

# INTEGRAL FORMULAS FOR A CLASS OF CURVATURE PDE'S AND APPLICATIONS TO ISOPERIMETRIC INEQUALITIES AND TO SYMMETRY PROBLEMS

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ABSTRACT. We prove integral formulas for closed hypersurfaces in  $\mathbb{C}^{n+1}$ , which furnish a relation between elementary symmetric functions in the eigenvalues of the complex Hessian matrix of the defining function and the Levi curvatures of the hypersurface. Then we follow the Reilly approach to prove an isoperimetric inequality. As an application, we obtain the “Soap Bubble Theorem” for star-shaped domains with positive and constant Levi curvatures bounding the classical mean curvature from above.

## 1. INTRODUCTION

The study of surfaces in the Euclidean space with either constant Gauss curvature or constant mean curvature received in the past a great amount of attention. In 1899 Liebmann [9] proved that the spheres are the only compact surfaces in  $\mathbb{R}^3$  with constant Gauss curvature. In 1952 Süß [20] extended the Liebmann result showing that a compact convex hypersurface in the Euclidean space must be a sphere, provided that for some  $j$  the  $j$ -th elementary symmetric polynomial in the principal curvatures is constant. In 1954 Hsiung [8] proved that the “convexity” assumption can be relaxed to the “star-shapedness” one. The proofs of the above results are based on the Minkowski formulae. A breakthrough for this sort of problems was made by Alexandrov [1] in 1956, who proved that the sphere is the only compact hypersurface embedded into the Euclidean space with constant mean curvature. Alexandrov method is completely different from the Liebmann-Süß method, and is based on the moving plane technique, on the interior maximum principle for elliptic

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Mathematics Subject Classification 2000: 32W20, 32W50, 35J70, 53C42.

equations and on the boundary maximum principle of Hopf type for uniformly elliptic equations. In 1978 Reilly [14] obtained another proof of the Alexandrov theorem combining the Minkowski formulae with some new elegant arguments.

It is well known that the Levi curvatures contain less geometric information than Euclidean curvatures, because the Levi form is only a part of the second fundamental form. However, in a joint paper with Lanconelli [12] we wrote the  $j$ -th Levi curvature for real hypersurfaces in  $\mathbb{C}^{n+1}$  in terms of elementary symmetric functions of the eigenvalues of the normalized Levi form, and we proved a strong comparison principle, leading to symmetry theorems for domains with constant curvatures.

Precisely:

**Theorem 1.1.** *Let  $D \subseteq \mathbb{C}^{n+1}$  be a strictly  $j$ -pseudoconvex domain with connected boundary,  $1 \leq j \leq n$ . Let  $B_R(z_0) \subseteq D$  be a tangent sphere to  $\partial D$  at some point  $p \in \partial D$ . If  $K_{\partial D}^{(j)}(z)$  is the  $j$ -th Levi curvature of  $\partial D$  at  $z \in \partial D$  and*

$$K_{\partial D}^{(j)}(z) \geq 1/R^j, \quad \forall z \in \partial D,$$

then  $D = B_R(z_0)$ .

In [12] we prove that if  $\Omega$  is a bounded domain of  $\mathbb{C}^{n+1}$ , with boundary a real hypersurface of class  $C^2$ , then the  $j$ -th Levi curvature of  $\partial\Omega$  at  $z = (z_1, \dots, z_{n+1}) \in \partial\Omega$  writes in term of defining function  $f$  of  $\Omega = \{f(z) < 0\}$  as

$$(1) \quad K_{\partial\Omega}^{(j)}(z) = -\frac{1}{\binom{n}{j}} \frac{1}{|\partial f|^{j+2}} \sum_{1 \leq i_1 < \dots < i_{j+1} \leq n+1} \Delta_{(i_1, \dots, i_{j+1})}(f)$$

for all  $j = 1, \dots, n$ , where  $|\partial f| = \sqrt{\sum_{j=1}^{n+1} |f_j|^2}$

$$(2) \quad \Delta_{(i_1, \dots, i_{j+1})}(f) = \det \begin{pmatrix} 0 & f_{\bar{i}_1} & \dots & f_{\bar{i}_{j+1}} \\ f_{i_1} & f_{i_1, \bar{i}_1} & \dots & f_{i_1, \bar{i}_{j+1}} \\ \vdots & \vdots & \ddots & \vdots \\ f_{i_{j+1}} & f_{i_{j+1}, \bar{i}_1} & \dots & f_{i_{j+1}, \bar{i}_{j+1}} \end{pmatrix}$$

and  $f_j = \frac{\partial f}{\partial z_j}$ ,  $f_{\bar{j}} = \overline{f_j}$ ,  $f_{j\bar{\ell}} = \frac{\partial^2 f}{\partial z_j \partial \bar{z}_\ell}$ .

Theorem 1.1 suggested the following question: are spheres the unique compact hypersurfaces with constant Levi curvatures? Klingenberg in [10] gave a first positive answer to this problem by showing that a compact and strictly pseudoconvex real hypersurface  $M \subset \mathbb{C}^{n+1}$  is isometric to a sphere, provided that  $M$  has constant horizontal mean curvature and the CR structure  $T_{1,0}(M)$  is parallel in  $T^{1,0}(\mathbb{C}^{n+1})$ . Later on in [11] we relaxed Klingenberg conditions and we proved that if the characteristic direction is a geodesic, then Alexandrov Theorem holds for hypersurfaces with positive constant Levi mean curvature.

The problem of characterizing hypersurfaces with constant Levi curvature has been studied by many authors. Hounie and Lanconelli in [6] showed the result for Reinhardt domain of  $\mathbb{C}^2$ , i.e. for domains  $D$  such that if  $(z_1, z_2) \in D$  then  $(e^{i\theta_1} z_1, e^{i\theta_2} z_2) \in D$  for all real  $\theta_1, \theta_2$ . Under this hypothesis, in a neighborhood of a point, there is a defining function  $F$  only depending on the radius  $r_1 = |z_1|$ ,  $r_2 = |z_2|$ ,  $F(r_1, r_2) = f(r_2^2) - r_1^2$  with  $f$  the solution of the ODE

$$(3) \quad s f f'' = s f'^2 - k(f + s f'^2)^{3/2} - f f'$$

Alexandrov Theorem follows from uniqueness of the solution of (3). Their technique has then been used in [7] to prove an Alexandrov Theorem for Reinhardt domains in  $\mathbb{C}^{n+1}$  with spherical symmetry for every  $n$ . Then in [13] Monti and Morbidelli proved a Darboux -type theorem for  $n \geq 2$ : the unique Levi umbelical hypersurfaces in  $\mathbb{C}^{n+1}$  with all constant Levi curvatures are spheres or cylinders.

In this paper we prove some integral formulas for compact hypersurfaces, which furnish a relation between elementary symmetric functions in the eigenvalues of the complex Hessian matrix of the defining function and the Levi curvatures of the hypersurface. Then we follow the Reilly approach to prove the following

**Theorem 1.2 (ISOPERIMETRIC ESTIMATES).** *Let  $\Omega$  be a bounded domain of  $\mathbb{C}^{n+1}$  with boundary a real hypersurface of class  $C^\infty$ . If  $K_{\partial\Omega}^{(j)}$  is non negative at every point*

of  $\partial\Omega$  then

$$(4) \quad \int_{\partial\Omega} \left( \frac{1}{K_{\partial\Omega}^{(j)}(x)} \right)^{1/j} d\sigma(x) \geq 2(n+1)|\Omega|$$

where  $|\Omega|$  is the Lebesgue measure of  $\Omega$ . If  $K_{\partial\Omega}^{(j)}$  is constant, then the equality holds in (4) if and only if  $\Omega$  is a ball of radius  $\left( \frac{1}{K_{\partial\Omega}^{(j)}} \right)^{1/j}$ .

The isoperimetric estimates (4) show that the Levi curvatures, which have been first introduced in analogy with Euclidean curvatures by Slodkowski and Tomassini in [21] and [19], have a deep geometric meaning, because they contain information about the measure of a bounded domain.

Let us remark that there are non spherical sets which satisfy the equality in (4) (see (18)). Thus, the class of sets which satisfy the equality in (4) is larger than the class of sets which satisfy the equality in the classical isoperimetric inequality and in the Alexandrov Fenchel inequalities for quermassintegrals ( see [3], [5] and [22]).

On the other side, if  $K_{\partial\Omega}^{(j)}$  is constant, then

$$(5) \quad \left( K_{\partial\Omega}^{(j)} \right)^{1/j} \leq \frac{|\partial\Omega|}{2(n+1)|\Omega|}$$

and an Alexandrov type theorem holds for star-shaped domains whose classical mean curvature is bounded from above by a constant  $j$ -Levi curvature. Precisly

**Corollary 1.1** (AN ALEXANDROV TYPE THEOREM). *Let  $\Omega \subset \mathbb{C}^{n+1}$  be a bounded star-shaped domain with boundary a smooth real hypersurface. If the  $j$ -Levi curvature is a positive constant  $K^{(j)}$  at every point of  $\partial\Omega$ , then the maximum of the mean curvature of  $\partial\Omega$  is bounded from below by  $(K^{(j)})^{(1/j)}$ . Moreover, if the mean curvature of  $\partial\Omega$  is bounded from above by  $(K^{(j)})^{(1/j)}$ , then  $\partial\Omega$  is a sphere and  $\Omega$  is a ball.*

Let us remark that, even if the Levi form is a part of the second fundamental form, in general it is not possible to bound from above the Levi curvatures with the Euclidean ones. Indeed, it is very easy to build a cylinder in  $\mathbb{C}^2$  whose Levi curvature is  $1/2$  while the mean curvature is  $1/3$ .

Our paper is organized as follows. In Section 2 we use the null Lagrangian property for elementary symmetric functions in the eigenvalues of the complex Hessian matrix and the classical divergence theorem to prove an integral formula (11) for a closed hypersurface in term of the  $j$ -th Levi curvature. In Section 3 we prove the isoperimetric estimates (4) and we use the Minkowski formula for the classical mean curvature to conclude the proof of Corollary 1.1.

**Acknowledgement** We would like to thank Ermanno Lanconelli and Daniele Morbidelli for some useful discussions about this argument.

## 2. NULL LAGRANGIAN PROPERTY FOR ELEMENTARY SYMMETRIC FUNCTIONS IN THE EIGENVALUES OF THE COMPLEX HESSIAN MATRIX

Given a Hermitian matrix  $A$ , let  $\sigma_j(A)$  be the  $j$ -th elementary symmetric function in the eigenvalues of  $A$ . Let us recall that if  $A$  is the  $(n+1) \times (n+1)$  matrix with eigenvalues  $\lambda_1, \dots, \lambda_{n+1}$  then

$$\sigma_j(A) = \sum_{1 \leq i_1 < \dots < i_j \leq n+1} \lambda_{i_1} \cdots \lambda_{i_j}.$$

If we choose  $A = [a_{\ell\bar{k}}] = \partial\bar{\partial}f$  the complex Hessian matrix of a smooth function  $f$  and we denote by  $\frac{\partial\sigma_j(A)}{\partial a_{\ell\bar{k}}}$  the partial derivative of the function  $\sigma_j$  with respect to the term of place  $\ell\bar{k}$  of the matrix  $A$ , then

$$(6) \quad \sum_{\ell} \partial_{\ell} \left( \frac{\partial\sigma_{j+1}(\partial\bar{\partial}f)}{\partial a_{\ell\bar{k}}} \right) = 0, \forall k = 1, \dots, n+1.$$

(see [16] for a similar argument for the real Hessian matrix).

Moreover, by using the notation (2), the following Lemma holds

**Lemma 2.1.** *For every  $f \in C^2$  and for every  $j = 1, \dots, n+1$*

$$(7) \quad \sum_{\ell, k=1}^{n+1} \frac{\partial\sigma_{j+1}}{\partial a_{\ell\bar{k}}} (\partial\bar{\partial}f) f_{\ell} f_{\bar{k}} = - \sum_{1 \leq i_1 < \dots < i_{j+1} \leq n+1} \Delta_{(i_1, \dots, i_{j+1})}(f)$$

*Proof.* By explicitly writing  $\sigma_{j+1}(\partial\bar{\partial}f)$  in the left hand side of (7) as

$$\sigma_{j+1}(\partial\bar{\partial}f) = \sum_{1 \leq i_1 < \dots < i_{j+1} \leq n+1} \begin{vmatrix} f_{i_1, \bar{i}_1} & \dots & f_{i_1, \bar{i}_{j+1}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1}, \bar{i}_1} & \dots & f_{i_{j+1}, \bar{i}_{j+1}} \end{vmatrix}$$

with  $|A| = \det A$  for every Hermitian matrix  $A$ , we get

$$(8) \quad \begin{aligned} \sum_{\ell, k=1}^{n+1} \frac{\partial \sigma_{j+1}}{\partial a_{\ell \bar{k}}} (\partial\bar{\partial}f) f_{\ell} f_{\bar{k}} &= \sum_{\ell, k=1}^{n+1} f_{\ell} f_{\bar{k}} \sum_{1 \leq i_1 < \dots < i_{j+1} \leq n+1} \frac{\partial}{\partial a_{\ell \bar{k}}} \begin{vmatrix} f_{i_1, \bar{i}_1} & \dots & f_{i_1, \bar{i}_{j+1}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1}, \bar{i}_1} & \dots & f_{i_{j+1}, \bar{i}_{j+1}} \end{vmatrix} \\ &= \sum_{1 \leq i_1 < \dots < i_{j+1} \leq n+1} \left( \sum_{\ell, k \in \{i_1, \dots, i_{j+1}\}} \frac{\partial}{\partial a_{\ell \bar{k}}} \begin{vmatrix} f_{i_1, \bar{i}_1} & \dots & f_{i_1, \bar{i}_{j+1}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1}, \bar{i}_1} & \dots & f_{i_{j+1}, \bar{i}_{j+1}} \end{vmatrix} f_{\ell} f_{\bar{k}} \right). \end{aligned}$$

On the other side, if we put  $F(\partial f, \bar{\partial} f, \partial\bar{\partial} f) = -\Delta_{(i_1, \dots, i_{j+1})}(f)$  and we twice differentiate it with respect to  $f_{i_{\ell}}$  and  $f_{\bar{i}_k}$  for every  $\ell, k = 1, \dots, j+1$ , we get

$$\frac{\partial F}{\partial f_{i_{\ell}}} = (-1)^{\ell+1} \begin{vmatrix} f_{\bar{i}_1} & \dots & f_{\bar{i}_{j+1}} \\ f_{i_1, \bar{i}_1} & \dots & f_{i_1, \bar{i}_{j+1}} \\ \vdots & \ddots & \vdots \\ f_{i_{\ell-1}, \bar{i}_1} & \dots & f_{i_{\ell-1}, \bar{i}_{j+1}} \\ f_{i_{\ell+1}, \bar{i}_1} & \dots & f_{i_{\ell+1}, \bar{i}_{j+1}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1}, \bar{i}_1} & \dots & f_{i_{j+1}, \bar{i}_{j+1}} \end{vmatrix}$$

$$\begin{aligned}
(9) \quad \frac{\partial^2 F}{\partial f_{i_\ell} \partial \bar{f}_{i_k}} &= (-1)^{\ell+k} \begin{vmatrix} f_{i_1, \bar{i}_1} & \cdots & f_{i_1, \bar{i}_{k-1}} & f_{i_1, \bar{i}_{k+1}} & \cdots & f_{i_1, \bar{i}_{j+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{i_{\ell-1}, \bar{i}_1} & \cdots & f_{i_{\ell-1}, \bar{i}_{k-1}} & f_{i_{\ell-1}, \bar{i}_{k+1}} & \cdots & f_{i_{\ell-1}, \bar{i}_{j+1}} \\ f_{i_{\ell+1}, \bar{i}_1} & \cdots & f_{i_{\ell+1}, \bar{i}_{k-1}} & f_{i_{\ell+1}, \bar{i}_{k+1}} & \cdots & f_{i_{\ell+1}, \bar{i}_{j+1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{i_{j+1}, \bar{i}_1} & \cdots & f_{i_{j+1}, \bar{i}_{k-1}} & f_{i_{j+1}, \bar{i}_{k+1}} & \cdots & f_{i_{j+1}, \bar{i}_{j+1}} \end{vmatrix} \\
&= \frac{\partial}{\partial a_{\ell \bar{k}}} \begin{vmatrix} f_{i_1, \bar{i}_1} & \cdots & f_{i_1, \bar{i}_{j+1}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1}, \bar{i}_1} & \cdots & f_{i_{j+1}, \bar{i}_{j+1}} \end{vmatrix}
\end{aligned}$$

Moreover,

$$\begin{aligned}
F(\partial f, \bar{\partial} f, \partial \bar{\partial} f) &= F(\partial f, \bar{\partial} f, \partial \bar{\partial} f) - F(0, \bar{\partial} f, \partial \bar{\partial} f) \\
&= \int_0^1 \frac{d}{ds} F(s \partial f, \bar{\partial} f, \partial \bar{\partial} f) ds \\
&= \int_0^1 \left( \sum_{\ell \in \{i_1, \dots, i_{j+1}\}} \frac{\partial F}{\partial f_\ell} (s \partial f, \bar{\partial} f, \partial \bar{\partial} f) f_\ell \right) ds \\
&= \int_0^1 ds \left( \sum_{\ell \in \{i_1, \dots, i_{j+1}\}} \frac{\partial F}{\partial f_\ell} (\partial f, \bar{\partial} f, \partial \bar{\partial} f) f_\ell \right) \\
&= \sum_{\ell \in \{i_1, \dots, i_{j+1}\}} \frac{\partial F}{\partial f_\ell} (\partial f, \bar{\partial} f, \partial \bar{\partial} f) f_\ell
\end{aligned}$$

and by the same argument and by (9)

$$\begin{aligned}
(10) \quad F(\partial f, \bar{\partial} f, \partial \bar{\partial} f) &= \sum_{\ell, k \in \{i_1, \dots, i_{j+1}\}} \frac{\partial^2 F}{\partial f_\ell \partial \bar{f}_k} (\partial f, \bar{\partial} f, \partial \bar{\partial} f) f_\ell \bar{f}_k \\
&= \sum_{\ell, k \in \{i_1, \dots, i_{j+1}\}} \frac{\partial}{\partial a_{\ell \bar{k}}} \begin{vmatrix} f_{i_1, \bar{i}_1} & \cdots & f_{i_1, \bar{i}_{j+1}} \\ \vdots & \ddots & \vdots \\ f_{i_{j+1}, \bar{i}_1} & \cdots & f_{i_{j+1}, \bar{i}_{j+1}} \end{vmatrix} f_\ell \bar{f}_k.
\end{aligned}$$

By substituting (10) in (8) we get (7). □

By using the null lagrangian property for elementary symmetric functions in the eigenvalues of the complex Hessian matrix and the classical divergence Theorem we get the following integral formulas for closed hypersurfaces.

**Theorem 2.1.** *Let  $\Omega$  be a bounded domain of  $\mathbb{C}^{n+1}$  with boundary a real hypersurface of class  $C^2$ . For every defining function  $f$  of class  $C^2$  of  $\Omega = \{f(z) < 0\}$  and for every  $j = 1, \dots, n$  we have*

$$(11) \quad \int_{\Omega} \sigma_{j+1}(\partial\bar{\partial}f) dx = \binom{n+1}{j+1} \frac{1}{2(n+1)} \int_{\partial\Omega} K_{\partial\Omega}^{(j)}(z) |\partial f|^{j+1} d\sigma(x),$$

where  $K_{\partial\Omega}^{(j)}$  is the  $j$ -th Levi curvature of  $\partial\Omega$ .

*Proof.* Since  $\sigma_j$  is a homogenous function of degree  $j$ , i.e.  $\sigma_j(tA) = t^j \sigma_j(A)$  for every real  $t$ , we get

$$(12) \quad \sigma_{j+1}(\partial\bar{\partial}f) = \frac{1}{j+1} \sum_{\ell, k=1}^{n+1} \frac{\partial \sigma_{j+1}(\partial\bar{\partial}f)}{\partial a_{\ell\bar{k}}} f_{\ell\bar{k}}$$

Let us set  $\nu_{\ell} = \frac{\partial_{\ell} f}{|\partial f|}$  and let us identify  $z \in \mathbb{C}^{n+1}$  with  $x \in \mathbb{R}^{2(n+1)}$ , then by (12), by (6), by the classical divergence theorem and by (7) we get

$$\begin{aligned} \int_{\Omega} \sigma_{j+1}(\partial\bar{\partial}f) dx &= \frac{1}{j+1} \int_{\Omega} \sum_{\ell, k=1}^{n+1} \partial_{\ell} \left( \frac{\partial \sigma_{j+1}(\partial\bar{\partial}f)}{\partial a_{\ell\bar{k}}} f_{\bar{k}} \right) dx \\ &= \frac{1}{2(j+1)} \int_{\partial\Omega} \sum_{\ell, k=1}^{n+1} \left( \frac{\partial \sigma_{j+1}(\partial\bar{\partial}f)}{\partial a_{\ell\bar{k}}} f_{\bar{k}} \nu_{\ell} \right) d\sigma(x) \\ &= \frac{1}{2(j+1)} \int_{\partial\Omega} \sum_{\ell, k=1}^{n+1} \frac{\left( \frac{\partial \sigma_{j+1}(\partial\bar{\partial}f)}{\partial a_{\ell\bar{k}}} f_{\bar{k}} f_{\ell} \right)}{|\partial f|} d\sigma(x) \\ &= -\frac{1}{2(j+1)} \int_{\partial\Omega} \frac{\sum_{1 \leq i_1 < \dots < i_{j+1} \leq n+1} \Delta_{(i_1, \dots, i_{j+1})}(f)}{|\partial f|} d\sigma(x) \\ &= \binom{n}{j} \frac{1}{2(j+1)} \int_{\partial\Omega} K_{\partial\Omega}^{(j)}(z) |\partial f|^{j+1} d\sigma(x) \\ &= \binom{n+1}{j+1} \frac{1}{2(n+1)} \int_{\partial\Omega} K_{\partial\Omega}^{(j)}(z) |\partial f|^{j+1} d\sigma(x). \end{aligned}$$

□

### 3. ISOPERIMETRIC ESTIMATES AND AN ALEXANDROV TYPE THEOREM

In this section we use the integral formula (11) to get an estimate of the  $j$ -th Levi curvature of a closed hypersurface and to show Theorem 1.2.

*Proof of Theorem 1.2.* If  $\int_{\partial\Omega} (1/K_{\partial\Omega}^{(j)})^{1/j} d\sigma(x) = +\infty$  then the inequality (4) is trivial. Thus, in the sequel we shall assume  $\int_{\partial\Omega} (1/K_{\partial\Omega}^{(j)})^{1/j} d\sigma(x) < +\infty$ . Let  $f : \bar{\Omega} \rightarrow \mathbb{R}$  be the  $C^2(\bar{\Omega})$  solution of the Dirichlet problem

$$(13) \quad \begin{cases} \text{trace } \partial\bar{\partial}f = 1, & \text{in } \Omega; \\ f = 0, & \text{on } \partial\Omega. \end{cases}$$

Let us remark that  $\text{trace } \partial\bar{\partial} = \frac{1}{4}\Delta$ , with  $\Delta$  the Laplacian operator in  $\mathbb{R}^{2n+2}$ . Hence, if  $\partial\Omega$  is of class  $C^{2,\alpha}$ , the Dirichlet problem (13) has a unique solution  $f \in C^2(\bar{\Omega})$ . Let us recall that for every  $(n+1) \times (n+1)$  Hermitian matrix  $A$  the Newton inequality holds

$$(14) \quad \sigma_j(A) \leq \binom{n+1}{j} \left( \frac{\text{trace } A}{n+1} \right)^j$$

for all  $j = 2, \dots, n+1$ . Moreover, in (14) equality holds iff the matrix  $A$  is proportional to the identity matrix.

By applying (14) to the complex Hessian matrix of  $f$ , with  $f$  a solution of (13), we get an estimate from above of the left hand side of (11)

$$(15) \quad \begin{aligned} \int_{\Omega} \sigma_{j+1}(\partial\bar{\partial}f) dx &\leq \binom{n+1}{j+1} \frac{1}{(n+1)^{j+1}} \int_{\Omega} (\text{trace } (\partial\bar{\partial}f))^{j+1} dx \\ &= \binom{n+1}{j+1} \frac{|\Omega|}{(n+1)^{j+1}}. \end{aligned}$$

By applying again the diverge Theorem we get

$$\int_{\partial\Omega} |\partial f| d\sigma(x) = \frac{1}{2} \int_{\partial\Omega} \langle \nabla f, N \rangle d\sigma(x) = \frac{1}{2} \int_{\Omega} \Delta f dx = 2|\Omega|$$

and by using the Cauchy-Schwarz inequality in the right hand side of (11) we get

$$(16) \quad \begin{aligned} \binom{n+1}{j+1} \frac{1}{2(n+1)} \int_{\partial\Omega} K_{\partial\Omega}^{(j)} |\partial f|^{j+1} d\sigma(x) &\geq \frac{\left(\int_{\partial\Omega} |\partial f| d\sigma(x)\right)^{j+1}}{2(n+1) \left(\int_{\partial\Omega} \left(\frac{1}{K_{\partial\Omega}^{(j)}}\right)^{1/j} d\sigma(x)\right)^j} \\ &= \frac{\binom{n+1}{j+1} (2|\Omega|)^{j+1}}{2(n+1) \left(\int_{\partial\Omega} \left(\frac{1}{K_{\partial\Omega}^{(j)}}\right)^{1/j} d\sigma(x)\right)^j}. \end{aligned}$$

In (16) the equality holds iff  $|\partial f|$  is proportional to  $\left(\frac{1}{K_{\partial\Omega}^{(j)}}\right)^{1/j}$ . By the equality (11) and by the inequalities (15) and (16) we deduce

$$\frac{(2|\Omega|)^j}{\left(\int_{\partial\Omega} \left(\frac{1}{K_{\partial\Omega}^{(j)}}\right)^{1/j} d\sigma(x)\right)^j} \leq \frac{1}{(n+1)^j}$$

and we get

$$\int_{\partial\Omega} \left(\frac{1}{K_{\partial\Omega}^{(j)}}\right)^{1/j} d\sigma(x) \geq 2(n+1)|\Omega|.$$

Moreover, in (15) the equality holds iff the complex Hessian matrix of  $f$  is proportional to the identity matrix. Since the defining function for  $\Omega$  has been chosen such that  $\text{trace } \partial\bar{\partial}f = 1$ , then it should be  $\partial\bar{\partial}f = \frac{1}{n+1}I$  in  $\bar{\Omega}$  and by (1) and (2) we have

$$(17) \quad (K_{\partial\Omega}^{(j)})^{1/j} = \frac{1}{(n+1)|\partial f|}$$

on  $\partial\Omega$ . The last equality does not seem enough to conclude that  $\Omega$  is a ball. Indeed, by the maximum principle for the Laplacian operator, the function  $f$  has an interior minimum at  $z_0 \in \Omega$ , and it is not restrictive to assume  $z_0 = 0$  because the Levi curvature equations are invariant with respect to translations. Then, for every pluriharmonic function  $h$  such that  $h(0) = 0$  and  $\partial h(0) = 0$ ,

$$(18) \quad f(z) = f(0) + \frac{1}{n+1}|z|^2 + h(z)$$

is a defining function for  $\Omega$ . For example, if in  $\mathbb{C}^2$  we choose

$$h(z_1, z_2) = \frac{\operatorname{Re}(z_1^2 + z_2^2)}{4}$$

then the set of zeros of the function  $f$  in (18) is not a sphere. However, if  $K_{\partial\Omega}^{(j)}$  is constant for some  $j$ , then by (17)  $|\partial f|$  should be constant on  $\partial\Omega$ . It follows that the Dirichlet problem (13) is over determinate and by Serrin Theorem [18] we can conclude that  $\Omega$  is a ball and  $\partial\Omega$  is a sphere.  $\square$

Let  $H$  be the Euclidean mean curvature of  $\partial\Omega$ . We recall that the celebrated Minkowski formula (see for instance [15]) asserts that

$$(19) \quad \int_{\partial\Omega} d\sigma = \int_{\partial\Omega} H(x) \langle N, x \rangle d\sigma(x),$$

where  $N$  is the outward unit normal.

We are now ready to conclude the proof of Corollary 1.1

*Proof of Corollary 1.1.* If  $\Omega$  is star-shaped with respect to a point then by (19) and by the divergence theorem we have

$$(20) \quad \begin{aligned} |\partial\Omega| &= \int_{\partial\Omega} d\sigma \leq \max_{\partial\Omega} H \int_{\partial\Omega} \langle N, x \rangle d\sigma(x) = \max_{\partial\Omega} H \int_{\Omega} \left( \sum_j \partial_{x_j} x_j \right) dx \\ &= 2(n+1) \max_{\partial\Omega} H |\Omega|. \end{aligned}$$

By (20) we have

$$\max_{\partial\Omega} H \geq \frac{|\partial\Omega|}{2(n+1)|\Omega|}$$

and since  $K^{(j)}$  is a positive constant, then by (5) we get

$$(K^{(j)})^{1/j} \leq \frac{|\partial\Omega|}{2(n+1)|\Omega|} \leq \max_{\partial\Omega} H.$$

Moreover, if  $\max_{\partial\Omega} H \leq (K^{(j)})^{1/j}$  then equality holds in (5) and by Theorem 1.2 we conclude that  $\Omega$  is a ball.  $\square$

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