The isometric deformability question for constant mean curvature surfaces with topology

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Abstract

The paper treats the isometric deformability of non-simply-connected constant mean curvature surfaces which are neither assumed embedded nor complete. We prove that if a smooth oriented surface M immersed in $\mathbb{R}^3$ admits a nontrivial isometric deformation with constant mean curvature H then every cycle in M has vanishing flux and, when $H \neq 0$, also vanishing torque. The vanishing of all fluxes implies the existence of such an isometric deformation when $H = 0$. Our work generalizes to constant mean curvature surfaces a well-known rigidity result for minimal surfaces (see for instance [4]).

1 Introduction.

It has long been wondered which smooth surfaces in $\mathbb{R}^3$ can undergo a nontrivial isometric deformation. Anderson [1] recently showed the nonexistence of such deformation for *compact embedded* surfaces in $\mathbb{R}^3$. Even if we allow immersions or non compactness, or both, the occurrence of such deformations for *non simply-connected* surfaces appears to be rare. The only place in surface theory where local isometric deformations overtly present themselves is with surfaces of constant mean curvature.

Within the class of isometric immersions of *simply-connected* surfaces, every one of constant mean curvature admits a canonical 2π-periodic isometric deformation — the associate deformation (see Section 3); this family also captures (to within a congruence of $\mathbb{R}^3$) all isometric immersions of that surface with this constant mean curvature. However once the constant mean curvature surface M has topology (i.e. $\pi_1(M) \neq 0$) there is of necessity a continuum of latent period conditions which must be satisfied for the canonical deformation, at the level of the universal cover $\widetilde{M}$, to descend to M. Our purpose here is to give necessary conditions for the associate isometric deformation to exist.

A countable set of invariants associated with constant mean curvature surfaces arises from two naturally defined closed vector-valued 1-forms ω and σ on M, called here flux and torque forms (see Section 4). Thus their periods over cycles γ in M, $W([\gamma]) = \int_\gamma \omega$ and $T([\gamma]) = \int_\gamma \sigma$, depend only on the homology class of γ and are called the flux and torque of that class. These quantities can be found in [10] (see also [9]).

Theorem 1.1 *Let $x\colon M \to \mathbb{R}^3$ be an isometric immersion of a smooth oriented surface with constant mean curvature H. To within congruences, the family of all isometric immersions of M with constant mean curvature H is either*

(a) finite, or

(b) a circle (see the associate deformation in Section 3).

If (b) then

(i) every cycle in M has vanishing flux, and

(ii) every cycle in M has vanishing torque if $H \neq 0$.

There are many results on the isometric indeformability of constant mean curvature surfaces with topology (see for instance [5, 10, 11, 12, 15, 16]. Typically, these follow from Theorem 1.1 bi exhibiting a cycle with nonzero flux (see Section 5).

> *Are there non-simply connected constant mean curvature surfaces satisfying Theorem 1.1 (i) — or even both Theorem 1.1 (i) and (ii)?*
>
> *Do these conditions then guarantee the existence of a constant mean curvature isometric deformation?*

These are the central questions arising from this work. Both questions are answered in the affirmative for minimal surfaces (see Section 4) but nothing is known on either when $H \neq 0$.

2 Isometric deformation of surfaces

Let $x\colon M \to \mathbb{R}^3$ be an immersion of a smooth oriented surface. The differential x_* of x is given by $x_*(X) = Xx$ where the right-hand side is the derivative of the vector-valued function x with respect to X. The induced metric g is given by

$$g(X, Y) = \langle x_*(X), x_*(Y)\rangle = \langle Xx, Yx\rangle$$

where $\langle\,,\rangle$ is the Euclidean metric on $\mathbb{R}^3$. Let J denote the complex structure induced by the orientation of M and let ξ be the oriented unit normal field to the immersion x. The second fundamental form A of x is defined by

$$X\xi = -x_*(AX).$$

The Gauss equation is

$$\det A = K$$

where K is the curvature of the metric g and Codazzi's equation is

$$(\nabla_X A)Y = (\nabla_Y A)X$$

where ∇ is the Levi-Civita connection of the metric g; from now on the metric will be denoted by $\langle\,,\rangle$. These equations come from differentiating the structure equation

$$XYx = x_*(\nabla_X Y) + \langle AX, Y\rangle\xi. \tag{1}$$

Now suppose $x_t\colon M \to \mathbb{R}^3$ is a smooth 1-parameter family of immersions each inducing the same metric $\langle\,,\rangle$; this is called an *isometric deformation*. The unit normal field and second fundamental form of each immersion x_t are denoted by ξ_t and A_t respectively. From now on prime denotes differentiation with respect to t.

Since $\langle Xx_t, Xx_t\rangle$ is independent of t we know that $\langle Xx_t', Xx_t\rangle = 0$. Hence $Xx_t' = k(x_t)_*(JX) + p(X)\xi_t$ where k is a function on M and p is a 1-form on M, both dependent on t.

Since $\langle \xi_t, \xi_t\rangle = 1$ it follows that $\xi_t' = (x_t)_*(Z)$ where Z is a vector field on M, dependent on t. Since $\langle Xx_t, \xi_t\rangle = 0$ it follows

$$\langle Xx_t', \xi_t\rangle + \langle Xx_t, \xi_t'\rangle = 0$$

and so $p(X) = -\langle X, Z\rangle$. Thus

$$Xx_t' = k\ (x_t)_*(JX) - \langle X, Z\rangle\xi_t. \tag{2}$$

In continuing the computation for the deformation we will drop the subscript t. From equation (1)

$$
\begin{aligned}
XYx' &= X(kx_*(JY) - \langle Y, Z\rangle\xi) = \\
&= X(k)x_*(JY) + k[x_*(\nabla_X JY) + \langle AX, JY\rangle\xi] - X\langle Y, Z\rangle\xi + \langle Y, Z\rangle x_*(AX) = \\
&= x_*(X(k)JY + kJ\nabla_X Y + \langle Y, Z\rangle AX) + (k\langle AX, JY\rangle - X\langle Y, Z\rangle)\xi. \quad (3)
\end{aligned}
$$

Differentiating equation (1) with respect to t and using equation (2) gives

$$
\begin{aligned}
XYx' &= \nabla_X Y(x') + \langle A'X, Y\rangle\xi + \langle AX, Y\rangle x_*(Z) = \\
&= kx_*(J\nabla_X Y) - \langle\nabla_X Y, Z\rangle\xi + \langle A'X, Y\rangle\xi + \langle AX, Y\rangle x_*(Z) = \\
&= x_*(kJ\nabla_X Y + \langle AX, Y\rangle Z) + (\langle A'X, Y\rangle - \langle\nabla_X Y, Z\rangle)\xi. \quad (4)
\end{aligned}
$$

Comparing normal components in equations (3) and (4) we obtain

$$A'X = -kJAX - \nabla_X Z \tag{5}$$

and comparing tangential components

$$X(k)JY = \langle AX, Y\rangle Z - \langle Y, Z\rangle AX$$

which is equivalent to

$$\nabla k = -AJZ. \tag{6}$$

Equations (5) and (6) are the integrability conditions of the deformation. Finally note

$$Xx' = (x_*(JZ) + k\xi) \times Xx_t \tag{7}$$

where $\times$ denotes the cross product on $\mathbb{R}^3$. The vector-valued function $\eta = x_*(JZ) + k\xi$ is called the Drehriss [3].

3 The associate surfaces of a constant mean curvature surface

Let $x\colon M \to \mathbb{R}^3$ be an oriented surface with constant mean curvature H then, for each $t \in [0, 2\pi]$, the symmetric tensor field

$$A_t = \cos(t)(A - HI) + \sin(t)J(A - HI) + HI \tag{8}$$

satisfies the Gauss-Codazzi equations with respect to the induced metric $\langle\,,\rangle$ and $\frac{\mathrm{Tr}A_t}{2} \equiv H$. If M is simply-connected then, by the fundamental theorem of surface theory, we obtain a one-parameter family of isometric immersions x_t

with second fundamental form A_t and therefore constant mean curvature H and $x_0 = x$. These immersions are uniquely determined to within a congruence or rigid motion of $\mathbb{R}^3$ that is an orientation preserving isometry of $\mathbb{R}^3$. Without loss of generality, we may assume $x(p_0) = 0$ for some $p_0 \in M$ and normalize the family by $x_t(p_0) = 0$, $\xi_t(p_0) = \xi(p_0)$ and $(x_t)_*(p_0) = x_*(p_0) \circ \mathrm{Rot}_{p_0}(-t)$ where $\mathrm{Rot}_{p_0}(\theta)$ denotes the oriented rotation of the tangent plane $T_{p_0}M$ (with the induced metric) through an angle θ. The resulting normalized isometric deformation $x_t \colon M \to \mathbb{R}^3$, $t \in [0, 2\pi]$ is called the *associate deformation* here (see also [2]).

For example, let $x \colon M \to \mathbb{R}^3$ be an oriented *simply-connected* minimal surface and choose the origin of $\mathbb{R}^3$ on the surface, i.e. $x(p_0) = 0$ for a certain $p_0 \in M$. Since $\Delta x \equiv 0$, where Δ is the Laplace operator of the induced metric on M, we have a complex conjugate $y \colon M \to \mathbb{R}^3$ of x, unique to within a translation of $\mathbb{R}^3$. We may therefore assume $y(p_0) = 0$ also. Then

$$x_t = \cos(t)x + \sin(t)y$$

is a 1-parameter family of minimal isometric immersions of M into $\mathbb{R}^3$ with second fundamental form A_t as given in equation (8) above. Since by the Cauchy-Riemann equations $x_*(X) = y_*(JX)$ and $x_*(JX) = -y_*(X)$ it is easy to see that ξ_t does not change with t and $(x_t)_{*_p} = x_{*_p} \circ \mathrm{Rot}_p(-t))$.

Returning now to the constant mean curvature case, if M is not simply-connected then we may lift x to $\widetilde{x} \colon \widetilde{M} \to \mathbb{R}^3$ and let $\widetilde{A}_t$ denote the lift of A_t to $\widetilde{M}$. By the earlier discussion, $\widetilde{A}_t$ is the second fundamental form of an isometric immersion $\widetilde{x}_t \colon \widetilde{M} \to \mathbb{R}^3$ with constant mean curvature H and is unique to within a motion of $\mathbb{R}^3$. Fixing $p_0 \in M$ and $\widetilde{p}_0 \in \widetilde{M}$ over p_0, we may assume $\widetilde{x}_t(\widetilde{p}_0) = 0$, $\widetilde{\xi}_t(\widetilde{p}_0) = \xi(p_0)$ and $(\widetilde{x}_t)_{*_{\widetilde{p}_0}} = \widetilde{x}_{*_{\widetilde{p}_0}} \circ \mathrm{Rot}_{\widetilde{p}_0}(-t)$ for all t. With this normalization we obtain a smooth isometric deformation

$$\widetilde{x}_t \colon \widetilde{M} \to \mathbb{R}^3, \quad 0 \leq t \leq 2\pi$$

with constant mean curvature H. Of course $\widetilde{x}_0$ projects to $x \colon M \to \mathbb{R}^3$. Let $S = \{t \in [0, 2\pi) \mid \widetilde{x}_t \text{ projects to } x_t \colon M \to \mathbb{R}^3\}$; this set of immersions of M in $\mathbb{R}^3$ will be called the *associate family* for $x \colon M \to \mathbb{R}^3$. We now consider the structure of S for a constant mean curvature surface M with topology. The first and most interesting question is whether S contains an open interval.

Lemma 3.1 *If $x_m \colon M \to \mathbb{R}^3$, $m = 1, 2$, are isometric immersions with constant mean curvature H then x_2 is congruent to an associate of x_1.*

Thus if $x \colon M \to \mathbb{R}^3$ is an immersion with constant mean curvature H, all other isometric immersions of M in $\mathbb{R}^3$ with the same constant mean curvature occur, to within congruences, in the family of associates of x.

Proof. Locally on M we may choose positive isothermal coordinates (u, v), i.e. $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$ is a positively oriented frame and the metric is of the form $\langle\, ,\rangle = e^{2\rho}(du^2+dv^2)$. The second fundamental form of x_m with respect to the coordinate frame is written

$$A_m = \begin{bmatrix} H+\alpha_m & \beta_m \\ \beta_m & H-\alpha_m \end{bmatrix}$$

Now, $\omega_m = \langle A_m \frac{\partial}{\partial w}, \frac{\partial}{\partial w}\rangle$, is a complex function in the coordinate $w = u+iv$, where $\frac{\partial}{\partial w} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})$ and $\langle\, ,\rangle$ is extended by complex linearity. Clearly, $\Omega_m = \omega_m dw^2$ is a well defined complex quadratic differential on M; a simple computation gives $\omega_m = -\frac{i}{2}e^{2\rho}(\beta_m + i\alpha_m)$. Codazzi's equation in these isothermal coordinates reads

$$(\omega_m)_{\overline{w}} = \frac{1}{2}e^{2\rho}H_w$$

and, since H is constant, ω_m is holomorphic.

Since $|\omega_m|^2 = \frac{e^{4\rho}}{4}(\beta_m^2+\alpha_m^2) = \frac{e^{4\rho}}{4}(H^2-K)$, by Gauss' equation, it follows that the meromorphic function $\frac{\omega_2}{\omega_1}$ is constant and of modulus one. Hence, $\omega_2 = e^{-it}\omega_1$. It follows easily that

$$A_2 = \cos(t)(A_1 - HI) + \sin(t)J(A_1 - HI) + HI.$$

Hence x_2 is congruent to an associate of x_1. $\square$

Assume S contains an interval $[0, \varepsilon)$ then we have the associate deformation

$$x_t \colon M \to \mathbb{R}^3$$

for $0 \leq t \leq \varepsilon$. Since, by (8), $A' = J(A - HI)$ the integrability condition (5) becomes

$$\nabla_X JZ = (k+1)AX - HX.$$

Replacing Y by JZ in the structure equation (1)

$$\begin{aligned} Xx_*(JZ) &= x_*(\nabla_X JZ) + \langle AX, JZ\rangle\xi = \\ &= x_*((k+1)AX - HX) + \langle X, AJZ\rangle\xi = \\ &= x_*((k+1)AX - HX) - \langle X, \nabla k\rangle\xi = \\ &= -X((k+1)\xi + Hx) \end{aligned}$$

so that $x_*(JZ) + (k+1)\xi + Hx = V$ is a constant vector field along each surface in the variation. From the normalization in the definition of the associate deformation we obtain $Z(p_0) = 0$ and $(x_t)_{*_{p_0}}X = x_{*_{p_0}} \circ \mathrm{Rot}_{p_0}(-t)X$ gives

$$\frac{d}{dt}(x_t)_{*_{p_0}}X = x_{*_{p_0}}(-\sin(t)X - \cos(t)JX) = -x_{*_{p_0}} \circ \mathrm{Rot}_{p_0}(-t)JX.$$

On the other hand, from equation (2),

$$\frac{d}{dt}(x_t)_{*p_0}X = X(x'_t)\mid_{p_0} = k(x_t)_{*p_0}(JX) - \langle X, Z\rangle \xi_t(p_0) = k(p_0)x_{*p_0} \circ \mathrm{Rot}_{p_0}(-t)JX.$$

Hence $k(p_0) = -1$ and $V \equiv 0$, i.e. $x_*(JZ) + (k+1)\xi + Hx \equiv 0$. Now $Xx'_t = (x_*(JZ) + k\xi) \times Xx_t = -(Hx + \xi) \times Xx_t$ and since $\langle Xx_t, \xi\rangle = 0$ it follows, also on differentiating with respect to t, that

$$\xi' = -(Hx + \xi) \times \xi.$$

We collect these facts in the following lemma

Lemma 3.2 *If the associate deformation of $x\colon M \to \mathbb{R}^3$ exists then*

$$Xx' = -(Hx + \xi) \times Xx,$$

$$\xi' = -(Hx + \xi) \times \xi.$$

4 The flux and torque of a cycle on an immersed surface of constant mean curvature

To motivate the notions of flux and torque take an embedded oriented surface

$$x\colon M \to \mathbb{R}^3$$

with oriented normal ξ and constant mean curvature $H \geq 0$. Imagining the surface as a liquid membrane in equilibrium under a constant normal pressure field F, the equilibrium equation is $F = -2H\tau\xi$ [18], where τ is the surface tension of the membrane. We may assume $\tau = 1$.

Take a compact domain D in M and along each oriented boundary component γ we insert a smooth embedded cap

$$k\colon K \to \mathbb{R}^3$$

that is $k(\partial K) = x(\gamma)$, then the orientation of γ determines an orientation on K. Let ν_k be the oriented unit normal to K and $\eta = J\dot{\gamma}$ the oriented unit normal to γ in M (see Figure 1).

Considering the domain D with caps inserted on each boundary component the resulting closed surface is maintained in equilibrium by the application of a total restorative force on each end —to counter the inherent forces due to pressure and surface tension—of that end which total

$$2H\int_K \nu_k da_k + \int_\gamma \eta ds$$

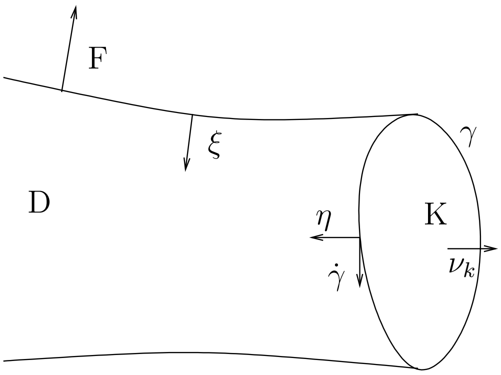

Figure 1: Flux and Torque

where da_k also denotes the area element in K.

Define $W([\gamma]) = 2H\int_K \nu_k da_k + \int_\gamma \eta ds$ as the flux of the component γ.

Let ω_0 be the 1-form on M defined by $\omega_0(X) = Hx \times x_*(X)$ then $d\omega_0 = 2H\xi da$ where da is the area element of M. The corresponding 1-form ω_0^k on K, defined by $\omega_0^k(X) = Hk \times k_*(X)$, satisfies $d\omega_0^k = 2H\nu_k da_K$. If ω is the 1-form defined on M by $\omega(X) = (Hx + \xi) \times x_*(X)$ then

$$\int_\gamma \omega = \int_\gamma \omega_0 + \int_\gamma \eta = \int_\gamma \omega_0^k + \int_\gamma \eta ds,$$

since $\omega_0^k = \omega_0$ along γ. By Stokes' theorem

$$\int_\gamma \omega = \int_K d\omega_0^k + \int_\gamma \eta = 2H \int_K \nu_k da_k + \int_\gamma \eta ds = W(\gamma).$$

Now ω is easily checked to be a closed 1-form on M for any immersed oriented surface

$$x \colon M \to \mathbb{R}^3$$

of constant mean curvature H. Thus the quantity

$$W([\gamma]) = \int_\gamma \omega$$

depends only on the homology class of the cycle γ. This will be called the *flux* or *force* of the immersion for the cycle γ.

Returning again to the domain D with ends capped as above the torque of the inherent forces of pressure and surface tension at the end γ totals

$$2H \int_K k \times \nu_k da_k + \int_\gamma x \times \eta ds.$$

Define the torque of γ by

$$T(\gamma) = 2H \int_K k \times \nu_k da_k + \int_\gamma x \times \eta ds.$$

Define $\sigma_0(X) = \frac{2}{3} Hx \times (x \times x_*(X))$. We can easily compute $d\sigma_0 = 2Hx \times \xi da$. The corresponding 1-form σ_0^k on K defined by $\sigma_0^k(X) = \frac{2}{3} Hk \times (k \times k_*(X))$ satisfies $d\sigma_0^k = 2Hk \times \nu_k da_k$. Thus

$$2H \int_K k \times \nu_k da_k = \int_{\partial K} \sigma_0^k = \int_\gamma \sigma_0$$

since $\sigma_0^k = \sigma_0$ along γ. Then $T(\gamma) = \int_\gamma \sigma_0 + \int x \times \eta ds = \int_\gamma \sigma$ where σ is the 1-form defined by

$$\sigma(X) = \frac{2}{3} Hx \times (x \times x_*(X)) + x \times x_*(JX) = \frac{1}{3} x \times [2(Hx + \xi) \times x_*(X) + x_*(JX)].$$

Again it is easy to check that σ is a closed 1-form on any immersed surface and so

$$T([\gamma]) = \int_\gamma \sigma$$

depends only on the homology class of γ.

To prove the consequences *(i)* and *(ii)* in Theorem 1.1 announced in the introduction we need only show

Lemma 4.1 *Let $x\colon M \to \mathbb{R}^3$ be an immersed surface of constant mean curvature H admitting a nontrivial isometric deformation through surfaces of constant mean curvature H then for the immersion x*

(i) ω is exact;

(ii) σ is exact if $H \neq 0$.

Proof. Consider the associate family $x_t\colon M \to \mathbb{R}^3$ with $x_0 = x$ then, from the assumption of the lemma, x_t is defined for $0 \le t < \varepsilon$, for some $\varepsilon > 0$. As we saw in Section 3,

$$Xx' = -(Hx + \xi) \times Xx = -\omega(X).$$

Hence ω is exact.

We begin by calculating Xx''. Write $P = Hx + \xi$

$$\begin{aligned} Xx'' &= -(P \times Xx)' = -P' \times Xx + P \times (P \times Xx) = \\ &= -Hx' \times Xx - \xi' \times Xx + P \times (P \times Xx) = \\ &= -HX(x' \times x) + HXx' \times x - \xi' \times Xx + P \times (P \times Xx) = \\ &= -HX(x' \times x) - H(P \times Xx) \times x + (P \times \xi) \times Xx + P \times (P \times Xx) \end{aligned}$$

$$
\begin{aligned}
X(x''+Hx' \times x) &= (Hx+P) \times (P \times Xx) + (P \times \xi) \times Xx = \\
&= (2Hx+\xi) \times (P \times Xx) - Xx \times (P \times \xi) = \\
&= 2Hx \times (P \times Xx) + \xi \times (P \times Xx) - Xx \times (P \times \xi) = \\
&= 2Hx \times (P \times Xx) - P \times (Xx \times \xi) - Xx \times (\xi \times P) - Xx \times (P \times \xi) \\
&= 2Hx \times (P \times Xx) + (Hx+\xi) \times x_*(JX)
\end{aligned}
$$

Thus

$$X(x'' + Hx' \times x + x) = Hx \times \{2P \times Xx + x_*(JX)\} = 3H\sigma(X).$$

Hence if $H \neq 0$ σ is exact if there exists an isometric deformation of x. □

To prove parts *(a)* and *(b)* of Theorem 1.1 we need:

Lemma 4.2 *Let $x\colon M \to \mathbb{R}^3$ be an immersion of a smooth oriented surface with constant mean curvature H. Then either*

(a) the associates $x_t\colon M \to \mathbb{R}^3$ exist for all $t \in [0, 2\pi]$, or

(b) there are only finitely many isometric immersions of constant mean curvature H.

Proof. Let $\widetilde{M}$ be the universal cover of M with the lifted metric and complex structure (denoted $\langle\, ,\rangle$ and J respectively), $\pi\colon \widetilde{M} \to M$ the projection and $\mathcal{D}$ be the group of deck transformations of this cover which are, of course, orientation preserving isometries. The lift $\widetilde{A}_t$ of

$$A_t = \cos t(A - HI) + \sin tJ(A - HI) + HI$$

to the universal cover is the second fundamental form of the associate family

$$\widetilde{x}_t\colon \widetilde{M} \to \mathbb{R}^3$$

defined above. Since each deck transformation $\sigma \in \mathcal{D}$ preserves the lifted second fundamental form $\widetilde{A}_t$, that is,

$$\sigma_{*_p}\widetilde{A}_t(p)(\sigma_{*_p})^{-1} = \widetilde{A}_t(\sigma(p))$$

for all $p \in \widetilde{M}$, it follows that $\widetilde{x}_t \circ \sigma$ and $\widetilde{x}_t$ have the same second fundamental form. Hence $\widetilde{x}_t \circ \sigma = \Phi_t(\sigma) \circ \widetilde{x}_t$, where $\Phi_t(\sigma) \in \mathcal{M}$ the group of motions of $\mathbb{R}^3$. It easy to see that

$$\Phi_t\colon \mathcal{D} \to \mathcal{M}$$

is a homomorphism for each $t \in [0, 2\pi]$ and $\widetilde{x}_t$ projects to $x_t \colon M \to \mathbb{R}^3$ if and only if $\Phi_t(\mathcal{D}) = \{I\}$.

Let $S = \{t \in [0, 2\pi] \mid \widetilde{x}_t$ projects to $M\}$. Assuming S is infinite there exists an infinite sequence of points $\{t_n\}$ of points in S with an accumulation point $t_0 \in S$. Then, for each $\sigma \in D$, $\Phi_{t_n}(\sigma) = I$ so, by continuity, $\Phi_{t_0}(\sigma) = I$; it is also easy to check all derivatives of $\Phi_t(\sigma)$ vanish at $t = t_0$. Since for each $\sigma \in \mathcal{D}$, $\Phi_t(\sigma)$ is an analytic curve in $\mathcal{M}$, it follows that $\Phi_t(\sigma) \equiv I$ for all t. Thus, $S = [0, 2\pi]$ if S is infinite. This proves that either *(a)* or *(b)* in Theorem 1.1 must hold. □

Proposition 4.3 *Let $I(M)$ (resp. $I_0(M)$) be the group of orientation preserving isometries of M (resp. the subgroup of such isometries extending under x to a congruence of $\mathbb{R}^3$). If $\sigma \in I(M)$ then $x \circ \sigma$ is congruent to an associate $x_{t(\sigma)}$ of x. The map $t \colon I(M) \to [0, 2\pi]$ is a group homomorphism with kernel $I_0(M)$.*

Proof.

Let $\sigma \in I(M)$. Comparing the isometric immersions x and $y = x \circ \sigma$ the respective oriented normals to these maps at p are $\xi(p)$ and $N(p) = \xi(\sigma(p))$. If B is the second fundamental form of y then, by its definition, $(XN)_p = -y_{*_p}(B(p)X) = -x_*(\sigma(p))\sigma_{*_p}(B(p)X)$. But, since $N = \xi \circ \sigma$,

$$(XN)_p = \xi_*(\sigma(p))(\sigma_{*_p}X) = -x_*(\sigma(p))(A(\sigma(p))\sigma_{*_p}X).$$

Taken together, these give

$$B(p) = (\sigma_{*_p})^{-1}A(\sigma(p))\sigma_{*_p}.$$

Thus y has constant mean curvature H. By Lemma 3.1, $y = x \circ \sigma$ is congruent to an associate $x_{t(\sigma)}$ of x. This defines a map $t \colon I(\mathcal{M}) \to [0, 2\pi)$. Obviously, $t(\sigma) = 0$ if and only if $\sigma \in I_0(M)$. To complete the proof of the theorem we must show that t is a homomorphism.

For $\sigma, \tau \in I(M)$ let C denote the second fundamental form of $x \circ \sigma \circ \tau$. As before,

$$\begin{aligned} C(p) &= ((\sigma \circ \tau)_{*_p})^{-1}A((\sigma \circ \tau)(p))(\sigma \circ \tau)_{*_p} = \\ &= (\tau_{*_p})^{-1}(\sigma_*(\tau(p)))^{-1}A(\sigma(\tau(p)))\sigma_*(\tau(p))\tau_{*_p} = \\ &= (\tau_{*_p})^{-1}B(\tau(p))\tau_{*_p} \end{aligned}$$

Since $x \circ \sigma$ is congruent to the associate $x_{t(\sigma)}$ of x we have $B = e^{it(\sigma)}(A - HI) + HI$, where $e^{it}(A - HI) = \cos t(A - HI) + \sin tJ(A - HI)$. Thus

$$\begin{aligned} C(p) &= e^{it(\sigma)}(\tau_{*_p})^{-1}(A - HI)(\tau(p))\tau_{*_p} + HI(p) = \\ &= e^{it(\sigma)}[(\tau_{*_p})^{-1}A(\tau(p))\tau_{*_p} - HI(p))] + HI(p) = \\ &= e^{it(\sigma)}\{e^{it(\tau)}(A - HI)(p)\} + HI(p), \end{aligned}$$

since $x \circ \tau$ is congruent to $x_t(\tau)$. Thus

$$C(p) = e^{i(t(\sigma)+t(\theta))}(A - HI(p)) + HI(p).$$

and hence $t(\sigma \circ \tau) = (t(\sigma) + t(\tau)) \mod 2\pi$ and the map t defines a homomorphism. $\square$

5 Application

We start this section by noticing that there exist complete, immersed minimal surfaces with genus zero and finitely many ends admitting a nontrivial isometric deformation through minimal surfaces. First, let us recall the "Weierstrass representation theorem"

Theorem 5.1 *Suppose M is a minimal surface in $\mathbb{R}^3$, M its Riemann surface, g the stereographic projection of its Gauss map, $dh = dx_3 - idx_3 \circ J$. Then M may be represented (up to a translation) by the conformal immersion*

$$x = Re \int \Phi, \text{ where} \tag{9}$$

$$\Phi = (\Phi_1, \Phi_2, \Phi_3) = ((g^{-1} - g)\frac{dh}{2}, i(g^{-1} + g)\frac{dh}{2}, dh). \tag{10}$$

Conversely, let M be a Riemann surface, $g\colon M \to \mathbb{S}$ a holomorphic function and dh a holomorphic one-form on M. Then, provided that $Re \int_\alpha \Phi = 0$ for all closed curves α on M, equation (9) and (10) define a conformal minimal mapping of M into $\mathbb{R}^3$, which is regular provided the poles and zeros of g coincide with the zeros of dh. The holomorphic function g and holomorphic one form dh are the so-called Weierstrass data.

With Theorem 5.1 in mind, it is easy to check that, for an immersed minimal surface M, admitting a nontrivial isometric deformation through minimal surfaces is equivalent to the condition that $Im \int_\alpha \Phi = 0$ for all closed curves α on M. Take $p_1, ..., p_n \in \mathbb{C}$ and, for any $k \in \mathbb{N}$, consider the following Weierstrass data

$$g = \sum_{i=1}^{n} (z - p_i)^{2k} \text{ and } dh = dz.$$

It is easy to check that this data yield non simply-connected, genus zero minimal surfaces admitting a nontrivial isometric deformation through minimal surfaces.[1]

[1]We thank Fred Xavier for helpful conversations.

There are many results on the isometric indeformability of a constant mean curvature surfaces with topology (see for instance [5, 10, 11, 12, 15, 16]. Typically, Theorem 1.1 may be used in this direction. In what follow, we give a criterion which guarantees the isometric indeformability of a constant mean curvature surfaces. In particular this result can be seen as a generalization of a rigidity theorem of Choi-Meeks-White for minimal surface, see Theorem 1.2 in [5].

Theorem 5.2 *Let $x\colon M \to \mathbb{R}^3$ be an isometric immersion of a smooth oriented surface with constant mean curvature H. Suppose that a plane π intersects $x(M)$ transversally in a closed curve $\gamma\colon [0, L] \to M$ then*

$$W([\gamma]) = \int_0^L \frac{1}{\langle x_*(J\dot\gamma), T\rangle}(\langle x_*(J\dot\gamma), T\rangle^2 + H\langle x\circ\gamma, \xi\rangle)ds,$$

where T is the unit vector normal to π. In particular, if

$$\int_0^L \frac{1}{\langle x_*(J\dot\gamma), T\rangle}(\langle x_*(J\dot\gamma), T\rangle^2 + H\langle x\circ\gamma, \xi\rangle)ds \neq 0$$

then M does not admit a nontrivial isometric deformation through surfaces of constant mean curvature H. (Note that transversality implies that $|\langle x_*(J\dot\gamma), T\rangle| > 0$ therefore when $H = 0$ that integral is never zero.)

Proof. Without loss of generality, we can assume that π is the xy-plane. Clearly,

$$x\circ\gamma = \langle x\circ\gamma, x_*(\dot\gamma)\rangle x_*(\dot\gamma) + \langle x\circ\gamma, x_*(J\dot\gamma)\rangle x_*(J\dot\gamma) + \langle x\circ\gamma, \xi\rangle\xi$$

and

$$0 = \langle x\circ\gamma, e_3\rangle = H\langle x\circ\gamma, x_*(J\dot\gamma)\rangle\langle x_*(J\dot\gamma), e_3\rangle + \langle x\circ\gamma, \xi\rangle\langle \xi, e_3\rangle.$$

Since the plane intersects $x(M)$ transversally $\langle x_*(J\dot\gamma), e_3\rangle \neq 0$ and consequently

$$\langle x\circ\gamma, x_*(J\dot\gamma)\rangle = -\frac{\langle x\circ\gamma, \xi\rangle\langle\xi, e_3\rangle}{\langle x_*(J\dot\gamma), e_3\rangle}.$$

The discussion above yields the following,

$$\begin{aligned}
&\langle(\xi + Hx\circ\gamma)\times x_*(\dot\gamma), e_3\rangle = \\
&\qquad = \langle x_*(J\dot\gamma) - H\langle x\circ\gamma, x_*(J\dot\gamma)\rangle\xi + H\langle x\circ\gamma, \xi\rangle x_*(J\dot\gamma), e_3\rangle = \\
&\qquad = \langle x_*(J\dot\gamma), e_3\rangle + H(-\langle x\circ\gamma, x_*(J\dot\gamma)\rangle\langle\xi, e_3\rangle + \langle x\circ\gamma, \xi\rangle\langle x_*(J\dot\gamma), e_3\rangle) = \\
&\qquad = \langle x_*(J\dot\gamma), e_3\rangle + H\langle x\circ\gamma, \xi\rangle(\frac{\langle\xi, e_3\rangle^2}{\langle x_*(J\dot\gamma), e_3\rangle} + \langle x_*(J\dot\gamma), e_3\rangle).
\end{aligned}$$

□

Corollary 5.3 *Let $x\colon M \to \mathbb{R}^3$ be an isometric immersion of a smooth oriented surface with constant mean curvature H. Suppose that M has a plane of symmetry π which intersects $x(M)$ in a closed curve $\gamma\colon [0, L] \to M$. If $x \circ \gamma$ lies in an open disk of radius $\frac{1}{H}$ then M does not admit a nontrivial isometric deformation through surfaces of constant mean curvature H.*

Proof. Since π is a plane of symmetry, we can assume that $\langle x_*(J\dot{\gamma}), T\rangle = 1$. Therefore Theorem 5.2 gives

$$W([\gamma]) = L + H \int_0^L \langle x \circ \gamma, \xi\rangle ds.$$

Suppose $x \circ \gamma$ lies in an open disk of radius $\frac{1}{H}$. Without loss of generality we can assume that the disk is centered at the origin. Then $|H \int_0^L \langle x \circ \gamma, \xi\rangle ds| < L$ and therefore $W([\gamma]) \neq 0$. □

Definition 5.4 *An embedding $x\colon M \to \mathbb{R}^3$ has a plane of Alexandrov symmetry π if π is a plane of symmetry for $x(M)$ and $x(M)\backslash\{x(M) \cap \pi\}$ consists of two graphs over π.*

Theorem 5.5 *Let $x\colon M \to \mathbb{R}^3$ be a complete proper isometric embedding of a smooth oriented surface with constant mean curvature H. Suppose that M has a plane of Alexandrov symmetry then M does not admit a nontrivial isometric deformation through surfaces of constant mean curvature H.*

In order to prove Theorem 5.5 we recall a result of Korevaar and Kusner, Theorem 1.12 in [8].

Theorem 5.6 *Let $x\colon M \to \mathbb{R}^3$ be an embedding of a smooth oriented surface with constant mean curvature H. Suppose that a plane π intersects $x(M)$ transversally in a closed curve $\gamma\colon [0, L] \to M$. Let Γ be the compact region in the plane π bounded by $x(\gamma)$. If*

(i) the projection of $\xi\big|_\gamma$ onto the plane π is pointing outside the region Γ, or

(ii) γ is homologically non-trivial.

then $W([\gamma]) \neq 0$.

Remark 5.7 *As a consequence of the results in this paper and Theorem 5.6, if $x\colon M \to \mathbb{R}^3$ satisfies the hypothesis of Theorem 5.6 then it does not admit a nontrivial isometric deformation through surfaces of constant mean curvature H.*

Here is the proof of Theorem 5.5.

Proof.

If M does not have genus zero then π intersects $x(M)$ in at least a simple closed curve. If this simple closed curve satisfies item *(i)* in Theorem 5.6 then, as pointed in Remark 5.7, we are done. If not, then $x(M)$ must be compact and therefore a round sphere (round sphere do not admit nontrivial isometric deformation).

If M has genus zero and bounded second fundamental form, then M does not admit such a deformation (see [10, 15]). In fact, M has an end asymptotic to an unduloid (see [9, 14]). A simple compactness argument and the fact that an unduloid does not admit such a deformation then proves this case.

If M has genus zero and unbounded second fundamental form, then for any $n \in \mathbb{N}$ there exists $p(n) \in x(M)$ such that $|A|(p(n)) > n$. Recall that there exists a constant C depending only on H such that $x_3|A| < C$ (see for instance [17]) therefore $|p_3(n)| < \frac{C}{|A|(p(n))}$. After a sequence of translations of $x(M)$ which take $p(n)$ to the origin, we obtain a sequence of constant mean curvature surfaces M_n with a plane of Alexandrov symmetry and whose norms of the second fundamental form is blowing up at the origin. Moreover, the distance from the origin to the plane of symmetry is bounded by $\frac{C}{|A_n|(o)}$.

Consider the non-negative function

$$F_n(x) = (1 - |x|)^2 |A_n(x)|^2$$

over the connected component of $M_n \cap B_1(0)$ containing the origin which we denote by $M_n(0)$. The function F_n is zero on the boundary of $M_n(0)$ and therefore it reaches its maximum at a point in the interior. Let $q(n)$ be such point, i.e.

$$F_n(q(n)) = \max_{M_n(0)} F_n(x) = (|q(n)| - 1)^2 |A|^2(q(n)) \geq F_n(0) = |A_n(0)|^2.$$

Fix $\sigma_n > 0$ such that $2\sigma_n < 1 - |q(n)|$ and

$$4\sigma_n^2 |A|^2(q(n)) = 4|A_n(0)|^2 = C_n^2.$$

Since F_n achieves its maximum at $q(n)$,

$$\sup_{B_{\sigma_n}(q(n)) \cap M_n(0)} \sigma_n^2 |A_n|^2 \leq \sup_{B_{\sigma_n}(q(n)) \cap M_n(0)} \sigma_n^2 \frac{F_n(x)}{(|x| - 1)^2} \leq$$
$$\leq \frac{4\sigma_n^2}{(|q(n)| - 1)^2} \sup_{B_{\sigma_n}(q(n)) \cap M_n(0)} F_n(x) = \frac{4\sigma_n^2}{(|q(n)| - 1)^2} F_n(q(n)) = 4\sigma_n^2 |A|^2(q(n)). \tag{11}$$

After translating the surfaces M_n so that the plane of Alexandrov symmetry is the xy-plane and $q(n)$ lies on the z axis, denote by $M(q(n))$ the translation of $B_{\sigma_n}(q(n)) \cap M_n(0)$. We have obtained the following

$$\sup_{M(q(n))} |A_n|^2 \leq 4|A|^2(q(n)),$$

$$4\sigma_n^2|A|^2(q(n)) = C_n^2, \text{ and}$$

$$|q_3(n)| < \frac{C}{|A|^2(q(n))}.$$

Let M_n' be the connected component of $M(q(n))$ rescaled by a factor of $|A|^2(q(n))$. Notice that $|q_3'(n)| < C$. A standard compactness argument implies that this sequence converges to a non-flat, genus zero, embedded minimal surface M_∞ with bounded second fundamental form and hence properly embedded (see [6, 13]). Since such a surface cannot be contained in a half-space (see [7]), the xy-plane must be a plane of Alexandrov symmetry. This being the case, M_∞ cannot be a helicoid and therefore it does not admit a nontrivial isometric deformation (see [15]). A simple compactness argument then implies that M does not admit a nontrivial isometric deformation. $\square$

References

[1] M. Anderson. Local rigidity of surfaces in space forms. Preprint, arXiv:0709.3202v1.

[2] O. Bonnet. Memoire sur la theorie des surfaces applicables. *J. Ecole Polytechnique*, 42, 1867.

[3] J. M. Burns and M. J. Clancy. Smyth surfaces and the drehriss. *Proc. Edinburgh Math. Soc. (2)*, 48(3):549–555, 2005.

[4] E. Calabi. Quelques applications de l'analyse complexe aux surfaces d'aire minima. In *Topics in Complex Manifolds*, pages 59–81. Les Presses de l'Université de Montréal, 1967. H. Rossi, editor.

[5] T. Choi, W. H. Meeks III, and B. White. A rigidity theorem for properly embedded minimal surfaces in $\mathbb{R}^3$. *J. of Differential Geometry*, 32:65–76, 1990. MR1064865, Zbl 0704.53008.

[6] T. H. Colding and W. P. Minicozzi II. The Calabi-Yau conjectures for embedded surfaces. To appear in Annals of Math., Preprint math. DG/0404197 (2004).

[7] D. Hoffman and W. H. Meeks III. The strong halfspace theorem for minimal surfaces. *Invent. Math.*, 101:373–377, 1990. MR1062966, Zbl 722.53054.

[8] N. Korevaar and R. Kusner. The global structure of constant mean curvature surfaces. *Invent. Math.*, 114:311–332, 1993.

[9] N. Korevaar, R. Kusner, and B. Solomon. The structure of complete embedded surfaces with constant mean curvature. *J. of Differential Geometry*, 30:465–503, 1989. MR1010168, Zbl 0726.53007.

[10] R. Kusner. *Global geometry of extremal surfaces in three-space*. PhD thesis, University of California, Berkeley, 1988.

[11] W. H. Meeks III. The theory of triply-periodic minimal surfaces. *Indiana Univ. Math. J.*, 39(3):877–936, 1990. MR1078743, Zbl 721.53057.

[12] W. H. Meeks III and H. Rosenberg. The global theory of doubly periodic minimal surfaces. *Invent. Math.*, 97:351–379, 1989. MR1001845, Zbl 676.53068.

[13] W. H. Meeks III and H. Rosenberg. The minimal lamination closure theorem. *Duke Math. Journal*, 133(3):467–497, 2006. MR2228460, Zbl 1098.53007.

[14] W. H. Meeks III and G. Tinaglia. The *CMC* dynamics theorem in $\mathbb{R}^3$. Preprint, available at `http://www.nd.edu/~gtinagli/papers1.html`.

[15] W. H. Meeks III and G. Tinaglia. The rigidity of constant mean curvature surfaces. Preprint, available at `http://www.nd.edu/~gtinagli/papers1.html`.

[16] J. Pérez. A rigidity theorem for periodic minimal surfaces. *Comm. in Analysis and Geom.*, 7(1):95–104, 1999. MR1674113, Zbl 0947.53006.

[17] A. Ros and H. Rosenberg. Properly embedded surfaces with constant mean curvature. Preprint.

[18] B. Smyth. Soliton surfaces in the mechanical equilibrium of closed membranes. *Communications in Mathematical Physics 250*, 1:81–94, 2004.