The CMC Dynamics Theorem in homogeneous n-manifolds

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Abstract

In this paper we generalize the Dynamics Theorem for nonzero CMC surfaces in $\mathbb{R}^3$ to a new Dynamics Theorem for nonzero CMC hyper-surfaces in a homogeneous manifold. In this case, the role of translations of $\mathbb{R}^3$ is played by a subgroup, G, of the isometry group of N which acts transitively on N.

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1 Introduction.

This paper is a preliminary version. Throughout this paper N will denote a non-compact homogeneous n-manifold[1]. For $H > 0$, we let $\mathcal{M}^H(N)$ denote the space of connected, non-compact, separating hypersurfaces of N which are properly embedded with constant mean curvature H. Recall that in a simply-connected manifold, any properly embedded hypersurface separates. The special case where N is $\mathbb{R}^3$ was considered in our previous paper [2].

Our first result is the following proposition.

Proposition 1.1 *Suppose $M \in \mathcal{M}^H(N)$ has bounded second fundamental form and G is a subgroup of the isometry group of N which acts transitively on N. For $p \in N$, any divergent sequence of points $p_n \in M$ and isometries $i_n \in G$ with $i(p_n) = p$, a subsequence of the surfaces $i_n(M)$ converges to a properly immersed surface in N with connected component M_∞ passing through p.*

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[1]A Riemannian manifold N is *homogeneous* if for any two points $p,\ q \in N$, there exists an isometry of N taking p to q.

In this paper we will obtain dynamics-type results for the set $\mathcal{T}_p^G(M)$ of limit surfaces M_∞ obtained in the above theorem. We consider every surface M_∞ in $\mathcal{T}_p^G(M)$ to be *pointed* in the following sense. Consider M_∞ together with a base point x and an isometric immersion $f\colon M_\infty \to N$ with $f(x) = p$. For a generic element M_∞ in $\mathcal{T}_p^G(M)$, $f^{-1}(p)$ consists of a single point of M_∞. We note that the maximum principle for constant mean curvature hypersurfaces implies $f^{-1}(p)$ never has more than two points. If $f^{-1}(p)$ consists of two points, then we consider the surface M_∞ to represent two distinct elements in $\mathcal{T}_p^G(M)$ corresponding to the two different base points in $f^{-1}(p)$.

We now state our main theorem. In what follows we let $\mathbb{B}_N(p,\varepsilon)$ denote the open ball in N centered at p with radius R; we let $\overline{\mathbb{B}}_N(p,\varepsilon)$ denote the corresponding closed ball.

Theorem 1.2 (*CMC* **Dynamics Theorem)** *Suppose $M \in \mathcal{M}^H(N)$ has bounded second fundamental form. Then the following statements hold.*

1. *M admits a uniform one-sided regular neighborhood on its mean convex side. In particular, there exists a constant C such that for all $q \in N$ and $R > 0$,*

$$\mathrm{Volume}_M(M \cap \mathbb{B}_N(q,R)) \leq C \cdot \mathrm{Volume}_N(\mathbb{B}_N(q,R))$$

2. *$\mathcal{T}_p^G(M)$ is a compact metric space with respect to a natural distance function induced by the Hausdorff distance function on compact subsets of N.*

3. *For any $\Sigma \in \mathcal{T}_p^G(M)$, $\mathcal{T}_p^G(\Sigma) \subseteq \mathcal{T}_p^G(M)$.*

4. *Every nonempty $\mathcal{T}_p^G$-invariant subset $\Delta \subset \mathcal{T}_p^G(M)$ contains a nonempty minimal, i.e. smallest, nonempty $\mathcal{T}_p^G$-invariant subset*[2]*. In particular, since $\mathcal{T}_p^G(M)$ is a nonempty $\mathcal{T}_p^G$-invariant set, $\mathcal{T}_p^G(M)$ contains minimal elements*[3].

5. *Let Σ be a minimal element of $\mathcal{T}_p^G(M)$. For all $\varepsilon > 0$, there exists a $d_\varepsilon > 0$ such that the following statements holds. For every smooth, connected compact domain $W \subset \Sigma$ and for all $q \in \Sigma$, there exists a compact smooth, connected domain $W' \subset \Sigma$ and an isometry $i \in G$ such that*

$$d_\Sigma(q, W') < d_\varepsilon \quad \text{and} \quad d_{\mathcal{H}}(W, i(W')) < \varepsilon,$$

   where d_Σ is distance function on Σ and $d_{\mathcal{H}}$ is the Hausdorff distance on compact sets in N.

We remark that the *CMC* Dynamics Theorem and its proof are motivated by the statement and proof of the Dynamics Theorem for Minimal Surfaces in $\mathbb{R}^3$ by Meeks, Perez and Ros [1] and by results in our previous paper [2].

[2]A subset $\Delta \subset \mathcal{T}_p^G(M)$ is $\mathcal{T}_p^G$-invariant if for any $\Sigma \in \Delta$, $\mathcal{T}_p^G(\Sigma) \subset \Delta$.
[3]A surface $\Sigma \in \mathcal{T}_p^G(M)$ is a *minimal element* if it lies in a minimal $\mathcal{T}_p^G$-invariant subset of $\mathcal{T}_p^G$.

2 The proof of the CMC Dynamics Theorem

Fix a $\delta > 0$ and a point $p \in N$. Let $\mathcal{M}^H_p(N,\delta) = \{M \in \mathcal{M}^H(N) \mid |A_M| \leq \delta \text{ and } p \in M\}$. In [3], it is proved that there exists an $\varepsilon > 0$ such that every $M \in \mathcal{M}^H_p(N,\delta)$ has a one-sided regular neighborhood on its mean convex side. Define $\Delta^H_p(N,\varepsilon)$ be the set of properly immersed pointed surfaces $f\colon (\Sigma, x) \to (N,p)$ which have constant mean curvature H and a one-sided open regular neighborhood of radius ε on their mean convex side. Since there is a uniform bound on the second fundamental form of every surface in $\Delta^H_p(N,\varepsilon)$, there exists and $\nu > 0$, so that for every $f\colon (\Sigma, x) \to (N,p)$ in $\Delta^H_p(N,\varepsilon)$, the component $D(\varepsilon, x)$ of $f^{-1}(\overline{\mathbb{B}}(p,\nu))$ with $x \in D(\Sigma, x)$ is a ball whose image under f can be considered to be a small graph over the tangent plane $T_p\Sigma$ at $p = f(x)$ in Fermi coordinates around $p \in N$.

We define the distance $d_\Delta(f,g)$ for $f\colon (\Sigma, x) \to (N,p)$ and $g\colon (\Sigma', x') \to (N,p)$ in $\Delta^H_p(N,\varepsilon)$ to be the Hausdorff distance between $f(D(\Sigma,x))$ and $g(D(\Sigma',x'))$.

Assertion 2.1 *With respect to the distance function d_Δ, $\Delta^H_p(N,\varepsilon)$ is a compact metric space.*

Proof. Since the Hausdorff distance is metric on compact subsets of N, one easily checks that d_Δ is a metric on $\langle \Delta^H_p(N,\varepsilon), d_\Delta\rangle$. We will prove that it is compact by checking that this space is sequentially compact.

Suppose $f_n\colon (\Sigma_n, x_n) \to (N,p)$ is a sequence in $\Delta^H_p(N,\varepsilon)$. Standard elliptic theory implies that a subsequence of the "graphs" $f_n(D(\Sigma_n, x_n))$ converges to a constant mean curvature graph D over its tangent space T_pD. A standard diagonal argument implies that D is contained in a complete, connected, immersed surface Σ in N of constant mean curvature H. Straightforward arguments prove that $\Sigma \in \Delta^H_p(N,\varepsilon)$, which completes the proof of the assertion. □

We now give - the proof of Proposition 1.1 stated in the introduction.

Proof of Proposition 1.1. Let $M \in \mathcal{M}^H(N)$ have bounded second fundamental form. Hence, $M \in \mathcal{M}^H_p(N,\delta)$ for some $\delta > 0$. Suppose $\{p_n\}_n$ is a divergent sequence of points in M and $i_n \in G$ is a sequence of isometries with $i_n(p_n) = p$. Then for δ and ε sufficiently small, we can consider the surface $M_n = i_n(M)$ to lie in $\Delta^H_p(N,\varepsilon)$. By Assertion 2.1, a subsequence of the M_n considered to lie in the metric space $\langle \Delta^H_p(N,\delta), d_\Delta\rangle$ converge to a surface M_∞ in $\Delta^H_p(H,\delta)$ satisfying the conclusions of Proposition 1.1. This completes the proof of the proposition.

We are now in a position to prove the CMC Dynamics Theorem stated in the introduction.

Proof of the CMC Dynamics Theorem. Let $M \in \mathcal{M}^H(N)$ have bounded second fundamental form. Then statements *1*, *2* and *3* in the theorem follows immediately from Assertion 2.1.

We next prove statement *4* holds. Assume now that $\Delta \subset \mathcal{T}_p^G(M)$ is a nonempty $\mathcal{T}_p^G$-invariant set. Let $\Sigma \in \Delta$ and note that $\mathcal{T}_p^G(\Sigma) \subset \Delta$ is a closed set in $\mathcal{T}_p^G(M)$, since the set of points limits of limit points of a set A in a metric space are themselves limit points of A. Consider the collection $\mathcal{C}_\Delta$ of all nonempty $\mathcal{T}_p^G$-invariant subsets A of Δ, which are closed subsets of $\mathcal{T}_p^G(M)$. Note that $\mathcal{C}_\Delta$ is nonempty since $\mathcal{T}_p^G(\Sigma) \in \mathcal{C}_\Delta$. Also note that $\mathcal{C}_\Delta$ is partially ordered by inclusion $\subset$. As we just observed, every nonempty $\mathcal{T}_p^G$-invariant set $\Delta' \subset \Delta$ contains a subset which is an element in $\mathcal{C}_\Delta$ and so, to prove statement *4*, it suffices to prove that $\mathcal{C}_\Delta$ contains a minimal element with respect to the partial ordering $\subset$. We will prove this fact by demonstrating that every nonempty totally ordered subset $T = \{\Delta_\alpha\}_{\alpha \in I}$ of $\mathcal{C}_\Delta$ has a lower bound in $\mathcal{C}_\Delta$ and then apply Zorn's lemma.

Claim 2.2 *Let $T = \{\Delta_\alpha\}_{\alpha \in I} \subset \mathcal{C}_\Delta$ be a nonempty totally ordered set. Then $\bigcap T = \bigcap_{\alpha \in I} \Delta_\alpha$ is an element in $\mathcal{C}_\Delta$.*

Proof. Since the collection $\{\Delta_\alpha\}_{\alpha \in I}$ of sets is totally ordered, they satisfy the finite intersection property[4] and since the sets Δ_α are also closed in the topological space $\mathcal{T}_p^G(M)$, then, by the compactness of $\mathcal{T}_p^G(M)$, $\bigcap_{\alpha \in I} \Delta_\alpha$ is nonempty. We now check that $\bigcap_{\alpha \in I} \Delta_\alpha$ is $\mathcal{T}_p^G$-invariant. Suppose $\Sigma \in \bigcap_{\alpha \in I} \Delta_\alpha$ and so, $\Sigma \in \Delta_\alpha$ for all α. Since each Δ_α is $\mathcal{T}_p^G$-invariant $\mathcal{T}_p^G(\Sigma) \subset \Delta_\alpha$ for each $\alpha \in I$. Hence, $\mathcal{T}_p^G(\Sigma) \subset \bigcap_{\alpha \in I} \Delta_\alpha$, which implies $\bigcap_{\alpha \in I} \Delta_\alpha$ is $\mathcal{T}_p^G$-invariant. Finally, since the intersection of closed sets in a topological space is always closed, $\bigcap_{\alpha \in I} \Delta_\alpha$ is a closed set in $\mathcal{T}_p^G(M)$. By definition of $\mathcal{C}_\Delta$, $\bigcap_{\alpha \in I} \Delta_\alpha$ is an element of $\mathcal{C}_\Delta$. This proves the claim, and, by Zorn's lemma completes the proof of statement *4*. $\square$

We next prove statement *5* holds. Arguing by contradiction, suppose $\Sigma \in \mathcal{T}_p^G(M)$ is a minimal element such that statement *5* fails to hold. In this case, there exists an $\varepsilon > 0$, a smooth, connected compact domain $W_\varepsilon \subset \Sigma$ and a sequence of points $q_n \in \Sigma$ such that there do not exist smooth, connected compact domains $W_\varepsilon(n) \subset \Sigma$ with

$$d_\Sigma(q_n, W_\varepsilon(n)) < n \quad \text{and} \quad d_{\mathcal{H}}(W_\varepsilon, i(W_\varepsilon(n)) < \varepsilon,$$

for some isometry $i \in G$.

First note that the sequence of points $q_n \in \Sigma$ is divergent in Σ and so, by the properness of Σ, is divergent in N. Let $i_n \in G$ be chosen so that $i_n(q_n) = p$ and let $\Sigma_\infty \in \mathcal{T}_p^G(\Sigma)$ be a related limit arising from the sequence of pointed surface $(i_n(\Sigma), i_n(q_n) = p)$. Since Σ is a minimal element of $\mathcal{T}_p^G(M)$, the definition of minimal $\mathcal{T}_p^G$-invariant set implies that for any $\Sigma' \in \mathcal{T}_p^G(\Sigma)$, then $\mathcal{T}_p^G(\Sigma') = \mathcal{T}_p^G(\Sigma)$ and so $\Sigma' \in \mathcal{T}_p^G(\Sigma')$. If also follows by similar reasoning that $\Sigma \in \mathcal{T}_p^G(\Sigma)$ and so $\Sigma \in \mathcal{T}_p^G(\Sigma_\infty)$.

Since $\Sigma \in \mathcal{T}_p^G(\Sigma_\infty)$, there exists a smooth, connected compact domain $W_\varepsilon(\infty) \subset \Sigma_\infty$ and an isometry $I \in G$ with $I(W_\varepsilon(\infty))$ being $\frac{\varepsilon}{2}$-close to W_ε. Suppose that the distance on Σ_∞ from p to $W_\varepsilon(\infty)$ is d_0. Since Σ_∞ is a limit of the sequence $i_n(\Sigma)$, for

[4] The intersection of any finite number of sets in $\{\Delta_\alpha\}_{\alpha \in I}$ is nonempty.

n large, there exist smooth, connected compact domains $W_\varepsilon(n) \subset i_n(\Sigma)$ of surface distance at most $2d_0$ from q_n and such that $i_n(W_\varepsilon(n))$ is $\frac{\varepsilon}{2}$-close to $W_\varepsilon(\infty)$. By the triangle inequality, $i = I \circ i_n(W_\varepsilon(n))$ is ε-close to W_ε with respect to $d_{\mathcal{H}}$. Since $d_\Sigma(q_n, W_\varepsilon(n)) \leq 2d_0$, we obtain a contradiction, thereby proving statement *5* holds. This completes the proof of Theorem 1.2.

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