{G}$* **is pointwise X-normal with respect to** *$\mathcal{I}_z(X)$ if for every $y \in \mathcal{I}_z(X)$ we have that $\gamma_{X_y}^{-1}\{\gamma_{X_y}(\mathbb{R}) \cap D\}$ is the empty set or a connected subset of $\mathbb{R}$ or, equivalently, if $\gamma_{X_y}(\mathbb{R}) \cap D$ is either empty or a connected subset of $\gamma_{X_y}(\mathbb{R})$. Moreover, we say that D* **is X-normal with respect to** *$\mathcal{I}_z(X)$ if D is L^1-equivalent to a subset of $\mathbb{G}$ that is pointwise X-normal with respect to $\mathcal{I}_z(X)$.*

Usually, we term this property *pointwise H-normality* (resp. *H-normality*) with respect to a vertical hyperplane. As already observed, for any point $x \in \mathbb{G}$ and for any horizontal direction $X \in H$ there exists a unique horizontal X-line passing from x. This implies that *H-normality is invariant under group translations*, as left translations send a vertical hyperplane orthogonal to $X \in H$ into a vertical hyperplane which is still orthogonal to X. Let now $z \in \mathbb{G}$ and consider the family $\mathcal{V}_z$ of vertical hyperplanes through z. The invariance under group translations of the notion of H-normality allows to see that the following two conditions are equivalent:

(i) *$D \subseteq \mathbb{G}$ is pointwise H-normal with respect to any vertical hyperplane $\mathcal{I}_z(X)$ of $\mathcal{V}_z$;*

(ii) *$D \subseteq \mathbb{G}$ is pointwise H-normal with respect to any vertical hyperplane $\mathcal{I}_z(Z)$ where $z \in \mathbb{G}$ and $Z \in H$.*

We emphasize that the notions introduced above generalize that corresponding euclidean because, if $(\mathbb{G}, \bullet) = (\mathbb{R}^n, +)$ they coincide, as it can be easily proved. Moreover the analogy with the euclidean case suggests the following.

Definition 3.16. *[H-convexity] We say that $D \subseteq \mathbb{G}$* **is H-convex** *if, for every $x \in \mathbb{G}$ and every $X \in H$, we have that $\gamma_{X_x}^{-1}\{\gamma_{X_x}(\mathbb{R}) \cap D\}$ is the empty set or a connected subset of $\mathbb{R}$ or, equivalently, if $\gamma_{X_x}(\mathbb{R}) \cap D$ is either empty or a connected subset of $\gamma_{X_x}(\mathbb{R})$.*

Also in this case, if the Carnot group reduces to $(\mathbb{R}^n, +)$, these definitions coincide. Moreover, *H-convexity turns out to be invariant under group translations* and it is *stable under intersection*, i.e. if D_1, $D_2 \subseteq \mathbb{G}$ are H-convex sets, then also $D_1 \cap D_2$ is a H-convex set.

We refer the reader to [24] and [65] for some different, but in fact equivalent, definitions of convexity in Carnot groups. See also [8] for a detailed discussion on this topic.

Remark 3.17. *Notice that, H-convexity turns out to be equivalent to condition (ii), i.e.*

($*$) D is H-convex if, and only if, D is pointwise H-normal with respect to every vertical hyperplane.

Clearly, if D is just H-normal with respect all of vertical hyperplanes of $\mathbb{G}$, then it is L^1-equivalent to a H-convex set.

To better explain the meaning of H-convexity we make use of the horizontal fibers, thought as family of moving m_1-planes on $\mathbb{G}$. More precisely, if $z \in \mathbb{G}$, we identify the horizontal fiber H_z at z with the left translated by z of the m_1-plane $exp\,(H_e) \subset \mathbb{G}$, i.e.

$$H_z \cong L_z\{\exp\,(H_e)\} \qquad (z \in \mathbb{G}),$$

and so H_z is viewed as the *horizontal m_1-plane through z of all horizontal lines starting from z.*

Proposition 3.18. *If $D \subseteq \mathbb{G}$, we have that D is H-convex if, and only if, $log\,(L_{-z}\{H_z \cap D\})$ is starshaped in H_e with respect to the identity $0 \in \mathfrak{g}$ for all $z \in D$. In particular, if for every $z \in D$ we have that $log\,(L_{-z}\{H_z \cap D\})$ is a* *euclidean convex in H_e, then D is H-convex. Finally, if $z \in \exp(V_{\mathrm{k}})$, where V_{k} is the center of the Lie algebra $\mathfrak{g}$, then any horizontal plane H_z through z is an affine*[3] *m_1-dimensional affine plane in $\mathbb{G} \cong_{\exp} \mathbb{R}^n$, and we get that, if D is H-convex, then $H_z \cap D$ is starshaped in H_z with respect to z for every $z \in \exp(V_k)$.*

Proof. Obvious from the previous definitions. □

Remark 3.19. *[H-convexity in 2-step Carnot groups] If $\mathbb{G}$ is a 2-step Carnot group, then its horizontal lines are also euclidean lines. This is a straightforward consequence of the group law that is completely determined by Campbell-Hausdorff formula, as we have seen in Section 1.1. Thus, from the definition of H-convexity, it follows that euclidean convex subsets of $\mathbb{G}$ are H-convex sets. In general,* **the converse it is not true**, *as proved in the next example.*

Example 3.20. [**A H-convex set in $\mathbb{H}^1$ that is not euclidean convex**] *Let us consider the Heisenberg group, here defined as follows: $\mathbb{H}^1 = (\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}, *)$, where $(z,t) * (z',t') = (z + z', t + t' + 2\Im(z\bar{z'}))$. Then, the truncated cone of width $\alpha > 0$, given by*

$$\mathcal{C}_\alpha = \Big\{(z,t) \in \mathbb{C} \times \mathbb{R} : |z| \leq \alpha\,|t|,\ |z| \leq 1,\ \alpha\,|t| \leq 1\Big\}$$

is an $\mathbb{H}^1$-convex set for any $\alpha \geq 2$ but it is not convex. This easily follows observing that the maximal slope of the horizontal lines having initial data in the cylinder $\{(z,t) \in \mathbb{H}^1 : |z| \leq 1\}$ is 2 so that any such a line intercepts $\mathcal{C}_\alpha$ in a segment line.

This definition of H-convexity can be used to generalize the *Cauchy's formula* for the area of euclidean convex sets. For the statement of this classical theorem see [13], [17], [86].

Theorem 3.21. *[Cauchy type formula] Let D be a H-convex subset of $\mathbb{G}$ and $z \in \mathbb{G}$. Then*

$$|\partial D|_H(\mathbb{G}) = \frac{1}{\kappa_{m_1-1}} \int_{UH_z} \mathcal{H}^{n-1}\Big(pr^X_{\mathcal{I}_z(X)}(D)\Big)\, d\,\mu_{0_z}(X) \tag{60}$$

[3] Since $\mathbb{G}$ is identified with $\mathbb{R}^n$ via exponential coordinates, it makes sense the notion of affine p-plane $(p = 1, ..., n)$.

where κ_{m_1-1} is the m_1-1-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^{m_1-1}$.

Proof. Using Proposition 3.6 and Proposition 3.14 we get the thesis observing that, since D is H-convex, then $var^1_X[\mathbf{1}_{D^X_y}](\gamma_{X_y}) = 2$ for $\mathcal{H}^{n-1}$-a.e. $y \in pr^X_{\mathcal{I}_z(X)}(D)$ for any $X \in UH_z$. $\square$

The above theorem, analogously to the euclidean case, allows to see that, in one sense, H-convex sets minimize the H-perimeter; indeed, as immediate consequence, we have the following.

Corollary 3.22. *If $D \subset \mathbb{G}$ is a H-convex set, then for any open set U containing D we have*

$$|\partial D|_H(\mathbb{G}) \leq |\partial U|_H(\mathbb{G}).$$

Proof. Fixing $z \in \mathbb{G}$, the claim follows by the previous Theorem 3.21 observing that, for every $X \in H$, one has $pr^X_{\mathcal{I}_z(X)}(D) \subseteq pr^X_{\mathcal{I}_z(X)}(U)$. $\square$

4 A Santaló type formula and related topics

Throughout this section will be discussing some result about the integration on UH, i.e. the unit horizontal bundle of a k-step Carnot group. The results here exposed rely on the definition of a canonical measure $d\mu_0$ on the unit horizontal bundle UH. We stress that this measure is just that defined in Section 3.2. We shall then prove *its invariance under the action of the horizontal flow*, i.e. the flow generated by restriction to H of the Riemannian geodesic flow. The measure $d\mu_0$ generalize to Carnot groups the classical notion of *Kinematic density*; see, for instance, [10] and [16]. More precisely, an integral formula is given in Theorem 4.5, which generalizes the well-known Santaló formula [86]. We emphasize that, in the case of the Heisenberg group $\mathbb{H}^1$, a Santaló-type formula was proved by Pansu, [81]. We then give some applications of Theorem 4.5. In particular, in Proposition 4.10 and Theorem 4.11, we find two lower bounds for the first eigenvalue of the Dirichlet problem for the Carnot sub-Laplacian Δ_H on smooth domains.

In the tangent bundle $T\mathbb{G}$ we use coordinates given by

$$(x; X) = ((x_1, ..., x_n); (a_1, ..., a_n)),$$

where $(x_1, ..., x_n)$ are the exponential coordinates of $x \in \mathbb{G}$ and $(a_1, ..., a_n)$ are the coordinates of X in the Lie algebra $\mathfrak{g}\,(= T_e\mathbb{G})$, i.e. $X_e = \sum_{i=1}^n a_i \mathrm{e}_i$. We remind that $\mathfrak{g}$ is endowed with an inner product denoted by $\langle \cdot, \cdot \rangle$ that is just that usual in $\mathbb{R}^n$. This uniquely determines the left invariant Riemannian metric on $\mathbb{G}$, also denoted by $\langle \cdot, \cdot \rangle$, that is defined by setting

$$\langle X, Y \rangle := \langle X_e, Y_e \rangle \quad \forall\; X, Y \in T\mathbb{G}.$$

The *energy function* of $X \in \mathfrak{X}(\mathbb{G})$ associated with $\langle \cdot, \cdot \rangle$ is given by

$$E(X) := \frac{1}{2}\langle X, X \rangle = \frac{1}{2}\sum_{i=1}^n a_i^2.$$

Moreover we denote by α *the canonical* 1-*form on the cotangent bundle* $T^*\mathbb{G}$, which is given, in this notation, by $\alpha := \sum_{i=1}^n a_i \omega_i$. Following Besse, [10], we call *geodesic* *vector field* on $T\mathbb{G}$ the solution of the equation

$$T \rfloor d\,\alpha = -dE. \tag{61}$$

We remind that, if $X \in T\mathbb{G}$, then $X \rfloor\, : \Lambda^k(\mathbb{G}) \to \Lambda^{k-1}(\mathbb{G})$ denotes the *interior product* with X, i.e. the linear map defined by ([54], [64]):

$$X \rfloor \psi(Y_1, ..., Y_{k-1}) := \psi(X, Y_1, ..., Y_{k-1}).$$

The *geodesic flow* is then the flow generated by T and we may explicitly compute it. We have

$$T = \sum_{i=1}^{n} a_i X_i - \frac{1}{2} \sum_{l,i,j,k=1}^{n} a_i\, a_l\, c_{jk}^i \left(\delta_{lj} \frac{\partial}{\partial a_k} - \delta_{lk} \frac{\partial}{\partial a_j}\right).$$

To prove this is enough to use the definitions of α, E and T and the above equation (61). The result then follows by applying Proposition 1.21. Now we shall prove that the restriction of the canonical 1-form α to the unit horizontal bundle is *invariant under the geodesic flow*, i.e. the Lie derivative by T of α is equal to 0. Indeed, using *Cartan's identity* (see [64]) we get

$$\mathcal{L}_T\, \alpha = T \rfloor\, d\alpha + d\,T \rfloor\, \alpha = -dE + 2 \sum_{i=1}^{n} a_i\, da_i = dE.$$

Now, since we consider unit horizontal vectors, the thesis follows observing that $a_i = 0$ for any $i = m_1 + 1, ..., n$, and that $\sum_{i=1}^{m_1} a_i^2 = 1$. Therefore, denoting by $\alpha_0 := \alpha_{|UH}$ the restriction of α to the unit horizontal bundle UH, we then get that α_0 is invariant under the restriction of T to the horizontal bundle. From now on we denote this vector by T_0, i.e. $T_0 = T_{|H}$, and we call *horizontal flow* the flow on H generated by T_0.

We want to show that there is a canonical measure on the unit horizontal bundle UH which turns out to be invariant under the horizontal flow. To this end we make use of the volume form on UH given by $\Omega^n \wedge \sigma_{m_1-1}$, where $\Omega^n = \omega_1 \wedge ... \wedge \omega_n$ is the bi-invariant volume form on $\mathbb{G}$ and σ_{m_1-1} is the volume form on the unit sphere $\mathbb{S}^{m_1-1}$ ($\hookrightarrow \mathbb{R}^{m_1}$) identified with the generic fiber of UH (see Section 3.2).

We stress that if $(x;X) \in UH\,(X_e = (a_1, ..., a_{m_1}, 0, ..., 0))$, then

$$\begin{aligned} \sigma_{m_1-1}(X) &= X \rfloor da_1 \wedge ... \wedge da_{m_1} \\ &= \sum_{i=1}^{m_1-1} (-1)^{i+1} a_i\, da_1 \wedge ... \wedge \widehat{da_i} \wedge ... \wedge da_{m_1} \end{aligned}$$

and also that

$$\begin{aligned} &(\Omega^n \wedge \sigma_{m_1-1})(x;X)(X_1, ..., X_n; Y_1, ..., Y_{m_1}) \\ &= \Omega^n(x)(X_1, ..., X_n) \cdot \sigma_{m_1-1}(X)(Y_1, ..., Y_{m_1}) \end{aligned}$$

for all $X_1, ..., X_n \in T_x\mathbb{G}$ and all $Y_1, ..., Y_{m_1} \in UH_x$.

Definition 4.1. *We denote by μ_0 the measure on UH obtained by integration of $\Omega^n \wedge \sigma_{m_1-1}$ and by $d\mu_{0x}$ the measure on the fiber at x, UH_x, obtained by integration of σ_{m_1-1}. Thus, for every function $f \in L^1(UH)$ we may write*

$$\int_{UH} f(x;X)\, d\,\mu_0(x;X) = \int_{\mathbb{G}} d\,\mathcal{L}^n(x) \int_{UH_x} f(x;X)\, d\,\mu_{0x}(X). \tag{62}$$

From now on, we set

$$\Omega_1 := \omega_1 \wedge ... \wedge \omega_{m_1},\ \Omega_2 := \omega_{m_1+1} \wedge ... \wedge \omega_{m_2},\ \Omega_k := \omega_{m_{k-1}+1} \wedge ... \wedge \omega_{m_k}$$

so that $\Omega^n = \Omega_1 \wedge ... \wedge \Omega_k$. Moreover, $* : \Lambda^k T^*\mathbb{G} \to \Lambda^{n-k} T^*\mathbb{G}$ denotes the Hodge star operator; we explicitly note that $*\Omega_1 = \Omega_2 \wedge ... \wedge \Omega_k$.

The next theorem asserts a *Liouville type property* of the measure μ_0.

Theorem 4.2. *The measure $d\mu_0$ on UH turns out to be invariant with respect to the horizontal flow on H associated with T_0 and we have that*

$$\Omega^n \wedge \sigma_{m_1} = \pm \frac{1}{(m_1-1)!}\, \alpha_0 \wedge (d\,\alpha_0)^{m_1-1} \wedge *\Omega_1.$$

The proof relies on the following lemmas.

Lemma 4.3. *With the previous notation we have*

$$\begin{aligned} &\alpha_0 \wedge (d\alpha_0)^{m_1-1} \\ =\ & (m_1-1)!\,(-1)^{\frac{(m_1-1)(m_1-2)}{2}}\, \Omega_1 \wedge \sigma_{m_1-1} \\ =\ & (m_1-1)! \sum_{i=1}^{m_1} (-1)^{\frac{(m_1-1)(m_1-2)}{2}} (-1)^i\, a_i\, \omega_1 \wedge ... \wedge \omega_{m_1} \wedge da_1 \wedge ... \wedge \widehat{da_i} \wedge ... \wedge da_{m_1}. \end{aligned}$$

Proof. One can prove this lemma by induction on $m_1(=\dim H)$, just by using the definitions, the expression of $\alpha_0 = \sum_{i=1}^{m_1} a_i\omega_i$ and that of $d\alpha_0 = \sum_{i=1}^{m_1} da_i \wedge \omega_i$. □

Lemma 4.4. *If $X \in \mathbf{C}^\infty(\mathbb{G}, H)$, then*

$$\Omega_1 \wedge i(X)(d * \Omega_1) = 0.$$

Proof. We have that

$$\begin{aligned} d\,(*\Omega_1) &= d\,(\Omega_2 \wedge ... \wedge \Omega_k) \\ &= \sum_{i=m_1+1}^{n} (-1)^{i+1}\omega_{m_1+1} \wedge ... \wedge \omega_{i-1} \wedge d\omega_i \wedge \omega_{i+1} \wedge ... \wedge \omega_n \\ &= -\frac{1}{2}\sum_{l=1}^{k}\sum_{i=m_1+1}^{n}\sum_{1\le j,\,h\le h_{l-1}} (-1)^{i+1} c_{jh}^{i}\,\omega_{m_1+1} \wedge ... \wedge \underbrace{(\omega_j \wedge \omega_h)}_{i-thplace} \wedge ... \wedge \omega_n. \end{aligned}$$

This formula, which is an easy consequence of Proposition 1.21 and Remark 1.22, enable us to say that $d\,(*\Omega_1)$ is a linear combination of $(n - m_1 + 1)$-forms of the type

$$(\omega_j \wedge \omega_h) \wedge \omega_{m_1+1} \wedge ... \wedge \omega_{i-1} \wedge \widehat{\omega_i} \wedge \omega_{i+1} \wedge ... \wedge \omega_n$$

for $i = m_1, ..., n,\ \ j\,h = 1, ..., n$, and $i \neq j, h$. Thus, by a direct computation it follows that $\Omega_1 \wedge i(X)(d * \Omega_1)$ is a linear combination of n-forms, each of which have the following expression

$$\omega_1 \wedge ... \wedge \omega_{s-1} \wedge (\omega_s)^2 \wedge \omega_{s+1} \wedge ... \wedge \omega_n \qquad (s = 1, ..., n)$$

and the claim follows since these terms are equal to 0. □

Proof of Theorem 4.2. We have to show that the Lie derivative along T_0 of $\Omega^n \wedge \sigma_{m_1-1}$ is 0. From Lemma 4.3 it follows that

$$\Omega^n \wedge \sigma_{m_1-1} = (-1)^{\frac{(m_1-1)(m_1-2)}{2}} \frac{1}{(m_1-1)!}\,\alpha_0 \wedge (d\alpha_0)^{m_1-1} \wedge (*\Omega_1).$$

Thus we need to compute the Lie derivative along T_0 of $\alpha_0 \wedge (d\alpha_0)^{m_1-1} \wedge (*\Omega_1)$ and using Cartan's identity and the invariance of α_0 under the horizontal flow induced by T_0 we get

$$\begin{aligned}
& \mathcal{L}_T\Big(\alpha_0\wedge(d\alpha_0)^{m_1-1}\wedge(*\Omega_1)\Big)\\
= \;& \mathcal{L}_T\Big(\alpha_0\wedge(d\alpha_0)^{m_1-1}\Big)\wedge(*\Omega_1)+\Big(\alpha_0\wedge(d\alpha_0)^{m_1-1}\Big)\wedge\mathcal{L}_T(*\Omega_1)\\
= \;& \Big(\alpha_0\wedge(d\alpha_0)^{m_1-1}\Big)\wedge\Big(T\rfloor(d*\Omega_1)+d\,(T\rfloor*\Omega_1)\Big)\\
= \;& \Big(\alpha_0\wedge(d\alpha_0)^{m_1-1}\Big)\wedge\Big(T\rfloor(d*\Omega_1)\Big)
\end{aligned}$$

and the thesis follows from Lemmas 4.3 and 4.4. □

Let $D\subset\mathbb{G}$ be a smooth, relatively compact domain (open and connected) and let us consider

$$UH D=\Big\{X\in UH:\pi_{UH}(X)\in D\Big\},$$

that is the restriction to D of the structure of unit horizontal bundle. If $(x;X)\in UH D$ we set

$$\ell_x(X):=\sup\Big\{s\in\mathbb{R}_+:\gamma_X(t)\in D,\,\forall\,t\in(0,s)\Big\},$$

where γ_X is the (unique) horizontal line satisfying $\gamma_X(0)=\pi_{UH}(X)$, $\dot\gamma_X(0)=X$. Notice that

$$\ell_x(X)=\mathcal{H}^1_c\Big(\gamma_X\Big(\,]0,\ell_x(X)[\,\Big)\Big).$$

By the boundedness of D we have $\ell_x(X)<\infty$, everywhere in D. Moreover $\gamma_X(\ell_x(X))$ is the first point of the horizontal line γ_X starting from $x=\pi_{UH}(X)$ to hit the boundary of D.

Let now ν_D be the unit inward $\mathbb{G}$-normal to ∂D and let us set

$$UH^+\partial D:=\Big\{X\in UH\overline{D}:\pi_{UH}(X)\in\partial D,\ \langle X,\nu_D\rangle_{H_x}>0\Big\}. \tag{63}$$

This is the set of inward pointing unit horizontal vectors along the boundary ∂D and, identifying the generic fiber with $\mathbb{S}^{m_1-1}$, we may think it as the hemisphere determined by ν_D which will be denoted by $\mathbb{U}^{m_1-1}$. We also provide $UH^+\partial D$ with the following measure

$$d\,\sigma(x;X):=d\,\mu_{0x}(X)\,d\,|\partial D|_H(x)\quad\forall\,(x;X)\in UH^+\partial D.$$

Clearly $d\,\mu_{0x}$ will be concentrated on the hemisphere $\mathbb{U}^{m_1-1} \cong UH^+\partial D_x$.

Below we shall denote by $C(\partial D)$ the so-called *characteristic set* of ∂D (see for instance [7], [42], [43], [45], [67]), i.e.

$$C(\partial D) := \Big\{x \in \partial D : \langle N(x), X(x)\rangle = 0 \;\; \forall\; X \in H_x\Big\}.$$

Moreover we shall set

$$D^* = \Big\{x \in D : \exists X \in H_x : \quad \gamma_X(\ell_x(X)) \in C(\partial D)\Big\}.$$

Along the lines of [16], [81] and [86] we may prove now the following:

Theorem 4.5. *Let D be a smooth relatively compact domain. For all $f \in L^1(UHD)$, we have*

$$\begin{aligned} & \int_{UHD} f(y;Y)\,d\,\mu_0(y;Y) \\ = & \int_{UH^+\partial D}\int_0^{\ell_x(X)} f(\gamma_X(t);X)\,\langle X,\nu_D\rangle_{H_x}\,dt\,d\,\sigma(x;X) \\ = & \int_{\partial D}\int_{UH^+\partial D_x}\int_0^{\ell_x(X)} f(\gamma_X(t);X)\,\langle X,\nu_D\rangle_{H_x}\,dt\,d\,\mu_{0x}(X)\,d\,|\partial D|_H(x). \end{aligned}$$

Proof. First we consider the following map

$$\mathbb{R}_+ \times UH^+\partial D \ni (t,(x;X)) \longmapsto (\gamma_X(t);X) \in UH,$$

that is nothing but the restriction to $UH^+\partial D$ of the horizontal flow. Denoting by $\Phi_t(X)$ this flow, we shall see how $\Phi_t(X)$ acts on the measure $d\,\mu_0$. To this end we have to compute the pull back by $\Phi_t(X)$ of the volume form of UH. Observing that $(\Phi_t(X))^*\sigma_{m_1-1} = \sigma_{m_1-1}$ we get

$$\Big(\Phi_t(X)\Big)^*(\Omega^n \wedge \sigma_{m_1-1}) = \Big((\gamma_X(t))^*\Omega^n\Big) \wedge \sigma_{m_1-1}(X).$$

Notice that we have already performed this computation in the proof of Lemma 2.11 by means of a local parametrization and so we have just to reformulate it. We have

$$(\gamma_X(t))^*\Omega^n = (X \rfloor \Omega^n)_x \wedge dt,$$

and explicitly this means that

$$(\gamma_X(t))^* d\,\mathcal{L}^n = \langle X, \nu_D\rangle_{H_x}\, d\,t\, d\,|\partial D|_H(x)$$

for $t > 0$ and $x \in \pi_{|UH}\Big(UH^+\partial D\Big)$.
Therefore

$$(\Phi_t(X))^* d\mu_0 = \langle X, \nu_D\rangle_{H_x}\, d\,t\, d\,|\partial D|_H(x)\, d\mu_{0x}(X). \tag{64}$$

Since D is a relatively compact domain, we can univocally associate to any

$$(y; Y) \in UH(D \setminus D^*)$$

the time $t = \ell_y(-Y) < \infty$ and the point $(x; X) = (\gamma_{-Y}(\ell_y(-Y)); -Y)$, so that x is the first point on the boundary of D reachable from y along the (unique) horizontal Y-line passing through y; furthermore $t < \ell_x(X)$. Thus we have that the map $\Phi_t(X)$ which takes $(t, (x; X))$ onto $(y; Y)$ is a diffeomorphism of the open set $\big\{(t, (x; X)) : 0 < t < \ell_x(X)\big\}$ of $\mathbb{R}_+ \times UH^+\partial D$ onto $UH(D \setminus D^*)$.

Finally, if $\mu_0(UH(D^*)) = 0$ then the thesis will hold multiplying both sides of (64) by f and then integrating. But we can get the last claim from the classical Area formula [33], by applying again the same computations of Lemma 2.11. □

Remark 4.6. *If D is H-convex then $D^* = \emptyset$ and the map $\Phi_t(X)$ defined in the above proof is a diffeomorphism onto UHD.*

From the last theorem we easily deduce an integral geometric formula that allows to compute the volume of a smooth relatively compact domain in a Carnot group.

Remark 4.7.

$$\mathcal{L}^n(D) = \frac{1}{O_{m_1-1}} \int_{\partial D} \int_{UH^+\partial D_x} \ell_x(X)\, \langle X, \nu_D\rangle_{H_x} d\,\mu_{0x}(X)\, d\,|\partial D|_H(x), \tag{65}$$

where O_{m_1-1} denotes the $(m_1 - 1)$-dimensional surface measure of the sphere $\mathbb{S}^{m_1-1}$.

We shall give now a first application of this theorem. To this end we need some preliminaries.

Let $(x;\widetilde{X}) \in UH$ be fixed and denotes by UH_x^+ the hemisphere determined by $\widetilde{X}$, i.e.

$$UH_x^+ := \Big\{X \in UH_x : \langle \widetilde{X}, X\rangle_{H_x} > 0\Big\}.$$

Lemma 4.8.

$$\int_{UH^+\mathbb{G}_x} \langle X, \widetilde{X}\rangle_{H_x}\, d\,\mu_{0x}(X) = \frac{O_{m_1-2}}{m_1-1}. \tag{66}$$

Proof. It is enough to observe that this integral is the measure of the projection of the (m_1-1)-dimensional hemisphere $\mathbb{U}^{m_1-1} \cong UH_x^+$ onto a diametral plane and so we may perform the computation using spherical coordinates. □

As above, let D be a smooth, relatively compact, open subset of $\mathbb{G}$ and denotes by $\mathrm{diam}_H(D)$ its *horizontal diameter,* that is the quantity defined as

$$\mathrm{diam}_H(D) := \sup_{(y;Y)\in UH^+\partial D} \ell_y(Y).$$

Denoting by $\mathrm{diam}_c(D)$ the diameter of D with respect to the Carnot-Carathéodory distance d_c, we have obviously

$$\mathrm{diam}_H(D) \leq \mathrm{diam}_c(D).$$

Corollary 4.9. *Let $D \subset \mathbb{G}$ be a smooth and relatively compact domain. Then we have*

$$\frac{\mathcal{L}^n(D)}{|\partial D|_H(\mathbb{G})} \leq \frac{O_{m_1-2}}{O_{m_1-1}\cdot(m_1-1)} \cdot \mathrm{diam}_c(D),$$

where, in general, O_k denotes the k-dimensional surface measure of the unit sphere $\mathbb{S}^k$ of $\mathbb{R}^{k+1}$.

Proof. From Remark 4.7 we get

$$\mathcal{L}^n(D) \leq \frac{\mathrm{diam}_H(D)}{O_{m_1-1}} \int_{UH^+\partial D} \langle X, \nu_D\rangle_{H_x} d\,\sigma(X)$$

$$\leq \frac{\mathrm{diam}_c(D)}{O_{m_1-1}} \int_{\partial D} d\,|\partial D|_H(x) \int_{UH^+\mathbb{G}_x} \langle X, \nu_D\rangle_{H_x} d\,\mu_{0x}(X)$$

and, using the foregoing lemma, we get the claim. □

We would now to show some applications of Theorem 4.5 to the Analysis in Carnot groups. For what follows, we refer the reader to [11], [23], [93], [94]. Here, more precisely, we give two explicit lower bounds for the first eigenvalue of the Dirichlet problem for the Carnot sub-Laplacian. To this end, we use Theorem 4.5 by also adapting some classical arguments of Riemannian geometry, for which we refer the reader to [16], [21], [22], [26].

We stress that in these inequalities, as well as in Corollary 4.9, *we do not characterize the equality cases and, in general, they are non-sharp.*

We recall that, with our notation , the Carnot sub-Laplacian of $\mathbb{G}$ is defined by

$$\Delta_H := \sum_{j=1}^{m_1} X_j^2, \qquad \Delta_H\psi(x) = \sum_{j=1}^{m_1} \frac{d^2}{dt^2}\bigg|_{t=0} \psi(x \bullet exp\,(tX_j)) \quad \forall\ \psi \in \mathbf{C}^\infty(\mathbb{G}).$$

Let us consider the *Dirichlet eigenvalue problem* for Δ_H on a smooth bounded domain D, i.e. we find all real numbers t for which there exist non trivial solutions $\phi \in W^{1,2}_{\mathbb{G}}(D)$ -the *horizontal* Sobolev space- of the problem

$$\Delta_H\phi + \lambda\phi = 0 \quad (x \in D) \tag{67}$$

satisfying the boundary condition $\phi_{|\partial D} = 0$. One can prove that the *eigenvalues* λ of this problem are strictly positive real numbers and that all the *eigenfunctions* ϕ can be choose to be real-valued. Moreover, eigenfunctions corresponding to distinct eigenvalues turn out to be orthogonal in $L^2(D)$, with respect to the usual inner product on $L^2(D)$. The main result that we use in what follows is the variational characterization of the *first eigenvalue* of (67) that we denote by $\lambda_1(D)$, i.e.

$$\lambda_1(D) = \inf_{\varphi \in \mathbf{C}_0^\infty(D)} \frac{\int_D |\nabla^H\varphi|_H^2\, d\mathcal{L}^n}{\int_D |\varphi|^2\, d\mathcal{L}^n}. \tag{68}$$

Notice that to prove (68) one uses the following Green's identity

$$\int_D \{\varphi\, \Delta_H\psi + \langle \nabla^H\varphi, \nabla^H\psi\rangle_H\}\, d\mathcal{L}^n = 0$$

whenever $\varphi,\, \psi : D \longrightarrow \mathbb{R}$ are smooth and with at least one of them compactly supported in D.

Proposition 4.10. *Let $D \subset \mathbb{G}$ be a smooth, relatively compact domain and let $\lambda_1(D)$ be the first eigenvalue of (67). Then we have*

$$\lambda_1(D) \geq \frac{\pi^2 \cdot m_1}{\mathrm{diam}_H(D)^2} \geq \frac{\pi^2 \cdot m_1}{\mathrm{diam}_c(D)^2}.$$

Proof. We have just to prove the first inequality since the second one is trivial. To this end we notice that for any $\varphi \in \mathbf{C}_0^\infty(D)$ we have

$$|\nabla^H \varphi|^2_{H_x} = \frac{m_1}{O_{m_1-1}} \int_{UH_x} (X\varphi)^2 d\,\mu_{0x}(X).$$

Moreover the *fixed-endpoint* version of the 1-dimensional Wirtinger's inequality says that

$$\int_0^l \dot{h}(t)^2 dt \geq \frac{\pi^2}{l^2} \int_0^l h(t)^2 dt \quad \forall\, h \in \mathbf{C}^1([0,l]),\;\; h(0) = h(l) = 0.$$

Therefore, using this remarks and Theorem 4.5 we get

$$\begin{aligned}
& \int_D |\nabla^H \varphi|^2_{H_x}\, d\mathcal{L}^n(x) \\
= \;& \frac{m_1}{O_{m_1-1}} \int_{UHD} (X\varphi)^2 d\,\mu_0(x;X) \\
= \;& \frac{m_1}{O_{m_1-1}} \int_{UH^+\partial D} \int_0^{\ell_{\widetilde{x}}(X)} \Big(\frac{d}{dt}\varphi(\gamma_X(t))\Big)^2 \langle X, \nu_D\rangle_{H_{\widetilde{x}}}\, dt\, d\,\sigma(\widetilde{x};X) \\
\geq \;& \frac{m_1}{O_{m_1-1}} \int_{UH^+\partial D} \frac{\pi^2}{\ell^2_{\widetilde{x}}(X)} \int_0^{\ell_{\widetilde{x}}(X)} |\varphi(\gamma_X(t))|^2 \langle X, \nu_D\rangle_{H_{\widetilde{x}}}\, dt\, d\,\sigma(\widetilde{x};X) \\
\geq \;& \frac{\pi^2 \cdot m_1}{O_{m_1-1} \cdot \mathrm{diam}^2_H(D)} \int_{UH^+\partial D} \int_0^{\ell_{\widetilde{x}}(X)} |\varphi(\gamma_X(t))|^2 \langle X, \nu_D\rangle_{H_{\widetilde{x}}}\, dt\, d\,\sigma(\widetilde{x};X) \\
= \;& \frac{\pi^2 \cdot m_1}{O_{m_1-1} \cdot \mathrm{diam}^2_H(D)} \int_{UHD} |\varphi(x)|^2 d\,\mu_0(x;X) \\
= \;& \frac{\pi^2 \cdot m_1}{\mathrm{diam}_H(D)^2} \int_D |\varphi(x)|^2\, d\mathcal{L}^n(x).
\end{aligned}$$

□

Now we state another similar result along the line of [22]; see also [16] and [26].

Theorem 4.11. *Let $D \subset \mathbb{G}$ and $\lambda_1(D)$ be defined as above. Then we have*

$$\lambda_1(D) \geq \frac{m_1 \cdot \pi^2}{O_{m_1-1}} \cdot \inf_{x \in D} \int_{UH_x} \frac{1}{\ell_x^2(X)}\, d\mu_{0x}(X).$$

Proof. Analogously to the previous proof we have

$$\begin{aligned}
& \int_D |\nabla^H \varphi|^2_{H_x}\, d\mathcal{L}^n(x) \\
= & \geq \frac{m_1}{O_{m_1-1}} \int_{UH^+\partial D} \frac{\pi^2}{\ell^2_{\widetilde{x}}(X)} \int_0^{\ell_{\widetilde{x}}(X)} \Big(\varphi(\gamma_X(t))\Big)^2 \langle X, \nu_D\rangle_{H_{\widetilde{x}}}\, dt\, d\,\sigma(\widetilde{x}; X) \\
= & \frac{m_1 \cdot \pi^2}{O_{m_1-1}} \int_{UH^+\partial D} \int_0^{\ell_{\widetilde{x}}(X)} \frac{\varphi^2(\gamma_X(t))}{\ell^2_{\widetilde{x}}(X)} \langle X, \nu_D\rangle_{H_{\widetilde{x}}}\, dt\, d\,\sigma(\widetilde{x}; X) \\
= & \frac{m_1 \cdot \pi^2}{O_{m_1-1}} \int_{UHD} \frac{\varphi^2(x)}{\ell^2_x(X)}\, d\mu(x; X) \\
= & \frac{m_1 \cdot \pi^2}{O_{m_1-1}} \int_D \varphi^2(x) \int_{UH_x} \frac{1}{\ell^2_x(X)}\, d\mu_{0x}(X)\, d\mathcal{L}^n(x) \\
\geq & \frac{m_1 \cdot \pi^2}{O_{m_1-1}} \times \inf_{x \in D} \int_{UH_x} \frac{1}{\ell^2_x(X)}\, d\mu_{0x}(X) \times \int_D |\varphi(x)|^2\, d\mathcal{L}^n(x)
\end{aligned}$$

and the claim follows. $\square$

5 Some remarks about the geometry of hypersurfaces in Carnot groups

5.1 The H-perimeter form σ_H on non-characteristic hypersurfaces

Let $\mathbb{G}$ be a k-step Carnot group. Below we shall adopt the notation of Section 1. We shall also use the following standard operations on differentials forms:

(a) if $X \in \mathfrak{X}(\mathbb{G})$, then $X \rfloor : \Lambda^k(\mathbb{G}) \longrightarrow \Lambda^{k-1}(\mathbb{G})$ denotes the *contraction with* X (or *interior multiplication with* X) of a k-form $\omega \in \Lambda^k(\mathbb{G})$, defined as

$$(X \rfloor \omega)(Y_1, ..., Y_{k-1}) := \omega(X, Y_1, ..., Y_{k-1})$$

for every $Y_1, ..., Y_{k-1} \in \mathfrak{X}(\mathbb{G})$;

(b) if $X \in \mathfrak{X}(\mathbb{G})$ and $\omega \in \Lambda^k(\mathbb{G})$ then $\mathcal{L}_X\omega$ denotes the *Lie derivative* of ω with respect to X and by Cartan's identity we have

$$\mathcal{L}_X\omega = d(X \rfloor \omega) + X \rfloor d\omega;$$

(c) $* : \Lambda^k(\mathbb{G}) \longrightarrow \Lambda^{n-k}(\mathbb{G})$ denotes the *Hodge star operator* (see [54], pp 142-143, or [64]).

Now we introduce the canonical Riemannian volume form on hypersurfaces, [64]. To this end, let $S \subset \mathbb{G}$ be a smooth immersed hypersurface –without boundary– and let N denote a smooth unit normal vector along S. In the sequel we shall denote by NS the (Riemannian) *normal bundle over* S. With respect to the orientation of S determined by N, the induced Riemannian volume form of S is canonically defined by

$$\sigma^{n-1} := (N \rfloor \Omega^n)|_S, \tag{69}$$

and we will often denote it by the symbol dA^{n-1}. We remind that the $n-1$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ associated with the Riemannian metric $\langle\cdot,\cdot\rangle$ on $\mathbb{G}$ and the volume form dA^{n-1} agree on smooth hypersurfaces; for a proof, see [33], or Theorem IV.1.8 of [17].

If $X \in \mathfrak{X}(\mathbb{G})$, then $(X \rfloor dV^n)|_S = \langle X, N\rangle\, dA^{n-1}|_S$ and since the Riemannian divergence operator div : $\mathfrak{X}(\mathbb{G}) \longrightarrow \mathbf{C}^{\infty}(\mathbb{G})$ satisfies $d(X \rfloor dV^n) = \mathrm{div} X\, dV^n$, by Stoke's Theorem we get the usual Riemannian divergence theorem for smooth domains contained in $\mathbb{G}$ having S as a boundary. If we assume S with boundary, an analogous construction enables us to define the $n-2$-dimensional Riemannian volume measure on ∂S. Therefore, if $\eta \in TS$ is the outward-pointing unit normal vector along ∂S, we shall set

$$\sigma^{n-2} := (\eta \rfloor \sigma^{n-1})|_{\partial S}. \tag{70}$$

Later on we shall introduce both the H-perimeter measure and a differential $n-1$-form σ_H, henceforth called *H-perimeter form*, that will be used as the "regular counterpart" of the H-perimeter measure. We remark right now that these notions coincide on "regular" hypersurfaces.

Let us consider the projection map onto the horizontal space

$$\mathcal{P}_H : T\mathbb{G} \longrightarrow H,$$

that is, the homomorphism given by $\mathcal{P}_H(X) := \sum_{i=1}^{m_1}\langle X, X_i\rangle X_i$ with respect to the coordinates of the frame $(X_1, ..., X_n)$. Hereafter, unless otherwise mentioned, we assume that $S \subset \mathbb{G}$ **is a smooth immersed hypersurface with a smooth unit normal vector** N**. We also assume that** S **is transversal to the horizontal distribution** H. In this case we say that S is *non-characteristic* and we set $S \pitchfork H$. On the contrary, if S have characteristic points, we shall denote by $C(S)$ the *characteristic set* of S, defined by

$$C(S) := \Big\{x \in S : dim\, H_x = dim(H_x \cap T_x S)\Big\}.$$

The transversality condition can be formulated by means of the projection map $\mathcal{P}_H$ of the unit normal N. Indeed, we easily get that

$$S \pitchfork H \Longleftrightarrow \mathcal{P}_H(N) \neq 0 \Longleftrightarrow \exists\, X \in H : \langle X_x, N_x\rangle \neq 0\ \forall\, x \in S.$$

Because our assumption that S is a smooth non-characteristic hypersurface, we may normalize the horizontal projection of the unit normal N and so we obtain a smooth unit section of H, called *horizontal unit normal along* S. More precisely, we set

$$\nu_{_H} := \frac{\mathcal{P}_H(N)}{|\mathcal{P}_H(N)|_H}. \tag{71}$$

We stress that we may equip S with the smooth bundle structure, denoted by HS, which is induced by H. We will refer to HS as the *horizontal bundle over* S, and it is just defined by

$$HS := \Big\{X \in H : \pi_{_H}(X) \in S\Big\}. \tag{72}$$

Note that $\nu_{_H}$ is a smooth unit section of HS. Together with HS, we may define the *vertical bundle over* S, denoted by VS, as follows. For $x \in S$ we define V_x as the $n-1$-dimensional vector subspace of $T_x\mathbb{G}$ such that $T_x\mathbb{G} = V_x \oplus \mathrm{span}\{(\nu_{_H})_x\}$, i.e. $V_x := (\nu_{_H})_x^{\perp}$, and so we may canonically construct VS as the smooth vector bundle over the base space S given by

$$VS := \coprod_{x\in S} V_x. \tag{73}$$

The bundle projection maps of HS and VS will be denoted, respectively, by $\pi_{_{HS}}$ and $\pi_{_{VS}}$. Finally, we shall define two bundle structures over S which are proper subbundles of HS. They are, respectively, the *horizontal tangent bundle* HTS and the *horizontal normal bundle* $\nu_{_H}S$ and they are associated with the decomposition of the horizontal space at $x \in S$ given by

$$H_xS = \mathrm{span}\{(\nu_{_H})_x\} \oplus \{(\nu_{_H})_x^{\perp} \cap H_x\} \qquad \forall\, x \in S.$$

We set $HT_xS := (\nu_{_H})_x^{\perp} \cap H_x$ $(x \in S)$. Note that $HT_xS = \mathcal{P}_H(V_x)$ $\forall\, x \in S$. Thus we define

$$HTS := \coprod_{x\in S} HT_xS, \qquad \nu_{_H}S := \coprod_{x\in S} (\nu_{_H})_x. \tag{74}$$

Remark 5.1. *For $x \in S$, let $X \in \mathfrak{X}(\mathbb{G})$ be such that $X_x \in T_xS \cap H_x$. We have $\langle X_x, N_x \rangle = 0$. Moreover we get that $(\mathcal{P}_{V_i}(X))_x = 0$ for any $i = 2, ..., \mathrm{k}$, where $\mathcal{P}_{V_i} : T\mathbb{G} \longrightarrow V_i$ denotes the projection map onto V_i that is the natural subbundle of $T\mathbb{G}$ associated with the i-th layer V_i of the stratification of $\mathfrak{g}$. But this implies that $X_x = (\mathcal{P}_H(X))_x$ and so*

$$\langle X_x, N_x \rangle = \langle X_x, (\mathcal{P}_H(N))_x \rangle = \langle X_x, (\nu_{_H})_x \rangle_H = 0.$$

Furthermore, it follows that $HT_xS = T_xS \cap H_x$.

The definition below allows us to regard the horizontal perimeter measure on non-characteristic hypersurfaces as a smooth (non-degenerate) differential $n-1$-form.

Definition 5.2. *[H-perimeter form $\sigma_{_H}$] Let $S \subset \mathbb{G}$ be a smooth, non-characteristic hypersurface with unit horizontal normal $\nu_{_H}$. Then the H***-perimeter form** *$\sigma_{_H}$ on S is the differential $n-1$-form on S given by contraction with $\nu_{_H}$ of the volume form Ω^n, i.e.*

$$\sigma_{_H}|_S := (\nu_{_H} \rfloor \Omega^n)|_S. \tag{75}$$

Remark 5.3. *By the previous Definition 5.2 we get*

$$\begin{aligned} \sigma_{_H}|_S &= \sum_{i=1}^{m_1} (\nu_{_H})_i \, (X_i \rfloor \Omega^n)|_S \\ &= \sum_{i=1}^{m_1} (\nu_{_H})_i * \omega_i \Big|_S \\ &= \sum_{i=1}^{m_1} (-1)^{m_1+1} (\nu_{_H})_i \, \omega_1 \wedge ... \wedge \widehat{\omega}_i \wedge ... \wedge \omega_n \Big|_S, \end{aligned}$$

where $(\nu_{_H})_i := \langle \nu_{_H}, X_i \rangle_H$ $(i = 1, ..., m_1)$. Note also that

$$\sigma_{_H}|_S = |\mathcal{P}_H(N)|_H \cdot \sigma^{n-1}|_S.$$

We also remind that if U is an open subset of $\mathbb{G}$ and E is a set of finite H-perimeter in U with $\mathbf{C}^1$-smooth boundary, then

$$|\partial E|_H(U) = \int_{\partial E \cap U} |\mathcal{P}_H(N)|_H \, dA^{n-1}, \tag{76}$$

where N denotes the outward-pointing unit normal vector along ∂E. Moreover we have $\nu_{_E} = \nu_{_H}$; see Proposition 1.34.

We state in the next Remark 5.4 some results about the representation of the H-perimeter measure and about characteristic points on regular submanifolds which can be found in [67, 68]; see also [41, 42, 43] and [45].

Remark 5.4. *Let U be an open subset of a* k*-step Carnot group and let $\Sigma \subset U$ be a $\mathbf{C}^1$ submanifold of codimension h. Then, the intrinsec $Q-h$-dimensional Hausdorff measure $\mathcal{H}_c^{Q-h}$ of the characteristic set of Σ is 0, i.e.*

$$\mathcal{H}_c^{Q-h}(C(\Sigma)) = 0.$$

If Σ is a $\mathbf{C}^1$-smooth hypersurface, then the Hausdorff dimension of Σ with respect to the cc-distance d_c is $Q-1$, i.e.

$$dim_{\mathcal{H}_c}(\Sigma) = Q - 1;$$

see [7], [41, 42, 43], [49], [67, 68], [81]. Furthermore, [67], if $\mathbb{G}$ is 2*-step and $\Sigma \subset \mathbb{G}$ is a $\mathbf{C}^{1,1}$-smooth hypersurface, then*

$$dim_{\mathcal{H}_c}(C(S)) \leq Q - 2.$$

Now let us suppose that $h = 1$, i.e. $\Sigma \subset U$ is a $\mathbf{C}^1$ submanifold of codimension 1*. Then we have*

$$|\mathcal{P}_H(N)|_H \cdot \sigma^{n-1} \llcorner \Sigma = k_{_{Q-1}} \, \mathcal{S}_c^{Q-1} \llcorner \Sigma,$$

where $k_{_{Q-1}}$ denotes the so-called **metric factor** *of d_c (see Definition 2.17 of [67]) that is a function depending on both the structure of the Lie algebra $\mathfrak{g}$ and on* *the direction of the unit H-normal ν_H along Σ* [4]. *In some particular cases, the metric factor k_{Q-1} reduces to an explicitly computable constant, as in the case of the Heisenberg groups $\mathbb{H}^k$ $(k \geq 1)$. Moreover, let $E \subset \mathbb{G}$ be such that ∂E is a $\mathbf{C}^1$ hypersurface with outward-pointing unit normal vector denoted by N. Then*

$$|\partial E|_H = k_{Q-1}\, \mathcal{S}_c^{Q-1} \llcorner \partial E = |\mathcal{P}_H(N)|_H \cdot \sigma^{n-1} \llcorner \partial E.$$

Finally, if E is a $\mathbf{C}^1$ closed subset of a k*-step Carnot group $\mathbb{G}$, the following version of the Divergence Theorem holds true:*

$$\begin{aligned}\int_E \mathrm{div}_H \psi \, dV^n &= -\int_{\partial E} \langle \psi, \nu_E \rangle_H \, |\mathcal{P}_H(N)|_H \, dA^{n-1} \\ &= -\int_{\partial E} \langle \psi, \nu_E \rangle_H \, k_{Q-1}\, \mathcal{S}_c^{Q-1} \qquad \forall\, \psi \in \mathbf{C}_0^1(\mathbb{G}, H).\end{aligned}$$

5.2 Geometry of 2-step Carnot groups

In this section we are mainly concerned with the study of 2-step Carnot groups. There are many reasons for this and one, for instance, is that many proofs can be given in a simpler way, by means of more explicit computations. Moreover, an important reason for the study of geometric properties of hypersurfaces in 2-step Carnot groups is that a remarkable rectifiability theory for sets of finite H-perimeter holds in this setting; see [42, 43]. Here below we shall simplify some of our previous notation.

Let $\mathbb{G}$ denote a 2-step Carnot group of dimension n and let $\mathfrak{g}$ be its Lie algebra. In this case the stratification of $\mathfrak{g}$ has only two layers: the first one, denoted by H, is the horizontal space of $\mathfrak{g}$, while the second one is the center of $\mathfrak{g}$, denoted by Z. We have $\mathfrak{g} := H \oplus Z$ and we put $m := \dim H$, so that $n - m = \dim Z$. According

[4] Let $B_c(e, 1)$ denote the open d_c-ball centered at the identity of $\mathbb{G}$ and of radius equal to 1 and let $\mathcal{I}_e(\nu_E)$ denote the maximal proper subgroup of $\mathbb{G}$ that is orthogonal to ν_E. Then the metric factor k_{Q-1} of d_c is given by

$$k_{Q-1} := \frac{1}{\kappa_{Q-1}} \mathcal{H}^{n-1}(\mathcal{I}_e(\nu_E) \cap B_c(e, 1)),$$

where κ_h denotes the h-dimensional Lebesgue measure of the Euclidean unit ball in $\mathbb{R}^h$.

to Remark 1.1 we denote by Z_x ($x \in \mathbb{G}$) the image in $T_x\mathbb{G}$ of Z through $(L_x)_*$. We shall denote by Z the smooth subbundle of $T\mathbb{G}$ defined by $Z := \coprod_{x\in\mathbb{G}} Z_x$ and by π_Z its bundle projection map.

Notation 5.5. *From now on we shall adopt the following convention on the range of indices in the case of* 2*-step Carnot groups:*

$$1 \leq I, J, H, ... \leq n = dim\,\mathfrak{g}; \quad 1 \leq i, j, h, ... \leq m = dim\,H; \quad m{+}1 \leq \alpha, \beta, \gamma, ... \leq n.$$

Moreover we shall set $I_1 := \{1, ..., m\}$ *and* $I_2 := \{m+1, ..., n\}$.

In the case of 2-step nilpotent Lie groups, one has many simplifications with respect to the general case. For example, the Campbell-Hausdorff formula takes the form

$$exp\,(X) \bullet exp\,(Y) = exp\left(X + Y + \frac{1}{2}\,[X, Y]\right) \qquad \forall\, X, Y \in \mathfrak{g},$$

and, for instance, one has $d\,exp_X(Y) = d\big(L_{exp_X}\big)_e\big(Y + \frac{1}{2}\,[Y, X]\big)$ for all $X, Y \in \mathfrak{g}$.

We now introduce a family of skew-symmetric linear transformations of H which "capture" all the geometry of 2-step nilpotent Lie groups equipped with a left invariant Riemannian metric; see [58], [31]. We emphasize that the metric adopted in the sequel is that already defined by formula (3) of Section 1.1. So let $Z \in Z$ and define $\mathtt{j}(Z) : H \longrightarrow H$ by

$$\mathtt{j}(Z)(X) := (\mathrm{ad}\,X)^* Z \qquad \forall\; X \in H,$$

where $(\mathrm{ad}\,X)^*$ is the adjoint linear transformation of $\mathrm{ad}\,X$. Equivalently we have

$$\langle \mathtt{j}(Z)X, Y\rangle = \langle [X, Y], Z\rangle = -\sum_{\alpha\in I_2} z_\alpha \langle \mathcal{C}^\alpha X, Y\rangle, \quad \forall\, X, Y \in H,\, \forall\, Z = \sum_{\alpha\in I_2} z_\alpha X_\alpha \in Z.$$

By means of these maps, we shall explicitly write the Levi-Civita connection ∇ for $\mathbb{G}$ and we will specify both the expression of the Riemannian curvature tensor of $\mathbb{G}$ and the values of its sectional curvatures. We have (see [31] pp. 620):

$$\begin{cases} \nabla_X Y = \frac{1}{2}[X, Y] & \forall\; X,\, Y \in H; \\ \\ \nabla_X Z = \nabla_Z X = -\frac{1}{2}\mathtt{j}(Z)X & \forall\; X \in H,\, Z \in Z; \\ \\ \nabla_Z \widetilde{Z} = 0 & \forall\; Z,\, \widetilde{Z} \in Z. \end{cases} \tag{77}$$

$$
\begin{cases}
a)\ \mathsf{R}(X,Y)\widetilde{X} = \frac{1}{2}\mathtt{j}([X,Y])\widetilde{X} - \frac{1}{4}\mathtt{j}([Y,\widetilde{X}])X + \frac{1}{4}\mathtt{j}([X,\widetilde{X}])Y & \forall\ X,\,Y,\,\widetilde{X} \in H;\\[2ex]
a')\ \mathsf{R}(X,Y)X = \frac{3}{4}\mathtt{j}([X,Y])X & \forall\ X,\,Y \in H;\\[2ex]
b)\ \mathsf{R}(X,Z)Y = -\frac{1}{4}[X,\mathtt{j}(Z)Y] & \forall\ X,\,Y \in H\\
 & \forall\ Z \in Z;\\
b')\ \mathsf{R}(X,Y)Z = -\frac{1}{4}[X,\mathtt{j}(Z)Y] + \frac{1}{4}[Y,\mathtt{j}(Z)X] & \forall\ X,\,Y \in H\\
 & \forall\ Z \in Z;\\
c)\ \mathsf{R}(X,Z)\widetilde{Z} = -\frac{1}{4}\{\mathtt{j}(Z)\circ\mathtt{j}(\widetilde{Z})\,X\} & \forall\ X \in H\\
 & \forall\ Z,\,\widetilde{Z};\\
c')\ \mathsf{R}(Z,\widetilde{Z})X = -\frac{1}{4}\{\mathtt{j}(\widetilde{Z})\circ\mathtt{j}(Z)\,X\} + -\frac{1}{4}\{\mathtt{j}(Z)\circ\mathtt{j}(\widetilde{Z})\,X\} & \forall\ X \in H\\
 & \forall\ Z,\,\widetilde{Z};\\
d)\ \mathsf{R}(Z_1,Z_2)Z_3 = 0 & \forall\ Z_1,Z_2,Z_3 \in Z.
\end{cases}
\tag{78}
$$

From (ii) we obtain a complete description of the curvature tensor but we may also compute the *Ricci tensor* of $\mathbb{G}$, defined, for $X, Y \in \mathfrak{g}$, by

$$
\mathsf{Ric}(X,Y) := \mathrm{Trace}(Z \longrightarrow \mathsf{R}(Z,X)Y)\ Z \in \mathfrak{g}.
$$

From (ii) we get the expression of the sectional curvature of two orthonormal vectors X, $Y \in \mathfrak{g}$, i.e. $\mathsf{K}(X,Y) = \langle \mathsf{R}(X,Y)Y, X\rangle$:

$$
\begin{cases}
\mathsf{K}(X,Y) = -\frac{3}{4}|[X,Y]|^2 & (X,\,Y \in H);\\[2ex]
\mathsf{K}(X,Z) = -\frac{1}{4}|\mathtt{j}(Z)X|^2 & (X \in H,\,Y \in Z);\\[2ex]
\mathsf{K}(Z,\widetilde{Z}) = 0 & (Z,\,\widetilde{Z} \in Z).
\end{cases}
\tag{79}
$$

In the 2-step case we may easily compute the 1-forms of the coframe $(\omega_1, ..., \omega_n)$ for $\mathbb{G}$. According to Remark 1.22, if $x = exp\,(X)$ $(X = \sum_{K=1}^{n} x_K\,\mathrm{e}_K)$ we get

$$
(\omega_I)_x = \sum_{H=1}^{n} B_{IH}(x)dx_H
$$

where

$$\begin{aligned}
B_{IH}(x) &= dx_I\Big(\frac{1-e^{-\mathrm{ad}X}}{\mathrm{ad}X}(\mathrm{e}_H)\Big) = dx_I\Big(\mathrm{e}_H + \frac{1}{2}[\mathrm{e}_H, X]\Big)\\
&= \delta_I^H + \frac{1}{2}\sum_{K=1}^{n} c_{HK}^I x_K \qquad \text{since } c_{HK}^I = \langle [\mathrm{e}_H, \mathrm{e}_K], \mathrm{e}_I\rangle.
\end{aligned}$$

Remark 5.6. *Putting $B(x) := [B_{IH}(x)]_{\{I,H=1,\dots,n\}}$ $(x \in \mathbb{G})$ by an easy calculation one gets $B(x) = A(-x)$ where $A(x)$ is the matrix representing the pushforward associated with L_x; see Section 1.1. To see this, let $x = \exp(X)$ and $y = \exp(Y)$, where $X = \sum_{H=1}^{n} x_H\, \mathrm{e}_H$, $Y = \sum_{H=1}^{n} y_H\, \mathrm{e}_H$; then*

$$\begin{aligned}
A_{IH}(-x) &= \frac{\partial \mathcal{P}_I(-x, 0)}{\partial y_H}\\
&= \frac{\partial}{\partial y_H}\Big(- x_I + y_I - \frac{1}{2}\sum_{R,S=1}^{n} c_{RS}^I x_R\, y_S\Big)\Big|_{y=e}\\
&= \delta_I^H - \frac{1}{2}\sum_{R,S=1}^{n} c_{RS}^I\, \delta_S^H\, x_R = \delta_I^H - \frac{1}{2}\sum_{R=1}^{n} c_{RH}^I\, x_R\\
&= B_{IH}(x),
\end{aligned}$$

where the last equality follows from (i) of (16) of Section 1.1.

By the stratification hypothesis on the Lie algebra (see Remark 1.22) we obtain also that $B_{ij} = \delta_i^j$ for any $i, j \in I_1$ and that $B_{\alpha\beta} = \delta_\alpha^\beta$ for any $\alpha, \beta \in I_2$. More generally, note that

$$c_{HK}^I \neq 0 \Longrightarrow H,\, K \in I_1,\ I \in I_2.$$

We now remind that the *connections* 1*-forms* $\{\omega_{ij}\}_{i,\,j=1,\dots,n}$ (see [16], [59], [88]) of the coframe $(\omega_1, \dots, \omega_n)$ for $\mathbb{G}$ are defined by

$$\omega_{ij}(X) := \langle \nabla_X X_i, X_j\rangle \qquad \forall\ X \in \mathfrak{X}(\mathbb{G}).$$

They are n^2 skew-symmetric 1-forms, i.e. $\omega_{ij} = -\omega_{j\,i}$ $(i,\, j = 1, \dots, n)$, satisfying the equations

$$d\omega_j = \sum_{k=1}^{n} \omega_k \wedge \omega_{kj} \quad (j = 1, \dots, n).$$

See also Proposition 1.21 of Section 1.1 for a more explicit result. By using this definition and either the expression of ∇ for 2-step Carnot groups (see item (i)), or both Proposition 1.11 and the system of equations (11) of Section 1.1, we obtain the system $\{\omega_{JK}\}_{J,\,K=1,\dots,n}$ of connection 1-forms for the coframe $(\omega_1, ..., \omega_n)$ of $\mathbb{G}$. More precisely, since by definition we have

$$\omega_{JK}(X) := \langle \nabla_X X_J, X_K \rangle \qquad \forall\ X \in \mathfrak{X}(\mathbb{G}) \ \ (J, K = 1, ..., n), \tag{80}$$

we get by an easy computation the following result:

$$\begin{cases} \omega_{jk}(X_i) = 0 & \forall\ i, j, k \in I_1; \\ \\ \omega_{jk}(X_\alpha) = -\frac{1}{2}\, c^\alpha_{jk} & \forall\ j, k \in I_1 \ \ \forall \alpha \in I_2; \\ \\ \omega_{j\alpha}(X_i) = \frac{1}{2}\, c^\alpha_{ij} & \forall\ i, j \in I_1 \ \ \forall \alpha \in I_2; \\ \\ \omega_{j\alpha}(X_\beta) = 0 & \forall\ j \in I_1 \ \ \forall \alpha, \beta \in I_2; \\ \\ \omega_{\alpha\beta}(X_I) = 0 & \forall\ \alpha, \beta \in I_2 \ \ \forall\ I \in I_1 \cup I_2. \end{cases} \tag{81}$$

We end this section by computing the covariant derivative of two smooth left invariant sections of $T\mathbb{G}$ with respect to the coordinates of the frame $(X_1, ..., X_n)$, i.e. if $X = \sum_I x_I X_I$ and $Y = \sum_J y_J X_J$, we shall compute $\nabla_X Y$ with respect to the vector basis $X_1, \dots, X_n$. We have

$$\begin{aligned} \nabla_X Y &= \sum_I x_I \nabla_{X_I}(y_J X_J) \\ &= \sum_{I,\,J} \Big\{ x_I\,(X_I y_J)\, X_J + x_I\, y_J \nabla_{X_I} X_J \Big\} \\ &= \sum_J (X y_J) X_J + \sum_{I,\,J,\,K} x_I\, y_J\, \omega_{JK}(X_I) X_K \end{aligned}$$

$$= \sum_J (X y_J) X_J + \sum_{I,\,j,\,k} x_I\, y_j\, \omega_{jk}(X_I) X_k + \sum_{I,\,j,\,\alpha} x_I\, y_j\, \omega_{j\alpha}(X_I) X_\alpha + \sum_{I,\,\alpha,\,k} x_I\, y_\alpha\, \omega_{\alpha k}(X_I) X_k.$$

Here above, the first summation is nothing but the image of X under the Jacobian matrix of Y with respect to the basis $(X_1, \dots, X_n)$, i.e.

$$\mathcal{J}_Y X = \sum_J (Xy_J) X_J = \sum_{j,\beta} \Big((Xy_j)X_j + (Xy_\beta)X_\beta \Big).$$

Now by (81) and the fact that the connection 1-forms are skew-symmetric, we get

$$\begin{aligned}
\nabla_X Y &= \mathcal{J}_Y X + \sum_{\beta,j,k} x_\beta\, y_j\, \omega_{jk}(X_\beta)X_k + \sum_{i,j,\alpha} x_i\, y_j\, \omega_{j\alpha}(X_i)X_\alpha + \sum_{i,\alpha,k} x_i\, y_\alpha\, \omega_{\alpha k}(X_i)X_k \\
&= \mathcal{J}_Y X - \frac{1}{2}\sum_{\beta,j,k} c^\beta_{jk} x_\beta\, y_j\, X_k + \frac{1}{2}\sum_{i,j,\alpha} c^\alpha_{ij} x_i\, y_j X_\alpha - \frac{1}{2}\sum_{i,\alpha,k} c^\alpha_{ik} x_i\, y_\alpha X_k \\
&= \mathcal{J}_Y X + \frac{1}{2}\sum_{j,k,\alpha} c^\alpha_{jk}\Big\{ x_j\, y_k\, X_\alpha - \big(x_\alpha\, y_j + x_j\, y_\alpha\big)X_k \Big\}. \qquad (82)
\end{aligned}$$

Definition 5.7. *We set*

$$\mathsf{C}^\alpha := [c^\alpha_{ik}]_{\{i,k\in I_1\}} \in \mathcal{M}_{m,m}(\mathbb{R}), \qquad \mathsf{C}_k := [c^\alpha_{ik}]_{\{i\in I_1,\,\alpha\in I_2\}} \in \mathcal{M}_{m,n-m}(\mathbb{R}).$$

Moreover we define another family of matrices $\mathsf{C}^k \in \mathcal{M}_{n,n}(\mathbb{R})$ $(k \in I_1)$ *by*

$$\mathsf{C}^k := \begin{bmatrix} 0_{m,m} & \mathsf{C}_k \\ & \\ \mathsf{C}_k^T & 0_{n-m,n-m} \end{bmatrix},$$

where C_k^T *denotes the matrix adjoint of* C_k. *We shall also denote by* $\mathcal{C}^\alpha : H \longrightarrow H$ *and, respectively, by* $\mathcal{C}^k : \mathfrak{g} \longrightarrow \mathfrak{g}$, *the linear operators associated with* C^α *and* C^k.

Warning. *Sometimes in some of the following computations we will use, with a slight abuse of notation, the symbol* $\langle\cdot,\cdot\rangle_H$ *to denote the inner product in* $\mathbb{R}^m \cong H$.

Note that $\mathcal{C}^\alpha, \alpha \in I_2$, are skew-symmetric linear operators while $\mathcal{C}^k, k \in I_1$, turn out to be symmetric and such that $\mathrm{Im}(\mathcal{C}^k) \subseteq \mathrm{Ker}(\mathcal{P}_H)$ $(k \in I_1)$. We may state the following:

Lemma 5.8. *Let* $(\mathbb{G}, \langle\cdot,\cdot\rangle)$ *be a* 2*-step Carnot group endowed with the Levi-Civita connection* ∇. *Let* $X, Y \in \mathfrak{X}(\mathbb{G})$ *be two left invariant vector fields. Then*

$$\nabla_X Y = \mathcal{J}_Y X - \frac{1}{2}\left\{ \sum_{k\in I_1} \langle \mathcal{C}^k X, Y\rangle\, X_k + \sum_{\alpha\in I_2} \langle (\mathcal{C}^\alpha \circ \mathcal{P}_H)(X), \mathcal{P}_H(Y)\rangle_H\, X_\alpha \right\}. \quad (83)$$

Proof. The equation (82) shows that, using the coordinates of the frame $(X_1, ..., X_n)$, we have

$$\begin{aligned}\nabla_X Y = \mathcal{J}_Y X \quad &- \quad \frac{1}{2}\sum_{k\in I_1} \langle \mathsf{C}_k (x_{m+1}, ..., x_n)^T, (y_1, ..., y_m)^T \rangle \, X_k \\ &- \quad \frac{1}{2}\sum_{k\in I_1} \langle \mathsf{C}_k (y_{m+1}, ..., y_n)^T, (x_1, ..., x_m)^T \rangle X_k \\ &+ \quad \frac{1}{2}\sum_{\alpha\in I_2} \langle \mathsf{C}^\alpha (x_1, ..., x_m)^T, (y_1, ..., y_m)^T \rangle_H \, X_\alpha,\end{aligned}$$

so the thesis follows from Definition 5.7. □

6 Regular non-characteristic hypersurfaces in 2-step Carnot groups

6.1 H-adapted moving frames and structure equations

In this section we shall introduce a moving frame in $\mathbb{G}$ adapted to a regular non-characteristic hypersurface $S \subset \mathbb{G}$; see Definition 6.1 below. This frame will enable us to get explicit computations about the local geometry of S and will be mainly used to understand the meaning of some variational formulas that we will prove in the sequel. Here we just remark that the choice of this frame is motivated by the fact that we cannot use the usual Riemannian approach (see [63], [16], [87], [88]) in stating variational formulas concerning the H-perimeter form $\sigma_{_H}$ as, for instance, divergence type theorems on hypersurfaces and the first and the 2nd variation of $\sigma_{_H}$. Indeed, it should be noted that the tangent space to a smooth hypersurface does not play the same role as in the Riemannian setting, so that we shall replace it by the vertical bundle VS over S defined in Section 1.2. Al least for 2-step Carnot groups this seems perhaps motivated by the *Blow-up* method. Indeed let $x \in \partial^* E$, where $\partial^* E$ denotes the *Reduced Boundary* of a (locally) finite H-perimeter set E. Then it can be proved that the (local) tangent structure at $x \in \partial^* E$ is the vertical hyperplane orthogonal to $\nu_{_E}(x)$ which turns out to be a maximal (proper) subgroup of $\mathbb{G}$; see [42, 43], [67, 68].

Let $\mathbb{G}$ be a 2-step Carnot group and $S \subset \mathbb{G}$ be a smooth immersed non-characteristic hypersurface. Since the following discussion is local we may assume also that S is imbedded. Let VS, HS, HTS and $\nu_{_H}S$ denote the vector bundles on the base space S, defined in Section 1.2. Moreover we shall denote by ZS the smooth vector bundle over the base space S whose fibre at any $x \in S$ is given by $Z_xS = (L_x)_*Z$. We have already seen that $T_x\mathbb{G} = \mathrm{span}\{(\nu_{_H})_x\} \oplus V_x$ and by Remark 5.1, for any $x \in S$, we get that $H_xS = \mathrm{span}\{(\nu_{_H})_x\} \oplus HT_xS$. Note also that $V_xS = HT_xS \oplus Z_xS$. We may start by giving the following:

Definition 6.1. *Let $\mathbb{G}$ be a 2-step Carnot group and $S \subset \mathbb{G}$ be a smooth immersed* *non-characteristic hypersurface. Fix an open set $U \subset \mathbb{G}$ such that $U \cap S \neq \emptyset$. A* H**-adapted moving frame for** S *on* U *is a smooth orthonormal frame* $(\tau_1, ..., \tau_n)$ *for* U *such that:*

(i) $(\tau_1)_x := (\nu_{_H})_x$ *and* $H_xTS = \mathrm{span}\{(\tau_2)_x, ..., (\tau_m)_x\}$ *for* $x \in \mathcal{U} = U \cap S$*;*

(ii) $\tau_\alpha := X_\alpha \quad (\alpha \in I_2 = \{m+1, ..., n\})$.

Note that

$$H_xS = \mathrm{span}\{(\tau_1)_x, ..., (\tau_m)_x\} = \mathrm{span}\{(X_1)_x, ..., (X_m)_x\} \qquad \forall\, x \in U \cap S,$$

and that $Z_xS = \mathrm{span}\{(\tau_{m+1})_x, ..., (\tau_n)_x\} = \mathrm{span}\{(X_{m+1})_x, ..., (X_n)_x\}$ for every $x \in U \cap S$. Thus, $\tau_1, ..., \tau_m$ are smooth left invariant horizontal sections that turn out to be homogeneous of degree 1 with respect to Carnot dilations

$$\{\delta_\lambda(x_1, \dots, x_n) = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n)\}_{t>0},$$

while, clearly, $\tau_{m+1}, ..., \tau_n$ are homogeneous of degree 2. The *H-adapted dual coframe* $(\phi_1, ..., \phi_n)$ *for* S in U is then (uniquely) determined by requiring that

$$\phi_I(\tau_J) = \delta_I^J \ (I, J = 1, ..., n).$$

By construction, the 1-forms $\phi_1, ..., \phi_m$, which are dual of the horizontal sections $\tau_1, ..., \tau_m$ of *HS*, are homogeneous of degree 1 with respect to Carnot dilations, i.e.

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n)^* \phi_i = t\, \phi_i \ (i \in I_1),$$

while the 1-forms $\phi_{m+1}, ..., \phi_n$ which are dual of the sections $\tau_{m+1}, ..., \tau_n$ of *ZS* are homogeneous of degree 2, i.e.

$$\delta_\lambda(x_1, \dots, x_n) = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_n} x_n)^* \phi_\alpha = t^2 \phi_\alpha \ (\alpha \in I_2).$$

Remark 6.2. *Since locally the Riemannian volume n-form Ω^n can be written in terms of the wedge product of the 1-forms $\phi_1, ..., \phi_n$, i.e.*

$$\Omega^n = \phi_1 \wedge ... \wedge \phi_n,$$

by setting $\mathcal{U} := U \cap S$, *from Definition 5.2 we get*

$$\begin{aligned}\sigma_{_H} \llcorner \mathcal{U} &:= (\nu_{_H} \rfloor \Omega^n)|_{\mathcal{U}} \\ &= (\tau_1 \rfloor \phi_1 \wedge ... \wedge \phi_n)|_{\mathcal{U}} \\ &= (\phi_2 \wedge ... \wedge \phi_n)|_{\mathcal{U}}.\end{aligned}$$

The following lemma will be useful in many computations of the next sections.

Lemma 6.3. *[Vanishing Lemma] Assume that* $S \subset \mathbb{G}$ *is a smooth non-characteristic hypersurface. Then, for any* $i \in I_1 \setminus \{1\} = \{2, ..., m\}$ *we have*

$$(\tau_i \rfloor \Omega^n)|_S = 0.$$

Proof. Obvious, since

$$(\tau_i \rfloor \Omega^n)|_S = (\tau_i \rfloor dV^n)|_S = \langle \tau_i, N \rangle \, dA^{n-1}|_S$$

and $\langle \tau_i, N \rangle = 0$. $\square$

From now on, we shall denote by ϕ_{IJ} $(I, J = 1, ..., n)$ the *connection* 1*-forms* of the H-adapted coframe $(\phi_1, ..., \phi_n)$, which are defined by

$$\phi_{IJ}(X) := \langle \nabla_X \tau_I, \tau_J \rangle \qquad \forall\; X \in \mathfrak{X}(\mathbb{G}),$$

where ∇ denotes the Levi-Civita connection of $\mathbb{G}$. $\{\phi_{IJ}\}_{\{I,\, J=1,...n\}}$ is a family of n^2 skew-symmetric 1-forms. Furthermore we denote by Φ_{JK} $(J,\, K = 1, ..., n)$ the *curvature* 2*-forms* of $(\phi_1, ..., \phi_n)$ (see [16], [59], [88]) defined by

$$\Phi_{JK}(X, Y) := \phi_K(\mathsf{R}(X, Y)\, \tau_J) = \langle \mathsf{R}(X, Y)\, \tau_J, \tau_K \rangle$$

for all $X,\, Y \in \mathfrak{X}(\mathbb{G})$ $(J,\, K = 1, ...n)$. Now we may write the (Riemannian) *structure equations* of the H-adapted coframe $(\phi_1, ..., \phi_n)$:

$$d\phi_J = \sum_{K=1}^{n} \phi_K \wedge \phi_{KJ} \quad (J = 1, ..., n) \qquad \text{(1st structure equation)}; \tag{84}$$

$$d\phi_{JK} = \sum_{L=1}^{n} \phi_{JL} \wedge \phi_{LK} - \Phi_{JK} \quad (J,\, K = 1, ..., n) \qquad \text{(2nd structure equation)}. \tag{85}$$

The H-adapted frame $(\tau_1, ..., \tau_n)$ to S is a smooth frame for $\mathbb{G}$ which is defined locally in a suitable open neighborhood of S and in the sequel we shall make use of it instead of the frame $(X_1, ..., X_n)$, and consequently we shall replace the coframe $(\omega_1, ..., \omega_n)$ with that H-adapted $(\phi_1, ..., \phi_n)$. Our point of view will be that of interpreting the results of the next sections in terms of that geometric invariants encapsulated by the H-adapted frame to S.

Below, we shall compute in detail the expression of the connection 1-forms ϕ_{IJ} $(I, J = 1, ..., n)$ of the coframe $(\phi_1, ..., \phi_n)$ by means of Lemma 5.8. To this aim, let $x \in \mathcal{U} := U \cap S$, and consider the orthogonal $n \times n$-matrix $\mathsf{O}(x) \in \mathbf{O}(n)$ given by

$$\mathsf{O}(x) = \begin{bmatrix} \mathsf{O}_H(x) & 0_{m,n-m} \\ 0_{n-m,m} & I_{n-m,n-m} \end{bmatrix},$$

where $x \in \mathcal{U}$ and $\mathsf{O}_H(x) \in \mathbf{O}(m)$ is the orthogonal $m \times m$-matrix given by

$$\mathsf{O}_H(x) := \Big[\langle \tau_i, X_j \rangle\Big]_{\{i,j=1,...,m\}} .$$

We set $o_{IJ}(x) := \langle (\tau_I)_x, (X_J)_x \rangle$ $(I, J = 1, ..., n)$ and denote by $o_L(x)$ the L-th column of $\mathsf{O}(x)$ representing, in the coordinates of the frame $(X_1, ..., X_n)$, the L-th vector τ_L of the H-adapted frame $(\tau_1, ..., \tau_n)$. Now if $\mathcal{O}(x) : T_x\mathbb{G} \longrightarrow T_x\mathbb{G}$ denotes the linear operator associated with $\mathsf{O}(x)$ and $\mathcal{O}_H(x) : H_x \longrightarrow H_x$ denotes the operator associated with $\mathcal{O}_H(x)$, by making use of Definition 6.1 we get that

$$\mathcal{O}(x)(X_I)_x = (\tau_I)_x \qquad (I = 1, ..., n).$$

Using Lemma 5.8 and the very definition of ϕ_{IJ}, get us

$$\begin{aligned} \phi_{IJ}(\tau_L) &= \langle \nabla_{\tau_L} \tau_I, \tau_J \rangle \\ = \langle [\mathcal{J}_{o_I}]_x\, o_L, o_J \rangle - \frac{1}{2}\Big\{ \sum_{k \in I_1} \langle \mathsf{C}^k o_L, o_I \rangle\, o_{Jk} &- \sum_{\alpha \in I_2} \Big\langle \mathsf{C}^\alpha (o_{1L}, ..., o_{mL})^T, (o_{1I}, ..., o_{mI})^T \Big\rangle_H o_{J\alpha} \Big\}, \end{aligned}$$

and by this formula we deduce the following:

Lemma 6.4. *If $x \in \mathcal{U}$, we have*

(i) $\phi_{ij}(\tau_k) = \langle [\mathcal{J}_{o_i}]_x\, o_k, o_j \rangle \quad (i,\, j,\, k \in I_1);$

(ii) $\phi_{ij}(\tau_\beta) = \langle \frac{\partial o_i}{\partial x_\beta}, o_j\rangle - \frac{1}{2}\sum_{k\in I_1}\langle \mathsf{C}^k o_\beta, o_i\rangle\, o_{jk} \quad (i,\, j \in I_1,\ \beta \in I_2);$

(iii) $\phi_{i\beta}(\tau_j) = -\frac{1}{2}\,\langle \mathsf{C}^\beta (o_{1j}, ..., o_{mj})^T, (o_{1i}, ..., o_{mi})^T\rangle_H \quad (i,\, j \in I_1,\ \beta \in I_2);$

(iv) $\phi_{i\alpha}(\tau_\beta) = 0 \quad (i \in I_1,\ \alpha,\, \beta \in I_2);$

(v) $\phi_{\alpha\beta}(\tau_L) = 0 \quad (\alpha,\, \beta \in I_2,\ L \in I_1 \cup I_2).$

Proof. We have just to use the above formula and the fact that the derivative along X_α ($\alpha \in I_2$) is nothing but the partial derivative $\frac{\partial}{\partial x_\alpha}$; see Remark 1.5 in Section 1.1. $\square$

Remark 6.5. *Let $S \subset \mathbb{G}$ be a smooth non-characteristic hypersurface and let us consider the tangent space TS of S. Suppose that $(\mathrm{t}_1..., \mathrm{t}_{n-1})$ is an orthonormal frame for S. Since S is non-characteristic, Frobenius' Theorem (see [64], [88]) implies that the Lie bracket of any two tangent vectors is still a tangent vector, i.e. if $x \in S$ and if $X, Y \in T_xS$, then there exist $\alpha_1, ..., \alpha_{n-1}$ such that*

$$[X, Y]_x = \sum_{i=1}^{n-1} \alpha_i \cdot (\mathrm{t}_i)_x.$$

Remark 6.6. *Let $x \in \mathcal{U} = U \cap S$ and consider the tangent space of S at x, T_xS. Moreover let N denote the Riemannian unit normal vector along S. Note that, with respect to the H-adapted frame $(\tau_1, ..., \tau_n)$ for U, we have that*

$$N = |\mathcal{P}_H(N)|_H\, \tau_1 + \sum_{\alpha\in I_2} n_\alpha\, \tau_\alpha = n_1\, \tau_1 + \sum_{\alpha\in I_2} n_\alpha\, \tau_\alpha.$$

By using the coordinates associated with a H-adapted frame, a vector basis for T_xS can be written by noting that:

(i) $(\tau_2)_x, ..., (\tau_m)_x$ *are $m-1$ tangent vectors to S at x; they form an orthonormal basis for the horizontal tangent space HT_xS to S (see Section 1.2);*

(ii) $\left(\tau_\alpha - \frac{n_\alpha}{n_1}\,\tau_1\right)_x$ *($\alpha \in I_2$) are $n-m$ linearly independent tangent vectors to S at x; they are orthogonal with each other vector $(\tau_j)_x$, for every $j \in I_1 \setminus \{1\}$.*

By normalizing the family of vectors introduced at item (ii), we would get a full basis for TS.

In the sequel we shall make use of the following notation:

$$\tau_\alpha^S := \tau_\alpha - \frac{n_\alpha}{n_1}\,\tau_1 \qquad \alpha \in I_2.$$

Remark 6.7. *We remind the definition of the* **horizontal tangential operators**

$$\delta_i^H \psi := X_i\psi - \langle \nabla^{H}\psi, \nu_{H}\rangle_H\, \nu_{H\,i} \qquad \forall\ \psi \in \mathbf{C}^\infty(U)\ \ (i \in I_1),$$

introduced by N. Garofalo and S. Pauls in [44]. We stress that, with our notation, one gets

$$\begin{aligned} \delta^H\psi &:= \nabla^{H}\psi - \langle \nabla^{H}\psi, \nu_{H}\rangle_H\ \nu_{H} = \nabla^{H}\psi - \langle \nabla^{H}\psi, \tau_1\rangle_H\, \tau_1 \\ &= \sum_{i=2}^{m} \tau_i(\psi)\,\tau_i \qquad \forall\ \psi \in \mathbf{C}^\infty(U). \end{aligned}$$

Remark 6.8. *There is no affine connection on the vertical bundle VS over S because, obviously, VS* **it is not a subbundle of** *TS; see Definition 1.12 of Section 1.2. Later on we shall define an HTS-restricted connection over HTS for S, in the sense of Definition 1.12 of Section 1.1; see Definition 6.12.*

However, we may give the following notion that will be useful in the sequel.

Definition 6.9. *From now on, we shall denote by $\mathcal{D}^{VS}$ the rule which assigns to each vector field $X \in \mathfrak{X}(\mathbb{G})$ the operator $\mathcal{D}^{VS}_X : \mathfrak{X}(\mathbb{G}) \longrightarrow \mathbf{C}^\infty(\mathbb{G}, VS)$ defined by*

$$\begin{aligned} \mathcal{D}^{VS}_X Y &= \mathcal{P}_{VS}(\nabla_X Y) \\ &= \nabla_X Y - \langle \nabla_X Y, \nu_{H}\rangle\, \nu_{H} \qquad X,\, Y \in \mathfrak{X}(\mathbb{G}). \end{aligned} \tag{86}$$

Remark 6.10. *We may explicitly write the rule $\mathcal{D}^{VS}$ by means of the frame $(\tau_2, ..., \tau_n)$ for VS. More precisely, for every $X \in \mathfrak{X}(\mathbb{G})$ we have*

$$\mathcal{D}^{VS}_X \tau_J = \sum_{H=2}^{n} \phi_{JH}(X)\,\tau_H\ \in VS \qquad \forall\ J = 2, ..., n. \tag{87}$$

Definition 6.11. *If $\psi \in \mathbf{C}^\infty(\mathbb{G})$ we denote by $\mathcal{D}^{VS}\psi$, the unique vector field of $\mathbf{C}^\infty(\mathbb{G}, VS)$ such that $\langle \mathcal{D}^{VS}\psi, X\rangle = d\psi(X) = X\psi$ ($\forall\, X \in VS$), and we call $\mathcal{D}^{VS}\psi$ the VS-***gradient of** *ψ. Also, we define the VS-***divergence of** *$X \in VS$, denoted by $\mathrm{div}_{VS} X$, to be the function given at each point $x \in S$ by*

$$\mathrm{div}_{VS} X := \mathrm{Trace}\Big(Y \longrightarrow \mathcal{D}^{VS}_Y X\Big)\ \ (Y \in V_x S).$$

By definition, since $(\tau_2, ..., \tau_n)$ is an orthonormal frame for VS, we get

$$\mathrm{div}_{VS} X = \sum_{J=2}^{n} \langle \mathcal{D}^{VS}_{\tau_J} X, \tau_J \rangle \qquad \forall\ X \in \mathbf{C}^\infty(\mathbb{G},\, VS). \tag{88}$$

Furthermore, for every $X \in \mathbf{C}^\infty(\mathbb{G},\, VS)$ and every $\phi,\, \psi \in \mathbf{C}^\infty(\mathbb{G})$ we get that

$$\mathcal{D}^{VS}(\phi\,\psi) = \phi\,\mathcal{D}^{VS}\psi + \psi\,\mathcal{D}^{VS}\phi \tag{89}$$

$$\mathrm{div}_{VS}(\psi\, X) = \langle \mathcal{D}^{VS}\psi, X\rangle + \psi\,\mathrm{div}_{VS} X = X\psi + \psi\,\mathrm{div}_{VS} X. \tag{90}$$

The above discussion shows, in a sense, a formal analogy with the classical *Gauss Formulas*; see [16], [59], [88]. Nevertheless, by making use of the horizontal connection, this analogy becomes more evident. Indeed, first note that $HTS \subset TS$ is a smooth subbundle of the tangent bundle of S, whose fiber at $x \in S$ is, by definition, an $m-1$-dimensional vector subspace of H_x. We shall now define an HTS-connection over S, which is naturally associated with the decomposition of the horizontal space at $x \in S$ given by

$$H_x = HT_xS \oplus \mathrm{span}\{(\nu_{_H})_x\}. \tag{91}$$

Definition 6.12. *Let $\overline{\nabla}$ denotes the Levi-Civita connection over TS induced by $\langle\cdot,\cdot\rangle$, that is, by definition, $\overline{\nabla} := (\nabla)^{TS}$. Then, we denote by ∇^{HTS} the HTS-* **restricted connection over** *HTS, in the sense of Definition 1.12, i.e.*

$$\nabla^{HTS} := \overline{\nabla}^{(HTS,HTS)}.$$

Definition 6.13. *We define the HTS-* **gradient of** *$\psi \in \mathbf{C}^\infty(S)$, denoted by the symbol $\nabla^{HTS}\psi$, to be the (unique) horizontal tangent vector field such that*

$$\langle \nabla^{HTS}\psi, X\rangle_H = d\psi(X) = X\psi\ (\forall\ X \in HTS).$$

Moreover, the HTS- **divergence of** *$X \in HTS$, denoted by $\mathrm{div}_{HTS} X$, is the function given at each point $x \in S$ by*

$$\mathrm{div}_{HTS} X := \mathrm{Trace}\Big(Y \longrightarrow \nabla^{HTS}_Y X\Big)\ (Y \in HT_xS).$$

Finally, we denote by Δ_{HTS} the HTS- **laplacian** *that is defined by*

$$\Delta_{HTS}\psi := \mathrm{div}_{HTS}(\nabla^{HTS}\psi) \qquad \forall\ \psi \in \mathbf{C}^\infty(S). \tag{92}$$

From Definition 6.11, using also (87) and (88), we easily get that

$$\mathrm{div}_{VS} X = \mathrm{div}_{HTS}(\mathcal{P}_{HTS}(X)) + \mathrm{div}_{Z}(\mathcal{P}_{Z}(X)) \qquad \forall\ X \in \mathbf{C}^{\infty}(\mathbb{G}, VS). \tag{93}$$

Remark 6.14. *Using* (91)*, if $X,\,Y \in \mathbf{C}^{\infty}(S, HTS)$, we may decompose $\nabla^{H}_{X} Y$ as follows:*

$$(\nabla^{H}_{X_x} Y)_x = (\mathcal{P}_{HTS}(\nabla^{H}_{X} Y))_x + (\mathcal{P}_{\nu_{H} S}(\nabla^{H}_{X} Y))_x \quad (x \in S).$$

It is easily verified that

$$\nabla^{HTS}_{X} = \mathcal{P}_{HTS}(\nabla^{H}_{X} Y). \tag{94}$$

Note also that

$$\begin{aligned}\nabla^{H}_{X}\tau_i &= \sum_{j\in I_1} \phi_{ij}(X)\,\tau_j = \phi_{i1}(X)\,\tau_1 + \sum_{j=2}^{m} \phi_{ij}(X)\,\tau_j \\ &= \langle \nabla^{H}_{X}\tau_i, \tau_1\rangle_H\,\tau_1 + \nabla^{HTS}_{X}\tau_i \qquad \forall\ X \in \mathbf{C}^{\infty}(S, HTS),\ i \in I_1 \setminus \{1\}.\end{aligned}$$

We therefore get that the horizontal connection ∇^{H} satisfies a generalized version of the classical Gauss Formulas. Before the statement of this result, we give the following:

Definition 6.15. *We define the* **horizontal second fundamental form of** S *to be the map $\mathtt{b}_H : HTS \times HTS \longrightarrow \nu_{H} S$ given by*

$$\mathtt{b}_H(X, Y) := \langle \nabla^{H}_{X} Y, \nu_{H}\rangle_H\,\nu_{H} \qquad \forall\ X,\,Y \in HTS.$$

The trace of $\mathtt{b}_H$, denoted by H*, is called the* **horizontal mean curvature of** S*. Finally, the quantity $\mathcal{H}^{sc}_{H} := \langle \mathrm{H}, \nu_{H}\rangle_H$ will be called the* **scalar horizontal mean curvature of** S*.*

Remark 6.16. *Clearly, we have that $\mathrm{H} \in \nu_{H} S$ and that*

$$\mathrm{H} := \sum_{j=2}^{m} \langle \nabla^{H}_{\tau_j}\tau_j, \tau_1\rangle_H\,\tau_1 = -\sum_{j=2}^{m} \langle \nabla^{H}_{\tau_j}\tau_1, \tau_j\rangle_H\,\tau_1 = -\sum_{j=2}^{m} \phi_{1j}(\tau_j)\,\tau_1.$$

By arguing as in the Riemannian case, we may prove that $\mathtt{b}_H(X, Y)$ is a $\mathbf{C}^{\infty}(S)$**-bilinear form** *in X and Y and that $\mathtt{b}_H(X, Y)$ only depends on X_x and Y_x; indeed* *to see this, we can proceed as in the proof of Proposition 3.2 of [59], Vol.II. More importantly, in general,* $\mathtt{b}_H$ **is not symmetric**. *The reason is the following. Symmetry of* $\mathtt{b}_H$ *is easily seen to be equivalent to the following condition:*

$$X, Y \in HTS \Longrightarrow \mathcal{P}_H[X, Y] \in HTS.$$

But this condition fails to be true, in general. For instance, this condition turns out to be trivially true, in the case of the Heisenberg group $\mathbb{H}^1$*, being HTS a 1-dimensional subbundle of TS, for any given non-characteristic surface* $S \subset \mathbb{H}^1$*. But, for example, the condition fails to hold, in general, for the case of* $\mathbb{H}^n$ $(n > 1)$*, as it can be easily proved by using a dimensional argument.*

Proposition 6.17. *[HTS-restricted Gauss Formulas] For each* $x \in S$ *we have*

$$\nabla^{H}_{X_x} Y = \nabla^{HTS}_{X_x} Y + \mathtt{b}_H(X_x, Y_x)$$

where $X_x \in HT_xS$ *and* Y *is any horizontal vector field which is tangent along* S*.*

Proof. Obvious by the previous discussion. □

We may give the following:

Definition 6.18. *We define the* **torsion** T^{HTS} *of the partial HTS-connection* ∇^{HTS}*, by*

$$\mathsf{T}^{HTS}(X, Y) := \nabla^{HTS}_X Y - \nabla^{HTS}_Y X - \mathcal{P}_H[X, Y] \quad (X, Y \in HTS).$$

From this definition, it follows that for every X, $Y \in HTS$ one has

$$\mathsf{T}^{HTS}(X, Y) = \mathtt{b}_H(Y, X) - \mathtt{b}_H(X, Y) = \langle \mathcal{P}_H[Y, X], \nu_H \rangle_H \nu_H. \tag{95}$$

Note also that the mapping $HS \ni X \longmapsto \nabla^H_X \nu_H$ is, in fact, the sub-Riemannian analogous of the usual Weingarten map. In the case of hypersurfaces, using the compatibility of ∇^H with the metric $\langle \cdot, \cdot \rangle_H$, we get that $(\nabla^H_X \nu_H)_p \in H_pS$. Indeed, by differentiating the identity $|\nu_H|^2_H = 1$, we obtain

$$X\langle \nu_H, \nu_H \rangle_H = 2\langle \nabla^H_X \nu_H, \nu_H \rangle_H = 0.$$

6.2 Gauss-Green type formulae on hypersurfaces

Below, we shall prove first a generalized version of the Riemannian divergence theorem on regular non-characteristic hypersurfaces and then some related Green's identities. The main proofs will be given in the next section.

Here we will just make a short comment. We remind that if $(M, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold and σ denotes the Riemannian volume form on M, then the divergence of a $\mathbf{C}^1$ vector field X on M satisfies the following well-known identities:

$$\mathcal{L}_X\,\sigma = \mathrm{Div}_\sigma(X)\,\sigma = d\,(X \rfloor \sigma) = \mathrm{div}X\,\sigma,$$

where $\mathrm{Div}_\sigma X$ denote the divergence of X with respect to σ (see [16], [88]). These relations allow to prove easily, via Stoke's Theorem, the Riemannian divergence theorem. Here, following the same approach, we state a generalized version of this theorem in the case of regular non-characteristic hypersurfaces in 2-step Carnot groups, endowed with the H-perimeter form $\sigma_{\!{}_H}$. However, we cannot expect such an extension of it to be trivial, as further terms will appear in it, due to the non-Abelian structure of the Lie algebra.

Let $\mathbb{G}$ be a 2-step Carnot group and let $S \subset \mathbb{G}$ be a smooth immersed non-characteristic hypersurface with unit normal vector along S denoted by N. Let $\mathcal{U} \subset S$ be compact and let us suppose that the boundary $\partial\mathcal{U}$ of $\mathcal{U}$ is a smooth (immersed) $n-2$-dimensional Riemannian submanifold with outward pointing unit normal η. Let $\overline{\nabla}$ denote the Levi-Civita connection induced by $\langle\cdot,\cdot\rangle$ on $\mathcal{U}$ (i.e. $\overline{\nabla} := (\nabla)^{T\mathcal{U}}$). Finally, we denote by $\overline{\nabla}\psi$ the gradient of $\psi \in \mathbf{C}^\infty(\mathcal{U})$ and by $\mathrm{div}_{T\mathcal{U}}$ the Riemannian divergence on $\mathcal{U}$. A first easy remark is contained in the following:

Proposition 6.19. *Let $\mathcal{U}$ be as above and let $X \in \mathfrak{X}(\mathcal{U})$ be a tangent vector field on $\mathcal{U}$. Then the following holds*

$$\int_{\mathcal{U}} \left\{ \mathrm{div}_{T\mathcal{U}}\, X + \langle X, \overline{\nabla} log(|\mathcal{P}_H(N)|_H)\rangle \right\} \sigma_{\!{}_H} = \int_{\partial\mathcal{U}} \langle X, \eta\rangle\, |\mathcal{P}_H(N)|_H\, \sigma^{n-2}. \qquad (96)$$

Proof. The proof it is a straightforward application of the Riemannian divergence theorem. Indeed, by definition of $\sigma_{_H}$ and a simple calculation, we have

$$\begin{aligned} d(X \rfloor \sigma_{_H}) &= d(|\mathcal{P}_H(N)|_H\, X \,\rfloor\, \sigma^{n-1}) = \mathrm{div}_{T\mathcal{U}}(|\mathcal{P}_H(N)|_H\, X)\, \sigma^{n-1} \\ &= \{\mathrm{div}_{T\mathcal{U}}\, X + \langle X, \overline{\nabla} log(|\mathcal{P}_H(N)|_H)\rangle\}\, \sigma_{_H} \end{aligned}$$

and the thesis now follows by Stoke's Theorem. $\square$

The definition of the H-perimeter form $\sigma_{_H}$ modify the role of the usual tangent space and we shall replace it, in our analysis, by using both the vertical bundle VS and the horizontal tangent bundle HTS. The starting point of our work will be that of stating the analogous version of the Riemannian divergence theorem for regular hypersurfaces endowed with the H-perimeter form $\sigma_{_H}$. Our first result in this direction is the following:

Lemma 6.20. *[Main Lemma] For every $X \in \mathbf{C}^{\infty}(\mathbb{G}, VS)$, $(X = \sum_{J=2}^{n} x_J\, \tau_J)$, we have*

$$d(X \rfloor \sigma_{_H})|_S$$

$$= \left\{ \sum_{J=2}^{n} \tau_J(x_J) + \sum_{j=2}^{m} \sum_{\alpha\in I_2} \left[x_j \Big[\sum_{h=2}^{m} \phi_{jh}(\tau_h) + \langle \mathcal{C}^{\alpha}(\tau_1), \tau_j\rangle_H \frac{n_\beta}{n_1} \Big] - \phi_{1j}(\tau_j)\, \frac{n_\alpha\, x_\alpha}{n_1} \right] \right\} (\sigma_{_H})|_S.$$

where $n_1 = \langle N, \tau_1\rangle = |\mathcal{P}_H(N)|_H$ and $n_\gamma = \langle N, \tau_\gamma\rangle$ $(\gamma \in I_2)$.

The proof will be given in Section 2.3. However, we may state the main consequences of Lemma 6.20. To this end, we make use of the VS-divergence operator $\mathrm{div}_{_{VS}}$, introduced in Definition 6.11. Now, let us state the following elementary fact:

Lemma 6.21. *For every $X \in \mathbf{C}^{\infty}(\mathbb{G}, VS)$, $X = \sum_{I=2}^{n} x_I\, \tau_I$, we have*

$$\mathrm{div}_{_{VS}} X = \sum_{I=2}^{n} \tau_I(x_I) + \sum_{h,\, j=2}^{m} x_h\, \phi_{hj}(\tau_j).$$

Proof. By definition of $\mathrm{div}_{_{VS}}$ we have

$$
\begin{aligned}
\operatorname{div}_{VS} X &= \sum_{J=2}^{n} \langle \mathcal{D}^{VS}_{\tau_J} X, \tau_J \rangle = \sum_{I,\,J=2}^{n} \langle \mathcal{D}^{VS}_{\tau_J} (x_I\, \tau_I), \tau_J \rangle \\
&= \sum_{I,\,J=2}^{n} \left\{ \tau_J(x_I) \delta^I_J + x_I \, \langle \mathcal{D}^{VS}_{\tau_J} \tau_I, \tau_J \rangle \right\} \\
&= \sum_{I=2}^{n} \tau_I(x_I) + \sum_{I,\,J=2}^{n} x_I \, \phi_{IJ}(\tau_J) \\
&= \sum_{I=2}^{n} \tau_I(x_I) + \sum_{h,\,j=2}^{m} x_h \, \phi_{hj}(\tau_j) \qquad \text{(by (iv) of Lemma 6.4).} \qquad (97)
\end{aligned}
$$

□

By this lemma we obtain a more concise formulation of Lemma 6.20:

$$
d\,(X \rfloor \sigma_{H})|_S = \left\{ \operatorname{div}_{VS} X + \sum_{\beta \in I_2} \langle \mathcal{C}^\beta \tau_1, \mathcal{P}_H(X) \rangle_H \frac{n_\beta}{n_1} + \mathcal{H}^{sc}_{H} \, \langle \mathcal{P}_Z(N), \mathcal{P}_Z(X) \rangle \, \frac{1}{n_1} \right\} \sigma_{H}|_S,
$$

for every $X \in \mathbf{C}^\infty(\mathbb{G}, VS)$, or equivalently

$$
d\,(X \rfloor \sigma_{H})|_S = \operatorname{div}_{VS} X \, \sigma_{H}|_S + \left\{ \Big\langle \sum_{\beta \in I_2} n_\beta \, \mathcal{C}^\beta \tau_1, \mathcal{P}_H(X) \Big\rangle_H + \mathcal{H}^{sc}_{H} \, \langle \mathcal{P}_Z(N), \mathcal{P}_Z(X) \rangle \right\} \sigma^{n-1}|_S.
$$

From equation (98) we then get the next two corollaries.

Corollary 6.22. *For every $X \in \mathbf{C}^\infty(S, HTS)$, we have*

$$
d\,(X \rfloor \sigma_{H})|_S = \operatorname{div}_{HTS} X \, \sigma_{H}|_S + \Big\langle \sum_{\beta \in I_2} n_\beta \, \mathcal{C}^\beta \tau_1, X \Big\rangle_H \sigma^{n-1}|_S.
$$

Proof. It is enough to use equation (98) together with the natural definition of HTS-divergence related to Definition 6.12 (see also Remark 6.13) and observing that $x_I = \langle X, \tau_I \rangle \neq 0$ if, and only if, $I \in I_1 \setminus \{1\}$. □

Corollary 6.23. *For every $X \in \mathfrak{X}(S) (= \mathbf{C}^\infty(S, TS))$, we have*

$$
d\,(X \rfloor \sigma_{H})|_S = \left\{ \operatorname{div}_{VS} X - x_1 \mathcal{H}^{sc}_{H} \right\} \sigma_{H}|_S + \Big\langle \sum_{\beta \in I_2} n_\beta \, \mathcal{C}^\beta \tau_1, \mathcal{P}_H(X) \Big\rangle_H \sigma^{n-1}|_S.
$$

Proof. Also in this case we use equation (98). Since, for $X \in TS$ ($X = \sum_I x_I \, \tau_I$), we have $\langle X, N \rangle = 0$, we obtain

$$\langle X, N \rangle = x_1 \, |\mathcal{P}_H(N)|_H + \sum_{\alpha \in I_2} x_\alpha \, n_\alpha = x_1 \, n_1 + \sum_{\alpha \in I_2} x_\alpha \, n_\alpha = 0.$$

□

Theorem 6.24. *[Divergence type theorems on regular hypersurfaces] Let $\mathbb{G}$ be a 2-step Carnot group and let $S \subset \mathbb{G}$ be a smooth immersed non-characteristic hypersurface with unit normal vector along S denoted by N. Let $\mathcal{U} \subset S$ be compact and suppose that the boundary $\partial\mathcal{U}$ is a smooth $n-2$-dimensional Riemannian submanifold with outward pointing unit normal η. Then the following hold:*

(i) For every smooth vector field $X \in \mathbf{C}^\infty(\mathbb{G}, VS)$ we have

$$\int_{\mathcal{U}} \mathrm{div}_{VS} X \; \sigma_{H} + \int_{\mathcal{U}} \Big\{ \Big\langle \sum_{\beta \in I_2} n_\beta \, C^\beta \nu_{H}, \mathcal{P}_H(X) \Big\rangle_H + \mathcal{H}_{H}^{sc} \, \langle \mathcal{P}_Z(N), \mathcal{P}_Z(X) \rangle \Big\} \, \sigma^{n-1}$$
$$= \int_{\partial\mathcal{U}} \langle X, \eta \rangle \, |\mathcal{P}_H(N)|_H \, \sigma^{n-2};$$

(ii) For every smooth vector field $X \in \mathbf{C}^\infty(S, HTS)$ we have

$$\int_{\mathcal{U}} \mathrm{div}_{HTS} X \; \sigma_{H} \quad + \quad \int_{\mathcal{U}} \Big\langle \sum_{\beta \in I_2} n_\beta \, C^\beta \nu_{H}, X \Big\rangle_H \sigma^{n-1} = \int_{\partial\mathcal{U}} \langle X, \eta \rangle \, |\mathcal{P}_H(N)|_H \, \sigma^{n-2};$$

(iii) For every smooth vector field $X \in \mathfrak{X}(S)$ we have

$$\int_{\mathcal{U}} \Big\{ \mathrm{div}_{VS} X \quad - \quad \mathcal{H}_{H}^{sc} \langle X, \nu_{H} \rangle \Big\} \, \sigma_{H} + \int_{\mathcal{U}} \Big\langle \sum_{\beta \in I_2} n_\beta \, C^\beta \nu_{H}, \mathcal{P}_H(X) \Big\rangle_H \sigma^{n-1}$$
$$= \quad \int_{\partial\mathcal{U}} \langle X, \eta \rangle \, |\mathcal{P}_H(N)|_H \, \sigma^{n-2}. \tag{98}$$

Proof. This follows from Stokes' Theorem, and the fact (easily verified) that

$$(X \rfloor \sigma_{H})|_{\partial\mathcal{U}} = |\mathcal{P}_H(N)|_H \,\langle X, \eta\rangle|_{\partial\mathcal{U}},$$

by using, respectively, Lemma 6.20, Corollary 6.22 and Corollary 6.23. $\square$

Corollary 6.25. *With the same hypothesis of Theorem 6.24 the following hold:*

(i) For every smooth vector field $X \in \mathbf{C}_0^{\infty}(\mathbb{G},\, VS|_{\mathcal{U}})$ *such that* $\mathrm{spt}(X) \cap S \Subset \mathcal{U}$ *we have*

$$\int_{\mathcal{U}} \mathrm{div}_{VS} X \,\sigma_{H} = -\int_{\mathcal{U}} \Big\{ \Big\langle \sum_{\beta \in I_2} n_\beta\, C^\beta \nu_{H}, \mathcal{P}_H(X) \Big\rangle_H + \mathcal{H}_{H}^{sc} \,\langle \mathcal{P}_Z(N), \mathcal{P}_Z(X)\rangle \Big\}\, \sigma^{n-1}.$$

(ii) For every smooth vector field $X \in \mathbf{C}_0^{\infty}(\mathcal{U}, HTS|_{\mathcal{U}})$ *we have*

$$\int_{\mathcal{U}} \mathrm{div}_{HTS} X \;\sigma_{H} \quad + \quad \int_{\mathcal{U}} \Big\langle \sum_{\beta \in I_2} n_\beta\, C^\beta \nu_{H}, X \Big\rangle_H \sigma^{n-1} = 0.$$

(iii) For every smooth vector field $X \in \mathbf{C}_0^{\infty}(\mathcal{U},\, TS|_{\mathcal{U}})$ *we have*

$$\int_{\mathcal{U}} \Big\{ \mathrm{div}_{VS} X \quad - \quad \mathcal{H}_{H}^{sc} \langle X, \nu_{H}\rangle \Big\}\, \sigma_{H} + \int_{\mathcal{U}} \Big\langle \sum_{\beta \in I_2} n_\beta\, C^\beta \nu_{H}, \mathcal{P}_H(X) \Big\rangle_H \sigma^{n-1} = 0.$$

Proof. It is obvious by Theorem 6.24. $\square$

Now let $\psi \in \mathbf{C}^{\infty}(\mathbb{G})$. By Definition 6.11 we may consider the VS-gradient of ψ, i.e. the (unique) vector field $\mathcal{D}^{VS}\psi$ of $\mathbf{C}^{\infty}(\mathbb{G},\, VS)$ such that

$$\langle \mathcal{D}^{VS}\psi, X\rangle = d\psi(X) = X\psi \qquad \forall\; X \in VS.$$

Clearly, we may compute also the VS-divergence of $\mathcal{D}^{VS}\psi$.

Definition 6.26. *We define the VS-laplacian of $\psi \in \mathbf{C}^{\infty}(\mathbb{G})$ to be the function given by $\Delta_{_{VS}}\psi := \mathrm{div}_{_{VS}}(\mathcal{D}^{_{VS}}\psi)$.*

From (93) we get that

$$\Delta_{_{VS}}\psi = \Delta_{_{HTS}}(\psi|_S) + \sum_{\alpha\in I_2} \tau_{\alpha}^{(2)}(\psi) \qquad \forall\ \psi \in \mathbf{C}^{\infty}(\mathbb{G}).$$

From equations (89) and (90) we get, by means of an easy computation, that the *VS*-laplacian satisfies the following properties:

$$\mathrm{div}_{_{VS}}(\phi\,\nabla^{_{HTS}}\psi) = \phi\,\Delta_{_{VS}}(\psi) + \langle \nabla^{_{HTS}}\phi, \nabla^{_{HTS}}\psi\rangle_H, \tag{99}$$

$$\Delta_{_{VS}}(\phi\,\psi) = \phi\,\Delta_{_{VS}}(\psi) + \psi\,\Delta_{_{VS}}(\phi) + 2\,\langle \mathcal{D}^{_{VS}}\phi, \mathcal{D}^{_{VS}}\psi\rangle \tag{100}$$

for all ϕ, $\psi \in \mathbf{C}^{\infty}(\mathbb{G})$. Analogous relations are satisfied if we consider the *HTS*-laplacian on S; see Definition 6.13. More precisely, we have:

$$\mathrm{div}_{_{HTS}}(\phi\,\nabla^{_{HTS}}\psi) = \phi\,\Delta_{_{HTS}}(\psi) + \langle \nabla^{_{HTS}}\phi, \nabla^{_{HTS}}\psi\rangle, \tag{101}$$

$$\Delta_{_{HTS}}(\phi\,\psi) = \phi\,\Delta_{_{HTS}}(\psi) + \psi\,\Delta_{_{HTS}}(\phi) + 2\,\langle \nabla^{_{HTS}}\phi, \nabla^{_{HTS}}\psi\rangle \tag{102}$$

for all ϕ, $\psi \in \mathbf{C}^{\infty}(S)$. These formulae allows us to state the announced Green's type identities for regular non-characteristic hypersurfaces.

Theorem 6.27. *[Green's type formulae:I] Let $\mathbb{G}$ be a 2-step Carnot group and let $S \subset \mathbb{G}$ be a smooth immersed non-characteristic hypersurface with unit normal vector along S denoted by N. Let $\mathcal{U} \subset S$ be compact and suppose that the boundary $\partial\mathcal{U}$ is a smooth $n-2$-dimensional Riemannian submanifold with outward pointing unit normal η. Then the following hold:*

(i) Let ϕ_1, $\phi_2 \in \mathbf{C}^{\infty}(\mathbb{G})$ and let us suppose that, for at least one $i \in \{1, 2\}$, we have

$$\mathrm{spt}(\phi_i) \cap S \Subset \mathcal{U}.$$

Then we have

$$\int_{\mathcal{U}} \Big\{ \phi_1 \, \Delta_{_{VS}} \phi_2 + \langle \mathcal{D}^{_{VS}} \phi_1, \mathcal{D}^{_{VS}} \phi_2 \rangle \Big\} \, \sigma_{_H}$$
$$+ \int_{\mathcal{U}} \phi_1 \Big\{ \Big\langle \sum_{\beta \in I_2} n_\beta \, \mathcal{C}^\beta \nu_{_H}, \mathcal{P}_H(\mathcal{D}^{_{VS}} \phi_2) \Big\rangle_H + \mathcal{H}^{sc}_{_H} \, \langle \mathcal{P}_Z(N), \mathcal{P}_Z(\mathcal{D}^{_{VS}} \phi_2) \rangle \Big\} \, \sigma^{n-1} = 0.$$

(ii) Let $\phi_1, \phi_2 \in \mathbf{C}^\infty(S)$, with at least one of them compactly supported on $\mathcal{U}$. Then we have

$$\int_{\mathcal{U}} \Big\{ \phi_1 \, \Delta_{_{HTS}} \phi_2 + \langle \nabla^{_{HTS}} \phi_1, \nabla^{_{HTS}} \phi_2 \rangle_H \Big\} \, \sigma_{_H} + \int_{\mathcal{U}} \phi_1 \, \Big\langle \sum_{\beta \in I_2} n_\beta \, \mathcal{C}^\beta \nu_{_H}, \nabla^{_{HTS}} \phi_2 \Big\rangle_H \, \sigma^{n-1} = 0.$$

(iii) Let $\phi_1, \phi_2 \in \mathbf{C}^\infty(S)$, with at least one of them compactly supported on $\mathcal{U}$. Then we have

$$\int_{\mathcal{U}} \Big\{ \phi_1 \, \Delta_{_{VS}} \phi_2 + \langle \mathcal{D}^{_{VS}} \phi_1, \mathcal{D}^{_{VS}} \phi_2 \rangle - \phi_1 \, \langle \mathcal{D}^{_{VS}} \phi_2, \nu_{_H} \rangle \, \mathcal{H}^{sc}_{_H} \Big\} \, \sigma_{_H}$$
$$+ \int_{\mathcal{U}} \phi_1 \, \Big\langle \sum_{\beta \in I_2} n_\beta \, \mathcal{C}^\beta \nu_{_H}, \mathcal{P}_H(\mathcal{D}^{_{VS}} \phi_2) \Big\rangle_H \, \sigma^{n-1} = 0.$$

Proof. It follows from Corollary 6.25 and from the identities (99) and (101). $\square$

Theorem 6.28. *[Green's type formulae:II] Under the hypotheses of Theorem 6.27 the following hold:*

(i) Let $\phi_1, \phi_2 \in \mathbf{C}^\infty(\mathbb{G})$ and let us suppose that, for at least one $i \in \{1, 2\}$, we have $\mathrm{spt}(\phi_i) \cap S \Subset \mathcal{U}$. Then we have

$$\int_{\mathcal{U}} \Big\{ \phi_1 \, \Delta_{_{VS}} \phi_2 + \langle \mathcal{D}^{_{VS}} \phi_1, \mathcal{D}^{_{VS}} \phi_2 \rangle \Big\} \, \sigma_{_H}$$
$$+ \int_{\mathcal{U}} \phi_1 \Big\{ \Big\langle \sum_{\beta \in I_2} n_\beta \, \mathcal{C}^\beta \nu_{_H}, \mathcal{P}_H(\mathcal{D}^{_{VS}} \phi_2) \Big\rangle_H + \mathcal{H}^{sc}_{_H} \, \langle \mathcal{P}_Z(N), \mathcal{P}_Z(\mathcal{D}^{_{VS}} \phi_2) \rangle \Big\} \, \sigma^{n-1}$$
$$= \int_{\partial \mathcal{U}} \phi_1 \, \langle \mathcal{D}^{_{VS}} \phi_2, \eta \rangle \, |\mathcal{P}_H(N)|_H \, \sigma^{n-2}.$$

(ii) *Let $\phi_1,\, \phi_2 \in \mathbf{C}^\infty(S)$, with at least one of them compactly supported on $\mathcal{U}$. Then we have*

$$\int_{\mathcal{U}} \Big\{\phi_1\, \Delta_{\scriptscriptstyle HTS}\phi_2 + \langle \nabla^{\scriptscriptstyle HTS}\phi_1, \nabla^{\scriptscriptstyle HTS}\phi_2\rangle_H \Big\}\, \sigma_{\!\scriptscriptstyle H} + \int_{\mathcal{U}} \phi_1 \Big\langle \sum_{\beta\in I_2} n_\beta\, \mathcal{C}^\beta \nu_{\!\scriptscriptstyle H}, \nabla^{\scriptscriptstyle HTS}\phi_2\Big\rangle_H \sigma^{n-1}$$

$$= \int_{\partial\mathcal{U}} \phi_1\, \langle \nabla^{\scriptscriptstyle HTS}\phi_2, \eta\rangle\, |\mathcal{P}_H(N)|_H\, \sigma^{n-2}.$$

(iii) *Let $\phi_1,\, \phi_2 \in \mathbf{C}^\infty(S)$, with at least one of them compactly supported on $\mathcal{U}$. Then we have*

$$\int_{\mathcal{U}} \Big\{\phi_1\, \Delta_{\scriptscriptstyle VS}\phi_2 + \langle \mathcal{D}^{\scriptscriptstyle VS}\phi_1, \mathcal{D}^{\scriptscriptstyle VS}\phi_2\rangle - \phi_1\, \langle \mathcal{D}^{\scriptscriptstyle VS}\phi_2, \nu_{\!\scriptscriptstyle H}\rangle\, \mathcal{H}^{sc}_{\scriptscriptstyle H}\Big\}\, \sigma_{\!\scriptscriptstyle H}$$

$$+ \int_{\mathcal{U}} \phi_1 \Big\langle \sum_{\beta\in I_2} n_\beta\, \mathcal{C}^\beta \nu_{\!\scriptscriptstyle H}, \mathcal{P}_H(\mathcal{D}^{\scriptscriptstyle VS}\phi_2)\Big\rangle_H \sigma^{n-1} = \int_{\partial\mathcal{U}} \phi_1\, \langle \mathcal{D}^{\scriptscriptstyle VS}\phi_2, \eta\rangle\, |\mathcal{P}_H(N)|_H\, \sigma^{n-2}.$$

Proof. It follows from Theorem 6.24 and identities (99) and (101). $\square$

Theorem 6.29. *[Green's type formulae: III] With the hypothesis of Theorem 6.27 the following hold:*

(i) *Let $\phi_1,\, \phi_2 \in \mathbf{C}^\infty(\mathbb{G})$ and let us suppose that, for at least one $i \in \{1, 2\}$, we have*

$$\mathrm{spt}(\phi_i) \cap S \Subset \mathcal{U}.$$

Then we have

$$\int_{\mathcal{U}} \Big\{\phi_1\, \Delta_{\scriptscriptstyle VS}\phi_2 - \phi_2\, \Delta_{\scriptscriptstyle VS}\phi_1\Big\}\, \sigma_{\!\scriptscriptstyle H} + \int_{\mathcal{U}} \Big\{\Big\langle \sum_{\beta\in I_2} n_\beta\, \mathcal{C}^\beta \nu_{\!\scriptscriptstyle H}, \mathcal{P}_H(\phi_1\mathcal{D}^{\scriptscriptstyle VS}\phi_2 - \phi_2\mathcal{D}^{\scriptscriptstyle VS}\phi_1)\Big\rangle_H$$

$$+ \Big\langle \mathcal{P}_Z(N), \mathcal{P}_Z(\phi_1\, \mathcal{D}^{\scriptscriptstyle VS}\phi_2 - \phi_2\, \mathcal{D}^{\scriptscriptstyle VS}\phi_1)\Big\rangle \mathcal{H}^{sc}_{\scriptscriptstyle H}\Big\}\, \sigma^{n-1} = 0.$$

(ii) Let $\phi_1,\, \phi_2 \in \mathbf{C}^\infty(S)$, with at least one of them compactly supported on $\mathcal{U}$. Then we have

$$\int_{\mathcal{U}} \Big\{\phi_1\, \Delta_{_{HTS}}\phi_2 - \phi_2\, \Delta_{_{HTS}}\phi_1\Big\}\, \sigma_{_H} + \int_{\mathcal{U}} \Big\langle \sum_{\beta\in I_2} n_\beta\, \mathcal{C}^\beta \nu_{_H},\, \Big[\phi_1\, \nabla^{_{HTS}}\phi_2 - \phi_2\, \nabla^{_{HTS}}\phi_1\Big]\Big\rangle_H \sigma^{n-1} = 0.$$

(iii) Let $\phi_1,\, \phi_2 \in \mathbf{C}^\infty(S)$, with at least one of them compactly supported on $\mathcal{U}$. Then we have

$$\int_{\mathcal{U}} \Big\{\Big[\phi_1\, \Delta_{_{VS}}\phi_2 - \phi_2\, \Delta_{_{VS}}\phi_1\Big] - \Big\langle\Big[\phi_1\, \mathcal{D}^{_{VS}}\phi_2 - \phi_2\, \mathcal{D}^{_{VS}}\phi_1\Big], \nu_{_H}\Big\rangle \langle \mathrm{H}, \nu_{_H}\rangle_H\Big\}\, \sigma_{_H}$$

$$+ \int_{\mathcal{U}} \Big\langle \sum_{\beta\in I_2} n_\beta\, \mathcal{C}^\beta \nu_{_H}, \mathcal{P}_H(\phi_1\, \mathcal{D}^{_{VS}}\phi_2 - \phi_2\, \mathcal{D}^{_{VS}}\phi_1)\Big\rangle_H \sigma^{n-1} = 0.$$

Proof. It follows immediately from Theorem 6.27. $\square$

Theorem 6.30. *[Green's type formulae: IV] With the hypothesis of Theorem 6.27 the following hold:*

(i) Let $\phi_1,\, \phi_2 \in \mathbf{C}^\infty(\mathbb{G})$ and let us suppose that, for at least one $i \in \{1, 2\}$, we have

$$\mathrm{spt}(\phi_i) \cap S \Subset \mathcal{U}.$$

Then we have

$$\int_{\mathcal{U}} \Big\{\phi_1\, \Delta_{_{VS}}\phi_2 - \phi_2\, \Delta_{_{VS}}\phi_1\Big\}\, \sigma_{_H} + \int_{\mathcal{U}} \Big\{\Big\langle \sum_{\beta\in I_2} n_\beta\, \mathcal{C}^\beta \nu_{_H}, \mathcal{P}_H(\phi_1 \mathcal{D}^{_{VS}}\phi_2 - \phi_2 \mathcal{D}^{_{VS}}\phi_1)\Big\rangle_H$$

$$+ \Big\langle \mathcal{P}_Z(N), \mathcal{P}_Z(\phi_1\, \mathcal{D}^{_{VS}}\phi_2 - \phi_2\, \mathcal{D}^{_{VS}}\phi_1)\Big\rangle \langle\, \mathrm{H}, \nu_{_H}\rangle_H\, \Big\}\, \sigma^{n-1}$$

$$= \int_{\partial\mathcal{U}} \Big\langle\Big[\phi_1\, \mathcal{D}^{_{VS}}\phi_2 - \phi_2\, \mathcal{D}^{_{VS}}\phi_1\Big], \eta\Big\rangle\, |\mathcal{P}_H(N)|_H\, \sigma^{n-2}.$$

(ii) Let ϕ_1, $\phi_2 \in \mathbf{C}^{\infty}(S)$, with at least one of them compactly supported on $\mathcal{U}$. Then we have

$$\int_{\mathcal{U}} \Big\{\phi_1\, \Delta_{_{HTS}}\phi_2 - \phi_2\, \Delta_{_{HTS}}\phi_1\Big\}\, \sigma_{_H} + \int_{\mathcal{U}} \Big\langle \sum_{\beta\in I_2} n_\beta\, \mathcal{C}^{\beta}\nu_{_H}, \Big[\phi_1\, \nabla^{_{HTS}}\phi_2 - \phi_2\, \nabla^{_{HTS}}\phi_1\Big]\Big\rangle_H \sigma^{n-1}$$

$$= \int_{\partial\mathcal{U}} \Big\langle \Big[\phi_1\, \nabla^{_{HTS}}\phi_2 - \phi_2\, \nabla^{_{HTS}}\phi_1\Big], \eta\Big\rangle\, |\mathcal{P}_H(N)|_H\, \sigma^{n-2}.$$

(iii) Let ϕ_1, $\phi_2 \in \mathbf{C}^{\infty}(S)$, with at least one of them compactly supported on $\mathcal{U}$. Then we have

$$\int_{\mathcal{U}} \Big\{\Big[\phi_1\, \Delta_{_{VS}}\phi_2 - \phi_2\, \Delta_{_{VS}}\phi_1\Big] - \Big\langle \Big[\phi_1\, \mathcal{D}^{_{VS}}\phi_2 - \phi_2\, \mathcal{D}^{_{VS}}\phi_1\Big], \nu_{_H}\Big\rangle \langle \mathrm{H}, \nu_{_H}\rangle_H\Big\}\, \sigma_{_H}$$

$$+ \int_{\mathcal{U}} \Big\langle \sum_{\beta\in I_2} n_\beta\, \mathcal{C}^{\beta}\nu_{_H}, \mathcal{P}_H(\phi_1\, \mathcal{D}^{_{VS}}\phi_2 - \phi_2\, \mathcal{D}^{_{VS}}\phi_1)\Big\rangle_H \sigma^{n-1}$$

$$= \int_{\partial\mathcal{U}} \Big\langle \Big[\phi_1\, \mathcal{D}^{_{VS}}\phi_2 - \phi_2\, \mathcal{D}^{_{VS}}\phi_1\Big], \eta\rangle\, |\mathcal{P}_H(N)|_H\, \sigma^{n-2}.$$

Proof. It follows immediately from Theorem 6.28. □

Example 6.31. *[Application: The Heisenberg group $\mathbb{H}^1$.] Let $\mathcal{U} \subset S$ be a compact subset of a non-characteristic surface $S \subset \mathbb{H}^1$. Suppose that the boundary $\partial\mathcal{U}$ ($\neq \emptyset$) is smooth and denote by $\imath$ the inclusion map of $\partial\mathcal{U}$. Then for every smooth vector field $X \in \mathfrak{X}(\mathbb{H}^1)$ we get*

$$\int_{\mathcal{U}} \Big[\tau_2(x_2) + \tau_3(x_3)\Big]\sigma_{_H} + \int_{\mathcal{U}} \Big[x_2 - x_3\phi_{12}(\tau_2)\Big]\phi_1 \wedge \phi_2 = \int_{\partial\mathcal{U}} \imath^*(x_2\phi_3 - x_3\phi_3),$$

where we have used the H-adapted coframe (ϕ_1, ϕ_2, ϕ_3), i.e. the dual coframe of the H-adapted frame (τ_1, τ_1, τ_1) ($\tau_1 := \nu_{_H}$, $\tau_2 = \nu_{_H}^{\perp_H}$, $\tau_3 = X_3$; $d\,\phi_3 = \phi_1 \wedge \phi_2$). By the above formula we also get

$$\int_{\mathcal{U}} \Big[\tau_3(x_3) - x_1\phi_{12}(\tau_2)\Big]\sigma_{_H} = -\int_{\partial\mathcal{U}} \imath^*(x_3\phi_2)$$

and this can be written more explicitly as follows

$$\int_{\mathcal{U}} \Big[\tau_3(x_3) + \langle \mathrm{H}, X\rangle\Big]\sigma_{_H} = -\int_{\partial\mathcal{U}} \imath^*(x_3\phi_2).$$

6.3 Proof of Lemma 6.20

Proof. For $X \in VS$, we compute the exterior derivative of the contraction by X of $\sigma_{_H}$, i.e.

$$d\,(X \rfloor \sigma_{_H})|_S = d\,(X \rfloor \phi_2 \wedge \ldots \wedge \phi_n)|_S.$$

So if $X = \sum_{J=2}^{n} x_J \tau_J$, then

$$\begin{aligned}
d\,(X \rfloor \sigma_{_H})|_S &= \sum_{J=2}^{n} d\,(x_J\,\tau_J \rfloor \phi_2 \wedge \ldots \phi_n)|_S = \sum_{J=2}^{n} d\,(x_J \tau_J \rfloor \sigma_{_H})|_S \\
&= \sum_{J=2}^{n} \tau_J(x_J)\,\sigma_{_H}|_S + \sum_{J=2}^{n} x_J\,d\,(\tau_J \rfloor \sigma_{_H})|_S \\
&= \sum_{J=2}^{n} \tau_J(x_J)\,\sigma_{_H}|_S + \sum_{j \in I_1 \setminus \{1\}} x_j\,d\,(\tau_j \rfloor \sigma_{_H})|_S + \sum_{\alpha \in I_2} x_\alpha\,d\,(\tau_\alpha \rfloor \sigma_{_H})|_S \\
&.
\end{aligned} \tag{103}$$

Thus the proof follows by computing the exterior derivative of the form $(\tau_I \rfloor \sigma_{_H})|_S$, i.e.

$$d\,(\tau_I \rfloor \sigma_{_H})|_S = (-1)^I d\,(\phi_2 \wedge \ldots \wedge \widehat{\phi_I} \wedge \ldots \wedge \phi_n)|_S \qquad (I = 2, \ldots, n).$$

Step 1. *Computation of*

$$\begin{aligned}
d\,(\tau_I \rfloor \sigma_{_H})|_S &= (-1)^i d\,(\phi_2 \wedge \ldots \wedge \widehat{\phi_i} \wedge \ldots \wedge \phi_n)|_S \\
&= (-1)^i d\left(\phi_2 \wedge \ldots \wedge \widehat{\phi_i} \wedge \ldots \wedge \phi_m \bigwedge_{\alpha \in I_2} \phi_\alpha\right)\Big|_S \qquad i = 2, \ldots, m.
\end{aligned}$$

Proof of Step 1. With no loss of generality we may suppose $i = 2$. We have

$$\begin{aligned}
A: &= d\,(\phi_3 \wedge \ldots \wedge \ldots \wedge \phi_n) \\
&= \sum_{J=3}^{n} (-1)^{J+1} \phi_3 \wedge \ldots \wedge d\phi_J \wedge \ldots \phi_n
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{J=3}^{n}(-1)^{J+1}\phi_3\wedge\ldots\bigwedge\Big(\sum_{J=1}^{n}\phi_I\wedge\phi_{IJ}\Big)\bigwedge\ldots\phi_n\\
&= -\sum_{J=3}^{n}(-1)^{J+1}\Big(\sum_{I=1}^{2}\phi_{IJ}\wedge\phi_I\Big)\bigwedge\widehat{\phi_2}\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\ldots\phi_n\\
&= -\sum_{J=3}^{n}(-1)^{J+1}(\phi_{1J}\wedge\phi_1)\wedge\widehat{\phi_2}\wedge\phi_3\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\ldots\phi_n\\
&\quad -\sum_{J=3}^{n}(-1)^{J+1}(\phi_{2J}\wedge\phi_2)\wedge\widehat{\phi_2}\wedge\phi_3\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\ldots\phi_n. \qquad (104)
\end{aligned}
$$

Here above we have used the 1st structure equation (84) of the H-adapted coframe $(\phi_1,\ldots,\phi_n)$ for S. Now we note that

$$
\phi_{1J}=\sum_{K=1}^{n}\phi_{1J}(\tau_K)\,\phi_K \qquad \text{and} \qquad \phi_{2J}=\sum_{K=1}^{n}\phi_{2J}(\tau_K)\,\phi_K.
$$

We have so

$$
\begin{aligned}
(A_1)_J:\ &= (\phi_{1J}\wedge\phi_1)\wedge\widehat{\phi_2}\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\phi_n\\
&= (\phi_{1J}(\tau_2)\,\phi_2+\phi_{1J}(\tau_J)\,\phi_J)\wedge\phi_1\wedge\widehat{\phi_2}\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\phi_n\\
&= -\phi_{1J}(\tau_2)\phi_1\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\phi_n+(-1)^J\phi_{1J}(\tau_J)\,\phi_1\wedge\widehat{\phi_2}\wedge\ldots\wedge\phi_n. \qquad (105)
\end{aligned}
$$

Now if $J\in I_1\setminus\{1\}$ Lemma 6.3 says that $(\phi_1\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\phi_n)|_S=0$. Moreover, by Lemma 6.4 of Section 2.1, if $J\in I_2$, the second expression in the above formula (105) is 0, while the first one is different from 0 only if $J\in I_2$. Analogously, we have that

$$
\begin{aligned}
(A_2)_J:\ &= (\phi_{2J}\wedge\phi_2)\wedge\widehat{\phi_2}\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\phi_n\\
&= (\phi_{2J}\,(\tau_1)\,\phi_1+\phi_{2J}\,(\tau_J)\,\phi_J)\wedge\phi_2\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\phi_n\\
&= \phi_{2J}\,(\tau_1)\,\phi_1\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\phi_n+(-1)^J\phi_{2J}(\tau_J)\phi_2\wedge\ldots\wedge\phi_n. \qquad (106)
\end{aligned}
$$

Using again Lemma 6.3 and Lemma 6.4 of Section 2 we get that the first term of (106) is different from 0 only if $J\in I_2$ while the second one is different from 0 only if $J\in I_1$ and so that cases are mutually exclusive. Now, using (104) and the expressions of $(A_1)_J$ and $(A_2)_J$, we may finish the computations of Step 1. More precisely, we have

$$\begin{aligned}
A|_S &= -\sum_{J=3}^{n}(-1)^{J+1}\Big((I_1)_J+(I_2)_J\Big)\Big|_S\\
&= -\sum_{J=3}^{n}(-1)^{J+1}\Big\{-\phi_{1J}(\tau_2)\phi_1\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\phi_n\\
&+ \phi_{2J}(\tau_1)\,\phi_1\wedge\ldots\wedge\widehat{\phi_J}\wedge\ldots\wedge\phi_n+(-1)^J\phi_{2J}(\tau_J)\,\phi_2\wedge\ldots\wedge\phi_n\Big\}\Big|_S\\
&= \sum_{J=3}^{n}\phi_{2J}(\tau_J)(\phi_2\wedge\ldots\phi_n)|_S\\
&\quad+\sum_{\beta\in I_2}(-1)^{\beta+1}(\phi_{1\beta}(\tau_2)-\phi_{2\beta}(\tau_1))(\phi_1\wedge\ldots\wedge\ldots\widehat{\phi_\beta}\wedge\ldots\wedge\phi_n)|_S\\
&= \sum_{J=3}^{n}\phi_{2J}(\tau_J)\,(\sigma_{H})|_S+\sum_{\beta\in I_2}(\phi_{1\beta}(\tau_2)-\phi_{2\beta}(\tau_1))\,(\tau_\beta\,\rfloor\,\Omega^n)|_S\\
&= \left\{\sum_{j=2}^{m}\phi_{2j}(\tau_j)+\sum_{\beta\in I_2}(\phi_{1\beta}(\tau_2)-\phi_{2\beta}(\tau_1))\frac{n_\beta}{n_1}\right\}\sigma_{H}|_S,
\end{aligned}$$

where we remind that $n_\beta=\langle N,\tau_\beta\rangle$ $(\beta\in I_2)$ and that $n_1=\langle N,\tau_1\rangle=|\mathcal{P}_H(N)|_H$. Notice that from item (iii) of 6.4 and Definition 5.7 we obtain

$$\phi_{1\beta}(\tau_2)=-\frac{1}{2}\langle\mathcal{C}^\beta(\tau_2),\tau_1\rangle_H,\qquad \phi_{2\beta}(\tau_1)=-\frac{1}{2}\langle\mathcal{C}^\beta\tau_1,\tau_2\rangle_H.$$

Since $\mathcal{C}^\beta$ $(\beta\in I_2)$ is a skew-symmetric linear operator[5] we get

$$\phi_{1\beta}(\tau_2)-\phi_{2\beta}(\tau_1)=\langle\mathcal{C}^\beta\tau_1,\tau_2\rangle_H.$$

Therefore

$$A|_S=\left\{\sum_{j=2}^{m}\phi_{2j}(\tau_j)+\sum_{\beta\in I_2}\langle\mathcal{C}^\beta\tau_1,\tau_2\rangle_H\frac{n_\beta}{n_1}\right\}(\sigma_{H})|_S,$$

and, in the general case, if $i\in I_1\backslash\{1\}$, we finally get

$$d(\tau_i\,\rfloor\,\sigma_{H})|_S=\left\{\sum_{j=2}^{m}\phi_{ij}(\tau_j)+\sum_{\beta\in I_2}\langle\mathcal{C}^\beta\tau_1,\tau_i\rangle_H\frac{n_\beta}{n_1}\right\}\sigma_{H}|_S. \tag{107}$$

[5]Notice that, with respect to the coordinates of the H-adapted frame $(\tau_1,...,\tau_n)$, the linear operator $\mathcal{C}^\beta$ corresponds to the matrix $\mathsf{O}^T\mathsf{C}^\beta\mathsf{O}$.

$\square$

Step 2. *Computations of*

$$d(\tau_\alpha \rfloor \sigma_{_H})|_S = (-1)^\alpha\, d(\phi_2 \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n)|_S \qquad \alpha = m+1, \ldots, n. \quad (108)$$

Proof of Step 2.

$$\begin{aligned} B: \; &= d(\phi_2 \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n) \\ &= \sum_{j=2}^{m} (-1)^j \phi_2 \wedge \ldots \wedge d\phi_j \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n \\ &\pm \sum_{\gamma \neq \alpha,\, \gamma \in I_2} \phi_2 \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge d\phi_\gamma \wedge \phi_n. \end{aligned} \quad (109)$$

We remark now that the second addend (109) of $B_1(j)$ must be 0. Indeed, by the first structure equation of the coframe $(\phi_1, \ldots, \phi_n)$, we get

$$d\phi_\gamma = \sum_K \phi_K \wedge \phi_{K\gamma} \neq 0 \quad \Longleftrightarrow \quad K \in I_1.$$

But (109) implies that $K = 1$. Moreover we have

$$\phi_1 \wedge \phi_{1\gamma} = \phi_1 \bigwedge \Big(\sum_R \phi_{1\gamma}(\tau_R)\phi_R \Big), \quad (110)$$

and if we substitute (110) in (109) the claim follows by item (iii) of 6.4, since R must be equal to $\alpha \in I_2$. Therefore

$$\begin{aligned} B|_S &= \sum_{j=2}^{m} (-1)^j \left\{ \phi_2 \wedge \ldots \wedge \Big(\sum_{k \in I_1} \phi_k \wedge \phi_{kj} \Big) \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n \right\}\Bigg|_S \\ &= -\sum_{j=2}^{m} (-1)^j \left\{ \Big(\sum_{k \in I_1} \phi_{kj} \wedge \phi_k \Big) \wedge \phi_2 \wedge \ldots \wedge \widehat{\phi_j} \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n \right\}\Bigg|_S. \end{aligned}$$

From the first structure equation we get that

$$\begin{aligned} B_1(j) &= \phi_{1j} \wedge \phi_1 \wedge \phi_2 \wedge \ldots \wedge \widehat{\phi_j} \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n \\ &= \Big(\sum_R \phi_{1j}(\tau_R)\, \phi_R \Big) \wedge \phi_1 \wedge \ldots \wedge \widehat{\phi_j} \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n \\ &= \phi_{1j}(\tau_j)\phi_j \wedge \phi_1 \wedge \ldots \wedge \widehat{\phi_j} \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n \\ &\quad + \phi_{1j}(\tau_\alpha)\, \phi_\alpha \wedge \phi_1 \wedge \ldots \wedge \widehat{\phi_j} \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n. \end{aligned} \quad (111)$$

By applying Lemma 6.3 we get that the second addend (111) of $B_1(j)$ is 0, if it is restricted to S. Furthermore

$$
\begin{aligned}
B_2(j) &= \phi_{\alpha j} \wedge \phi_\alpha \wedge \phi_2 \wedge \ldots \widehat{\phi_j} \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n \\
&= \Big(\sum_R \phi_{\alpha j}(\tau_R)\, \phi_R \Big) \wedge \phi_\alpha \wedge \ldots \wedge \widehat{\phi_j} \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n \\
&= \phi_{\alpha j}(\tau_1)\phi_1 \wedge \phi_\alpha \wedge \phi_2 \wedge \ldots \wedge \widehat{\phi_j} \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n \qquad (112) \\
&\quad + \phi_{\alpha j}(\tau_j)\phi_j \wedge \phi_\alpha \wedge \phi_2 \wedge \ldots \wedge \widehat{\phi_j} \wedge \ldots \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n.
\end{aligned}
$$

By applying again Lemma 6.3, we get that the first addend (112) of $B_2(j)$ is 0 when restricted to S. Thus we get

$$
B|_S = d(\phi_2 \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n)|_S
$$

$$
= -\sum_{j=2}^{m} (-1)^j \Big\{ (-1)^{j-1} \phi_{1j}(\tau_j)\phi_1 \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n + (-1)^{j+\alpha-1} \phi_{\alpha j}(\tau_j)\phi_2 \wedge \ldots \wedge \phi_n \Big\}\Big|_S
$$

$$
\begin{aligned}
&= (-1)^{\alpha-1} \sum_{j=2}^{m} \phi_{1j}(\tau_j)(\tau_\alpha \,\rfloor\, \Omega^n)|_S + (-1)^\alpha \sum_{j=2}^{m} \phi_{\alpha j}(\tau_j)(\sigma_{H})|_S \\
&= (-1)^{\alpha-1} \sum_{j=2}^{m} \Big\{ \phi_{1j}(\tau_j) \frac{n_\alpha}{n_1} - \phi_{\alpha j}(\tau_j) \Big\} (\sigma_{H})|_S \\
&= (-1)^{\alpha-1} \Bigg\{ - \Bigg[\sum_{j=2}^{m} \phi_{\alpha j}(\tau_j) \Bigg] + \Bigg[\sum_{j=2}^{m} \phi_{1j}(\tau_j) \Bigg] \frac{n_\alpha}{n_1} \Bigg\} (\sigma_{H})|_S.
\end{aligned}
$$

By the characterization of the connection 1-forms $\phi_{\alpha j}$ given in Lemma 6.4 we get

$$
\phi_{\alpha j}(\tau_j) = \langle \mathcal{C}^\alpha \tau_j, \tau_j \rangle = 0
$$

by the skew-symmetry of C^α, so that

$$
\begin{aligned}
d(\phi_2 \wedge \ldots \wedge \widehat{\phi_\alpha} \wedge \ldots \wedge \phi_n)|_S &= (-1)^{\alpha-1} \Bigg[\sum_{j=2}^{m} \phi_{1j}(\tau_j) \Bigg] (\tau_\alpha \,\rfloor\, \Omega^n)|_S \\
= (-1)^{\alpha-1} \Bigg[\sum_{j=2}^{m} \phi_{1j}(\tau_j) \Bigg] n_\alpha\, (\sigma^{n-1})|_S &= (-1)^{\alpha-1} \Bigg[\sum_{j=2}^{m} \phi_{1j}(\tau_j) \Bigg] \frac{n_\alpha}{n_1}\, (\sigma_{H})|_S
\end{aligned}
$$

and finally

$$d(\tau_\alpha \rfloor \sigma_{H})|_S = -\Big\{ \sum_{j=2}^{m} \phi_{1j}(\tau_j) \Big\} \frac{n_\alpha}{n_1} (\sigma_{H})|_S.$$

$\square$

At this point we may achieve the proof, by substituting into (103) the results of the computations of Step 1 and Step 2. More precisely, we have

$$\begin{aligned} & d(X \rfloor \sigma_{H})|_S \\ &= \sum_{J=2}^{n} \tau_J(x_J)\,(\sigma_{H})|_S + \sum_{j=2}^{m} x_j\, d(\tau_j \rfloor \sigma_{H})|_S + \sum_{\alpha\in I_2} x_\alpha\, d\,(\tau_\alpha \rfloor \sigma_{H})|_S \\ &= \left\{ \sum_{J=2}^{n} \tau_J(x_J) + \sum_{j=2}^{m} \left[x_j \Big[\sum_{h=2}^{m} \phi_{jh}(\tau_h) + \sum_{\beta\in I_2} \frac{n_\beta}{n_1} \langle \mathcal{C}^\beta \tau_1, \tau_j \rangle_H \Big] - \phi_{1j}(\tau_j) \sum_{\alpha\in I_2} x_\alpha \frac{n_\alpha}{n_1} \right] \right\} (\sigma_{H})|_S \end{aligned}$$

that is equivalent to the thesis. $\square$

7 1st and 2nd variation of σ_H in 2-step Carnot groups

7.1 Preliminaries

Before the statement of our results we would like to make a short comment, suggested by a paper of Hermann, [55].

Let M be a smooth manifold and let $\chi : N \longrightarrow M$ define N as a submanifold of M. Assume that N is an oriented manifold with oriented boundary ∂N. Suppose that ω is a p-form on N and denote by χ_t a family of of 1-parameter smooth deformations of N fixing the boundary of N and which is just the identity for $t = 0$. Then a very general variational problem is that to compute the 1st and the 2nd variation of the functional $L(\chi_t) = \int_N \chi_t^* \omega$, i.e.

$$\frac{d}{dt} \int_N \chi_t^* \omega, \qquad \frac{d^2}{dt^2} \int_N \chi_t^* \omega.$$

Assuming that $t \longrightarrow \chi_t$ is the integral curve of a fixed vector field $X \in \mathfrak{X}(M)$, we can prove, by using Cartan's formula and Stokes' Theorem, that:

$$\frac{d}{dt} \int_N \chi_t^*(\omega) = \int_N \chi_t^*(X \rfloor d\,\omega) + \int_{\partial N} \chi_t^*(X \rfloor \omega).$$

From this we obtain that χ is an *extremal of* $L(\chi_t) = \int_N \chi_t^* \omega$ if

$$\chi^*(X \rfloor d\,\omega) = 0 \qquad \forall\ X \in \mathfrak{X}(M).$$

Moreover we obtain the condition that X must be *transversal to the boundary*, i.e.

$$\chi^*(X \rfloor \omega)|_{\partial N} = 0.$$

Now the 2nd variation turns out to be given by

$$\frac{d^2}{dt^2} \int_N \chi_t^* \omega = \int_N \chi_t^*(X \rfloor d\,(X \rfloor \omega)) + \int_{\partial N} \chi_t^*(X \rfloor d(X \rfloor \omega)).$$

This kind of analysis goes back to Cartan, and applies as well to the case of the H-perimeter form σ_H in general k-step Carnot groups, but, of course, it can be used in studying more general variational problems in the subriemannian setting.

Below we shall introduce basic tools and definitions that nedeed to compute the 1st and the 2nd variation of the H-perimeter form σ_H on regular non-characteristic hypersurfaces of 2-step Carnot groups. We stress that we are dealing with 2-step Carnot groups, because in this case, we have previously developed the useful method of H-adapted moving frames. Actually, we will see in section 7.4, how stating some of these results for the case of k-step groups without using such H-adapted moving frames. For many calculations and well-known results needed in this section we will follow, in many respects, the classical Spivak's book, [88].

We now begin by quoting the following standard fact:

Proposition 7.1. *[Leibnitz' rule] Let N be a compact oriented n-dimensional $\mathbf{C}^\infty$-smooth manifold with or without boundary, and $\mathbb{R} \ni t \longmapsto \omega(t) \in \Lambda^n(N)$ a $\mathbf{C}^\infty$ 1-parameter family of n-forms on N. Then*

$$\frac{d}{dt}\Big|_{t=t_0} \int_N \omega(t) = \int_N \dot{\omega}(t).$$

Proof. See [88], Proposition 10, Chapter 9, vol. IV. □

This elementary proposition can directly be applied to the case of a regular hypersurface immersed in a k-step Carnot group.

Throughout this section let $\mathbb{G}$ denote a 2-step n-dimensional Carnot group and S be a smooth immersed non-characteristic hypersurface with unit normal vector along S denoted by N. Moreover, let $\mathcal{U} \subset S$ be compact and suppose that the boundary $\partial\mathcal{U}$ is a smooth $n-2$-dimensional Riemannian submanifold with outward pointing unit normal η.

Definition 7.2 (Smooth variation). *Let $\imath : \mathcal{U} \longrightarrow \mathbb{G}$ denote the inclusion of $\mathcal{U}$ into $\mathbb{G}$, and let $\vartheta : (-\epsilon, \epsilon) \times \mathcal{U} \longrightarrow \mathbb{G}$ be a $\mathbf{C}^\infty$ map. We say that ϑ is a* **smooth variation of** $\imath$ *if the following hold:*

(i) Each $\vartheta_t := \vartheta(t, \cdot) : \mathcal{U} \longrightarrow \mathbb{G}$ is an immersion;

(ii) $\vartheta_0 = \imath$;

(iii) $\vartheta_t|_{\partial\mathcal{U}} = \imath|_{\partial\mathcal{U}}$ for each $t \in (-\epsilon, \epsilon)$.

We remark that the pull-back metric $\vartheta_t^*\langle\cdot,\cdot\rangle$ on $\mathcal{U}$ determines a volume element on $\mathcal{U}$ which can be regarded as an $n-1$-form on $\mathcal{U}$, denoted by σ_t^{n-1}. The classical approach (see, for instance, [16], [63], [88]) to the calculation of either the first or the 2nd variation of the Riemannian volume form, is that to applying the above Lemma 7.2 to compute

$$\frac{d}{dt}\Big|_{t=0}\int_M \sigma_t^{n-1}, \qquad \frac{d^2}{dt^2}\Big|_{t=0}\int_M \sigma_t^{n-1}.$$

In the sequel, if $t\in(-\epsilon,\epsilon)$, we denote by N_t the unit normal vector along $\mathcal{U}_t:=\vartheta_t(\mathcal{U})$. Notice that, if $\{\mathrm{t}_1,...,\mathrm{t}_{n-1}\}$ is any orthonormal frame for $\mathcal{U}$, then

$$N_t=\frac{\vartheta_{t_n *}\mathrm{t}_1\wedge...\wedge\vartheta_{t_n *}\mathrm{t}_{n-1}}{|\vartheta_{t_n *}\mathrm{t}_1\wedge...\wedge\vartheta_{t_n *}\mathrm{t}_{n-1}|}.$$

Remark 7.3. *Let $\mathbb{G}$, S, $\mathcal{U}$ and ϑ be as above. If $\mathcal{U}$ and ϵ are small enough, then* $\mathcal{U}_t=\vartheta_t(\mathcal{U})$ **turn out to be non-characteristic** *for every $t\in(-\epsilon,\epsilon)$. Obviously, this is just a local property and this fact can easily be proved by a contradiction argument.*

From now on we choose $\mathcal{U}$ and ϵ so that any $\mathcal{U}_t$ is non-characteristic. Therefore, according to Definition 5.2, we may define the H-perimeter form $\sigma_{H,t}$ on $\mathcal{U}_t$ as

$$(\sigma_{H,t})|_{\mathcal{U}_t}=(\nu_{H,t}\,\lrcorner\,\Omega^n)|_{\mathcal{U}_t}\in\Lambda^{n-1}(\mathcal{U}_t), \qquad t\in(-\epsilon,\epsilon),$$

where we have set

$$\nu_{H,t}:=\frac{\mathcal{P}_H(N_t)}{|\mathcal{P}_H(N_t)|_H}. \tag{113}$$

Clearly, the family of $n-1$-forms

$$\Gamma(t):=\vartheta_t^*\sigma_{H,t}\in\Lambda^{n-1}(\mathcal{U}), \qquad t\in(-\epsilon,\epsilon), \tag{114}$$

is a $\mathbf{C}^\infty$ 1-parameter family of $n-1$-forms on $\mathcal{U}$ satisfying Proposition 7.2. Thus, if we want to determine the 1st variation of σ_H on $\mathcal{U}$ given by

$$I_{\mathcal{U}}(\sigma_H):=\frac{d}{dt}\Big|_{t=0}\int_{\mathcal{U}}\Gamma(t), \tag{115}$$

it suffices to determine $\dot\Gamma(0)$. From now on, let $\frac{\partial}{\partial t}$ denote the *canonical vector field* along the 1st factor in $(-\epsilon,\epsilon)\times\mathcal{U}$ and denote by W its variation vector field, defined as $W:=\vartheta_*\frac{\partial}{\partial t}\big|_{t=0}$.

7.2 1st variation of $\sigma_{_H}$ in 2-step Carnot groups

The main result that we shall prove in this section is the computation of $\dot{\Gamma}(t)|_{t=0}$. The result turns out to be a generalized version of the corresponding Riemannian one, but we cannot expect such an extension of it to be too trivial, because the different features between the Riemannian volume form on hypersurfaces and the H-perimeter form. We refer the reader to Section 6.1 for definitions and notation which will be used in sequel.

Our main result of this section reads as follows:

Theorem 7.4. *Let $\mathbb{G}$ be a 2-step Carnot group and let $\imath : \mathcal{U} \longrightarrow \mathbb{G}$ denote the inclusion into $\mathbb{G}$ of a smooth non-characteristic hypersurface $\mathcal{U}$, with boundary $\partial\mathcal{U}$. Moreover, let $\vartheta : (-\epsilon, \epsilon)\times\mathcal{U} \longrightarrow \mathbb{G}$ be a smooth variation of $\imath$, with variation vector field W, and assume that $\mathcal{U}_t = \vartheta_t(\mathcal{U})$ is non-characteristic for every $t \in (-\epsilon, \epsilon)$. Finally, let $\Gamma(t) = \vartheta_t^*\sigma_{_{H,t}}$ denote the $\mathbf{C}^\infty$ 1-parameter family of $n-1$-forms on $\mathcal{U}$ defined by* (114)*. Then the following hold:*

(i)

$$\dot{\Gamma}(0) = \Big\{ - \mathcal{H}_{_H}^{sc} \langle \mathcal{P}_H(W), \nu_{_H}\rangle \, \sigma_{_H} - \mathcal{H}_{_H}^{sc} \langle \mathcal{P}_Z(W), \mathcal{P}_Z(N)\rangle \, \sigma^{n-1} + d\left(|\mathcal{P}_H(N)|_H \, (W \rfloor \sigma^{n-1})\right)\Big\}\Big|_{\mathcal{U}}. \tag{116}$$

(ii)

$$I_{\mathcal{U}}(\sigma_{_H}) = - \int_{\mathcal{U}} \mathcal{H}_{_H}^{sc} \langle \mathcal{P}_H(W), \nu_{_H}\rangle_H \, \sigma_{_H} - \int_{\mathcal{U}} \mathcal{H}_{_H}^{sc} \langle \mathcal{P}_Z(W), \mathcal{P}_Z(N)\rangle \, \sigma^{n-1} + \int_{\partial\mathcal{U}} \langle W, \eta\rangle \, |\mathcal{P}_H(N)|_H \, \sigma^{n-2}. \tag{117}$$

Proof. Let U be an open set containing $\mathrm{Im}(\vartheta)$. We now fix an *H-adapted moving frame* $(\zeta_1, ..., \zeta_n)$ *for S on U* (see Definition 6.1) such that:

(i) $\zeta_1(\vartheta(t, x)) := \nu_{_{H,t}}(x)$ (see (113) of Section 7.1);

(ii) $H_{\vartheta(t,x)}T\mathcal{U}_t = \mathrm{span}\{\zeta_2(\vartheta(t, x)), ..., \zeta_m(\vartheta(t, x))\} \quad \forall\, x \in \mathcal{U}$;

(iii) $\zeta_\alpha := X_\alpha \qquad \forall\ \alpha \in I_2 = \{m+1, ..., n\}$.

Condition (ii) above means that $\{\zeta_2, ..., \zeta_m\}$ is a family of linearly independent horizontal vector fields which span $HT\mathcal{U}_t$, i.e. the *horizontal tangent bundle of* $\mathcal{U}_t$. Furthermore, we shall denote by $(\varphi_1, ..., \varphi_n)$ the corresponding dual coframe (i.e. $\varphi_I(\zeta_J) = \delta_I^J,\ I,\ J = 1, ..., n$). Clearly, by construction, *this frame and its associated coframe, satisfy all the properties discussed in Section 6.1.* We stress that, at $t = 0$, the orthonormal moving frame now defined, is an H-adapted moving frame along $\mathcal{U}$ (so that $\zeta_I = \tau_I$ and $\varphi_I = \phi_I$ for every $I = 1, ..., n$). Note that

(iv) $\sigma_{H,t}|_{\mathcal{U}_t} = (\zeta_1 \rfloor \varphi_1 \wedge ... \wedge \varphi_n)|_{\mathcal{U}_t} = (\varphi_2 \wedge ... \wedge \varphi_n)|_{\mathcal{U}_t}$;

(v) $\Gamma(t) = \vartheta_t^*(\varphi_2 \wedge ... \wedge \varphi_n)$.

The variation vector field W on $\mathcal{U}$ is the restriction of the vector field $\widetilde{W} = \frac{\partial\vartheta}{\partial t}$, which can be extended on some open set $U \subset \mathbb{G}$ containing $\mathrm{Im}(\vartheta)$. Clearly the integral curve of $\widetilde{W}$ that starts at a point $x \in \mathcal{U}$ is just $t \longmapsto \vartheta_t(x)$.

Step 1. *We claim that*

$$\dot{\Gamma}(0) = \imath^*(\mathcal{L}_{\widetilde{W}}(\sigma_{H,t})) = \imath^*(\mathcal{L}_{\widetilde{W}}(\varphi_2 \wedge ... \wedge \varphi_n)).$$

Proof of Step 1. The proof of this fact is standard and it can be found in [88]. Denoting by $\gamma_{\widetilde{W}_x}(t)$ the integral curve of $\widetilde{W}$ starting at $x \in U$, if $x \in \mathcal{U}$ and $Y \in T_x\mathcal{U}$, we get

$$\gamma_{\widetilde{W}_x}(t)_*(\imath_* Y) = \vartheta_{t*}Y.$$

So let $Y_1, ..., Y_{n-1}$ be tangent vectors along $\mathcal{U}$. Then

$$\begin{aligned}
\dot{\Gamma}(0)(Y_1, ..., Y_{n-1}) &= \lim_{s\to 0} \frac{1}{s}\{\Gamma(s)(Y_1, ..., Y_{n-1}) - \Gamma(0)(Y_1, ..., Y_{n-1})\}\\
&= \lim_{s\to 0} \frac{1}{s}\{{\vartheta_s}^*\sigma_{H,t}(Y_1, ..., Y_{n-1}) - \imath^*\sigma_{H,t}(Y_1, ..., Y_{n-1})\}\\
&= \lim_{s\to 0} \frac{1}{s}\{\sigma_{H,t}(\vartheta_{s*}Y_1, ..., \vartheta_{s*}Y_{n-1}) - \sigma_{H,t}(\imath_*Y_1, ..., \imath_*Y_{n-1})\}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{s\to 0}\frac{1}{s}\{\sigma_{H,t}(\theta_{t*}(\imath_* Y_1),...,\theta_{t*}(\imath_* Y_{n-1})) - \sigma_{H,t}(\imath_* Y_1,...,\imath_* Y_{n-1})\} \\
&= \mathcal{L}_{\widetilde{W}}\sigma_{H,t}(\imath_* Y_1,...,\imath_* Y_{n-1}) \qquad \text{(by definition of Lie derivative).}
\end{aligned}
$$

□

By using Cartan's identity we get

$$
\mathcal{L}_{\widetilde{W}}\sigma_{H,t} = \widetilde{W} \rfloor d\,\sigma_{H,t} + d\,(\widetilde{W} \rfloor \sigma_{H,t})
$$

and therefore by Step 1.

$$
\dot{\Gamma}(0) = \imath^*(\widetilde{W} \rfloor d\,\sigma_{H,t} + d\,(\widetilde{W} \rfloor \sigma_{H,t})). \tag{118}
$$

Now we have

$$
\begin{aligned}
d\,\sigma_{H,t} &= d\,(\varphi_2 \wedge ... \wedge \varphi_n) \\
&= \sum_{I=2}^{n}(-1)^I \varphi_2 \wedge ... \wedge d\,\varphi_I \wedge ... \wedge \varphi_n \\
&= \sum_{I=2}^{n}(-1)^I \varphi_2 \wedge ... \wedge \Big(-\sum_{J=1}^{n}\varphi_{JI}\wedge\varphi_J\Big) \wedge ... \wedge \varphi_n & (119)\\
&= -\sum_{I=2}^{n}(-1)^I \varphi_2 \wedge ... \wedge (\varphi_{1I}\wedge\varphi_1) \wedge ... \wedge \varphi_n. & (120)
\end{aligned}
$$

Note that equality (119) is the 1st structure equation of the coframe $(\varphi_1, ..., \varphi_n)$, while equality (120) comes from the fact that J can only be equal to 1. Also, we have

$$
\varphi_{1I} = \sum_{K=1}^{n}\varphi_{1I}(\zeta_K)\,\varphi_K.
$$

Therefore, by substituting this identity into (120) we obtain

$$
\begin{aligned}
d\,\sigma_{H,t} &= -\sum_{I=2}^{n}(-1)^I(-1)^{I-1}\varphi_1 \wedge ... \wedge \varphi_{1I} \wedge ... \wedge \varphi_n \\
&= \sum_{I=2}^{n}\varphi_{1I}(\zeta_I)\,\varphi_1 \wedge ... \wedge \varphi_I \wedge ... \wedge \varphi_n \qquad \text{(since } K \text{ must be equal to } I\text{)} \\
&= \sum_{i=2}^{m}\varphi_{1i}(\zeta_i)\,\varphi_1 \wedge ... \wedge \varphi_n, & (121)
\end{aligned}
$$

where (121) follows since the H-adapted coframe $(\varphi_1, ..., \varphi_n)$ for S on U satisfies, by construction, item (iii) of Lemma 6.4, i.e.:

$$\varphi_{i\alpha}(\zeta_\beta) = 0 \qquad \forall\, i \in I_1 = \{1, ..., m\},\ \forall\, \alpha,\, \beta \in I_2 = \{m+1, ..., n\}.$$

Thus we get

$$\begin{aligned} \imath^*\big(\widetilde{W} \rfloor d\,\sigma_{_{H,t}}\big) &= \big(\widetilde{W} \rfloor d\,\sigma_{_{H,t}}\big)|_{\mathcal{U}} \\ &= \Big\{\sum_{i=2}^{m} \varphi_{1i}(\zeta_i)\Big(\widetilde{W} \rfloor \varphi_1 \wedge ... \wedge \varphi_n\Big)\Big\}\Big|_{\mathcal{U}} \\ &= \left\{\left[\sum_{i=2}^{m} \varphi_{1i}(\zeta_i)\right] \langle \widetilde{W}, N_t\rangle\, \sigma_t^{n-1}\right\}\Bigg|_{\mathcal{U}} \\ &= -\mathcal{H}^{sc}_{_H} \langle W, N\rangle\, \sigma^{n-1}|_{\mathcal{U}} \end{aligned} \tag{122}$$

since $\imath^*\zeta_1 = \vartheta_0^*\zeta_1 = \nu_{_H}$. Note that at the last line we have used Definition 6.15 and Remark 6.16. Now the second term in (118) is easily computed by using the fact that

$$\imath^*(d\,(\widetilde{W} \rfloor \sigma_{_{H,t}})) = d\,(\imath^*(\widetilde{W} \rfloor \sigma_{_{H,t}})). \tag{123}$$

Moreover

$$\imath^*(\widetilde{W} \rfloor \sigma_{_{H,t}}) = \imath^*(\widetilde{W} \rfloor |\mathcal{P}_H(N_t)|_H\, \sigma_t^{n-1}) = (W \rfloor |\mathcal{P}_H(N)|_H\, \sigma^{n-1})|_{\partial\mathcal{U}} = \Big\{|\mathcal{P}_H(N)|_H\, (W \rfloor \sigma^{n-1})\Big\}\Big|_{\partial\mathcal{U}}.$$

Finally, using the last relation and equalities (118) and (122) we get

$$\dot{\Gamma}(0) = -\mathcal{H}^{sc}_{_H} \langle W, N\rangle\, \sigma^{n-1} + d\,(|\mathcal{P}_H(N)|_H\, (W \rfloor \sigma^{n-1}))$$

and item (i) of the theorem follows by Remark 6.6 of Section 6.1. Now item (ii) easily follows by using (115), Leibnitz' rule, and then by integrating both sides of (116). Finally, for the second term, we use Stokes' Theorem and the fact that

$$(W \rfloor \sigma^{n-1})|_{\partial\mathcal{U}} = \langle W, \eta\rangle\, (\sigma^{n-2})|_{\partial\mathcal{U}}.$$

□

We now state a definition of "divergence" of a vector field on $\mathbb{G}$ with respect to the H-perimeter form on a regular hypersurface.

Definition 7.5. *Let $\mathbb{G}$ be a* k*-step Carnot group and let $S \subset \mathbb{G}$ be a smooth immersed non-characteristic hypersurface with unit normal vector along S denoted by N. If $X \in \mathfrak{X}(\mathbb{G})$ we shall hereafter denote by $\mathrm{Div}_{\sigma_H} X$ the* **divergence of** X **with respect to** σ_H*, that is, the function satisfying*

$$(\mathrm{Div}_{\sigma_H} X)\, \sigma_H|_S = \mathcal{L}_X \sigma_H|_S.$$

Notice that $X \in \mathfrak{X}(\mathbb{G})$, i.e. X is any smooth section of $T\mathbb{G}$ and it is not necessarily tangent along S.

Proposition 7.6. *Let $\mathbb{G}$ be a 2-step Carnot group and let $S \subset \mathbb{G}$ be a smooth immersed non-characteristic hypersurface with unit normal vector along S denoted by N. Let $\mathcal{U} \subset S$ be compact and suppose that the boundary $\partial\mathcal{U}$ is a smooth $n-2$-dimensional Riemannian submanifold with outward pointing unit normal η. Then the following two items hold:*

(i) For every smooth vector field $X \in \mathfrak{X}(\mathbb{G})$ we have

$$(\mathrm{Div}_{\sigma_H} X)\, \sigma_H|_S$$
$$= \left\{ \mathrm{div}_{VS} X + \frac{1}{|\mathcal{P}_H(N)|_H} \Big\langle \sum_{\beta \in I_2} n_\beta\, C^\beta \nu_H, \mathcal{P}_H(X) \Big\rangle_H - \mathcal{H}^{sc}_H \langle \mathcal{P}_H(X), \nu_H \rangle_H \right\} \sigma_H|_S;$$

(ii) Let $x \in S$ and suppose that $X_x \in H_x S$. Since $H_x S = HT_x S \oplus \mathrm{span}\{(\nu_H)_x\}$, we set $X_x := (X^{\nu_H})_x + (X^{HTS})_x$. Then, for every $X \in \mathbf{C}^\infty(\mathbb{G}, H)$ we have

$$(\mathrm{Div}_{\sigma_H} X)\, \sigma_H|_S$$
$$= \left\{ \mathrm{div}_{HTS}(X^{HTS}) + \frac{1}{|\mathcal{P}_H(N)|_H} \Big\langle \sum_{\beta \in I_2} n_\beta\, C^\beta \nu_H, X \Big\rangle_H - \mathcal{H}^{sc}_H \langle X, \nu_H \rangle_H \right\} \sigma_H|_S$$

Proof. The proof is a consequence of Lemma 6.20 of Section 6.2 and of the previous proof of Theorem 7.4. Indeed, using Cartan's formula we get

$$(\mathrm{Div}_{\sigma_H} X)\,\sigma_H|_S = (X \rfloor d\,\sigma_H)|_S + d\,(X \rfloor \sigma_H)|_S. \tag{124}$$

Now the second addend of the left hand side can easily be computed from Lemma 6.20 by noting that it does not depend on the ν_H-component of X. Moreover, we have already proved that, for every $X \in \mathfrak{X}(\mathbb{G})$ the first addend of (124) turns out to be $(X \rfloor d\,\sigma_H)|_S = -\mathcal{H}_H^{sc}\,\langle X, N\rangle\,\sigma^{n-1}|_S$. Therefore, the proof follows by adding these two terms using Remark 6.6 and definitions.

□

We can finally state the following (see also Theorem 6.24):

Corollary 7.7. *Let $\mathbb{G}$ be a 2-step Carnot group and let $S \subset \mathbb{G}$ be a smooth immersed non-characteristic hypersurface with unit normal vector along S denoted by N. Let $\mathcal{U} \subset S$ be compact and suppose that the boundary $\partial\mathcal{U}$ is a smooth $n-2$-dimensional Riemannian submanifold with outward pointing unit normal η. Let $U \subset \mathbb{G}$ be an open neighborhood of $\mathcal{U} = U \cap S$. Then for every smooth vector field $X \in \mathfrak{X}(U)$ we have*

$$\begin{aligned}\int_{\mathcal{U}} \mathrm{Div}_{\sigma_H}(X)\,\sigma_H = \;& -\int_{\mathcal{U}} \mathcal{H}_H^{sc}\,\langle \mathcal{P}_H(X), \nu_H\rangle_H\,\sigma_H - \int_{\mathcal{U}} \mathcal{H}_H^{sc}\,\langle \mathcal{P}_Z(X), \mathcal{P}_Z(N)\rangle\,\sigma^{n-1} \\ & + \int_{\partial\mathcal{U}} \langle X, \eta\rangle\,|\mathcal{P}_H(N)|_H\,\sigma^{n-2}.\end{aligned} \tag{125}$$

Proof. This is obvious from Theorem 7.4. □

7.3 2nd variation of σ_H in 2-step Carnot groups

In the present section we derive the formula for the 2nd variation of σ_H for regular non-characteristic hypersurfaces in 2-step Carnot groups, according to the general discussion of Section 7.1. The calculation itself is quite difficult and also the result has a very complicated expression. Thus we do not compute the boundary term of the variation, since we make use only of compactly supported vector fields. Also in this case we shall use H-adapted moving frames in doing computations. The proof of the following Theorem 7.8 can be regarded as a continuation of the proof of Theorem 7.4 and we refer to Section 7.2 for notation and previous results.

Moreover, since Theorem 7.4 implies that the 1st variation of σ_H only depend on the normal component of W (i.e. the component of W along N) we restrict ourselves to the case of *normal variations* of $\mathcal{U}$, that is, smooth variations with variation vector field which is normal along $\mathcal{U}$ and to the case of *horizontal normal variations* (see Theorem 7.10 below).

Theorem 7.8. *Let $\mathbb{G}$ be a 2-step Carnot group and $\imath : \mathcal{U} \longrightarrow \mathbb{G}$ be the inclusion into $\mathbb{G}$ of a smooth non-characteristic hypersurface $\mathcal{U}$. Let $\vartheta : (-\epsilon, \epsilon)\times\mathcal{U} \longrightarrow \mathbb{G}$ be a* **smooth normal variation of** *$\imath$, with variation vector field $W \in \mathbf{C}_0^{\infty}(\mathbb{G}, T\mathbb{G})$ such that $\mathrm{spt}(W)\cap\mathcal{U} \Subset \mathcal{U}$ and $W_x \in N_x\mathcal{U}\ \ \forall\, x \in \mathrm{Int}(\mathcal{U})$. Assume that $\mathcal{U}_t = \vartheta_t(\mathcal{U})$ is non-characteristic for every $t \in (-\epsilon, \epsilon)$. Furthermore, let $\Gamma(t) = \vartheta_t^*\sigma_{H,t}$ and $(\zeta_1, ..., \zeta_n)$ be an* **orthonormal moving frame for** *U, where U is an open set containing* $\mathrm{Im}(\vartheta)$ *(see Section 7.1). Below we shall denote by w the function $w := \frac{\langle W,N\rangle}{|\mathcal{P}_H(N)|_H}$. Then*

(i)

$$\ddot{\Gamma}(0) = \Big\{ - W(w)\mathcal{H}_H^{sc} + w\Big[w_1\Big(\langle \mathrm{H}, \nu_H\rangle_H^2 - \|\mathsf{b}_H\|^2_{Gram}\Big) - \mathcal{H}_H^{sc}\mathrm{div}_Z(\mathcal{P}_Z W)\Big]$$
$$-w\Big[\Delta_{HTS} w_1 + \sum_{\alpha\in I_2}\Big\langle \nabla_{\tau_\alpha^S}\nu_H, \Big(\nabla^{HTS}w_\alpha + w_1 C^\alpha\nu_H\Big)\Big\rangle - \frac{w_1}{2}\sum_{\alpha\in I_2}\|C^\alpha\nu_H\|^2_{Gram}$$
$$+\sum_{\alpha\in I_2}\Big(\frac{n_\alpha}{n_1}\Delta_{HTS}w_\alpha + \Big\langle\nabla^H\Big(\frac{n_\alpha}{n_1}\Big), \nabla^{HTS}w_\alpha\Big\rangle\Big) + \mathrm{div}_{HTS}(\mathcal{P}_{HTS}(CW))\Big]\Big\}\sigma_H;$$

(ii)

$$II_{\mathcal{U}}^{int}(\sigma_H) = \int_{\mathcal{U}}\Big\{ - W(w)\mathcal{H}_H^{sc} + w\Big[w_1\Big(\langle \mathrm{H}, \nu_H\rangle_H^2 - \|\mathsf{b}_H\|^2_{Gram}\Big) - \mathcal{H}_H^{sc}\mathrm{div}_Z(\mathcal{P}_Z W)\Big]$$
$$-w\Big[\Delta_{HTS} w_1 + \sum_{\alpha\in I_2}\Big\langle \nabla_{\tau_\alpha^S}\nu_H, \Big(\nabla^{HTS}w_\alpha + w_1 C^\alpha\nu_H\Big)\Big\rangle - \frac{w_1}{2}\sum_{\alpha\in I_2}\|C^\alpha\nu_H\|^2_{Gram}$$
$$+\sum_{\alpha\in I_2}\Big(\frac{n_\alpha}{n_1}\Delta_{HTS}w_\alpha + \Big\langle\nabla^H\Big(\frac{n_\alpha}{n_1}\Big), \nabla^{HTS}w_\alpha\Big\rangle\Big) + \mathrm{div}_{HTS}(\mathcal{P}_{HTS}(CW))\Big]\Big\}\sigma_H.$$

Preliminarily, we give a definition that will be used trough the proof of the next results.

Definition 7.9. *Let S be a smooth non-characteristic hypersurface and let $(\tau_1, ..., \tau_n)$ be an H-adapted frame for S with H-adapted coframe $(\phi_1, ..., \phi_n)$. Then we define $\mathsf{Ric}_H : NS \longrightarrow \nu_{_H} S$ by the following*

$$\mathsf{Ric}_H(X) := \sum_{i \in I_1} \langle \mathsf{R}(\tau_i, X)\, \tau_i, \tau_1 \rangle_H \;\tau_1 \qquad \forall\; X \in NS.$$

We also set

$$Ric_H(X) = \langle \mathsf{Ric}_H(X), \tau_1 \rangle \qquad (X \in NS).$$

Moreover we define $\overline{\mathsf{Ric}}_H : \nu_{_H} S \longrightarrow \nu_{_H} S$ by the following

$$\overline{\mathsf{Ric}}_H(X) := \sum_{i \in I_1} \langle \mathsf{R}(\tau_i, X)\, \tau_i, \tau_1 \rangle_H \;\tau_1 \qquad \forall\; X \in \nu_{_H} S,$$

and we set

$$\overline{Ric}_H(X) = \langle \mathsf{Ric}_H(X), \tau_1 \rangle \qquad (X \in NS).$$

Proof of Theorem 7.8. Since the variation vector field W along $\mathcal{U}$ is a normal vector field, by using the coordinates given by the orthonormal frame $(\zeta_1, ..., \zeta_n)$ for the open set U (see, for instance, the proof of Theorem 7.4) we get that $\widetilde{W} = \widetilde{w}_1 \zeta_1 + \sum_{\beta \in I_2} \widetilde{w}_\beta \zeta_\beta$.

Using the hypothesis and Theorem 7.4 gets us the local expression of the 1st variation of $\sigma_{_H}$ at the interior of $\mathcal{U}$, i.e.

$$\begin{aligned}
\dot{\Gamma}(0) &= -\mathcal{H}^{sc}_{_H} \langle \mathcal{P}_H(W), \nu_{_H} \rangle\, \sigma_{_H} - \mathcal{H}^{sc}_{_H} \langle \mathcal{P}_Z(W), \mathcal{P}_Z(N) \rangle\, \sigma^{n-1} \\
&= -\mathcal{H}^{sc}_{_H} \frac{\langle W, N \rangle}{|\mathcal{P}_H(N)|_H} \sigma_{_H}. \qquad (126)
\end{aligned}$$

More generally, we remark that we have already proved that on $\mathrm{Int}(\mathcal{U})$ the 1st variation at $t \in (-\epsilon, \epsilon)$ is given by

$$\begin{aligned} \dot{\Gamma}(t) &= i^*(\widetilde{W} \rfloor d\,\sigma_{_{H,t}}) \\ &= \left\{ \sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \Big(\widetilde{W} \rfloor \varphi_1 \wedge ... \wedge \varphi_n \Big) \right\}\Big|_{\mathcal{U}} \\ &= \left\{ \left[\sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \right] \langle \widetilde{W}, N_t \rangle\, \sigma_t^{n-1} \right\}\Bigg|_{\mathcal{U}}. \end{aligned}$$

We then get for every $t \in (-\epsilon, \epsilon)$ the relation

$$\begin{aligned} \dot{\Gamma}(t) &= \left\{ \left[\sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \right] \frac{\langle \widetilde{W}, N_t \rangle}{|\mathcal{P}_H(N_t)|_H}\, \sigma_{_{H,t}} \right\}\Bigg|_{\mathcal{U}} \\ &= \left\{ \left[\sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \right] \frac{\langle \widetilde{W}, N_t \rangle}{|\mathcal{P}_H(N_t)|_H}\, \varphi_2 \wedge ... \wedge \varphi_n \right\}\Bigg|_{\mathcal{U}}. \end{aligned} \tag{127}$$

Throughout this proof, to sake of simplicity, we shall set:

$$w := \frac{\langle W, N \rangle}{|\mathcal{P}_H(N)|_H}, \qquad w_t := \frac{\langle \widetilde{W}, N_t \rangle}{|\mathcal{P}_H(N_t)|_H}.$$

According to the proof of Theorem 7.4 we then have to compute

$$\ddot{\Gamma}(0)|_{\mathcal{U}} \;=\; i^*\{\mathcal{L}_{\widetilde{W}}(\widetilde{W} \rfloor d\,\sigma_{_{H,t}})\}. \tag{128}$$

To this aim we make use of (137) by noting that

$$\begin{aligned} \dot{\Gamma}(t) &= \left\{ \left[\sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \right] w_t\, \varphi_2 \wedge ... \wedge \varphi_n \right\}\Bigg|_{\mathcal{U}} \\ &= \sum_{j=2}^{m} \left\{ w_t\, \varphi_2 \wedge ... \wedge \underbrace{\varphi_{1j}}_{j-th\,place} \wedge ... \wedge \varphi_n \right\}\Bigg|_{\mathcal{U}}. \end{aligned} \tag{129}$$

This easily follows from definitions and Lemma 6.3. Therefore we preliminarily have to compute the following expressions:

(i) $\mathcal{L}_{\widetilde{W}}(\varphi_h)$ for $h \in I_1 \setminus \{1\} = \{2, ..., m\}$;

(ii) $\mathcal{L}_{\widetilde{W}}(\varphi_\alpha)$ for $\alpha \in I_2 = \{m+1, ..., n\}$;

(iii) $\mathcal{L}_{\widetilde{W}}(\varphi_{1j})$ for $j \in I_1 \setminus \{1\}$.

This can be done by means of Cartan's formula and of the structure equations for our H-adapted coframe $(\varphi_1, ..., \varphi_n)$. For the term appearing at item (i) we get

$$\mathcal{L}_{\widetilde{W}}(\varphi_h) = \sum_{J \neq h} \Big\{ \varphi_J(\widetilde{W})\varphi_{Jh} - \varphi_{Jh}(\widetilde{W})\varphi_J \Big\}.$$

Similarly, for the term appearing at item (ii), we have

$$\mathcal{L}_{\widetilde{W}}(\varphi_\alpha) = \widetilde{w}_1 \sum_{h=2}^{m} \Big(\varphi_{1\alpha}(\zeta_h) - \varphi_{h\alpha}(\zeta_1) \Big) \varphi_h + d\,\widetilde{w}_\alpha.$$

Finally (iii) can be obtained as follows:

$$\begin{aligned}
\mathcal{L}_{\widetilde{W}}(\varphi_{1j}) &= \widetilde{W} \rfloor d\,\varphi_{1j} + d\,(\widetilde{W} \rfloor \varphi_{1j}) \\
&= \sum_{L \neq 1, j} \left\{ \Big(\varphi_{1L}(\widetilde{W})\varphi_{Lj} - \varphi_{Lj}(\widetilde{W})\varphi_{1L} \Big) - \varPhi_{1j}(\widetilde{W}) + d\,(\varphi_{1j}(\widetilde{W})) \right\} \\
&= \sum_{L \neq 1, j, K} \left\{ \Big(\varphi_{1L}(\widetilde{W})\varphi_{Lj} - \varphi_{Lj}(\widetilde{W})\varphi_{1L} \Big) - \langle \mathsf{R}(\zeta_K, \widetilde{W})\,\zeta_j, \zeta_1 \rangle \varphi_K + \, d\,(\varphi_{1j}(\widetilde{W})) \right\}
\end{aligned} \tag{130}$$

where in (130) the 2-forms $\varPhi_{JK}$ are the curvature 2-forms of the coframe $(\varphi_1, ..., \varphi_n)$ for U, which are defined by

$$\begin{aligned}
\varPhi_{JK}(X, Y) &:= \varphi_K(\mathsf{R}(X, Y)\,\zeta_J) \\
&= \langle \mathsf{R}(X, Y)\,\zeta_J, \zeta_K \rangle
\end{aligned}$$

for all $X, Y \in \mathfrak{X}(\mathbb{G})$ $(J, K = 1, ...n)$.

Now from (128) and (129) we get

$$\begin{aligned}
\ddot{\Gamma}(t) &= \sum_{j=2}^{m} \mathcal{L}_{\widetilde{W}} \Big\{ w_t\, \varphi_2 \wedge ... \wedge \varphi_{1j} \wedge ... \wedge \varphi_n \Big\} \\
&= \widetilde{W}(w_t) \left[\sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \right] \sigma_{H,t} + w_t \sum_{j=2}^{m} \mathcal{L}_{\widetilde{W}} \Big\{ \varphi_2 \wedge ... \wedge \varphi_{1j} \wedge ... \wedge \varphi_n \Big\} \\
&= I + II,
\end{aligned}$$

where

$$II := w_t \sum_{j=2}^{m} \mathcal{L}_{\widetilde{W}} \Big\{ \varphi_2 \wedge ... \wedge \varphi_{1j} \wedge ... \wedge \varphi_n \Big\}$$

$$= w_t \sum_{j,\, h=2}^{m} \varphi_2 \wedge ... \wedge \mathcal{L}_{\widetilde{W}}(\varphi_h) \wedge ... \wedge \varphi_{1j} \wedge ... \wedge \varphi_n \tag{131}$$

$$+ w_t \sum_{j=2}^{m} \sum_{\alpha \in I_2} \varphi_2 \wedge ... \wedge \varphi_{1j} \wedge ... \wedge \mathcal{L}_{\widetilde{W}}(\varphi_\alpha) \wedge ... \wedge \varphi_n \tag{132}$$

$$+ w_t \sum_{j=2}^{m} \varphi_2 \wedge ... \wedge \mathcal{L}_{\widetilde{W}}(\varphi_{1j}) \wedge ... \wedge \varphi_n = II_1 + II_2 + II_3. \tag{133}$$

To compute each of these terms we make use of the previous computations of the terms (i), (ii) and (iii). More precisely, we get

$$\begin{aligned}
II_1 : \ &= \sum_{j,h \in I_1 \setminus \{1\}} w_t \Big\{ \varphi_2 \wedge ... \wedge \big(\varphi_{1j}(\zeta_j)\varphi_j + \varphi_{1j}(\zeta_h)\varphi_h \big) \wedge ... \\
&\quad ... \wedge \Big[\sum_{L \neq h} \big(\varphi_{hL}(\widetilde{W})\varphi_L - w_L \varphi_{hL} \big) \Big] \wedge ... \wedge \varphi_n \Big\} \\
&= \sum_{j,h \in I_1 \setminus \{1\}} w_t \Big\{ \varphi_{1j}(\zeta_j) \Big[- \sum_{L \neq h} w_L \varphi_{hL}(\zeta_h) \Big] - \varphi_{1j}(\zeta_h) \Big[\varphi_{hj}(\widetilde{W}) - \sum_{L \neq h} w_L \varphi_{hL}(\zeta_j) \Big] \Big\} \sigma_{H,t} \\
&= \sum_{j,h \in I_1 \setminus \{1\}} w_t \Big\{ w_1 \varphi_{1j}(\zeta_j)\varphi_{1h}(\zeta_h) - \varphi_{1j}(\zeta_h) \Big(\sum_{L \neq h} w_L \langle [\zeta_j, \zeta_L], \zeta_h \rangle \Big) \Big\} \sigma_{H,t} \\
&= \sum_{j,h \in I_1 \setminus \{1\}} w_t w_1 \Big\{ \varphi_{1j}(\zeta_j)\varphi_{1h}(\zeta_h) - \varphi_{1j}(\zeta_h) \langle [\zeta_j, \zeta_1], \zeta_h \rangle \Big\} \sigma_{H,t} \\
&= \sum_{j,h \in I_1 \setminus \{1\}} w_t w_1 \Big\{ \varphi_{1j}(\zeta_j)\varphi_{1h}(\zeta_h) - \varphi_{1j}(\zeta_h) (\varphi_{1h}(\zeta_j) - \varphi_{jh}(\zeta_1))\rangle \Big\} \sigma_{H,t};
\end{aligned}$$

$$\begin{aligned}
II_2 := w_t \sum_{j \in I_1 \setminus \{1\}} \sum_{\alpha \in I_2} &\Big\{ \varphi_{1j}(\zeta_1) \Big[\zeta_1(\widetilde{w}_\alpha) + \widetilde{w}_1 \Big(\varphi_{1\alpha}(\zeta_j) - \varphi_{j\alpha}(\zeta_1) \Big) \Big] \frac{n_{t\,\alpha}}{n_{t\,1}} \\
&+ \varphi_{1j}(\zeta_j)\zeta_\alpha(\widetilde{w}_\alpha) - \varphi_{1j}(\zeta_\alpha) \Big[\zeta_j(\widetilde{w}_\alpha) + \widetilde{w}_1 \Big(\varphi_{1\alpha}(\zeta_j) - \varphi_{j\alpha}(\zeta_1) \Big) \Big] \Big\} \sigma_{H,t}
\end{aligned}$$

$$= w_t \sum_{j \in I_1 \setminus \{1\}} \sum_{\alpha \in I_2} \Big\{ \varphi_{1j}(\zeta_j)\zeta_\alpha(\widetilde{w}_\alpha) + - \varphi_{1j} \Big(\zeta_\alpha - \frac{n_{t\,\alpha}}{n_{t\,1}} \zeta_1 \Big) \Big[\zeta_j(\widetilde{w}_\alpha) + \widetilde{w}_1 \Big(\varphi_{1\alpha}(\zeta_j) - \varphi_{j\alpha}(\zeta_1) \Big) \Big] \Big\} \sigma_{H,t}.$$

Now we put $II_3 := (II_3)_a + (II_3)_b$, where

$$\begin{aligned}(II_3)_a &= w_t \sum_{j\in I_1\setminus\{1\}} \Big\{\varphi_2\wedge ... \wedge \Big\{ -\sum_K \langle \mathsf{R}(\zeta_K, \widetilde{W})\,\zeta_j, \zeta_1\rangle\, \varphi_K \\ &\quad + \sum_{L\neq 1,j} \Big[\varphi_{1L}(\widetilde{W})\varphi_{Lj} - \varphi_{Lj}(\widetilde{W})\varphi_{1L}\Big]\Big\}\wedge ... \wedge \varphi_n\Big\}\end{aligned}$$

and, by using Lemma 6.3 and Definition 7.9, we get

$$(II_3)_a = -w_t \Big\{ Ric_H(\widetilde{W}) - \sum_{j\in I_1\setminus\{1\}}\sum_{L\neq 1,j}\Big[\varphi_{1L}(\widetilde{W})\varphi_{Lj}(\zeta_j) - \varphi_{Lj}(\widetilde{W})\varphi_{1L}(\zeta_j)\Big]\Big\}\,\sigma_{_{H,t}}.$$

Furthermore we have

$$\begin{aligned}(II_3)_b &:= w_t \sum_{j=2}^m \varphi_2\wedge ... \wedge d\,(\varphi_{1j}(\widetilde{W}))\wedge ... \wedge\varphi_n \\ &= w_t\sum_{j=2}^m \zeta_j(\varphi_{1j}(\widetilde{W}))\sigma_{_{H,t}}. \qquad (134)\end{aligned}$$

Claim 1. *We claim that the connection 1-forms φ_{ij} are 0 whenever $j, h \in I_1\setminus\{1\}$.*

Proof. Consider a Riemannian orthonormal moving frame on U *H-adapted to the open set* $\mathcal{U} = U\cap S$. This means that we have an orthonormal frame $\underline{\xi} = \{\xi_1, ..., \xi_n\}$ on U, satisfying $\xi_1(p) = N(p)$ (N is the Riemannian unit normal along S) and such that

$$\underline{\xi^S} = \mathrm{span}_{\mathbb{R}}\{\xi_2(p), ..., \xi_n(p)\} = T_pS$$

for every $p\in\mathcal{U}\subset S$. Moreover let us denote by $\underline{\varepsilon} = \{\varepsilon_1, ..., \varepsilon_n\}$ its dual co-frame. Claim: *It is always possible to choose another Riemannian orthonormal moving frame $\underline{\widetilde{\xi}}$ for U H-adapted to $\mathcal{U}$ satisfying:*

(i) $\underline{\widetilde{\xi}}(p_0) = \underline{\xi}(p_0)$*;*

(ii) *The connection 1-forms $\widetilde{\varepsilon}_{IJ} = \langle\nabla\widetilde{\xi}_I, \widetilde{\xi}_J\rangle$ $(I, J = 1, ..., n)$ for $\underline{\widetilde{\xi}}$ satisfies $\widetilde{\varepsilon}_{ij}(p_0) = 0$ for every $i, j = 2, ..., n$.*

Here again, $\underline{\widetilde{\xi}^S} = \{\widetilde{\xi}_2, ..., \widetilde{\xi}_n\}$ is a tangent orthonormal frame for $\mathcal{U}$. We stress that the proof of this claim is standard and it can be found, for instance, in [88], pag.

517-519, eq.(17). Therefore, from this fact the thesis easily follows by assuming that at p_0 the frame $\underline{\xi}$ satisfy $\xi_i(p_0) = \zeta_i(p_0)$ for every $i \in I_1 \setminus \{1\}$, i.e. the set of vectors $\{\xi_2(p_0), ..., \xi_m(p_0)\}$ is an orthonormal basis of the horizontal tangent space $H_{p_0}S$ at p_0, coinciding with that given at the beginning. In this case we get, in particular, that

$$\widetilde{\varepsilon}_{ij}(p_0) = \langle \nabla_{X_{p_0}} \widetilde{\xi}_i, \widetilde{\xi}_j \rangle (p_0) = 0 \qquad \text{for every} \quad i, j \in I_1 \setminus \{1\}.$$

By extending the orthonormal frame $\{\widetilde{\xi}_2, ..., \widetilde{\xi}_m\}$ for the horizontal tangent space to a full H-adapted frame $\underline{\zeta}$ we get our initial claim. $\square$

Claim 2. *We claim that* $\langle [\widetilde{W}, \vartheta_{t*}X], N_t \rangle = 0$ *for every* $X \in \mathbf{C}^\infty(\mathcal{U}, HT\mathcal{U})$.

Proof. A proof of this claim can be found in Spivak, [88], Ch. 9, pag. 521-522. $\square$

Claim 3. *Let us set* $\mathcal{C}^t := \sum_{\alpha \in I_2} \frac{n_{t\alpha}}{n_{t1}} \mathcal{C}^\alpha$. *Then we have*

$$\nabla^H_{\widetilde{W}} \zeta_1 = -\nabla^{HT\mathcal{U}_t} \widetilde{w}_1 - \sum_{\alpha \in I_2} \frac{n_{t\alpha}}{n_{t1}} \nabla^{HT\mathcal{U}_t} \widetilde{w}_\alpha - \mathcal{P}_{HT\mathcal{U}_t}(\mathcal{C}^t \widetilde{W}). \tag{135}$$

Proof. Using the previous Claim 3 we get $\langle [\widetilde{W}, \zeta_j], N_t \rangle = 0$ for every $j \in I_1 \setminus \{1\}$. Therefore

$$\langle \nabla_{\widetilde{W}} \zeta_j, N_t \rangle = \langle \nabla_{\zeta_j} \widetilde{W}, N_t \rangle \qquad (j \in I_1 \setminus \{1\}).$$

This implies that

$$\begin{aligned}
-\langle \nabla^H_{\widetilde{W}} \nu_{H,t}, \zeta_j \rangle &= \langle \nabla_{\zeta_j} \widetilde{W}, \nu_{H,t} \rangle + \sum_{\alpha \in I_2} \frac{n_{t\alpha}}{n_{t1}} \big(\langle \nabla_{\zeta_j} \widetilde{W}, \zeta_\alpha \rangle - \langle \nabla_{\widetilde{W}} \zeta_j, \zeta_\alpha \rangle \big) \\
&= \zeta_j(\widetilde{w}_1) + \sum_{\alpha \in I_2} \frac{n_{t\alpha}}{n_{t1}} \zeta_j(\widetilde{w}_\alpha) + \sum_{\alpha \in I_2} \sum_I \widetilde{w}_I \frac{n_{t\alpha}}{n_{t1}} \big(\langle \nabla_{\zeta_j} \zeta_I, \zeta_\alpha \rangle - \langle \nabla_{\zeta_I} \zeta_j, \zeta_\alpha \rangle \big) \\
&= \zeta_j(\widetilde{w}_1) + \sum_{\alpha \in I_2} \frac{n_{t\alpha}}{n_{t1}} \zeta_j(\widetilde{w}_\alpha) + \sum_{\alpha \in I_2} \sum_I \widetilde{w}_I \frac{n_{t\alpha}}{n_{t1}} \mathcal{C}^\alpha_{jI} \\
&= \zeta_j(\widetilde{w}_1) + \sum_{\alpha \in I_2} \frac{n_{t\alpha}}{n_{t1}} \zeta_j(\widetilde{w}_\alpha) + \langle \mathcal{C}^t \widetilde{W}, \zeta_j \rangle \quad (j \in I_1 \setminus \{1\} = \{2, ..., m\})
\end{aligned}$$

which is equivalent to the claim. $\square$

At this point, by making use of Claim 3, we get that (134) can be computed as follows:

$$\begin{aligned}
(II_3)_b &= w_t \sum_{j=2}^{m} \zeta_j(\varphi_{1j}(\widetilde{W}))\sigma_{H,t} = w_t \mathrm{div}_{HT\mathcal{U}_t}(\nabla^{H}_{\widetilde{W}}\zeta_1)\\
&= w_t \mathrm{div}_{HT\mathcal{U}_t}\Big(-\nabla^{HT\mathcal{U}_t}\widetilde{w}_1 - \sum_{\alpha\in I_2}\frac{n_{t\alpha}}{n_{t1}}\nabla^{HT\mathcal{U}_t}\widetilde{w}_\alpha - \mathcal{P}_{HT\mathcal{U}_t}(\mathcal{C}^t\widetilde{W})\Big)\sigma_{H,t}\\
&= -w_t \mathrm{div}_{HT\mathcal{U}_t}\Big(\nabla^{HT\mathcal{U}_t}\widetilde{w}_1 + \sum_{\alpha\in I_2}\frac{n_{t\alpha}}{n_{t1}}\nabla^{HT\mathcal{U}_t}\widetilde{w}_\alpha + \mathcal{P}_{HT\mathcal{U}_t}(\mathcal{C}^t\widetilde{W})\Big)\sigma_{H,t}\\
&= -w_t\Big\{\Delta_{HT\mathcal{U}_t}\widetilde{w}_1 + \sum_{\alpha\in I_2}\Big(\frac{n_{t\alpha}}{n_{t1}}\Delta_{HT\mathcal{U}_t}\widetilde{w}_\alpha + \Big\langle \nabla^{H}\Big(\frac{n_{t\alpha}}{n_{t1}}\Big), \nabla^{HT\mathcal{U}_t}\widetilde{w}_\alpha\Big\rangle\Big)\\
&\quad +\mathrm{div}_{HT\mathcal{U}_t}(\mathcal{P}_{HT\mathcal{U}_t}(\mathcal{C}^t\widetilde{W}))\Big\}\sigma_{H,t}.
\end{aligned}$$

In sequel we shall set

$$\mathcal{C} := \sum_{\alpha\in I_2}\frac{n_\alpha}{n_1}\mathcal{C}^\alpha.$$

Now, tacking into account all the above computations, if we restrict to $\mathcal{U}$ by assuming $t=0$, we get that[6]

$$\begin{aligned}
\ddot{\Gamma}(0) &= \{I + II_1 + II_2 + (II_3)_a + (II_3)_b\}|_{t=0} = \Big\{ -W(w)\langle \mathrm{H}, \tau_1\rangle_H\\
&+ \sum_{j,h\in I_1\setminus\{1\}} w w_1\Big\{\phi_{1j}(\tau_j)\phi_{1h}(\tau_h) - \phi_{1j}(\tau_h)(\phi_{1h}(\tau_j) - \phi_{jh}(\tau_1))\Big\}\\
&+ w\sum_{j\in I_1\setminus\{1\}}\sum_{\alpha\in I_2}\Big\{\phi_{1j}(\tau_j)\tau_\alpha(w_\alpha) - \phi_{1j}\Big(\tau_\alpha - \frac{n_\alpha}{n_1}\tau_1\Big)\Big[\tau_j(w_\alpha) + w_1\Big(\phi_{1\alpha}(\tau_j) - \phi_{j\alpha}(\tau_1)\Big)\Big]\Big\}\\
&- w\Big\{Ric_H(W) - \sum_{j\in I_1\setminus\{1\}}\sum_{L\neq 1,j}\Big[\phi_{1L}(W)\phi_{Lj}(\tau_j) - \phi_{Lj}(W)\phi_{1L}(\tau_j)\Big]\Big\}\\
&- w\Big\{\Delta_{HTS}w_1 + \sum_{\alpha\in I_2}\Big(\frac{n_\alpha}{n_1}\Delta_{HTS}w_\alpha + \Big\langle\nabla^{H}\Big(\frac{n_\alpha}{n_1}\Big), \nabla^{HTS}w_\alpha\Big\rangle\Big) + \mathrm{div}_{HTS}(\mathcal{P}_{HTS}(\mathcal{C}W))\Big\}\Big\}\sigma_H
\end{aligned}$$

[6]Note that, at $t=0$, we have $\zeta_I = \tau_I$ and $\varphi_I = \phi_I$ for $I = 1, ..., n$. Moreover, remind that $\tau_1 = \nu_H$

$$
\begin{aligned}
= \quad & \Big\{ - W(w)\mathcal{H}_{_H}^{sc} + \sum_{j,h\in I_1\setminus\{1\}} ww_1 \Big\{\phi_{1j}(\tau_j)\phi_{1h}(\tau_h) - \phi_{1j}(\tau_h)\phi_{1h}(\tau_j)\Big\} \\
& + w \sum_{j\in I_1\setminus\{1\}} \sum_{\alpha\in I_2} \Big\{\phi_{1j}(\tau_j)\tau_\alpha(w_\alpha) - \phi_{1j}(\tau_\alpha^S)\Big[\tau_j(w_\alpha) + w_1\Big(\phi_{1\alpha}(\tau_j) - \phi_{j\alpha}(\nu_{_H})\Big)\Big]\Big\} \\
& - w\Big\{Ric_H(W) - \sum_{j\in I_1\setminus\{1\}} \sum_{L\neq 1,j} \Big[\phi_{1L}(W)\phi_{Lj}(\tau_j) - \phi_{Lj}(W)\phi_{1L}(\tau_j)\Big]\Big\} \\
& - w\Big\{\Delta_{_{HTS}} w_1 + \sum_{\alpha\in I_2}\Big(\frac{n_\alpha}{n_1}\Delta_{_{HTS}} w_\alpha + \Big\langle \nabla^{_H}\Big(\frac{n_\alpha}{n_1}\Big), \nabla^{_{HTS}} w_\alpha\Big\rangle\Big) + \mathrm{div}_{_{HTS}}(\mathcal{P}_{HTS}(\mathcal{C}W))\Big\}\Big\}\sigma_{_H} \\
= \quad & \Big\{ - W(w)\mathcal{H}_{_H}^{sc} + w\Big[w_1\Big(\langle \mathrm{H}, \nu_{_H}\rangle_H^2 - \|\mathfrak{b}_H\|_{Gram}^2\Big) - \mathcal{H}_{_H}^{sc}\mathrm{div}_{_Z}(\mathcal{P}_Z W)\Big] \\
& - w\sum_{\alpha\in I_2}\Big(\langle \nabla_{\tau_\alpha^S}\nu_{_H}, \nabla^{_{HTS}} w_\alpha\rangle + w_1\langle \mathcal{C}^\alpha \nu_{_H}, \nabla^{_H}_{\tau_\alpha^S}\nu_{_H}\rangle\Big) \\
& - w\Big\{Ric_H(W) + w_1 \sum_{j\in I_1\setminus\{1\}} \sum_{\alpha\in I_2} \phi_{\alpha j}(\nu_{_H})\phi_{1\alpha}(\tau_j)\Big\} \\
& - w\Big\{\Delta_{_{HTS}} w_1 + \sum_{\alpha\in I_2}\Big(\frac{n_\alpha}{n_1}\Delta_{_{HTS}} w_\alpha + \Big\langle \nabla^{_H}\Big(\frac{n_\alpha}{n_1}\Big), \nabla^{_{HTS}} w_\alpha\Big\rangle\Big) + \mathrm{div}_{_{HTS}}(\mathcal{P}_{HTS}(\mathcal{C}W))\Big\}\Big\}\sigma_{_H}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \Big\{ - W(w)\mathcal{H}_{_H}^{sc} + w\Big[w_1\Big(\langle \mathrm{H}, \nu_{_H}\rangle_H^2 - \|\mathfrak{b}_H\|_{Gram}^2\Big) - \mathcal{H}_{_H}^{sc}\mathrm{div}_{_Z}(\mathcal{P}_Z W)\Big] \\
& - w\sum_{\alpha\in I_2}\Big(\langle \nabla_{\tau_\alpha^S}\nu_{_H}, \nabla^{_{HTS}} w_\alpha\rangle + w_1\langle \mathcal{C}^\alpha \nu_{_H}, \nabla^{_H}_{\tau_\alpha^S}\nu_{_H}\rangle\Big) \\
& - w\Big\{Ric_H(W) + w_1 \sum_{j\in I_1\setminus\{1\}} \sum_{\alpha\in I_2} \frac{1}{4}\langle[\tau_1, \tau_j], \tau_\alpha\rangle^2\Big\} \\
& - w\Big\{\Delta_{_{HTS}} w_1 + \sum_{\alpha\in I_2}\Big(\frac{n_\alpha}{n_1}\Delta_{_{HTS}} w_\alpha + \Big\langle \nabla^{_H}\Big(\frac{n_\alpha}{n_1}\Big), \nabla^{_{HTS}} w_\alpha\Big\rangle\Big) + \mathrm{div}_{_{HTS}}(\mathcal{P}_{HTS}(\mathcal{C}W))\Big\}\Big\}\sigma_{_H} \\
= \quad & \Big\{ - W(w)\mathcal{H}_{_H}^{sc} + w\Big[w_1\Big(\langle \mathrm{H}, \nu_{_H}\rangle_H^2 - \|\mathfrak{b}_H\|_{Gram}^2\Big) - \mathcal{H}_{_H}^{sc}\mathrm{div}_{_Z}(\mathcal{P}_Z W)\Big] \\
& - w\sum_{\alpha\in I_2}\Big\langle \nabla_{\tau_\alpha^S}\nu_{_H}, \Big(\nabla^{_{HTS}} w_\alpha + w_1\mathcal{C}^\alpha\nu_{_H}\Big)\Big\rangle - \frac{ww_1}{2}\sum_{j\in I_1\setminus\{1\}} \sum_{\alpha\in I_2}\langle[\tau_1, \tau_j], \tau_\alpha\rangle^2 \\
& - w\Big\{\Delta_{_{HTS}} w_1 + \sum_{\alpha\in I_2}\Big(\frac{n_\alpha}{n_1}\Delta_{_{HTS}} w_\alpha + \Big\langle \nabla^{_H}\Big(\frac{n_\alpha}{n_1}\Big), \nabla^{_{HTS}} w_\alpha\Big\rangle\Big) + \mathrm{div}_{_{HTS}}(\mathcal{P}_{HTS}(\mathcal{C}W))\Big\}\Big\}\sigma_{_H}
\end{aligned}
$$

where in the last equality we have used the explicit expression of $Ric_H(W)$, i.e. $Ric_H(W) = -w_1\frac{3}{4}\sum_{j\in I_1\setminus\{1\}}\sum_{\alpha\in I_2}\langle[\tau_1, \tau_j], \tau_\alpha\rangle^2$. Indeed this result can easily be obtained by using (a') of (78). The last expression achieves the proof once we note that $\sum_{j\in I_1\setminus\{1\}}\langle[\tau_1,\tau_j],\tau_\alpha\rangle^2=\|\mathcal{C}^\alpha\nu_{_H}\|^2_{Gram}$.

$\square$

At this point we may state another similar result which give us the second horizontal normal variation of $\sigma_{_H}$ on regular non-characteristic hypersurfaces in 2-step Carnot groups. Also this formula is stated without boundary terms.

Theorem 7.10. *Let $\mathbb{G}$ be a 2-step Carnot group. Let $\imath:\mathcal{U}\longrightarrow\mathbb{G}$ denote the inclusion into $\mathbb{G}$ of a smooth non-characteristic hypersurface $\mathcal{U}$ and $\vartheta:(-\epsilon,\epsilon)\times\mathcal{U}\longrightarrow\mathbb{G}$ be a* **smooth normal H-variation of** *$\imath$, with variation vector field $W\in\mathbf{C}_0^\infty(\mathbb{G},H)$ such that* $\mathrm{spt}(W)\cap\mathcal{U}\Subset\mathcal{U}$ *and $W_x\in(\nu_{_H})_x\mathcal{U}\ \forall\, x\in\mathrm{Int}(\mathcal{U})$. Assume that $\mathcal{U}_t=\vartheta_t(\mathcal{U})$ is non-characteristic for every $t\in(-\epsilon,\epsilon)$. Finally, let $\Gamma(t)=\vartheta_t^*\sigma_{_{H,t}}$ and $(\zeta_1,...,\zeta_n)$ be an* **orthonormal moving frame for** *U, where U is an open set containing* $\mathrm{Im}(\vartheta)$ *(see Section 7.1). Then we have*

(i)

$$\begin{aligned}\ddot{\Gamma}(0)\;=\;&\Big\{-W(w)\mathcal{H}^{sc}_{_H}+w^2\Big(\langle\mathrm{H},\nu_{_H}\rangle^2_H-\|\mathfrak{b}_H\|^2_{Gram}\Big)-w\,\Delta_{_{HTS}}w\\&-w^2\sum_{\alpha\in I_2}\langle\nabla_{\tau^S_\alpha}\nu_{_H},\mathcal{C}^\alpha\nu_{_H}\rangle+\frac{w^2}{2}\sum_{\alpha\in I_2}\|\mathcal{C}^\alpha\nu_{_H}\|^2_{Gram}-w\,\mathrm{div}_{_{HTS}}(w\mathcal{C}\nu_{_H})\Big\}\sigma_{_H}\end{aligned}$$

(ii)

$$\begin{aligned}II^{int}_{\mathcal{U}}(\sigma_{_H})\;=\;&\int_{\mathcal{U}}\Big\{-W(w)\mathcal{H}^{sc}_{_H}+w^2\Big(\langle\mathrm{H},\nu_{_H}\rangle^2_H-\|\mathfrak{b}_H\|^2_{Gram}\Big)-w\,\Delta_{_{HTS}}w\\&-w^2\sum_{\alpha\in I_2}\langle\nabla_{\tau^S_\alpha}\nu_{_H},\mathcal{C}^\alpha\nu_{_H}\rangle+\frac{w^2}{2}\sum_{\alpha\in I_2}\|\mathcal{C}^\alpha\nu_{_H}\|^2_{Gram}-w\,\mathrm{div}_{_{HTS}}(w\mathcal{C}\nu_{_H})\Big\}\sigma_{_H}\end{aligned}$$

Proof. This proof can be regarded as a continuation of that of Theorem 7.8 and we refer to it for the notation used in the sequel.

We have by hypothesis that the variation vector field W along $\mathcal{U}$ is a horizontal normal vector field, and so using the coordinates given by the H-adapted frame

$(\zeta_1, ..., \zeta_n)$ for the open set U we just get that $\widetilde{W} = w_t\zeta_1$. By using the hypothesis and Theorem 7.4 we get the local expression of the *first H-variation* of σ_H at the interior of $\mathcal{U}$, i.e.

$$\dot{\Gamma}(0) \quad = \quad -\mathcal{H}^{sc}_{H} \langle W, \nu_H \rangle \, \sigma_H = -\langle \mathrm{H}, W \rangle_H \, \sigma_H .$$

Notice that we have to calculate

$$\ddot{\Gamma}(0)|_{\mathcal{U}} \quad = \quad \imath^*\{\mathcal{L}_{\widetilde{W}}(\widetilde{W} \rfloor d\,\sigma_{H,t})\}. \tag{136}$$

We also have that the 1st variation on $\mathrm{Int}(\mathcal{U})$ at $t \in (-\epsilon, \epsilon)$ is given by

$$\begin{aligned}
\dot{\Gamma}(t) \quad &= \quad \imath^*(\widetilde{W} \rfloor d\,\sigma_{H,t}) \\
&= \quad \Big\{ \sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \Big(\widetilde{W} \rfloor \, \varphi_1 \wedge ... \wedge \varphi_n \Big) \Big\} \Big|_{\mathcal{U}} \\
&= \quad \Big\{ \Big[\sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \Big] \, \widetilde{w}_1 \, \sigma_{H,t} \Big\} \Big|_{\mathcal{U}}
\end{aligned}$$

and then we have, for every $t \in (-\epsilon, \epsilon)$, the relation

$$\begin{aligned}
\dot{\Gamma}(t) \quad &= \quad \Big\{ \Big[\sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \Big] \, \widetilde{w}_1 \, \sigma_{H,t} \Big\} \Big|_{\mathcal{U}} \\
&= \quad \Big\{ \Big[\sum_{i=2}^{m} \varphi_{1i}(\zeta_i) \Big] \, \widetilde{w}_1 \, \varphi_2 \wedge ... \wedge \varphi_n \Big\} \Big|_{\mathcal{U}} \\
&= \quad \sum_{j=2}^{m} \Big\{ \widetilde{w}_1 \, \varphi_2 \wedge ... \wedge \varphi_{1j} \wedge ... \wedge \varphi_n \Big\} \Big|_{\mathcal{U}}.
\end{aligned}$$

As in the previous theorem we have to compute the following quantities

(i) $\mathcal{L}_{\widetilde{W}}(\varphi_h)$ for $h \in I_1 \setminus \{1\} = \{2, ..., m\}$;

(ii) $\mathcal{L}_{\widetilde{W}}(\varphi_\alpha)$ for $\alpha \in I_2 = \{m+1, ..., n\}$;

(iii) $\mathcal{L}_{\widetilde{W}}(\varphi_{1j})$ for $j \in I_1 \setminus \{1\}$.

Also in this case, we can do this by Cartan's formula and the structure equations for the H-adapted coframe $(\varphi_1, ..., \varphi_n)$. Clearly, we have many simplifications, because $\widetilde{W} = w_t\zeta_1$. For the term appearing at item (i) we have

$$\mathcal{L}_{\widetilde{W}}(\varphi_h) = \sum_{J\neq h}\Big\{\varphi_J(\widetilde{W})\varphi_{Jh} - \varphi_{Jh}(\widetilde{W})\varphi_J\Big\} = w_t\sum_{J\neq h}\Big\{\varphi_{1h} - \varphi_{Jh}(\zeta_1)\varphi_J\Big\}$$

$$= w_t\sum_{J\neq h}\Big\{\varphi_{1h}(\zeta_J) - \varphi_{Jh}(\zeta_1)\Big\}\varphi_J.$$

For the term appearing at item (ii), we have

$$\mathcal{L}_{\widetilde{W}}(\varphi_\alpha) = w_t\sum_{h=2}^{m}\Big(\varphi_{1\alpha}(\zeta_h) - \varphi_{h\alpha}(\zeta_1)\Big)\varphi_h.$$

Finally (iii) can be obtained as follows.

$$\begin{aligned}
\mathcal{L}_{\widetilde{W}}(\varphi_{1j}) &= \widetilde{W}\,\rfloor\, d\,\varphi_{1j} + d\,(\widetilde{W}\,\rfloor\,\varphi_{1j})\\
&= w_t\sum_{L\neq 1,j}\Big\{\Big[\varphi_{1L}(\zeta_1)\varphi_{Lj} - \varphi_{Lj}(\zeta_1)\varphi_{1L}\Big] - \sum_K\langle\mathsf{R}(\zeta_K,\zeta_1)\,\zeta_j,\zeta_1\rangle\varphi_K\Big\} + d\,(\varphi_{1j}(\widetilde{W})).
\end{aligned}$$

We therefore have

$$\begin{aligned}
\ddot{\Gamma}(t) &= \sum_{j=2}^{m}\mathcal{L}_{\widetilde{W}}\Big\{w_t\,\varphi_2\wedge\ldots\wedge\varphi_{1j}\wedge\ldots\wedge\varphi_n\Big\}\\
&= \widetilde{W}(w_t)\left[\sum_{i=2}^{m}\varphi_{1i}(\zeta_i)\right]\sigma_{H,t} + w_t\sum_{j=2}^{m}\mathcal{L}_{\widetilde{W}}\Big\{\varphi_2\wedge\ldots\wedge\varphi_{1j}\wedge\ldots\wedge\varphi_n\Big\}\\
&= I + II,
\end{aligned}$$

where

$$II := w_t\sum_{j=2}^{m}\mathcal{L}_{\widetilde{W}}\Big\{\varphi_2\wedge\ldots\wedge\varphi_{1j}\wedge\ldots\wedge\varphi_n\Big\}$$

$$= w_t\sum_{j,\,h=2}^{m}\varphi_2\wedge\ldots\wedge\mathcal{L}_{\widetilde{W}}(\varphi_h)\wedge\ldots\wedge\varphi_{1j}\wedge\ldots\wedge\varphi_n$$

$$+ w_t\sum_{j=2}^{m}\sum_{\alpha\in I_2}\varphi_2\wedge\ldots\wedge\varphi_{1j}\wedge\ldots\wedge\mathcal{L}_{\widetilde{W}}(\varphi_\alpha)\wedge\ldots\wedge\varphi_n$$

$$+ w_t\sum_{j=2}^{m}\varphi_2\wedge\ldots\wedge\mathcal{L}_{\widetilde{W}}(\varphi_{1j})\wedge\ldots\wedge\varphi_n =: II_1 + II_2 + II_3.$$

We substitute into this expression (i), (ii), (iii) above. This way we get

$$\begin{aligned} II_1 : &= \sum_{j,h\in I_1\setminus\{1\}} w_t\Big\{\varphi_2\wedge...\wedge\big(\varphi_{1j}(\zeta_j)\varphi_j+\varphi_{1j}(\zeta_h)\varphi_h\big)\wedge...\\ &\qquad ...\wedge\Big[w_t\sum_{J\neq h}\Big\{\varphi_{1h}(\zeta_J)-\varphi_{Jh}(\zeta_1)\Big\}\varphi_J\Big]\wedge...\wedge\varphi_n\Big\}\\ &= \sum_{j,h\in I_1\setminus\{1\}} w_t^2\Big\{\varphi_{1j}(\zeta_j)\varphi_{1h}(\zeta_h)-\varphi_{1j}(\zeta_h)\varphi_{1h}(\zeta_j)\Big\}\sigma_{H,t}; \end{aligned}$$

$$\begin{aligned} II_2 : &= w_t^2\sum_{j\in I_1\setminus\{1\}}\sum_{\alpha\in I_2}\Big\{\varphi_{1j}(\zeta_1)\Big(\varphi_{1\alpha}(\zeta_j)-\varphi_{j\alpha}(\zeta_1)\Big)\frac{n_{t\,\alpha}}{n_{t\,1}}-\varphi_{1j}(\zeta_\alpha)\Big(\varphi_{1\alpha}(\zeta_j)-\varphi_{j\alpha}(\zeta_1)\Big)\Big\}\,\sigma_{H,t}\\ &= -w_t^2\sum_{j\in I_1\setminus\{1\}}\sum_{\alpha\in I_2}\varphi_{1j}\Big(\zeta_\alpha-\frac{n_{t\,\alpha}}{n_{t\,1}}\zeta_1\Big)\Big(\varphi_{1\alpha}(\zeta_j)-\varphi_{j\alpha}(\zeta_1)\Big)\sigma_{H,t}. \end{aligned} \tag{137}$$

We put $II_3 := (II_3)_a + (II_3)_b$, where

$$\begin{aligned} (II_3)_a &= w_t\sum_{j\in I_1\setminus\{1\}}\Big\{\varphi_2\wedge...\wedge\Big\{-\sum_K\langle\mathsf{R}(\zeta_K,\widetilde{W})\,\zeta_j,\zeta_1\rangle\,\varphi_K\\ &\qquad+\sum_{L\neq 1,j}\Big[\varphi_{1L}(\widetilde{W})\varphi_{Lj}-\varphi_{Lj}(\widetilde{W})\varphi_{1L}\Big]\Big\}\wedge...\wedge\varphi_n\Big\} \end{aligned} \tag{138}$$

and, by using Lemma 6.3 and Definition 7.9, we get

$$(II_3)_a = -w_t^2\Big\{Ric_H(\zeta_1)-\sum_{j\in I_1\setminus\{1\}}\sum_{L\neq 1,j}\Big[\varphi_{1L}(\zeta_1)\varphi_{Lj}(\zeta_j)-\varphi_{Lj}(\zeta_1)\varphi_{1L}(\zeta_j)\Big]\Big\}\,\sigma_{H,t}.$$

Furthermore we have

$$(II_3)_b \;:=\; w_t\sum_{j=2}^m\varphi_2\wedge...\wedge d\,(\varphi_{1j}(\widetilde{W}))\wedge...\wedge\varphi_n = w_t\sum_{j=2}^m\zeta_j(\varphi_{1j}(w_t\zeta_1))\sigma_{H,t} \tag{139}$$

and we may compute this term by arguing exactly as in the previous proof of Theorem 7.8 and, more precisely, by making use of Claim 1, Claim 2 and Claim 3 we obtain

$$
\begin{aligned}
(II_3)_b &= w_t \sum_{j=2}^{m} \zeta_j(\varphi_{1j}(\widetilde{W}))\sigma_{H,t} = w_t \mathrm{div}_{HT\mathcal{U}_t}(\nabla^{H}_{\widetilde{W}}\zeta_1)\\
&= w_t \mathrm{div}_{HT\mathcal{U}_t}\Big(- \nabla^{HT\mathcal{U}_t} w_t - w_t \mathcal{P}_{HT\mathcal{U}_t}(\mathcal{C}^t\zeta_1)\Big)\sigma_{H,t}\\
&= -w_t \mathrm{div}_{HT\mathcal{U}_t}\Big(\nabla^{HT\mathcal{U}_t} w_t + w_t \mathcal{P}_{HT\mathcal{U}_t}(\mathcal{C}^t\zeta_1)\Big)\sigma_{H,t}\\
&= -w_t\Big\{\Delta_{HT\mathcal{U}_t} w_t + \mathrm{div}_{HT\mathcal{U}_t}(w_t \mathcal{P}_{HT\mathcal{U}_t}(\mathcal{C}^t\zeta_1))\Big\}\sigma_{H,t}.
\end{aligned}
$$

From what we have previously seen and by using Claim 1 of the proof of Theorem 7.8, we get that

$$
\begin{aligned}
\ddot{\Gamma}(0) &= \{I + II_1 + II_2 + (II_3)_a + (II_3)_b\}|_{t=0}\\
&= \Big\{ - w\frac{\partial w}{\partial \nu_H}\mathcal{H}^{sc}_H + \sum_{j,h\in I_1\setminus\{1\}} w^2 \Big\{\phi_{1j}(\tau_j)\phi_{1h}(\tau_h) - \phi_{1j}(\tau_h)\phi_{1h}(\tau_j)\Big\}\\
&\quad + w^2 \sum_{j\in I_1\setminus\{1\}}\sum_{\alpha\in I_2}\Big\{ - \phi_{1j}\Big(\tau_\alpha - \frac{n_\alpha}{n_1}\nu_H\Big)\Big(\phi_{1\alpha}(\tau_j) - \phi_{j\alpha}(\nu_H)\Big)\Big\}\\
&\quad - w^2\Big\{Ric_H(\nu_H) - \sum_{j\in I_1\setminus\{1\}}\sum_{L\neq 1,j}\Big[\phi_{1L}(\nu_H)\phi_{Lj}(\tau_j) - \phi_{Lj}(\nu_H)\phi_{1L}(\tau_j)\Big]\Big\}\\
&\quad - w\Big\{\Delta_{HT\mathcal{U}_t} w + \mathrm{div}_{HT\mathcal{U}_t}(w \mathcal{P}_{HT\mathcal{U}_t}(\mathcal{C}^t\zeta_1))\Big\}\Big\}\sigma_H
\end{aligned}
$$

$$
\begin{aligned}
&= \Big\{ - W(w)\mathcal{H}^{sc}_H + w^2\Big(\langle \mathrm{H}, \nu_H\rangle_H^2 - \|\mathsf{b}_H\|^2_{Gram}\Big)\\
&\quad - w^2 \sum_{\alpha\in I_2}\langle \nabla_{\tau^S_\alpha}\nu_H, \mathcal{C}^\alpha\nu_H\rangle + \frac{w^2}{2}\sum_{j\in I_1\setminus\{1\}}\sum_{\alpha\in I_2}\langle[\tau_1,\tau_j],\tau_\alpha\rangle^2\\
&\quad - w\Big\{\Delta_{HTS} w + \mathrm{div}_{HTS}(w\mathcal{C}\zeta_1)\Big\}\Big\}\sigma_H\\
&= \Big\{ - W(w)\mathcal{H}^{sc}_H + w^2\Big(\langle \mathrm{H}, \nu_H\rangle_H^2 - \|\mathsf{b}_H\|^2_{Gram}\Big) - w\Delta_{HTS} w\\
&\quad - w^2 \sum_{\alpha\in I_2}\langle \nabla_{\tau^S_\alpha}\nu_H, \mathcal{C}^\alpha\nu_H\rangle + \frac{w^2}{2}\sum_{\alpha\in I_2}\|\mathcal{C}^\alpha\nu_H\|^2_{Gram} - w\,\mathrm{div}_{HTS}(w\mathcal{C}\nu_H)\Big\}\sigma_H
\end{aligned}
$$

which is the thesis. $\square$

7.4 Addendum: Integration by parts and 1st variation of σ_H in k-step Carnot groups

The results previously stated can be generalized for k-step Carnot groups. Throughout this Addendum, we shall generalize some of them, and, more precisely, the theorem about the 1st variation of the H-perimeter form σ_H and the integration by parts formulae for smooth non-characteristic hypersurfaces.

However, many proofs given below will turn out to be slightly different from the previous ones because we will make use just of the fixed left invariant frame $(X_1, ..., X_n)$.

Let $\mathbb{G}$ be a k-step n-dimensional Carnot group and let $S \subset \mathbb{G}$ be a smooth immersed non-characteristic hypersurface with unit normal vector along S denoted by N. Let $\mathcal{U} \subset S$ be compact and suppose that the boundary $\partial\mathcal{U}$ is a smooth $n-2$-dimensional submanifold with outward pointing unit normal η. Finally, $U \subset \mathbb{G}$ will denote an open set having non-empty intersection with S and $\mathcal{U} := U \cap S$. In this section **we do not make use** of the indices convention used for 2-step Carnot groups.

In the sequel we shall set

$$I_1 := \{1, ..., m_1\},\ I_2 := \{m_1 + 1, ..., m_2\}, ..., I_{\mathrm{k}} := \{m_{\mathrm{k}-1} + 1, ..., n\}$$

$$\Omega_1 := \omega_1 \wedge ... \wedge \omega_{m_1}, \Omega_2 := \omega_{m_1+1} \wedge ... \wedge \omega_{m_2}, ..., \Omega_{\mathrm{k}} := \omega_{m_{\mathrm{k}-1}+1} \wedge ... \wedge \omega_{m_{\mathrm{k}}},$$

so that $\Omega^n = \Omega_1 \wedge ... \wedge \Omega_{\mathrm{k}}$.

The next notions of *horizontal second fundamental form* and of *horizontal mean curvature*, are analogous to the ones given in Definition 6.15.

Definition 7.11. *We define the* **horizontal second fundamental form of** S *to be the map* $\mathtt{b}_H : HTS \times HTS \longrightarrow \nu_H S$ *given by*

$$\mathtt{b}_H(X, Y) := \langle \nabla^H_X Y, \nu_H \rangle_H \, \nu_H \qquad \forall\ X,\, Y \in HTS.$$

The trace of $\mathtt{b}_H$*, denoted by* H*, is called the* **horizontal mean curvature of** S*. Finally, the quantity* $\mathcal{H}_H^{sc} = \langle \mathrm{H}, \nu_H \rangle_H$ *is the* **scalar horizontal mean curvature**

of S. *Clearly,* $\mathrm{H} \in \nu_{_H} S$ *and*

$$\mathrm{H} := \sum_{j=1}^{m_1} \langle \nabla^{_H}_{X_j} X_j, \nu_{_H} \rangle_H \, \nu_{_H} = - \sum_{j=1}^{m_1} \langle \nabla^{_H}_{X_j} \nu_{_H}, X_j \rangle_H \, \nu_{_H} = -\mathcal{H}^{sc}_{_H} \nu_{_H}.$$

We begin by stating some preliminary lemmas.

Lemma 7.12. *Let* $j \in I_1$. *Then we have*

$$d\,(X_i \rfloor X_j \rfloor \Omega^n)|_S = \begin{cases} \sum_{k=m_1+1}^{n} c^k_{ji}\, n_k\, \sigma^{n-1}|_S & (i \in I_1); \\ \\ \sum_{k=i+1}^{n} c^k_{ji}\, n_k\, \sigma^{n-1}|_S & (i \geq m_1 + 1 \Leftrightarrow i \in I_2 \cup ... \cup I_{\mathrm{k}}). \end{cases}$$

Definition 7.13. *Throughout this section we shall set* $\mathsf{C}^k_H := [c^k_{ij}]_{\{i,\, j \in I_1\}} \in \mathcal{M}_{m_1, m_1}(\mathbb{R})$ *and* $\mathsf{C}^k_V := [c^k_{ji}]_{\{j \in I_1,\, i \in I_2 \cup ... \cup I_{\mathrm{k}}\}} \in \mathcal{M}_{m_1, n-m_1}(\mathbb{R})$. *Moreover we shall denote by* $\mathcal{C}^k_H : H \longrightarrow H$ *and, respectively, by* $\mathcal{C}^k_V : V_2 \oplus ... \oplus V_{\mathrm{k}} \longrightarrow H$, *the linear operators associated with* C^α_H *and* C^k_V. *We shall also denote by*

$$\mathcal{P}_V : \mathfrak{g} \longrightarrow V_2 \oplus ... \oplus V_{\mathrm{k}}$$

the projection map onto $V_2 \oplus ... \oplus V_{\mathrm{k}}$ *given, for* $X \in \mathfrak{g}$, *by*

$$\mathcal{P}_V(X) := \sum_{i=m_1+1}^{n} \langle X, X_i \rangle X_i$$

.

Proof of Lemma 7.12. This proof will be divided into two steps. We start with the following:

Step 1. *Computations of*

$$d\,(X_i \rfloor X_j \rfloor \Omega^n)|_S \qquad \text{for } i \in I_1 \;\; (j = 1, ..., m_1). \tag{140}$$

Proof. We have

$$
\begin{aligned}
\alpha : \;&=\; d\,(X_i \rfloor X_j \rfloor \Omega_1 \wedge \Omega_2 \wedge ... \wedge \Omega_{\mathrm{k}})|_S \\
&=\; d\Big((X_i \rfloor X_j \rfloor \Omega_1) \wedge (\Omega_2 \wedge ... \wedge \Omega_{\mathrm{k}})\Big)\Big|_S \\
&=\; \Big(d\,(X_i \rfloor X_j \rfloor \Omega_1) \wedge (\Omega_2 \wedge ... \wedge \Omega_{\mathrm{k}}) \\
&\quad\; +(-1)^{m_1-2}(X_i \rfloor X_j \rfloor \Omega_1) \wedge d\,(\Omega_2 \wedge ... \wedge \Omega_{\mathrm{k}})\Big)\Big|_S \\
&=\; (-1)^{m_1}\Big((X_i \rfloor X_j \rfloor \Omega_1) \wedge d\,(\Omega_2 \wedge ... \wedge \Omega_{\mathrm{k}})\Big)\Big|_S
\end{aligned}
\tag{141}
$$

since $d\omega_k = 0 \;\forall\; k \in I_1$. Now setting $\beta := d\,(\Omega_2 \wedge ... \wedge \Omega_{\mathrm{k}})$ we get

$$
\begin{aligned}
\beta \;&=\; d\,(\omega_{m_1+1} \wedge ... \wedge \omega_n) \\
&=\; \sum_{k=m_1+1}^{n} (-1)^{k+m_1+1}\omega_{m_1+1} \wedge ... \wedge d\,\omega_k \wedge ... \wedge \omega_n \\
&=\; \sum_{k=m_1+1}^{n} (-1)^{k+m_1+1}\omega_{m_1+1} \wedge ... \wedge \Big(-\frac{1}{2}\sum_{1\le r,s\le h_{l-1}} c^k_{rs}\,\omega_r \wedge \omega_s\Big) \wedge ... \wedge \omega_n
\end{aligned}
\tag{142}
$$

whenever $h_{l-1} < k < h_{l+1} (\Leftrightarrow k \in I_l)$; see Remark 1.22 of Section 1.2. From (142) we get

$$
\begin{aligned}
\beta \;&=\; -\frac{1}{2}\sum_{k=m_1+1}^{n}\;\sum_{1\le r,s\le h_{l-1}} c^k_{rs}\,(-1)^{k+m_1+1}\omega_{m_1+1} \wedge ... \wedge \underbrace{\omega_r \wedge \omega_s}_{k-th\,place} \wedge ... \wedge \omega_n \qquad (143) \\
&=\; -\frac{1}{2}\sum_{k=m_1+1}^{n}\;\sum_{1\le r,s\le h_{l-1}} c^k_{rs}\,(-1)^{k+m_1+1}\,(\omega_r \wedge \omega_s) \wedge (\omega_{m_1+1} \wedge ... \wedge \widehat{\omega_k} \wedge ... \wedge \omega_n).
\end{aligned}
$$

Now we note that $\alpha \neq 0$ if, and only if, we have

$$\Omega_1 = \pm(X_i \rfloor X_j \rfloor \Omega_1) \wedge (\omega_r \wedge \omega_s).$$

This implies $r = i$ and $s = j$ or $r = j$ and $s = i$. So, tacking into account (141), (142), (143), using the skew-symmetry on the lower indices of the structural constants c^k_{rs}, one gets

$$
\begin{aligned}
\alpha &= -\sum_{k=m_1+1}^{n} c^k_{ij}\,(-1)^{k+1}\omega_1 \wedge ... \wedge \widehat{\omega_k} \wedge ... \wedge \omega_n|_S \\
&= \sum_{k=m_1+1}^{n} c^k_{ji}\,(X_k \rfloor \Omega^n)|_S \\
&= \sum_{k=m_1+1}^{n} c^k_{ji}\, n_k\, \sigma^{n-1}|_S.
\end{aligned}
$$

$\square$

Step 2. *Computations of*

$$
d(X_i \rfloor X_j \rfloor \Omega^n)|_S \qquad \textit{for } i \geq m_1+1 \Leftrightarrow i \in I_2 \cup ... \cup I_{\mathrm{k}} \ \ (j = 1, ..., m_1). \tag{144}
$$

Proof. We have

$$
\begin{aligned}
\alpha &= d\,(X_i \rfloor X_j \rfloor \Omega^n)\Big|_S = d\left(X_i \rfloor \left((X_j \rfloor \Omega_1) \bigwedge (\Omega_2 \wedge ... \wedge \Omega_{\mathrm{k}})\right)\right)\Big|_S \\
&= (-1)^{j+1} d\left(X_i \rfloor \left((\omega_1 \wedge ... \wedge \widehat{\omega_j} \wedge ... \wedge \omega_{m_1}) \bigwedge (\Omega_2 \wedge ... \wedge \Omega_{\mathrm{k}})\right)\right)\Big|_S \\
&= (-1)^{i+j} d\,\big(\underbrace{\omega_1 \wedge ... \wedge \widehat{\omega_j} \wedge ... \wedge \omega_{m_1}}_{j \in I_1} \wedge \underbrace{\omega_{m_1+1} \wedge ... \wedge \widehat{\omega_i} \wedge ... \wedge \omega_n}_{i \in I_2 \cup ... \cup I_{\mathrm{k}}}\big)|_S \\
&= (-1)^{i+j+m_1-1}(\omega_1 \wedge ... \wedge \widehat{\omega_j} \wedge ... \wedge \omega_{m_1}) \bigwedge d\,(\omega_{m_1+1} \wedge ... \wedge \widehat{\omega_i} \wedge ... \wedge \omega_n)|_S \\
&.
\end{aligned} \tag{145}
$$

since $d\omega_k = 0 \ \ \forall\ k \in I_1$. Now, setting

$$
\beta = \omega_{m_1+1} \wedge ... \wedge \widehat{\omega_i} \wedge ... \wedge \omega_n
$$

we get

$$
\alpha = (-1)^{i+j+m_1-1}(\omega_1 \wedge ... \wedge \widehat{\omega_j} \wedge ... \wedge \omega_{m_1}) \wedge d\,\beta|_S
$$

so that we have to compute $d\,\beta$. We note that $d\,\beta$ is a finite sum of terms of the following type:

$$\begin{aligned}\gamma_1 &= \omega_{m_1+1}\wedge ...\wedge d\,\omega_k\,\wedge ...\wedge\,\widehat{\omega_i}\,\wedge ...\wedge\omega_n,\\ \gamma_2 &= \omega_{m_1+1}\wedge ...\wedge\,\widehat{\omega_i}\,\wedge ...\wedge d\;\omega_k\,\wedge ...\wedge\omega_n.\end{aligned}$$

But each $n-m_1$-form of the type γ_1 must be 0. Indeed $d\,\omega_k$ is a finite sum of 2-forms $\omega_r\wedge\omega_s$, with $1\leq r,s\leq h_{l-1}$ whenever $h_{l-1}<k<h_{l+1}(\Leftrightarrow k\in I_l)$, and so in the wedge product which defines α, there is at least one term ω_r^2 (or ω_s^2). Therefore, setting $\beta:=\beta_1\wedge\,\widehat{\omega_i}\,\wedge\beta_2$, where $\beta_1:=\omega_{m_1+1}\wedge ...\wedge\omega_{i-1}$ and $\beta_2:=\omega_{i+1}\wedge ...\wedge\omega_n$, we get

$$d\,\beta=(-1)^{i-m_1-1}\beta_1\wedge\,\widehat{\omega_i}\,\wedge d\,\beta_2. \tag{146}$$

Now we have

$$\begin{aligned}d\,\beta_2 &= d\,(\omega_{i+1}\wedge ...\wedge\omega_n)\\ &= \sum_{k=i+1}^{n}(-1)^{k+i+1}\omega_{i+1}\wedge ...\wedge d\,\omega_k\wedge ...\wedge\omega_n\\ &= \sum_{k=i+1}^{n}(-1)^{k+i+1}\omega_{m_1+1}\wedge ...\wedge\Big(-\frac{1}{2}\sum_{1\leq r,s\leq h_{l-1}}c_{rs}^k\,\omega_r\wedge\omega_s\Big)\wedge ...\wedge\omega_n\end{aligned} \tag{147}$$

whenever $h_{l-1}<k<h_{l+1}(\Leftrightarrow k\in I_l)$; see Remark 1.22. From (147) we get

$$d\,\beta_2 = -\frac{1}{2}\sum_{k=i+1}^{n}\sum_{1\leq r,s\leq h_{l-1}}c_{rs}^k\,(-1)^{k+i+1}\omega_{i+1}\wedge ...\wedge\underbrace{\omega_r\wedge\omega_s}_{k-th\,place}\wedge ...\wedge\omega_n. \tag{148}$$

Then, tacking into account equations (145), (146) and (148) we obtain

$$\begin{aligned}
\alpha &= (-1)^{i+j+m_1-1}(-1)^{i-m_1-1}\omega_1 \wedge ... \wedge \widehat{\omega_j} \wedge ... \wedge \omega_{m_1} \wedge ... \wedge \widehat{\omega_i} \wedge d\,\beta_2|_S \\
&= (-1)^{i+j+m_1-1}(-1)^{i-m_1-1}\omega_1 \wedge ... \wedge \widehat{\omega_j} \wedge ... \wedge \omega_{m_1} \wedge ... \wedge \widehat{\omega_i} \\
&\quad \bigwedge \Big(-\frac{1}{2}\sum_{k=i+1}^{n} \sum_{1\le r,s\le h_{l-1}} (-1)^{k+i+1}\, c^k_{rs}\, \omega_{i+1} \wedge ... \wedge \underbrace{\omega_r \wedge \omega_s}_{k-th\,place} \wedge ... \wedge \omega_n \Big)\Big|_S \\
&= \frac{1}{2}\sum_{k=i+1}^{n} \sum_{1\le r,s\le h_{l-1}} (-1)^{i+j+k}\, c^k_{rs}\, \omega_1 \wedge ... \wedge \widehat{\omega_j} \wedge ... \wedge \widehat{\omega_i} \wedge ... \wedge (\omega_r \wedge \omega_s) \wedge ... \wedge \omega_n|_S \\
&= \sum_{k=i+1}^{n} (-1)^{k+1}\, c^k_{ji}\, \omega_1 \wedge ... \wedge \widehat{\omega_k} \wedge ... \wedge \omega_n|_S, \qquad (149)
\end{aligned}$$

where the last equality follows from the the skew-symmetry on the lower indices of the structural constants. Finally, from (149) we get

$$\alpha = \sum_{k=i+1}^{n} c^k_{ji}\,(X_k \rfloor \Omega^n)|_S = -\sum_{k=m_1+1}^{n} c^k_{ij}\, n_k\, \sigma^{n-1}|_S. \qquad (150)$$

□

Now the proof of Lemma 7.12 follows by applying both Step 1 and Step 2. □

By using the previous lemma we can prove the analogous of Lemma 6.20; see Section 6.2.

Lemma 7.14. *For every $X \in \mathfrak{X}(\mathbb{G})$ we have*

$$\begin{aligned}
d\,(X \rfloor \sigma_H)|_S &= \left\{ \mathrm{div} X + \left[\langle \mathrm{H}, \nu_H \rangle \frac{\langle X, N\rangle}{|\mathcal{P}_H(N)|_H} - \frac{\langle \mathcal{J}_X \nu_H, N\rangle}{|\mathcal{P}_H(N)|_H} \right] \right\} (\sigma_H)|_S \\
&- \sum_{k\in I_2\cup...\cup I_{\mathrm{k}}} \Big(\langle n_k\, \mathcal{C}^k_H \nu_H, \mathcal{P}_H(X)\rangle_H + \langle n_k\, \mathcal{C}^k_V \nu_H, \mathcal{P}_V(X)\rangle \Big) (\sigma^{n-1})|_S.
\end{aligned}$$

Proof. Let $(\nu_H)_j$ $(j = 1, ..., m_1)$ denote the j-th horizontal component of the unit H-normal of S, with respect to the frame $(X_1, ..., X_n)$, i.e. $(\nu_H)_j := \langle \nu_H, X_j\rangle$ and

$\nu_{_H} = ((\nu_{_H})_1, ..., (\nu_{_H})_{m_1}, 0, ..., 0)$ and let $X = \sum_{i=1}^n x_i X_i$. First we note that, by definition, $\sigma_{_H} = (\nu_{_H} \rfloor \Omega^n)|_S$. So we have

$$d\,(X \rfloor \sigma_{_H})|_S = d\,(X \rfloor \nu_{_H} \rfloor \Omega^n)|_S = \sum_{i=1}^{n}\sum_{j=1}^{m_1} d\,(x_i(\nu_{_H})_j\, X_i \rfloor X_j \rfloor \Omega^n)|_S \tag{151}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m_1}\Big\{X_i(x_i\,(\nu_{_H})_j)\,(X_j \rfloor \Omega^n)|_S - X_j(x_i\,(\nu_{_H})_j)(X_i \rfloor \Omega^n)|_S + x_i\,(\nu_{_H})_j\, d\,(X_i \rfloor X_j \rfloor \Omega^n)|_S\Big\}.$$

Now we remark that for Carnot groups we have

$$\begin{aligned}\mathrm{div}X &= \sum_{h=1}^{n}\langle \nabla_{X_h} X, X_h\rangle = \sum_{i,\,h=1}^{n}\Big(X_h(x_i)\delta_i^h + x_i\langle \nabla_{X_h} X_i, X_h\rangle\Big)\\ &= \sum_{i=1}^{n} X_i(x_i) + \frac{1}{2}\sum_{i,\,h=1}^{n} x_i(c_{ij}^i - c_{ji}^i)\\ &= \sum_{i=1}^{n} X_i(x_i),\end{aligned} \tag{152}$$

where the last equality follows from the stratification hypothesis on the Lie algebra, which implies that $c_{ij}^i = -c_{ji}^i = 0$ $(i, j = 1, ..., n)$. Moreover we note that

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m_1} X_i(x_i\,(\nu_{_H})_j)\,(X_j \rfloor \Omega^n)|_S\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m_1}\Big(X_i(x_i)\,(\nu_{_H})_j\ + x_i\, X_i(\nu_{_H})_j\Big)\,(X_j \rfloor \Omega^n)|_S\\ &= \mathrm{div}X\,(\sigma_{_H})|_S + \sum_{i=1}^{n}\Big(\sum_{j=1}^{m_1} x_i\,(\nu_{_H})_j\, X_i(\nu_{_H})_j\Big)\,(\sigma_{_H})|_S\\ &= \mathrm{div}X\,(\sigma_{_H})|_S\end{aligned} \tag{153}$$

since $\sum_j (\nu_{_H})_j\, X_i(\nu_{_H})_j = \frac{1}{2}X_i\Big(\sum_j (\nu_{_H})_j^2\Big) = 0$. Therefore, using (151), (152) and (153), we get

$$\begin{aligned}d\,(X \rfloor \sigma_{_H})|_S &= \mathrm{div}X\,(\sigma_{_H})|_S - \sum_{i=1}^{n}\sum_{j=1}^{m_1}\Big(x_i\, X_j(\nu_{_H})_j + (\nu_{_H})_j\, X_j(x_i)\Big)\, n_i\,(\sigma^{n-1})|_S\\ &+ \sum_{i=1}^{n}\sum_{j=1}^{m_1} x_i\,(\nu_{_H})_j\, d\,(X_i \rfloor X_j \rfloor \Omega^n)|_S\end{aligned}$$

$$= \left\{ \mathrm{div} X - \left[\mathrm{div}_{\scriptscriptstyle H} \nu_{\scriptscriptstyle H} \; \frac{\langle X, N \rangle}{|\mathcal{P}_H(N)|_H} + \frac{\langle \mathcal{J}_X \nu_{\scriptscriptstyle H}, N \rangle}{|\mathcal{P}_H(N)|_H} \right] \right\} (\sigma_{\scriptscriptstyle H})|_S$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m_1} x_i \, (\nu_{\scriptscriptstyle H})_j \, d\,(X_i \rfloor X_j \rfloor \Omega^n)|_S. \tag{154}$$

Notice that $-\mathrm{div}_{\scriptscriptstyle H} \nu_{\scriptscriptstyle H} = \langle \mathrm{H}, \nu_{\scriptscriptstyle H} \rangle = \mathcal{H}^{sc}_{\scriptscriptstyle H}$ is the *scalar horizontal mean curvature of* S. Now, by using Lemma 7.12 and Definition 7.13 we finally get

$$\begin{aligned} d\,(X \rfloor \sigma_{\scriptscriptstyle H})|_S &= \left\{ \mathrm{div} X + \left[\mathcal{H}^{sc}_{\scriptscriptstyle H} \frac{\langle X, N \rangle}{|\mathcal{P}_H(N)|_H} - \frac{\langle \mathcal{J}_X \nu_{\scriptscriptstyle H}, N \rangle}{|\mathcal{P}_H(N)|_H} \right] \right\} (\sigma_{\scriptscriptstyle H})|_S \\ &- \sum_{k \in I_2 \cup \ldots \cup I_{\mathrm{k}}} \left[\langle \mathcal{C}^k_H \nu_{\scriptscriptstyle H}, \mathcal{P}_H(X) \rangle_H + \langle \mathcal{C}^k_V \nu_{\scriptscriptstyle H}, \mathcal{P}_V(X) \rangle_H \right] n_k \, (\sigma^{n-1})|_S. \end{aligned}$$

□

Therefore, as an application of the previous lemma, it follows a divergence-type theorem for regular non-characteristic hypersurfaces in k-step Carnot groups.

Theorem 7.15 (Divergence-type Theorem). *Let* $\mathbb{G}$ *be a* k*-step Carnot group and let* $S \subset \mathbb{G}$ *be a smooth immersed non-characteristic hypersurface with unit normal vector along* S *denoted by* N*. Let* $\mathcal{U} \subset S$ *be compact and suppose that the boundary* $\partial\mathcal{U}$ *is a smooth* $n-2$*-dimensional Riemannian submanifold with outward pointing unit normal* η*. Then for every smooth vector field* $X \in \mathfrak{X}(\mathbb{G})$ *we have*

$$\begin{aligned} & \int_{\mathcal{U}} \mathrm{div} X \; \sigma_{\scriptscriptstyle H} \\ + \;\; & \int_{\mathcal{U}} \left\{ \mathcal{H}^{sc}_{\scriptscriptstyle H} \langle X, N \rangle - \langle \mathcal{J}_X \nu_{\scriptscriptstyle H}, N \rangle - \sum_{k=m_1+1}^{n} n_k \left(\left\langle \mathcal{C}^k_H \nu_{\scriptscriptstyle H}, \mathcal{P}_H(X) \right\rangle_H + \left\langle \mathcal{C}^k_V \nu_{\scriptscriptstyle H}, \mathcal{P}_V(X) \right\rangle \right) \right\} \sigma^{n-1} \\ = \;\; & \int_{\partial\mathcal{U}} \langle X, \eta \rangle \, |\mathcal{P}_H(N)|_H \, \sigma^{n-2}. \end{aligned} \tag{155}$$

Proof. The proof follows by Lemma 7.14, using Stokes' Theorem and the fact that

$$(X \rfloor \sigma_{\scriptscriptstyle H})|_{\partial\mathcal{U}} = (|\mathcal{P}_H(N)|_H \, \langle X, \eta \rangle)|_{\partial\mathcal{U}}.$$

□

We now state the analogous formulae for the 1st variation of σ_{H} in k-step Carnot groups. We refer to Section 7.1 for an introduction to the problem and for some further notation that we will use below.

According to [55], we will consider deformations of the following type:

Definition 7.16. *Let $X \in \mathfrak{X}(\mathbb{G})$ be a fixed smooth vector field. We say that $\vartheta : (-\epsilon, \epsilon) \times \mathcal{U} \longrightarrow U$ is a* **smooth deformation of $\imath : \mathcal{U} \longrightarrow \mathbb{G}$ generated by** *X if, for each $x \in \mathcal{U}$, the mapping*

$$(-\epsilon, \epsilon) \ni t \longmapsto \vartheta_t(x) := \vartheta(t, x)$$

is the integral curve of X starting at $x \in \mathcal{U}$.

Theorem 7.17. *Let $\mathbb{G}$ be a* k*-step Carnot group and let $\imath : \mathcal{U} \longrightarrow \mathbb{G}$ denote the inclusion into $\mathbb{G}$ of a smooth non-characteristic hypersurface $\mathcal{U}$ with boundary $\partial\mathcal{U}$. Moreover, let $\vartheta : (-\epsilon, \epsilon) \times \mathcal{U} \longrightarrow \mathbb{G}$ be be a smooth deformation of $\mathcal{U}$ generated by $X \in \mathfrak{X}(\mathbb{G})$ and assume that $\mathcal{U}_t = \vartheta_t(\mathcal{U})$ is non-characteristic for every $t \in (-\epsilon, \epsilon)$. Finally, let $\Gamma(t) = \vartheta_t^{*}\sigma_{H,t}$ denote the $\mathbf{C}^{\infty}$ 1-parameter family of $n-1$-forms on $\mathcal{U}$ associated with ϑ_t. Then the following hold:*

(i)

$$\begin{aligned}\dot{\Gamma}(0) &= \Big\{ -\mathcal{H}_{H}^{sc}\, \langle X, N\rangle\, \sigma^{n-1} + d\left(|\mathcal{P}_H(N)|_H\, (X \,\rfloor\, \sigma^{n-1})\right)\Big\}\Big|_{\mathcal{U}}\\ &= \Big\{ -\mathcal{H}_{H}^{sc}\, \langle \mathcal{P}_H(X), \nu_{H}\rangle_H\, \sigma_{H} - \mathcal{H}_{H}^{sc}\, \langle \mathcal{P}_Z(X), \mathcal{P}_Z(N)\rangle\, \sigma^{n-1} + d\left(|\mathcal{P}_H(N)|_H\, (X \,\rfloor\, \sigma^{n-1})\right)\Big\}\Big|_{\mathcal{U}};\end{aligned}$$

(ii)

$$\begin{aligned}I_{\mathcal{U}}(\sigma_{H}) = &-\int_{\mathcal{U}} \mathcal{H}_{H}^{sc}\, \langle \mathcal{P}_H(X), \nu_{H}\rangle_H\, \sigma_{H} - \int_{\mathcal{U}} \mathcal{H}_{H}^{sc}\, \langle \mathcal{P}_Z(X), \mathcal{P}_Z(N)\rangle\, \sigma^{n-1}\\ &+ \int_{\partial\mathcal{U}} \langle X, \eta\rangle\, |\mathcal{P}_H(N)|_H\, \sigma^{n-2},\end{aligned}$$

where $I_{\mathcal{U}}(\sigma_{H}) := \frac{d}{dt}\Big|_{t=0} \int_{\mathcal{U}} \Gamma(t)$ is the 1st variation of σ_{H} on $\mathcal{U}$.

(iii) For every smooth vector field $X \in \mathfrak{X}(U)$ we have

$$\int_{\mathcal{U}} \mathrm{Div}_{\sigma_{H}}(X)\,\sigma_{H} = -\int_{\mathcal{U}} \mathcal{H}_{H}^{sc}\,\langle \mathcal{P}_H(X), \nu_{H}\rangle_H\,\sigma_{H} - \int_{\mathcal{U}} \mathcal{H}_{H}^{sc}\,\langle \mathcal{P}_Z(X), \mathcal{P}_Z(N)\rangle\,\sigma^{n-1}$$
$$+ \int_{\partial\mathcal{U}} \langle X, \eta\rangle\,|\mathcal{P}_H(N)|_H\,\sigma^{n-2},$$

where $\mathrm{Div}_{\sigma_{H}}$ (see Definition 7.5) is the divergence operator with respect to σ_{H} which turns out to be defined by the following identity:

$$\mathrm{Div}_{\sigma_{H}} X = \mathrm{div}\, X + \frac{1}{|\mathcal{P}_H(N)|_H}\Big\langle \sum_{k=m_1+1}^{n} n_k\, \mathcal{C}_H^k \nu_{H}, \mathcal{P}_H(X)\Big\rangle_H$$
$$+\frac{1}{|\mathcal{P}_H(N)|_H}\Big\langle \sum_{k=m_1+1}^{n} n_k\, \mathcal{C}_V^k \nu_{H}, \mathcal{P}_V(X)\Big\rangle_H - \langle \mathcal{J}_X \nu_{H}, N\rangle.$$

A posteriori will be clear that the previous item (iii) it is just an *independent* reformulation of the previous Theorem 7.15. Now, to prove the theorem, it suffices to determine $\dot{\Gamma}(0)$. Also, we note that if $\frac{\partial}{\partial t}$ denote the *canonical vector field* along the 1st factor in $(-\epsilon, \epsilon) \times \mathcal{U}$ then X turns out to be its variation vector field, i.e. $X := \vartheta_* \frac{\partial}{\partial t}\big|_{t=0}$.

We may start with the following useful remark:

Lemma 7.18. *For any $X \in \mathfrak{X}(\mathbb{G})$ we have*

$$\begin{aligned}\dot{\Gamma}(0) = \mathcal{L}_X(\sigma_{H,t})|_{\mathcal{U}} &= \underbrace{(X \rfloor d\,(\nu_{H\,t} \rfloor \Omega^n))|_{\mathcal{U}}}_{a_1} + \underbrace{d\,(X \rfloor \sigma_{H,t})|_{\mathcal{U}}}_{a_2}\\ &= \underbrace{([X, \nu_{H\,t}] \rfloor \Omega^n)|_{\mathcal{U}}}_{b_1} + \underbrace{(\nu_{H\,t} \rfloor d\,(X \rfloor \Omega^n))|_{\mathcal{U}}}_{b_2}.\end{aligned}$$

Proof. The first identity follows by applying Cartan's formula and using the very definition of $\sigma_{H,t}$, while for the second, we have to use a well-know characterization of the Lie derivative[7]. $\square$

Remark 7.19. *The terms a_1, a_2, b_1, b_2, which appear in the above Lemma 7.18, can be computed as follows. First we note that the Riemannian divergence theorem give us*

$$(X \rfloor d\,(Y \rfloor \Omega^n))|_{\mathcal{U}_t} = (\operatorname{div} Y \rfloor \Omega^n)|_{\mathcal{U}_t} = \operatorname{div} Y \,\langle X, N_t\rangle\,(\sigma^{n-1})|_{\mathcal{U}_t} \qquad \forall\; X,\, Y \in \mathfrak{X}(\mathbb{G}).$$

Thus we immediately see that:

(i) $a_1 := \operatorname{div} \nu_{_H} \langle X, N\rangle\,(\sigma^{n-1})|_{\mathcal{U}}$;

(ii) $b_2 := (\operatorname{div} X\, \nu_{_H t} \rfloor \Omega^n)|_{\mathcal{U}} = \operatorname{div} X\,(\sigma_{_H})|_{\mathcal{U}}$.

Note that Definition 7.11 implies that $\operatorname{div} \nu_{_H} = -\langle \mathrm{H}, \nu_{_H}\rangle = -\mathcal{H}^{sc}_{_H}$. *Moreover, by using Stoke's Theorem, the term a_2 can be computed in terms of a boundary integral, if $\mathcal{U}$ is with boundary, or also, by a direct computation, as in the next Lemma 7.14 which is the* k*-step analogous to Lemma 6.20.*

Lemma 7.20. *For any $X \in \mathfrak{X}(\mathbb{G})$, $X = \sum_{i=1}^{n} x_i X_i$, we have*

$$([X, \nu_{_H t}] \rfloor \Omega^n)|_{\mathcal{U}_t}$$

$$= \Big\{ \sum_{k \in I_2 \cup \ldots \cup I_{\mathrm{k}}} \Big[\langle n_{tk}\, \mathcal{C}^k_{t\,H} \nu_{_H t}, \mathcal{P}_H(X)\rangle_H + \langle n_{tk}\, \mathcal{C}^k_{t\,V} \nu_{_H t}, \mathcal{P}_V(X)\rangle \Big] - \langle \mathcal{J}_X \nu_{_H t}, N_t\rangle \Big\}\,(\sigma^{n-1})|_{\mathcal{U}_t}.$$

Proof. We first notice that

$$[X, \nu_{_H t}] \;=\; \left[\sum_{i=1}^{n} x_i X_i, \sum_{j=1}^{m_1} (\nu_{_H t})_j X_j\right] = \sum_{i=1}^{n}\sum_{j=1}^{m_1} [x_i X_i, (\nu_{_H t})_j X_j]$$

[7] *if $\omega \in \Lambda^p(M)$ and $X,\, Y \in (M)$, where M is a n-dimensional smooth manifold oriented by Ω^n, then*

$$\mathcal{L}_X(Y \rfloor \omega) = [X, Y] \rfloor \Omega^n + Y \rfloor \mathcal{L}_X \omega.$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{m_1} \Big\{ x_i X_i (\nu_{H\,t})_j \, X_j - (\nu_{H\,t})_j \, X_j(x_i) X_i + x_i (\nu_{H\,t})_j \, [X_i, X_j] \Big\} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m_1}\sum_{k=m_1+1}^{n} \Big\{ x_i \, X_i (\nu_{H\,t})_j \, X_j - (\nu_{H\,t})_j X_j(x_i) X_i + \, c_{ij}^k x_i \, (\nu_{H\,t})_j \, X_k \Big\}.
\end{aligned}
\tag{156}
$$

From (156) we thus get

$$
\begin{aligned}
&([X, \nu_{H\,t}] \rfloor \Omega^n)|_{\mathcal{U}_t} \\
&= \left\{ \sum_{i=1}^{n}\sum_{j=1}^{m_1}\sum_{k=m_1+1}^{n} \Big\{ x_i X_i((\nu_{H\,t})_j) \, X_j - (\nu_{H\,t})_j X_j(x_i) X_i + \, c_{ij}^k x_i (\nu_{H\,t})_j \Big\} \rfloor \Omega^n \right\}\Bigg|_{\mathcal{U}_T} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m_1}\sum_{k=m_1+1}^{n} \Big\{ x_i X_i (\nu_{H\,t})_j \, n_{tj} - (\nu_{H\,t})_j \, X_j(x_i) n_{ti} + c_{ij}^k x_i (\nu_{H\,t})_j n_{tk} \Big\} \, (\sigma^{n-1})|_{\mathcal{U}_t} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m_1}\sum_{k=m_1+1}^{n} \Big\{ -(\nu_{H\,t})_j X_j(x_i) n_{ti} + c_{ij}^k x_i (\nu_{H\,t})_j n_{tk} \Big\} \, (\sigma^{n-1})|_{\mathcal{U}_t} \qquad (157) \\
&= \Big\{ \sum_{k \in I_2 \cup \ldots \cup I_{\mathrm{k}}} \Big[\langle n_{tk} \, \mathcal{C}_{t\,H}^k (\nu_{H\,t}), \mathcal{P}_H(X) \rangle_H + \langle n_{tk} \, \mathcal{C}_{t\,V}^k \nu_{H\,t}, \mathcal{P}_V(X) \rangle \Big] - \langle \mathcal{J}_X \nu_{H\,t}, N_t \rangle \Big\} \, (\sigma^{n-1})|_{\mathcal{U}_t},
\end{aligned}
$$

where equality (157) follows, since $X\left(\frac{|\nu_{H\,t}|_H^2}{2}\right) = 0$; see the previous equation (153). Note also that in the last line we have used Definition 7.13.

$\square$

Proof of Theorem 7.17. Tacking into account the previous discussion, from Lemma 7.18 and Remark 7.19 we easily get that

$$
\begin{aligned}
\dot{\Gamma}(0) = \mathcal{L}_X(\sigma_{H,t})|_{\mathcal{U}} &= (X \rfloor d\,(\nu_{H\,t} \rfloor \Omega^n))|_{\mathcal{U}} + d\,(X \rfloor \sigma_{H,t})|_{\mathcal{U}} =: I \\
&= ([X, \nu_{H\,t}] \rfloor \Omega^n)|_{\mathcal{U}} + (\nu_{H\,t} \rfloor d\,(X \rfloor \Omega^n))|_{\mathcal{U}} =: II
\end{aligned}
$$

and since $(X \rfloor \sigma^{n-1})|_{\partial\mathcal{U}} = \langle X, \eta \rangle \, (\sigma^{n-2})|_{\partial\mathcal{U}}$, we have

$$
\begin{aligned}
I &= \operatorname{div} \nu_H \langle X, N \rangle \, (\sigma^{n-1})|_{\mathcal{U}} + d\,(|\mathcal{P}_H(N)|_H \, (X \rfloor \sigma^{n-1})) \qquad (158) \\
&= ([X, \nu_{H\,t}] \rfloor \Omega^n)|_{\mathcal{U}} + \operatorname{div} X \, (\sigma_H)|_{\mathcal{U}} = II.
\end{aligned}
$$

Therefore (i) follows by applying Definition 7.11, while (ii) follows immediately from Stokes' Theorem and the very definition of σ_{H}. Now, using Definition 7.5, item (ii) of Remark 7.19 and the previous Lemma 7.20, we get

$$(\mathrm{Div}_{\sigma_{H}} X)\, \sigma_{H}|_{\mathcal{U}} = \mathrm{div}\, X\, \sigma_{H}|_{\mathcal{U}}$$

$$+ \sum_{k \in I_2 \cup \ldots \cup I_{\mathrm{k}}} \Big\{ \langle n_k\, \mathcal{C}^k_H \nu_{H}, \mathcal{P}_H(X)\rangle_H + \langle n_k\, \mathcal{C}^k_V \nu_{H}, \mathcal{P}_V(X)\rangle - \langle \mathcal{J}_X \nu_{H}, N\rangle \Big\}\, \sigma^{n-1}|_{\mathcal{U}}$$

$$= \mathrm{div}\, X\, \sigma_{H}|_{\mathcal{U}}$$

$$+ \Big\{ \Big\langle \sum_{k=m_1+1}^{n} n_k\, \mathcal{C}^k_H \nu_{H}, \mathcal{P}_H(X) \Big\rangle_H + \Big\langle \sum_{k=m_1+1}^{n} n_k\, \mathcal{C}^k_V \nu_{H}, \mathcal{P}_V(X) \Big\rangle - \langle \mathcal{J}_X \nu_{H}, N\rangle \Big\}\, \sigma^{n-1}|_{\mathcal{U}}$$

and so (iii) follows by using (158) integrating both sides and applying again Stokes' Theorem for the second addend of I.

$\square$

References

[1] L. Ambrosio, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*, Adv. in Math., 2001.

[2] L. Ambrosio, N. Fusco, & D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, 2000.

[3] L. Ambrosio & B. Kirchheim, *Rectifiable sets in metric and Banach spaces*, Math.Ann. **318**, 2000, 527-555.

[4] ——, *Current in metric spaces*, Acta Math.**185**, 2000, 1-80.

[5] L. Ambrosio & V. Magnani, *Some fine properties of BV functions on sub-Riemannian groups*, To appear in Math.Z.

[6] L. Ambrosio & P. Tilli, *Selected topics on "Analysis in Metric Spaces"* , Scuola Normale Superiore, Pisa, 2000.

[7] Z.M. Balogh, *Size of characteristic sets and functions with prescribed gradients*, To appear on J. Reine Angew. Math.

[8] Z.M. Balogh & M. Rickly, *Regularity of convex functions on Heisenberg groups*, Preprint 2003.

[9] A. Bellaïche, *The tangent space in subriemannian geometry*, in *Subriemannian Geometry*, Progress in Mathematics, **144**. ed. by A.Bellaiche and J.Risler, Birkhauser Verlag, Basel, 1996.

[10] A.L. Besse, *Manifolds all of whose Geodesics are closed*, Berlin: Springer Verlag, 1978.

[11] I. Birindelli & J. Prajapat, *Monotonicity and simmetry results for degenerate elliptic equations on nilpotent Lie groups*, Pacific Journal of Mathemathics, **204**, No. 1, 2002.

[12] E. Bombieri, *An Introduction to Minimal Currents and Parametric Variational Problems*, Mathematical Reports, 1985, J. Dieudonnè (esd.), vol. 2, Part 3, pp. 285-384.

[13] Yu.D. Burago & V.A. Zalgaller, *Geometric Inequalities*, Berlin: Springer Verlag, 1980.

[14] L. Capogna, D. Danielli, & N. Garofalo, *The geometric Sobolev embedding for vector fields and the isoperimetric inequality*, Comm. Anal. Geom. **12**, 1994.

[15] É. Cartan, *Sur la represéntation géométrique des systèmes matériels non holonomes*, Proc. Int. Congr. Math., vol. **4**, Bologna, 253-261, 1928.

[16] I. Chavel, *Riemannian Geometry: a modern introduction*, Cambridge University Press, 1994.

[17] ——, *Isoperimetric Inequalities*, Cambridge University Press, 2001.

[18] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geometry and Functional Analysis, **9**, 428-517, 1999.

[19] G. Citti, M. Manfredini, & A. Sarti, *Minimum of the Mumford-Shah functional in a conctact manifold on the Heisenberg space.* Preprint, Bologna, 2003.

[20] L.J. Corvin & F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications*, Cambridge University Press, 1984.

[21] C.B. Crooke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. Éc. Norm. Sup., Paris **13**, 1980, 419-435.

[22] C.B. Crooke & A. Derdziński, *A lower bound for λ_1 on manifolds with boundary*, Comment. Math. Helv. **59**, 1984, 187-192.

[23] T. Coulhon & L. Saloff-Coste, *Isopérimétrie pour les groupes et les variétés,*, Revista Mathematica Iberoamericana, **9**, 293-314, 1993.

[24] D. Danielli, N. Garofalo, & D.M. Nhieu, *Notions of Convexity in Carnot groups*, preprint 2002.

[25] G.David & S.Semmes, *Fractured Fractals and Broken Dreams. Self-Similar Geometry through Metric and Measure*, Oxford University Press, 1997.

[26] E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press,1989.

[27] E.De Giorgi, *Sulla proprietá isoperimetrica dell' ipersfera*, Mem. Accad. Lincei **5**, 8, 33-44, 1958.

[28] E.De Giorgi,*Un progetto di teoria delle correnti, forme differenziali e varietà non orientate in spazi metrici*, in *Variational Methods, Non Linear Analysys and Differential Equations in Honour of J.P. Cecconi*, M.Chicco et al. Eds. ECIG, Genova, 67-71, 1993.

[29] ——, *Un progetto di teoria unitaria delle correnti, forme differenziali, varietà ambientate in spazi metrici, funzioni a variazione limitata*, Manuscript, 1995.

[30] ——,*Problema di Plateau generale e funzionali geodetici*, Atti Sem. Mat. Fis.Univ. Modena, XLIII, 1995.

[31] P. Eberlein, *Geometry of* 2*-step Nilpotent Lie Groups with a Left Invariant Metric*, Ann. Scient. Éc. Nor. Sup., 4 série, **27**, 1994, pp. 611-660.

[32] L.C. Evans & R.F. Gariepy, *Measure Theory and Fine Properties of functions*, CRC Press, Boca Raton, (1992).

[33] H. Federer, *Geometric Measure Theory*, Springer Verlag, 1969.

[34] G.B.Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat., **13**, 161-207, 1975.

[35] G.B.Folland & E.M.Stein, *Hardy spaces on homogeneous groups*, Princeton University Press, 1982.

[36] B. Franchi, S. Gallot, & R.L. Wheeden, *Sobolev and isoperimetric inequalities for degenerate metrics*, Math. Ann.,
300, 557-571, 1994.

[37] B. Franchi & E. Lanconelli, *Hölder regularity theorem for a class of non uniformly elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa, **10**, 1983, 523-541.

[38] B. Franchi, G. Lu, & R.L.Wheeden, *Representation formulas and weighted Poincaré inequalities for Hörmander vector fields*, Ann. Inst. Fourier, **45**, 577-604, Grenoble, 1995.

[39] B. Franchi, R. Serapioni, & F. S. Cassano, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*, Houston Journal of Math. Vol. 22, **4**, 1996.

[40] ——, *Approximation and imbedding theorems for wheighted Sobolev spaces associated with Lipschitz continous vector fields,* Boll. Un. Mat. Ital., **7**, 11-B 1997, 83-117.

[41] ——, *Rectifiability and Perimeter in the Heisenberg Group*, Math. Annalen, **321**, 2001, 479-531.

[42] ——, *Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups*, Comm. Anal. Geom., to appear.

[43] ——, *On the structure of finite perimeter sets in step 2 Carnot groups*, J. Geometric Anal., to appear.

[44] N. Garofalo & S. Pauls, *The Berstein problem in the Heisenberg group,* submitted paper.

[45] N. Garofalo & D.M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math., **49**, 1081-1144, 1996.

[46] M. Gobbino, *Finite difference approximation of the Mumford Shah functional,* Comm. Pure. Appl. Math., **51**, 197-228, 1998.

[47] N. Goodman, *Nilpotent Lie groups,* Springer Lecture notes in Mathematics, Vol. 562, 1976.

[48] A. Gray & B. E, *Tubes*, in *Bull...*, Progress in Mathematics, **144**. ed. by A.Bellaiche and J.Risler, Birkhauser Verlag, Basel, 1996.

[49] M. Gromov, *Carnot-Carathéodory spaces seen from within*, in *Subriemannian Geometry*, Progress in Mathematics, **144**. ed. by A.Bellaiche and J.Risler, Birkhauser Verlag, Basel, 1996.

[50] ——, *Metric structures for Riemannian and Non Riemannian Spaces*, Progress in Mathematics, **153**, Birkhauser Verlag, Boston, 1999.

[51] C.E. Gutierrez & A. Montanari, *Convexity solutions of Monge Ampère Equations on the Heisenberg group,* preprint 2002.

[52] P.Hajłasz, & P.Koskela, *Sobolev met Poincare*, Mem. Am. Math. Soc., **688**, 2000.

[53] J. HEINONEN, *Calculus on Carnot groups*, Ber. Univ. Jyväskylä Math. Inst., **68**, 1-31, 1995.

[54] S. HELGASON, *Differential geometry, Lie groups, and symmetric spaces* , Academic Press, New York, 1978.

[55] R. HERMANN, *The 2nd variation for Minimal Submanifolds*, Journal of Mathematics and Mechanics, vol. 16, n. **5**, 1966.

[56] M. HIRSCH, *Differential Topology*, Springer-Verlag, New York, 1976.

[57] D. JERISON, *The Poincaré inequality for vector fields satisfying Hörmander condition*, Duke Math. Journal, **53**, 503-523, 1986.

[58] A. KAPLAN, *On the Geometry of the Groups of Heisenberg Type* Bull. London Math. Soc., Vol.**15**, 1983, pp.35-42.

[59] S. KOBAYASHI & K. NOMIZU, *Foundations of Differential geometry*, Vol. I, Vol. II, Tract in Math., n. **15**, Wiley (Interscience), New York, 1963-1969.

[60] J. KOILLER & P.R. RODRIGUES & P. PITANGA, *Non-holonomic connections following Élie Cartan*, An. Acad. Bras. Cienc. 2001, **7** (2), pp. 165-190.

[61] A. KORÁNYI & H.M. REIMANN, *Foundation for the Theory of Quasiconformal Mapping on the Heisenberg Group,* Advances in Mathematics, **111**, 1-85, 1995.

[62] S. LANG, *Differential and Riemannian Manifolds* , Springer Verlag, 1994.

[63] B. LAWSON, *Minimal Varieties in Riemannian Manifolds*, Publisch or Perisch, n.7, 1979.

[64] J.M. LEE, *Introduction to Smooth Manifolds* , Springer Verlag, 2003.

[65] G. LU, J. J. MANFREDI, & B. STROFFOLINI, *Convex functions on the Heisenberg group*, To appear in Calculus of Variations.

[66] V.MAGNANI, *Differentiability and Area Formula on Statified Lie Groups*, Houston Jour. Math., **27**, 2001.

[67] ——,*Characteristic point, rectifiability and perimeter measure on stratified groups*, Preprint 2003.

[68] ——,*Elements of Geometric Measure Theory on sub-Riemannian groups*, PhD Thesis, Scuola Normale Superiore, Pisa (2002).

[69] G.A. MARGULIS & G.D. MOSTOW, *The Differential of a Quasi-Conformal Mapping of a Carnot-Carathéodory Spaces*, Geometric and Functional Analysis, **5**, 2, 1995.

[70] P. MATTILA, *Geometry of sets and measures in euclidean spaces*, Cambridge University Press, 1995.

[71] J. MILNOR, *Curvatures of left invariant Riemannian metrics*, Advances in Math. **21**, 293-329, 1976.

[72] J. MILNOR & J.D. STASHEFF, *Characteristic classes*, Ann. of Math. Studies, Princeton University Press, 1974.

[73] J. MITCHELL, *On Carnot-Carathèodory metrics*, J.Differ. Geom. **21**, 35-45, 1985.

[74] F. MONTEFALCONE, *Sets of finite perimeter associated with vector fields and polyhedral approximation.* , Rend. Mat. Lincei s.9, v. 14:279-295 (2003).

[75] ——, *Some relations among volume, intrinsic perimeter and one-dimensional restrictions of BV functions in Carnot groups*, Preprint 2003.

[76] ——, *Some remarks about the geometry of non-characteristic hypersurfaces in Carnot groups*, Preprint 2004-unpublished.

[77] R. MONTGOMERY, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, American Mathematical Society, Math. Surveys and Monographs, Vol. 91, 2002.

[78] R. MONTI, *Distances, Boundaries and surface measures in Carnot Carathéodory Spaces*, PhD Thesis, 2001.

[79] R. MONTI & F. SERRA CASSANO, *Surface measures in Carnot-Carathéodory spaces*, Calc. Var. and Pde, **13**.

[80] A. NAGEL, E.M. STEIN, & S. WAINGER, *Balls and metrics defined by vector fields I: Basic properties,* Acta Math., **155**, 103-147, 1985.

[81] P. PANSU, *Geometrie du Group d'Heisenberg*, These pour le titre de Docteur, 3ème cycle, Universite Paris VII, 1982.

[82] ——,*Métriques de Carnot Carathéodory et quasi-isométries des espaces symmétriques de rang un* , Ann. of Math. 2 **129**, 1-60, 1989.

[83] S. Pauls, *Minimal surfaces in the Heisenberg group*, submitted paper.

[84] D. Preiss & J. Tisêr, *On Besicovitch* 1/2*-problem*, J. London Math. Soc., 45, (1992)

[85] L.P. Rothschild & E.M. Stein, *Hypoelliptic differential operators and nilpotent groups,* Acta Math., **137**, 247-320, 1976.

[86] L.A. Santaló, *Integral Geometry and Geometric Probability* , Enc.Math., **1**, 1976.

[87] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. Math. **88**, (1968), 62-105.

[88] M. Spivak, *Differential Geometry*, vol. IV, Publisch or Perish, 1964.

[89] E.M. Stein, *Harmonic Analysis*, Princeton University Press, 1993.

[90] R.S. Strichartz, *Sub-Riemannian geometry,* J. Diff. Geom., **24**, 221-263, 1986. Corrections: J. Diff. Geom., **30**, 595-596, 1989.

[91] G. Talenti, *The standard isoperimetric theorem,* in Handbook of Convexity, Vol. **A**, P.M.Gruber & J.M.Wills, eds, 73-123. Amsterdam:North Holland, 1993.

[92] V.S. Varadarajan, *Lie Groups, Lie Algebras, and their Representations*, Springer, 1984.

[93] N.Th. Varopoulos, *Analysis on Lie groups*, J. Funct.Anal., **76**, 346-410, 1988.

[94] N.Th. Varopoulos, L. Saloff-Coste, & T. Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, 1992.

[95] A.M. Vershik, & V.Ya Gershkovich, *Nonholonomic Dynamical systems, Geometry of Distributions and Variationals Problems*, in V.I. Arnold, S.P. Novikov (eds.), Dynamical systems VII, Springer-Verlag, 1996.

[96] S.K. Vodop'yanov, *$\mathcal{P}$-differentiability on Carnot Groups in different topologies and related topics*, Proc. on Analysis and Geometry , 603-670, Sobolev Inst.Press, Novosibirsk, 2000.

[97] W.P. Ziemer, *Weakly Differentiable Functions*, Springer Verlag, 1989.

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