The Eulerian distribution on self evacuated involutions

Marilena Barnabei, Flavio Bonetti, and Matteo Silimbani *

Abstract. We present an extensive study of the Eulerian distribution on the set of self evacuated involutions, namely, involutions corresponding to standard Young tableaux that are fixed under the Schützenberger map. We find some combinatorial properties for the generating polynomial of such distribution, together with an explicit formula for its coefficients. Afterwards, we carry out an analogous study for the subset of self evacuated involutions without fixed points.

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1 Introduction

The distribution of the descent statistic (classically known as *Eulerian distribution*) on peculiar subsets of permutations has been object of intensive studies in recent years (see e.g. [2] and [8]) . In particular, several authors examined the properties of the polynomial $I_n(x) = \sum_{j=0}^{n-1} i_{n,j} x^j$, where $i_{n,j}$ denotes the number of involutions on $[n] = \{1, 2, \ldots, n\}$ with j descents. More specifically, V. Strehl [7] proved that the coefficients of this polynomial are symmetric, and recently V.J. Guo and J. Zeng [3] showed that the polynomial $I_n(x)$ is unimodal. In a previous paper [1] the present authors proved that the polynomial $I_n(x)$ is not log-concave. The proof of this property, that has been an open problem for some years, lies upon a (not bijective) correspondence between involutions on $[n]$ with j descents and generalized

*Dipartimento di Matematica - Università di Bologna

involutions on length n on m symbols, with $m > j$. This correspondence yields an explicit formula for the coefficients $i_{n,j}$ of the polynomial $I_n(x)$.

In this paper we study the polynomial $S_n(x) = \sum_{j=0}^{n-1} s_{n,j} x^j$, where $s_{n,j}$ denotes the number of *self evacuated* involutions on $[n]$ with j descents, namely, involutions that correspond (via the Robinson-Schensted algorithm) to standard Young tableaux that are fixed under the action of the Schützenberger map. This class of tableaux has been formerly studied by M.A.A. van Leeuwen [9], who characterized the set of self evacuated tableaux of given shape by means of domino tilings.

First of all, we exhibit an explicit formula and a recursive rule for the total number of self evacuated involutions on $[n]$. Following along the lines of [1], we obtain some enumerative results for the sequence $s_{n,j}$ by exploiting a map that associates a self evacuated involution with a suitable set of generalized involutions. In particular, we deduce an explicit formula for the integers $s_{n,j}$, which allows to prove that the polynomials $I_n(x)$ and $S_n(x)$ share some properties, such as the symmetry of the coefficients and the non log-concavity.

The last section is devoted to the study of the Eulerian distribution on self evacuated involutions without fixed points, that is symmetric, as in the general case. Also in this case, we find an explicit formula for the number $s^*_{n,j}$ of self evacuated involutions on $[n]$ without fixed points and j rises.

2 Tableaux and involutions

In this section, we give some definitions and general results about tableaux, involutions and generalized involutions.

Consider the set $\mathscr{T}_n$ of standard Young tableaux on n boxes. It is well known that the Robinson-Schensted algorithm establishes a bijection $\rho : \mathscr{I}_n \to \mathscr{T}_n$, where $\mathscr{I}_n$ is the set of involutions over $[n] := \{1, 2, \dots, n\}$.

We recall that the *descent set* of a permutation σ is defined as $\text{des}(\sigma) = \{1 \leq i < n : \sigma(i) > \sigma(i+1)\}$. An analogous definition can be given for the *rise set* of a permutation, by replacing "$\sigma(i) > \sigma(i+1)$" by "$\sigma(i) < \sigma(i+1)$".

Given a Ferrers diagram λ, a *semistandard tableau* of shape λ over the alphabet $[m]$ is an array obtained by placing into each box of the diagram λ an integer in $[m]$ so that the entries are strictly increasing by rows and weakly increasing by columns.

A *generalized involution* is defined to be a biword:

$$\alpha = \binom{x}{y} = \left(\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{array} \right),$$

such that:

- for every $1 \leq i \leq n$, there exists an index j with $x_i = y_j$ and $y_i = x_j$,
- $x_i \leq x_{i+1}$,
- $x_i = x_{i+1} \Longrightarrow y_i \geq y_{i+1}$.

The word $x = x_1 \cdots x_n$ is called the *content* of the generalized involution, and the integer n is called its *length*.

The Robinson-Schensted-Knuth (RSK) algorithm (see [4]) associates bijectively a semistandard tableaux S with a generalized involution $\mathrm{inv}(S)$.

We say that an integer a is a *repetition* of multiplicity r for the generalized involution α if

$$x_i = y_i = x_{i+1} = y_{i+1} = \cdots = x_{i+r-1} = y_{i+r-1} = a.$$

We define a map Π from the set of generalized involutions to the set of involutions as follows: if

$$\alpha = \left(\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{array} \right),$$

then $\Pi(\alpha)$ is the involution σ

$$\sigma = \left(\begin{array}{cccc} 1 & 2 & \cdots & n \\ y'_1 & y'_2 & \cdots & y'_n \end{array} \right),$$

where $y'_i = 1$ if y_i is the least symbol occurring in the word y, $y'_j = 2$ if y_j is the second least symbol in y and so on. In the case $y_i = y_j$, with $i > j$, we consider y_i to be less then y_j. We will call the involution $\sigma = \Pi(\alpha)$ the *polarization* of α.

For example, the polarization of the generalized involution

$$\alpha = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 4 & 6 \\ 4 & 3 & 2 & 1 & 6 & 4 & 1 & 4 \end{pmatrix}$$

is the involution

$$\Pi(\alpha) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 3 & 2 & 8 & 6 & 1 & 5 \end{pmatrix}.$$

Note that the map Π is not injective, since, for any given involution σ, there are infinitely many generalized involutions whose polarization is σ. For example, the generalized involution

$$\beta = \begin{pmatrix} 1 & 1 & 1 & 3 & 4 & 4 & 5 & 6 \\ 5 & 3 & 1 & 1 & 6 & 4 & 1 & 4 \end{pmatrix}$$

has the same polarization as α in the previous example.

We will denote by $\mathrm{Gen}_m(\sigma)$ the set of generalized involutions, with symbols taken from $[m]$, whose polarization is σ. Remark that two generalized involutions in $\mathrm{Gen}_m(\sigma)$ can not have the same content. For this reason, the set $\mathrm{Gen}_m(\sigma)$ corresponds bijectively with the set of contents of its elements.
We will say that a content x is *compatible with* σ if there exists a generalized involution in some $\mathrm{Gen}_m(\sigma)$ whose content is x.
It is easy to check that a content $x = x_1 \cdots x_n$ is compatible with an involution σ if and only if we have

$$x_i < x_{i+1} \Longleftrightarrow \sigma \text{ has a rise at position } i.$$

The key tool in the present paper is the interplay between involutions and generalized involutions. For this reason, we need to evaluate the cardinality of the set $\mathrm{Gen}_m(\sigma)$, for any given involution σ. It turns out that this cardinality depends only on the number of rises of σ. In fact, we have the following result, formerly stated in [1]:

Proposition 1 *Let $\sigma \in \mathscr{I}_n$ be an involution with t rises. Then,*

$$|Gen_m(\sigma)| = \binom{n+m-t-1}{n}. \tag{1}$$

Proof Choose an involution $\sigma \in \mathscr{I}_n$ with t rises. As we remarked above, the set $\mathrm{Gen}_m(\sigma)$ corresponds bijectively to the set of contents $x = x_1 \dots x_n$ with $1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq m$, where the inequalities are strict in correspondence of the rises of σ. Every such content is uniquely determined by the sequence $\delta := \delta_0\delta_1 \dots \delta_n$, with

$$\delta_0 = x_1 - 1, \quad \delta_1 = x_2 - x_1, \quad \dots, \quad \delta_n = m - x_n$$

which is a composition of the integer $m-1$ such that its i-th component δ_i is at least one whenever σ has a rise at the i-th position. For this reason, we can consider the word $\delta' = \delta'_0\delta'_1 \dots \delta'_n$ defined as follows:

$$\delta'_i = \begin{cases} \delta_i - 1 & \text{if } \sigma \text{ has a rise at the i-th position} \\ \delta_i & \text{otherwise} \end{cases},$$

which is a composition of the integer $m-t-1$ in $n+1$ parts. This gives the assertion.

$\diamond$

3 Self evacuated standard tableaux

We are interested in some enumerative problems concerning Young tableaux which are fixed by the well known Schützenberger map (or *evacuation*). First of all we recall the definition of this map.

Given a standard Young tableau T with n boxes (on $[n]$), we construct a new tableau $ev(T)$ of the same shape as follows: we remove the symbol 1 from the tableau T, leaving an empty box. We now move into this box the smallest of the integers contained into its two neighbor boxes. This creates a new empty box into T. The process is repeated with this box according to the same rule. It continues until there are no neighbors to slide into the current empty box b_1, in which case we delete the box b_1 from T and we insert the symbol n at the same position in $ev(T)$. We repeat this procedure, removing from T the symbol 2 and placing the integer $n-1$ into the box b_2 of $ev(T)$. We proceed until the tableau T is empty. It is well known (see [5]) that $ev(T)$ is a standard tableau and $ev(ev(T)) = T$.

An alternative, and even simpler, description of the Schützenberger map can be given in terms of involutions of the symmetric group. If σ is the involution associated with T, then the tableau $ev(T)$ corresponds to the involution $ev(\sigma) = \psi\sigma\psi$, where ψ is the involution that maps the integer i into its *complement* $n+1-i$.
This means that an involution is a fixed point under the Schützenberger map if and only if it is contained in the centralizer of ψ. We will call such involutions *self evacuated involutions*, and the corresponding tableaux will be called *self evacuated tableaux*.

The involution point of view allows to give a simple characterization of the fixed points of the Schützenberger map:

Proposition 2 *An involution σ is self evacuated if and only if, for every $1 \leq i \leq n$,*

$$\sigma(i) + \sigma(n+1-i) = n+1.$$

Proof The statement is a straightforward consequence of the fact that σ must commute with the map ψ.

$\diamond$

Recall that σ is an involution if and only if its disjoint cycle decomposition consists uniquely of fixed points and transpositions. We will write $(i,j)|\,\sigma$ whenever the transposition (i,j) appears in the cycle decomposition of σ. We will say that (i,j) is a *smooth transposition* of S_n if $i \neq n+1-j$. From this perspective, Proposition 2 can be restated as follows:

Proposition 3 *An involution $\sigma \in \mathscr{I}_n$ is self evacuated if and only if:*

$$\sigma(i) = i \iff \sigma(n+1-i) = n+1-i, \tag{2}$$

$$(i,j)|\,\sigma \iff (n+1-i, n+1-j)|\,\sigma. \tag{3}$$

$\diamond$

Note that Proposition 3 implies that whenever a smooth transposition divides an involution σ, this forces four values of σ, while if a non-smooth transposition divides σ, it forces only two values of σ.

Denote by $\mathscr{S}_n$ the set of self evacuated involutions on n letters and by s_n its cardinality.

First of all, remark that $s_{2k} = s_{2k+1}$. In fact, if n is odd, Proposition 2 implies that $\sigma(\frac{n+1}{2}) = \frac{n+1}{2}$, for every $\sigma \in \mathscr{S}_n$. Hence, an involution in $\mathscr{S}_{2k+1}$ is associated to a unique involution in $\mathscr{S}_{2k}$ obtained by deleting the central symbol.

The characterization given in Proposition 3 allows us to give both a recurrence (Theorem 4) and an explicit formula (Theorem 5) for the integers s_{2k}.

Theorem 4 *We have:*

$$s_{2k} = 2s_{2k-2} + (2k-2)s_{2k-4} \tag{4}$$

Proof Let $\sigma \in \mathscr{S}_{2k}$. If $\sigma(1) = 1$ or $\sigma(1) = 2k$ (and hence $\sigma(2k) = 2k$ or $\sigma(2k) = 1$, respectively) the restriction of σ to the set $\{2, \ldots, 2k-1\}$ belongs to $\mathscr{S}_{2k-2}$. Otherwise, if $\sigma(1) = j$, with $j \neq 1, 2k$, we must have

$$\sigma(j) = 1 \qquad \sigma(2k+1-j) = 2k \qquad \sigma(2k) = 2k+1-j.$$

Also in this case, the restriction of σ to the set $\{2, \ldots, 2k-1\} \setminus \{j, 2k+1-j\}$ belongs to $\mathscr{S}_{2k-4}$. Remarking that there are $2k-2$ possible choices for the integer j, we get the assertion.

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Theorem 5 *The number of self evacuated involutions on $2k$ symbols is*

$$s_{2k} = \sum_{h=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(2k)!!}{(k-2h)!h!2^{2h}}.$$

Proof Fix an integer $h \leq \lfloor \frac{k}{2} \rfloor$. We count the number of involutions in $\mathscr{S}_{2k}$ with exactly $2h$ smooth transpositions. Choose a word $w = w_1 \cdots w_k$ consisting of k different letters taken from the alphabet $[2k]$ such that w does not contain simultaneously an integers i and its complement $2k+1-i$. We have $(2k)(2k-2)\cdots(2) = (2k)!!$ choices for such a word. This word corresponds to a unique self evacuated involution τ with $2h$ smooth transpositions defined by the following conditions:

$$\tau(w_1) = w_2, \quad \dots \quad , \tau(w_{2h-1}) = w_{2h};$$

$$\tau(w_{2h+j}) = \begin{cases} w_{2h+j} & \text{if } w_{2h+j} \leq k \\ 2k+1-w_{2h+j} & \text{otherwise} \end{cases},$$

with $0 < j \leq k-2h$. It is easily checked that the involution τ arises from $(k-2h)!h!2^{2h}$ different words w. This completes the proof.

$\diamond$

4 Self evacuated generalized involutions

The involution approach suggests how to extend the Schützenberger map to the set of semistandard tableaux on a given alphabet $[m]$, as follows: let S be a semistandard tableau on $[m]$, with associated generalized involution

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

Then the evacuated semistandard tableau $ev(S)$ is defined to be the semistandard tableau associated with the generalized involution

$$ev(\alpha) = \begin{pmatrix} m+1-x_n & m+1-x_{n-1} & \cdots & m+1-x_1 \\ m+1-y_n & m+1-y_{n-1} & \cdots & m+1-y_1 \end{pmatrix}.$$

Clearly, the generalized involutions α and $ev(\alpha)$ may have different content. More precisely, the integer i occurs in the content of α as many times as $m+1-i$ occurs in $ev(\alpha)$.

For example, consider the semistandard tableau

$$S = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & \\ 3 & 4 & & \\ 4 & & & \end{matrix}$$

corresponding to the generalized involution

$$\alpha = \left(\begin{array}{cccccccccc} 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \end{array} \right).$$

The evacuated tableau is

$$ev(S) = \begin{array}{llll} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & \\ 1 & 2 & & \\ 1 & & & \end{array}$$

corresponding to the generalized involution

$$ev(\alpha) = \left(\begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 \end{array} \right).$$

From now on, extending the previous notation, we will write $(i, j)|\alpha$ whenever the pair (i, j) appears in the generalized involution α. Also in this case, we will say that (i, j) is a smooth transposition if $i \neq m + 1 - j$ and $i \neq j$.

The fixed point of the Schützenberger map on generalized involutions, called *self evacuated generalized involutions*, can be easily characterized as follows:

Proposition 6 *A generalized involution α is self evacuated if and only if, whenever $(i, j)|\alpha$, we have also $(m + 1 - j, m + 1 - i)|\alpha$.*

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Remark that the Schützenberger map commutes with the polarization Π, namely, if α is a generalized involution, we have:

$$\Pi(ev(\alpha)) = ev(\Pi(\alpha)).$$

For instance, if α is the generalized involution of the previous exapmle, we have:

$$\sigma_1 = \Pi(\alpha) = \left(\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 3 & 2 & 6 & 5 & 4 & 10 & 9 & 8 & 7 \end{array} \right)$$

and

$$\sigma_2 = \Pi(ev(\alpha)) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 3 & 2 & 1 & 7 & 6 & 5 & 9 & 8 & 10 \end{pmatrix}.$$

It is easily checked that $ev(\sigma_1) = \sigma_2$.

Proposition 6 yields a further characterization of self evacuated generalized involutions, which will be useful in the following sections.

Proposition 7 *A generalized involution α is self evacuated if and only if it satisfies the following properties:*

- *the content $x = x_1 \dots x_n$ of α is symmetric, namely $x_i + x_{n+1-i} = m+1$,*
- *$\Pi(\alpha)$ is a self evacuated involution.*

$\diamond$

We denote by $c_{n,m}$ the number of generalized involutions of length n over the alphabet $[m]$.

Setting $n = 2k + 1$, straightforward considerations lead to the following properties:

- if $m = 2h$, $c_{2k+1,m} = 0$;
- if $m = 2h+1$, the central pair $\binom{x_{k+1}}{y_{k+1}}$ of every self evacuated generalized involution of length n over the alphabet $[m]$ is necessarily the pair $(h+1, h+1)$. This implies that $c_{2k+1,m} = c_{2k,m}$.

Hence, the values of the sequences $c_{2k+1,m}$ can derived from the sequences $c_{2k,m}$. For this reason, we restrict to the even case.

Theorem 8 *The number of self evacuated generalized involutions of length $2k$ over $[m]$ is:*

$$c_{2k,m} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\frac{\binom{m}{2} - \lfloor \frac{m}{2} \rfloor}{2} + j - 1}{j} \binom{m + k - 2j - 1}{k - 2j}. \tag{5}$$

Proof Fix $h \leq \lfloor \frac{k}{2} \rfloor$. We count the number of self evacuated generalized involutions of length $2k$ and m symbols with exactly $2h$ smooth transpositions which, in the present case, can or can not be different. The set A of all possible smooth transposition has cardinality

$$\binom{m}{2} - \left\lfloor \frac{m}{2} \right\rfloor .$$

Remark that, given a generalized involution α and a smooth transposition $\tau = (i, j)$, we have that $\tau | \, \alpha$ if and only if $\tau' | \, \alpha$, where $\tau' = (m+1-j, m+1-i)$. It is evident that τ can be chosen in

$$\frac{\binom{m}{2} - \lfloor \frac{m}{2} \rfloor}{2}$$

ways. Such choices determine $4h$ pairs of α. The remaining $2k - 4h$ pairs can be chosen to be either fixed points or non-smooth transpositions. This completes the proof.

◇

5 The Eulerian distribution on self evacuated involutions

In this section, we study the distribution of the descent statistic on the set of involutions. The combinatorial relations between involutions and generalized involutions pointed out in the previous sections will play a crucial role for this analysis.

The distribution of the descent statistic on the set of involutions behaves properly with respect to the action of the Schützenberger map. In fact:

Proposition 9 *For every involution σ on $[2k]$, we have:*

$$|Des(\sigma)| = |Des(ev(\sigma))|.$$

Moreover, the descent sets $Des(\sigma)$ and $Des(ev(\sigma))$ are mirror symmetric, i.e. σ has a descent at position i if and only if $ev(\sigma)$ has a descent at position $2k - i$.

Proof Suppose that σ has a descent at position i, namely, $\sigma(i) > \sigma(i+1)$. Then,

$$ev(\sigma)(2k-i) = 2k+1-\sigma(i+1) > 2k+1-\sigma(i) = ev(\sigma)(2k+1-i).$$

$\diamond$

For example, let

$$\sigma = \begin{pmatrix} \mathbf{1} & \mathbf{2} & 3 & 4 & \mathbf{5} & 6 & 7 & 8 \\ \mathbf{3} & \mathbf{2} & 1 & 4 & \mathbf{6} & 5 & 7 & 8 \end{pmatrix},$$

where, from now on, the bold-faced numbers denote the descent positions. Then,

$$ev(\sigma) = \begin{pmatrix} 1 & 2 & \mathbf{3} & 4 & 5 & \mathbf{6} & \mathbf{7} & 8 \\ 1 & 2 & \mathbf{4} & 3 & 5 & \mathbf{8} & \mathbf{7} & 6 \end{pmatrix}.$$

In particular, if σ is a self evacuated involution, then its descent set must be mirror symmetric with respect to the k-th entry.

We are now interested in finding an explicit formula for the number $s_{2k,d}$ of self evacuated involutions with d rises. First of all, we have:

Proposition 10 *The sequence $s_{2k,d}$ is symmetric, namely,*

$$s_{2k,i} = s_{2k,2k-1-i}.$$

Proof Given a self evacuated involution σ, it is easily checked that the permutation $\tau = \psi\sigma$ satisfies the following properties:

- τ is an involution;
- τ is self evacuated;
- τ has a descent at position i whenever σ has a rise at the same position.

$\diamond$

For example, let

$$\sigma = \begin{pmatrix} 1 & \mathbf{2} & 3 & \mathbf{4} & 5 & \mathbf{6} & 7 & 8 \\ 1 & \mathbf{7} & 5 & \mathbf{6} & 3 & \mathbf{4} & 2 & 8 \end{pmatrix}.$$

Then,

$$\psi\sigma = \begin{pmatrix} \mathbf{1} & 2 & \mathbf{3} & 4 & \mathbf{5} & 6 & \mathbf{7} & 8 \\ \mathbf{8} & 2 & \mathbf{4} & 3 & \mathbf{6} & 5 & \mathbf{7} & 1 \end{pmatrix}.$$

The preceding result shows that the integer $s_{2k,d}$ counts simultaneously the involutions in $\mathscr{S}_{2k}$ with d descents and those with d rises.

Now we want to express the number $c_{2k,m}$ of self evacuated generalized involutions of length $2k$ over $[m]$ in terms of the sequence $s_{2k,d}$ by exploiting the combinatorial relations between involutions and generalized involutions. As in the general case (Proposition 2), it turns out that the number of self evacuated generalized involutions on m symbols whose polarization is a given involution σ depends only on the number of rises of σ. In fact:

Theorem 11 *We have:*

$$c_{2k,m} = \sum_{j=0}^{m-1} \binom{k + \lfloor \frac{j}{2} \rfloor}{\lfloor \frac{j}{2} \rfloor} s_{2k,m-1-j}. \tag{6}$$

Proof Let $\sigma \in \mathscr{I}_{2k}$ a self evacuated involution with t rises. As remarked in proposition 2, σ corresponds to

$$\binom{2k+m-1-t}{m-1-t}$$

generalized involutions with m symbols, but only

$$\binom{k + \lfloor \frac{m-1-t}{2} \rfloor}{\lfloor \frac{m-1-t}{2} \rfloor}$$

of these are self evacuated. In fact, by Proposition 7, a generalized involution with m symbols in the set $\mathrm{Gen}_m(\sigma)$ is self evacuated if only if the corresponding composition δ' of the integer $m-1-t$ into $2k+1$ satisfies the condition $\delta'_{k-i} = \delta'_{k+i}$. By setting $j = m-1-t$, we get the assertion.

◇

We now exploit the described combinatorial relation between generalized involutions and involutions to determine an explicit formula for $s_{2k,d}$.

Theorem 12 *The number of self evacuated involutions of length $2k$ with d rises is:*

$$s_{2k,d} = \sum_{j=1}^{d+1} (-1)^{\lfloor \frac{d-j}{2}+1 \rfloor} \binom{k}{\lfloor \frac{d+1-j}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\frac{\binom{j}{2}-\lfloor \frac{j}{2} \rfloor}{2}+i-1}{i} \binom{j+k-2i-1}{k-2i}. \tag{7}$$

Proof Formula (6) yields, by inversion:

$$s_{2k,d} = \sum_{j=1}^{d+1} (-1)^{\lfloor \frac{d-j}{2}+1 \rfloor} \binom{k}{\lfloor \frac{d+1-j}{2} \rfloor} c_{2k,j}. \tag{8}$$

Then, combining Formulae (5) and (8), we derive (7).

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Moreover, this explicit formula allows to check that the polynomials $S_{2k}(x) = \sum_{j=0}^{2k-1} s_{2k,j} x^j$ are not, in general, log-concave, since we have, for example:

$$s_{100,0} \cdot s_{100,2} = 11950 > 2500 = s_{100,1}^2.$$

The first values of $s_{2k,d}$ are shown in the following table:

n/d	0	1	2	3	4	5	6	7	8	9
0	1									
1	1									
2	1	1								
3	1	0	1							
4	1	2	2	1						
5	1	0	4	0	1					
6	1	3	6	6	3	1				
7	1	0	9	0	9	0	1			
8	1	4	13	20	20	13	4	1		
9	1	0	17	0	40	0	17	0	1	
10	1	5	23	49	78	78	49	23	5	1

These first values seem to suggest that the polynomials $S_{2k}(x)$ are unimodal for every $k \in \mathbb{N}$. It would be interesting to find a combinatorial proof of this property.

6 Self evacuated involutions without fixed points

In this section, we extend the study of the Eulerian distribution to the set of self evacuated involutions on $[n]$ without fixed points. Obviously, such involutions exist only if n is even.
Denote by $\mathscr{S}^*_{2k}$ the set of self evacuated involutions on $2k$ objects without fixed points and by s^*_{2k} the cardinality of $\mathscr{S}^*_{2k}$. Then:

Theorem 13 *We have:*

$$s^*_{2k} = \sum_{h=0}^{\lfloor \frac{k}{2} \rfloor} \frac{k!}{(k-2h)!h!}, \tag{9}$$

and

$$s^*_{2k} = s^*_{2k-2} + (2n-2)s^*_{2k-4}. \tag{10}$$

Proof Following along the lines of the proof of Theorem 5, we count the number of self evacuated involutions without fixed points with exactly $2h$ smooth transpositions, $2h \leq k$. Choose a word $w = w_1 \cdots w_k$ consisting of k different letters taken from the alphabet $1, \ldots, 2k$ such that w does not contain simultaneously the integers i and $2k+1-i$. We have $(2k)!!$ choices for such a word. This word corresponds to a unique self evacuated involution τ without fixed points with $2h$ smooth transpositions defined by the following conditions:

$$\tau(w_1) = w_2, \quad \ldots, \quad \tau(w_{2h-1}) = w_{2h},$$
$$\tau(w_{2h+j}) = 2k+1-w_{2h+j}, \quad \text{for } 0 < j \leq k-2h.$$

It is easily checked that the involution τ arises from $(k-2h)!h!2^k$ different words w. Hence:

$$s^*_{2k} = \sum_{h=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(2k)!!}{(k-2h)!h!2^k},$$

which is equivalent to (9).

Let now $\sigma \in \mathscr{S}^*_{2k}$. If $\sigma(1) = 2k$, and hence $\sigma(2k) = 1$, the restriction of σ to the set $\{2, \ldots, 2k-1\}$ is a self evacuated involution on $2k-2$ symbols without fixed points. If $\sigma(1) = j$, with $j < 2k$, the symbol 1 is involved in a smooth transposition, hence we must have $\sigma(j) = 1$, $\sigma(2k+1-j) = 2k$ and $\sigma(2k) = 2k+1-j$. Then, the restriction of σ to the set $\{2, \ldots, 2k-1\} \setminus \{j, 2k+1-j\}$ is a self evacuated involution on $2k-4$ symbols without fixed points. Remarking that there are $2k-2$ possible choices for the integer j, we get (10).

$\diamond$

Denote by $s^*_{2k,d}$ the number of involutions in $\mathscr{S}^*_{2k}$ with d rises. Then:

Proposition 14 *The sequence $s^*_{2k,d}$ is symmetric, namely,*

$$s^*_{2k,d} = s^*_{2k,2k-d}.$$

Proof Denote by $\mathscr{I}^*_{2k}$ the set of involutions on $2k$ objects without fixed points. In [7], V. Strehl proved the symmetry of the Eulerian distribution on $\mathscr{I}^*_{2k}$ by means of a bijection:

$$\theta : \mathscr{I}^*_{2k} \to \mathscr{I}^*_{2k},$$

which maps an involutions σ with j rises to an involutions $\theta(\sigma)$ with $2k-j$ rises. It is easily checked that the restriction of θ to the set $\mathscr{S}^*_{2k}$ is a bijections of $\mathscr{S}^*_{2k}$ into itself. This gives the assertion.

$\diamond$

Once more, in order to find an explicit formula for the integers $s^*_{2k,d}$, we need to establish a connection between self evacuated involutions without fixed points and a suitable set of generalized involutions. Remarked that this set contains only self evacuated generalized involutions with repetitions of even multiplicity. Denote by $c^*_{2k,m}$ the number of such involutions of length $2k$ on the alphabet $[m]$. Then:

Theorem 15 *We have:*

$$c^*_{2k,m} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\frac{\binom{m}{2}+\lfloor \frac{m}{2} \rfloor}{2} + j - 1}{j} \binom{\lceil \frac{m}{2} \rceil + k - 2j - 1}{k - 2j}. \qquad (11)$$

Proof Remark that, given a generalized involution σ and a smooth transposition $\tau = (i\ j)$, we have that $(i\ j)|\,\sigma$ if and only if $\tau' = (m+1-j\ \ m+1-i)|\,\sigma$. Similarly, every *non central fixed point*, namely, an occurrence of a pair $(i\ i)$ in σ, with $i \neq \frac{m+1}{2}$, implies a second occurrence of the same pair.
Fix now $j \leq \lfloor \frac{k}{2} \rfloor$. We count the number of generalized involutions of length $2k$ on the alphabet $[m]$ containing only repetitions of even multiplicity, such that exactly $2j$ of its pairs are either non central fixed points or smooth transpositions. We can choose a non central fixed point in $\lfloor \frac{m}{2} \rfloor$ ways and a smooth transposition in $\frac{\binom{m}{2}+\lfloor \frac{m}{2} \rfloor}{2}$ ways. The remaining pairs must be chosen to be either cental fixed points or a non smooth transpositions. This completes the proof.

◇

Repeating the same argumentations as in the proof of Theorem 11, we obtain the following result:

Theorem 16 *We have:*

$$c^*_{2k,m} = \sum_{j=0}^{m-1} \binom{k + \lfloor \frac{j}{2} \rfloor}{\lfloor \frac{j}{2} \rfloor} s^*_{2k,m-1-j}. \tag{12}$$

Hence:

$$s^*_{2k,d} = \sum_{j=1}^{d+1} (-1)^{\lfloor \frac{d-j}{2}+1 \rfloor} \binom{k}{\lfloor \frac{d+1-j}{2} \rfloor} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{\frac{\binom{j}{2}+\lfloor \frac{j}{2} \rfloor}{2} + i - 1}{i} \binom{\lceil \frac{j}{2} \rceil + k - 2i - 1}{k - 2i}. \tag{13}$$

◇

The present table contains the first values of the sequences $s^*_{2k,d}$:

n/d	0	1	2	3	4	5	6	7	8	9
0	1									
2	1									
4	1	1	1							
6	1	1	3	1	1					
8	1	2	7	5	7	2	1			
10	1	2	12	12	27	12	12	2	1	

This table shows that the polynomial $S^*_{2k}(x)$ is not in general unimodal, and hence not log-concave.

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