ONTO INTERPOLATING SEQUENCES FOR THE DIRICHLET SPACE

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Abstract. We describe a class of "onto interpolating" sequences for the Dirichlet space and give a complete description of the analogous sequences for a discrete model of the Dirichlet space.

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1. Preliminaries

1.1. **Hardy Space Background.** In the 1950's Buck [Bu] raised the question of whether or not there exists an *infinite* subset $Z = \{z_j\}_{j=1}^{\infty}$ of $\mathbb{D}$ that is interpolating for $H^{\infty} = H^{\infty}(\mathbb{D})$, i.e. for every bounded sequence $\xi = \{\xi_j\}_{j=1}^{\infty}$ of complex numbers there is $f \in H^{\infty}(\mathbb{D})$ such that $f(z_j) = \xi_j$ for $1 \leq j < \infty$. In 1958 Carleson [C] gave an affirmative answer and moreover characterized all such interpolating sequences in the disk.

Implicit in Carleson's solution, and explicitly realized by Shapiro and Shields in 1961 [SS], is the equivalence of this problem with certain Hilbert space analogues. Let $H^2 = H^2(\mathbb{D})$ denote the classical Hardy space and $\tilde{k}_z(w) = (1-\bar{z}w)^{-1}$ its reproducing kernel. For $z \in \mathbb{D}$, the linear functional $f \to \left\langle f, \tilde{k}_z \right\rangle = f(z)$ is continuous on $H^2(\mathbb{D})$ with norm $\left\|\tilde{k}_z\right\| = \sqrt{\left\langle \tilde{k}_z, \tilde{k}_z \right\rangle} = \sqrt{\tilde{k}_z(z)} = \left(1 - |z|^2\right)^{-1/2}$. Thus the map $f \to \left\{ \left\|\tilde{k}_z\right\|^{-1} f(z_j) \right\}_{j=1}^{\infty}$ is bounded with norm 1 from H^2 to $\ell^{\infty}(Z)$. Shapiro and Shields then asked when this map takes H^2 onto (respectively into and onto) the smaller Hilbert space $\ell^2(Z)$? In terms of the restriction map $\tilde{R}f = \{f(z_j)\}_{j=1}^{\infty}$ their question is what conditions insure that $\tilde{R}$ maps H^2 onto (respectively into and onto) the Hilbert space $\ell^2(\tilde{\mu})$ where $\tilde{\mu}$ is the measure given by

$$\tilde{\mu} = \sum_{j=1}^{\infty} \tilde{\mu}_j \delta_{z_j} \text{ with}$$
$$\tilde{\mu}_j = 1 - |z_j|^2 = \tilde{k}_{z_j}(z_j)^{-1} = \left\|\tilde{k}_z\right\|^{-2}?$$

Two conditions on the sequence Z are easily seen to be necessary for R to map into and onto $\ell^2(\tilde{\mu})$. The first is a reformulation to the statement that the map is into; $\tilde{\mu}$ must be what is now called a *Carleson measure* (for the Hardy space).

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That is, there is a $C > 0$ so that for all $f \in H^2$

$$(\widetilde{Car}) \qquad \int |f|^2 \, d\tilde{\mu} \leq C \, \|f\|_{H^2}^2 .$$

The second condition is that there be a lower bound on the hyperbolic distance between distinct points of Z. We can reformulate this separation condition in terms of the normalized reproducing kernels $\tilde{K}_z = \left\| \tilde{k}_z \right\|^{-1} \tilde{k}_z$. These are unit vectors hence their inner products are at most one; the separation condition is $\exists \varepsilon > 0, \forall i, j \ i \neq j$

$$(\widetilde{Sep}) \qquad \left| \left\langle \tilde{K}_{z_i}, \tilde{K}_{z_j} \right\rangle \right| < 1 - \varepsilon.$$

The following summarizes the results of Carleson and Shapiro-Shields.

Theorem 1. *Suppose that* $Z = \{z_j\}_{j=1}^{\infty} \subset \mathbb{D}$ *and let* $\tilde{R}$ *be the restriction map* $\tilde{R}f = \{f(z_j)\}_{j=1}^{\infty}$. *Then the following conditions are equivalent:*

(1) $\tilde{R}$ maps $H^\infty(\mathbb{D})$ onto $\ell^\infty(Z)$ (*i.e.* Z is an *interpolating sequence* for $H^\infty(\mathbb{D})$),
(2) $\tilde{R}$ maps $H^2(\mathbb{D})$ into and onto $\ell^2(\tilde{\mu})$ (*i.e.* Z is an *interpolating sequence* for $H^2(\mathbb{D})$),
(3) Z satisfies $(\widetilde{Sep})$ and $\tilde{\mu}$ satisfies $(\widetilde{Car})$,
(4) Z satisfies $(\widetilde{Sep})$ and for some $C > 0$, and all arcs $I \subset \mathbb{T}$; $\tilde{\mu}(T(I)) \leq C|I|$.
(5) R maps $H^2(\mathbb{D})$ onto $\ell^2(\tilde{\mu})$.

Here $T(I)$ denotes the *tent* over I; let α_I be the midpoint of I, then $T(I)$ is the closed convex hull of I and the point $(1 - |I|)\alpha_I$. For a point w in $\mathbb{D}$ we let $T(w)$ be the tent associated with the boundary arc I_w centered at $|w|^{-1} w$ and having length $1 - |w|$.

1.2. **Dirichlet Space Background.** Marshall and Sundberg [MS] identified the crucial interplay between the spaces $H^\infty(\mathbb{D})$ and $H^2(\mathbb{D})$ that was at work here, namely $H^\infty(\mathbb{D})$ is the multiplier algebra $M_{H^2(\mathbb{D})}$ of the Hilbert space $H^2(\mathbb{D})$. They then studied the interpolation questions for the classical Dirichlet space and its multiplier algebra. At the same time overlapping work was done independently by Bishop [Bi]. The first published results were by Böe [Bo] using different proofs.

Let $B_2(\mathbb{D})$ be the Dirichlet space and $M_{B_2(\mathbb{D})}$ its multiplier algebra. We denote the reproducing kernels for $B_2(\mathbb{D})$ by $k_z(w) = -\bar{z}w \log(1 - \bar{z}w)$ and the normalized kernels by $K_z(w) = \|k_z\|^{-1} k_z(w)$. Note that $\|k_z\|^2 = -|z|^2 \log(1 - |z|^2)$ and that for z near the boundary, the only case of interest for us,

$$\|k_z\|^2 \sim -\log(1 - |z|^2).$$

We associate with $Z = \{z_j\}_{j=1}^{\infty} \subset \mathbb{D}$ the restriction map $Rf = \{f(z_j)\}_{j=1}^{\infty}$ and the measure

$$\mu = \mu_Z = \sum_{j=1}^{\infty} \mu_j \delta_{z_j} \text{ with} \tag{1.1}$$

$$\mu_j = -\log\left(1 - |z|^2\right) \sim k_{z_j}(z_j)^{-1}$$

The natural target for R is the weighted space $\ell^2(\mu_Z)$.

As before there are two conditions on the sequence Z which are easily seen to be necessary for R to map into and onto $\ell^2(\mu_Z)$. First μ_Z must be a *Carleson measure* for the the Dirichlet space. That is, there is a $C > 0$ so that for all $f \in B_2(\mathbb{D})$

$$(Car) \qquad \int |f|^2 \, d\mu_Z \le C \, \|f\|^2_{B_2(\mathbb{D})} .$$

The second condition again can be formulated using the normalized reproducing kernels: $\exists \varepsilon > 0, \forall i, j \; i \ne j$

$$(Sep) \qquad \left| \left\langle K_{z_i}, K_{z_j} \right\rangle \right| < 1 - \varepsilon.$$

A more geometric reformulation of this condition is given in (1.3) below.

The following summarizes the results of Marshall-Sundberg, Bishop, and Böe.

Theorem 2. *Suppose that $Z = \{z_j\}_{j=1}^{\infty} \subset \mathbb{D}$ and R is the restriction map. The following conditions are equivalent:*

1. *R maps $M_{B_2(\mathbb{D})}$ onto $\ell^{\infty}(Z)$ (i.e. Z is an* interpolating sequence *for $M_{B_2(\mathbb{D})}$),*
2. *R maps $B_2(\mathbb{D})$ into and onto $\ell^2(\mu_Z)$ (i.e. Z is an* interpolating sequence *for $B_2(\mathbb{D})$),*
3. *Z satisfies (Sep) and μ_Z satisfies (Car).*

This is a satisfying analog of the first three parts of Theorem 1, however there are differences. The condition $(\widetilde{Sep})$ has a straightforward formulation in the hyperbolic geometry of the disk and there is also a geometric formulation of (Sep), (1.3) below. The situation is more complicated when considering conditions $(\widetilde{Car})$ and (Car). A simple geometric description of the measures that satisfy $(\widetilde{Car})$ is given in (4) of Theorem 1 but geometric descriptions of the measures which satisfy (Car) are more complicated. In particular the condition (1.4) below, which is the analog of the condition in Theorem 1 and which is easily seen to be necessary in order for μ_Z to satisfy (Car), is not sufficient. These issues are however relatively well understood; precise statements can be found in [Bo] and [ArRoSa].

The fact that condition (5) in Theorem 1 is equivalent to the others is one of the deeper parts of that theorem and the analogous statement fails for the Dirichlet space. Bishop had noted that there are sequences for which the restriction map is onto, i.e. $\ell^2(\mu_Z) \subset R(B_2(\mathbb{D}))$, but the restriction map is not bounded; hence those sequences are not an interpolating sequences. We call sequences Z for which the restriction map is onto but not necessarily bounded *onto interpolating sequences* (for the Dirichlet space). The study of such sequences is the main theme of this paper.

1.3. **The Contents.**

1.3.1. *Dirichlet Space Interpolation.* If Z is an onto interpolating sequence then it is a consequence of the closed graph theorem that the interpolation can be done with norm control. In fact, if M is the closed subspace of $B_2(\mathbb{D})$ consisting of those f that vanish on Z, then there is a unique linear map $\Lambda : \ell^2(\mu) \to M^{\perp}$ such that $R\Lambda$ is the identity on $\ell^2(\mu)$. By the closed graph theorem Λ is continuous. Thus we see that a sequence $Z = \{z_j\}_{j=1}^{\infty}$ in the disk $\mathbb{D}$ is onto interpolating for the Dirichlet space $B_2(\mathbb{D})$ if and only if there is a positive constant C such that

$$(1.2) \qquad \text{for every sequence } \{\xi_j\}_{j=1}^{\infty} \text{ with } \left\| \left\{ \|k_{z_j}\|^{-1}_{B_2} \xi_j \right\}_{j=1}^{\infty} \right\|_{\ell^2} = 1,$$
$$\text{there is } f \in B_2(\mathbb{D}) \text{ with } \|f\|_{B_2} \le C \text{ and } f(z_j) = \xi_j,\ j \ge 1.$$

A necessary condition for (1.2) to hold is the separation condition (*Sep*). That analytic condition has a geometric reformulation in terms of the Bergman (Poincaré) metric on the disk, $\beta\left(z,w\right).$ It is shown in [MS], see [S, pg 22-23], that the sequence Z satisfies (*Sep*) if and only if $\exists c>0, \forall z_i, z_j \in Z$, $z_i \neq z_j$,

$$\beta\left(z_i,z_j\right)\geq c\left(1+\beta\left(0,z_j\right)\right). \tag{1.3}$$

Bishop showed that if Z satisfies (*Sep*) then the restriction map was onto if μ satisfies the *simple condition;* $\exists c>0$ $\forall w\in\mathbb{D}$

$$\mu(T(w))=\sum_{z\in T(w)\cap Z}\mu\left(z\right)\leq C\left(\log\frac{1}{1-\left|w\right|^2}\right)^{-1}. \tag{1.4}$$

In [Bo] Böe gave another proof of this and extended the result to $p\neq 2$. In Section 2 we improve this result by showing that the condition (1.4) can be replaced by the weaker condition (1.6).

We will use Bergman trees in our analysis. We describe them now informally, the detailed description is in [ArRoSa] or [ArRoSa2]. A Bergman tree is a subset $\mathcal{T}=\{\alpha_i\}\subset\mathbb{D}$ for which there is a positive lower bound on the hyperbolic distances between distinct points and so that for some constant C the union of hyperbolic balls, $\bigcup_i B(\alpha_i,C)$, cover $\mathbb{D}$. Each point α_i of $\mathcal{T}$, except the point closest to the origin, is connected by an edge to its predecessor α_i^-, a nearby point closer to the origin. We will assume that each $\alpha\in\mathcal{T}$ is the predecessor of exactly two other points of $\mathcal{T}$, the successors, $\alpha_\pm$. This assumption is a notational convenience; our trees automatically have an upper bound on their branching number and all our discussions extend to that case by just adding notation.

If $Z\subset\mathbb{D}$ satisfies (1.3) then there is a positive lower bound on the hyperbolic distance between distinct points of Z and, given this, it is easy to see that we can construct a Bergman tree $\mathcal{T}$ for the disk that contains Z. So, without loss of generality, we may assume that $Z\subset\mathcal{T}$. When the points of Z are regarded as elements of $\mathcal{T}$ we will often denote them with lower case Greek letters. Recall that for $\alpha\in\mathcal{T}$ we define $d(\alpha)$ to be the number of tree elements on the tree geodesic connecting α to the root o. In particular, for any α, $d(\alpha)\geq 1$.

For $\alpha\in\mathcal{T}$ the successor set,

$$S(\alpha)=\left\{\beta\in\mathcal{T}:\beta\geq\alpha\right\},$$

is the tree analog of the tent $T(\alpha)$. If μ_Z satisfies (1.4) then, regarded as a measure on $\mathcal{T}$, the measure satisfies $\exists c>0$ $\forall\alpha\in\mathcal{T}$

$$\mu(S(\alpha))=\sum_{\beta\in Z,\ \beta\geq\alpha}\mu\left(\beta\right)\leq Cd\left(\alpha\right)^{-1}. \tag{1.5}$$

We prove two results about onto interpolating sequences for the Dirichlet space. First, we show that a sequence Z whose associated measure μ_Z is *finite* is onto interpolating if Z satisfies the separation condition (1.3) and the *weak simple condition,* $\exists c>0$ $\forall\alpha\in\mathcal{T}$

$$\sum_{\substack{\beta\in Z,\ \beta\geq\alpha\\ \mu(\gamma)=0\text{ for }\alpha<\gamma<\beta}}\mu\left(\beta\right)\leq Cd\left(\alpha\right)^{-1}. \tag{1.6}$$

In particular this provides a geometric sufficient condition for interpolation that improves on the simple condition (1.4) of Bishop [Bi] - see Subsubsection 3.2 below for examples satisfying (1.6) but not (1.4) in this context.

In fact we show more in this case. We show that the interpolating functions can be taken from a special closed subspace $B_{2,Z}(\mathbb{D})$ of the Dirichlet space that we call the Böe space; and, furthermore, if μ_Z is finite and Z satisfies the separation condition (1.3) then the interpolation can be done using functions from the Böe space *if and only if* Z satisfies the weak simple condition. We also note that the role of (1.4) is elucidated by considering the Böe space; for Z satisfying (1.3), the restriction map R takes the Böe space $B_{2,Z}(\mathbb{D})$ *into* $\ell^2(\mu)$ if (1.4) holds, while conversely, if R maps into $\ell^2(\mu)$, then a weaker version of (1.4) is necessary. See Subsubsection 2.1.4 for this.

For our second result, we suppose that Z satisfies the separation condition (1.3), the weak simple condition (1.6), and the following tree-like condition: there is $\beta \in (1 - c/2, 1)$ where c is the constant in (1.3) such that

$$z_j \in T(z_k) \text{ whenever } |z_j| \geq |z_k|, \left|z_j - \frac{z_k}{|z_k|}\right| \leq \left(1 - |z_k|^2\right)^\beta \text{ and } z_j, z_k \in Z. \tag{1.7}$$

Under these conditions Z is an onto interpolating sequence for the Dirichlet space $B_2(\mathbb{D})$ even if the measure μ is *infinite*. We construct Cantor-like examples of such Z, thus demonstrating that onto interpolation for the Dirichlet space can hold even when $\|\mu_Z\| = \infty$, thus resolving a question raised by Bishop [Bi].

1.3.2. *Relations Between the Conditions.* In the final section we present two examples to help clarify the relationships between the various conditions we consider. We give an example of an onto interpolating subtree with infinite measure. There is also an example of a sequence Z satisfying the strong separation condition (1.3), with $\|\mu_Z\| < \infty$ and satisfying the weak simple condition (1.6), but not the simple condition (1.4).

2. Dirichlet Space Interpolation

2.1. **The Interpolation Theorem.** In fact the hypotheses (1.3), (1.6) and $\|\mu\| < \infty$ yield a stronger onto interpolation which in turn implies both (1.3) and (1.6).

Suppose Z is given and fixed. For $w \in Z$ we denote by φ_w the function introduced by Böe in [Bo] in his work on interpolation. By construction φ_w is a function in $B_2(\mathbb{D})$ which is essentially 1 on the tent $T(w)$ and small away from that region. The details of the construction and properties are recalled in Lemma 2 below. Actually there are various choices in Lemmas 1 and 2 below. We assume that allowable choices have been made once and for all. Also, we further require that the chosen parameters satisfy

$$\begin{gathered} \beta < \alpha < \frac{2\beta\eta}{(\eta+1)} \\ s > \frac{(\alpha-\rho)}{(\rho-\beta)} \end{gathered} \tag{2.1}$$

as we need that in our proof of the necessity of (1.6) for a certain type of interpolation, Proposition 1 below.

We define the *Böe space,* $B_{2,Z}(\mathbb{D})$, to be the closed linear span in $B_2(\mathbb{D})$ of the functions $\{\varphi_w\}_{w\in Z}$. It follows from (2.45) in Lemma 4 and Proposition 1 below that for an appropriate cofinite subset $\{\zeta_j\}_{j=1}^{\infty}$ of Z we have

$$B_{2,Z}(\mathbb{D}) = \left\{ \varphi = \sum_{j=1}^{\infty} a_j \varphi_{\zeta_j} : \sum_{j=1}^{\infty} |a_j|^2 \mu(\zeta_j) < \infty \right\}, \tag{2.2}$$

$$\|\varphi\|_{B_{2,Z}(\mathbb{D})} \approx \left\| \{a_j\}_{j=1}^{\infty} \right\|_{\ell^2(\mu)}.$$

We will say Z is an onto interpolating sequence for $B_{2,Z}(\mathbb{D})$ if it is an onto interpolating sequence for the Dirichlet space $B_2(\mathbb{D})$ and if, further, the interpolating functions can all be selected from $B_{2,Z}(\mathbb{D})$.

Theorem 3. *Let $Z \subset \mathbb{D}$ and suppose $\|\mu_Z\| < \infty$. Then Z is an onto interpolating sequence for the Böe space $B_{2,Z}(\mathbb{D})$ if and only if both the separation condition (1.3) and the weak simple condition (1.6) hold.*

Corollary 1. *Let $Z \subset \mathbb{D}$ and suppose $\|\mu_Z\| < \infty$. If Z satisfies the separation condition (1.3) and the weak simple condition (1.6) then Z is an onto interpolating sequence for the Dirichlet space $B_2(\mathbb{D})$.*

We will need the following lemma from [MS], (see also [ArRoSa2]). Let c be the constant in (1.3). For $w \in \mathbb{D}$ and $1 - \frac{c}{2} < \beta < 1$, define

$$V_w = V_w^\beta = \left\{ z \in \mathbb{D} : |z - w^*| \le (1 - |w|^2)^\beta \right\},$$

where $w^* = \frac{w}{|w|}$ is radial projection of w onto the circle $\mathbb{T} = \partial\mathbb{D}$.

Lemma 1. *Suppose the separation condition in (1.3) holds. Then for every β satisfying $1 - \frac{c}{2} < \beta < 1$ there is $\eta > \beta\eta > 1$ such that if $V_{z_i}^\beta \cap V_{z_j}^\beta \ne \phi$ and $|z_j| \ge |z_i|$, then $z_i \notin V_{z_j}^\beta$ and*

$$(1 - |z_j|) \le (1 - |z_i|)^\eta. \tag{2.3}$$

We have the following useful consequence of Lemma 1. If $\sigma > 0$ and μ satisfies (1.6), then

$$\sum_{z_j \ge z_k} (1 - |z_j|)^\sigma \le C_\sigma (1 - |z_k|)^\sigma. \tag{2.4}$$

Indeed, if $\mathcal{G}_1(z_k) = \left\{\alpha_m^1\right\}$ consists of the minimal elements in $[S(z_k) \setminus \{z_k\}] \cap Z$, $\mathcal{G}_2(z_k) = \cup_m \mathcal{G}_1(\alpha_m^1)$, etc., we have using (2.3) and (1.6),

$$\begin{aligned}
\sum_{z_j \ge z_k} (1 - |z_j|)^\sigma &= (1 - |z_k|)^\sigma + \sum_{\ell=1}^{\infty} \sum_{\beta \in \mathcal{G}_{\ell-1}(z_k)} \sum_{\alpha \in \mathcal{G}_1(\beta)} (1 - |\alpha|)^\sigma \\
&\le (1 - |z_k|)^\sigma + C_\delta \sum_{\ell=1}^{\infty} \sum_{\beta \in \mathcal{G}_{\ell-1}(z_k)} \sum_{\alpha \in \mathcal{G}_1(\beta)} (1 - |\alpha|)^{\sigma-\delta} \left(\log \frac{1}{1 - |\alpha|} \right)^{-1} \\
&\le (1 - |z_k|)^\sigma + C_\delta \sum_{\ell=1}^{\infty} \sum_{\beta \in \mathcal{G}_{\ell-1}(z_k)} (1 - |\beta|)^{(\sigma-\delta)\eta} C \left(\log \frac{1}{1 - |\beta|} \right)^{-1} \\
&\le (1 - |z_k|)^\sigma + C_\delta \sum_{z_j > z_k} (1 - |z_j|)^{(\sigma-\delta)\eta}.
\end{aligned}$$

Now we can choose $\delta > 0$ so small that $(\sigma - \delta)\eta - \sigma = \theta > 0$, and R such that $C_\delta (1-R)^\theta = \frac{1}{2}$, so that for $|z_k| \geq R$ we have

$$C_\delta \sum_{z_j > z_k} (1-|z_j|)^{(\sigma-\delta)\eta} \leq \left\{ C_\delta \sup_{j\geq 1} (1-|z_j|)^\theta \right\} \sum_{z_j > z_k} (1-|z_j|)^\sigma \leq \frac{1}{2} \sum_{z_j > z_k} (1-|z_j|)^\sigma .$$

Thus $\sum_{z_j \geq z_k} (1-|z_j|)^\sigma \leq 2(1-|z_k|)^\sigma$, proving (2.4) for $|z_k| \geq R$. Now the number of points z_k in the ball $B(0,R)$ depends only on R and the separation constant c in (1.3), and it is now easy to obtain (2.4) in general.

We will also use a lemma from [Bo] which constructs a holomorphic function $\varphi_w = \Gamma_s g_w$, where Γ_s is the projection operator below, that is close to 1 on the Carleson region associated to a point $w \in \mathbb{D}$, and decays appropriately away from the Carleson region. Again let $1 - \frac{c}{2} < \beta < 1$ where c is as in (1.3). Given $\beta < \rho < \alpha < 1$, we will use the cutoff function $c_{\rho,\alpha}$ defined by

$$c_{\rho,\alpha}(\gamma) = \begin{cases} 0 & \text{for} \quad \gamma < \rho \\ \frac{\gamma-\rho}{\alpha-\rho} & \text{for} \quad \rho \leq \gamma \leq \alpha \\ 1 & \text{for} \quad \alpha < \gamma \end{cases} . \tag{2.5}$$

Lemma 2. *(Lemma 4.1 in* [Bo]*) Suppose $s > -1$, c is as in (1.3), and $1 - \frac{c}{2} < \beta < 1$. There are β_1, ρ and α satisfying $\beta < \beta_1 < \rho < \alpha < 1$ such that for every $w \in \mathbb{D}$, we can find a function g_w so that*

$$\varphi_w(z) = \Gamma_s g_w(z) = \int_{\mathbb{D}} \frac{g_w(\zeta)(1-|\zeta|^2)^s}{(1-\overline{\zeta}z)^{1+s}} d\zeta$$

satisfies

$$\left\{ \begin{array}{lcll} \varphi_w(w) & = & 1 & \\ \varphi_w(z) & = & c_{\rho,\alpha}(\gamma_w(z)) + O\left(\left(\log \frac{1}{1-|w|^2}\right)^{-1}\right), & z \in V_w^\beta \\ |\varphi_w(z)| & \leq & C\left(\log \frac{1}{1-|w|^2}\right)^{-1} \left(1-|w|^2\right)^{(\rho-\beta_1)(1+s)}, & z \notin V_w^{\beta_1} \end{array} \right. , \tag{2.6}$$

where $\gamma_w(z)$ is defined by

$$|z - w^*| = \left(1-|w|^2\right)^{\gamma_w(z)},$$

w^* is radial projection of w to ∂D, and $c_{\rho,\alpha}$ is as in (2.5). Furthermore we have the estimate

$$\int_{\mathbb{D}} |g_w(\zeta)|^2 d\zeta \leq C \left(\log \frac{1}{1-|w|^2}\right)^{-1} . \tag{2.7}$$

2.1.1. *Sufficiency.* Order the points $\{z_j\}_{j=1}^\infty$ so that $1-|z_{j+1}| \leq 1-|z_j|$ for $j \geq 1$. We now define a "forest structure" on the index set $\mathbb{N}$ by declaring that j is a child of i (or that i is a parent of j) provided that

$$\begin{aligned} & i < j, \\ & V_{z_j} \subset V_{z_i}, \\ & V_{z_j} \subsetneq V_{z_k} \text{ for } i < k < j. \end{aligned} \tag{2.8}$$

Note if we have competing indices i and i' with $V_{z_j} \subset V_{z_i} \cap V_{z_{i'}}$ then the child j chooses the "nearest" parent i. We define a partial order associated with this parent-child relationship by declaring that j is a successor of i (or that i is a predecessor of j) if there is a "chain" of indices $\{i = k_1, k_2, ..., k_m = j\} \subset \mathbb{N}$ such that $k_{\ell+1}$ is a child of k_ℓ for $1 \leq \ell < m$. Under this partial ordering, $\mathbb{N}$ decomposes into a disjoint union of trees. Thus associated to each index $\ell \in \mathbb{N}$, there is a unique tree containing ℓ and, unless ℓ is the root of the tree, a unique parent $P(\ell)$ of ℓ in that tree. Denote by $\mathcal{G}_\ell$ the unique geodesic joining the root of the tree to ℓ. We will usually identify ℓ with z_ℓ and thereby transfer the forest structure $\mathcal{F}$ to Z as well.

Remark 1. *It is easy to see that one may discard finitely many points from the sequence $Z = \{z_j\}_{j=1}^\infty$ without loss of generality. Indeed, if Z is an onto interpolating sequence and $w \notin Z$, then we may choose $f \in B_2(\mathbb{D})$ such that $f(z_j) = \begin{cases} 1 & \text{if} \quad j = 1 \\ 0 & \text{if} \quad j > 1 \end{cases}$, and g linear such that $g(z_1) = 0$. Then $h = fg \in B_2(\mathbb{D})$ and there is m such that $H(z) = \frac{h(z)}{(z-w)^m} \in B_2(\mathbb{D})$, $H(w) \neq 0$ and $H(z_j) = 0$ for $j \geq 1$. It is now immediate that $Z \cup \{w\}$ is onto interpolating.*

We will need to discard finitely many points from Z so that

$$\|\mu\| = \sum_{j=1}^\infty \mu(z_j) = \sum_{j=1}^\infty \left(\log \frac{1}{1-|z_j|^2}\right)^{-1} < \varepsilon. \tag{2.9}$$

This can be achieved using our assumption that $\|\mu\| < \infty$. With this done we now suppose that the sequence $\{z_j\}_{j=1}^J$ is finite, and obtain an appropriate estimate independent of $J \geq 1$. Fix $\alpha, s > -1$ and a sequence of complex numbers $\{\xi_j\}_{j=1}^J$ in $\ell^2(\mu)$ where

$$\left\|\{\xi_j\}_{j=1}^J\right\|_{\ell^2(\mu)} = \left\|\left\{\frac{\xi_j}{\left\|k_{z_j}^{\alpha,2}\right\|_{B_2}}\right\}_{j=1}^J\right\|_{\ell^2}.$$

We will define a function $\varphi = \mathcal{S}\xi$ on the disk $\mathbb{D}$ by

$$\varphi(z) = \mathcal{S}\xi(z) = \sum_{j=1}^J a_j \varphi_{z_j}(z), \qquad z \in \mathbb{D}, \tag{2.10}$$

that will be our candidate for the interpolating function of ξ. We follow the inductive scheme of Böe that addresses the main difficulty in interpolating holomorphic functions, namely that the building blocks φ_{z_j} take on essentially *all* values in $[0,1]$ (rather than just 0 and 1 as in the tree analogue) on the sequence Z.

Recall that Pz_j denotes the parent of z_j in the forest structure $\mathcal{F}$ and that $\mathcal{G}_\ell$ is the geodesic from the root to z_ℓ in the tree containing z_ℓ. In order to define the coefficients a_j we will use the doubly indexed sequence $\{\beta_{i,j}\}$ of numbers given by

$$\beta_{i,j} = \varphi_{Pz_j}(z_i). \tag{2.11}$$

We consider separately the indices in each tree of the forest $\{1, 2, ..., J\}$, and define the coefficients inductively according to the *natural* ordering of the integers. So let $\mathcal{Y}$ be a tree in the forest $\{1, 2, ..., J\}$ with root k_0. Define $a_{k_0} = \xi_{k_0}$. Suppose that $k \in \mathcal{Y} \setminus \{k_0\}$ and that the coefficients a_j have been defined for $j \in \mathcal{Y}$ and $j < k$. Let

$$\mathcal{G}_k = [k_0, k] = \{k_0, k_1, ..., k_{m-1}, k_m = k\}$$

be the geodesic $\mathcal{G}_k$ in $\mathcal{Y}$ joining k_0 to k, and note that $\mathcal{G}_k = \mathcal{G}_{k_{m-1}} \cup \{k\}$. Define

$$f_k(z) = f_{k_m}(z) = \sum_{i=1}^{m} a_{k_i} \varphi_{z_{k_i}}(z) = f_{k_{m-1}}(z) + a_k \varphi_{z_k}(z)$$

and

$$\omega_k = f_{k_{m-1}}(z_k) = \sum_{i=1}^{m} a_{k_{i-1}} \varphi_{z_{k_{i-1}}}(z_k) = \sum_{i=1}^{m} \beta_{k,i} a_{k_{i-1}}, \qquad k \geq 1.$$

Then define the coefficient a_k by

$$a_k = \xi_k - \omega_k, \qquad k \geq 1. \tag{2.12}$$

This completes the inductive definition of the sequence $\{a_k\}_{k\in\mathcal{Y}}$, and hence defines the entire sequence $\{a_i\}_{i=1}^{J}$.

We first prove the following $\ell^2(d\mu)$ estimate for the sequence $\left\{a_j^m\right\}_{j=1}^{J}$ given in terms of the data $\left\{\xi_j^m\right\}_{j=1}^{J}$ by the scheme just introduced. This is the difficult step in the proof of sufficiency.

Lemma 3. *The sequence $\{a_i\}_{i=1}^{J}$ constructed in (2.12) above satisfies*

$$\left\|\{a_j\}_{j=1}^{J}\right\|_{\ell^2(d\mu)} \leq C \left\|\{\xi_j\}_{j=1}^{J}\right\|_{\ell^2(d\mu)}. \tag{2.13}$$

Proof: Without loss of generality, we may assume for the purposes of this proof that the forest of indices $\{j\}_{j=1}^{J}$ is actually a single tree $\mathcal{Y}$. Now fix ℓ. At this point it will be convenient for notation to momentarily relabel the points $\{z_j\}_{j\in\mathcal{G}_\ell} = \{z_{k_0}, z_{k_1}, ..., z_{k_m}\}$ as $\{z_0, z_1, ..., z_m\}$, and similarly relabel $\{a_0, a_1, ..., a_m\}$, $\{\xi_0, \xi_1, ..., \xi_m\}$ and $\{\beta_0, \beta_1, ..., \beta_m\}$ so that

$$a_k = \xi_k - \sum_{i=1}^{k} \beta_{k,i} a_{i-1}, \qquad 0 \leq k \leq \ell.$$

We also have $d\mu(j) = \left(\log \frac{1}{1-|z_j|^2}\right)^{-1}$ where z_j now denotes the point z_{k_j} in the ball corresponding to k_j before the relabelling. In other words, we are restricting attention to the geodesic $\mathcal{G}_\ell$ and relabeling sequences so as to conform to the ordering in the geodesic. We also rewrite $f_k(z)$ and ω_k as

$$f_k(z) = \sum_{i=1}^{k} a_i \varphi_{z_i}(z) = f_{k-1}(z) + a_k \varphi_{z_k}(z)$$

and

$$\omega_k = f_{k-1}(z_k) = \sum_{i=1}^{k} a_{i-1} \varphi_{z_{i-1}}(z_k) = \sum_{i=1}^{k} \beta_{k,i} a_{i-1}, \qquad k \geq 1. \tag{2.14}$$

so that the coefficients a_k are given by

$$\begin{aligned} a_0 &= \xi_0, \\ a_k &= \xi_k - \omega_k, \qquad k \geq 1. \end{aligned} \tag{2.15}$$

We now claim that

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_k \end{pmatrix} = \begin{bmatrix} b_{1,1} & 0 & \cdots & 0 \\ b_{2,1} & b_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,1} & b_{k,2} & \cdots & b_{k,k} \end{bmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{k-1} \end{pmatrix}, \qquad 1 \leq k \leq \ell, \tag{2.16}$$

where

$$\begin{aligned} b_{i,j} &= 0, \qquad i < j, \\ b_{i,i} &= \beta_{i,i}, \\ b_{i,j} &= b_{i-1,j}^* - b_{i-1,j}\beta_{i,i}, \qquad i > j, \end{aligned} \tag{2.17}$$

and the $b_{i,j}^*$ are defined in the following calculations. We also claim that the $b_{i,j}$ are bounded:

$$|b_{i,j}| \leq C. \tag{2.18}$$

For this we will use the estimate (see Lemma 5 below)

$$|\varphi_w'(z)| \leq \left(1 - |w|^2\right)^{-\alpha}, \qquad z \in \mathbb{D}. \tag{2.19}$$

Note first that

$$b_{1,1} = \beta_{1,1} = \varphi_{z_0}(z_1)$$

since then (2.14) and (2.15) yield

$$b_{1,1}\xi_0 = \omega_1,$$

which is (2.16) for $k = 1$. We also have (2.18) for $1 \leq j \leq i = 1$ since (2.6) yields

$$|b_{1,1}| \leq 1 + \lambda(z_0),$$

where we have introduced the convenient notation

$$\lambda(z_j) = \left(\log \frac{1}{1 - |z_j|^2}\right)^{-1}.$$

We now define a function $b_{1,1}(z)$ by

$$b_{1,1}(z) = \varphi_{z_0}(z),$$

i.e. we replace z_1 by z throughout the formula for $b_{1,1}$. If we then define

$$b_{1,1}^* = b_{1,1}(z_2) = \varphi_{z_0}(z_2),$$

we readily obtain

$$\begin{aligned} b_{2,1} &= b_{1,1}^* - b_{1,1}\beta_{2,2} = \varphi_{z_0}(z_2) - \varphi_{z_0}(z_1)\varphi_{z_1}(z_2), \\ b_{2,2} &= \beta_{2,2} = \varphi_{z_1}(z_2). \end{aligned}$$

Indeed, from (2.14), (2.15) and the equality $\xi_1 = a_1 + \omega_1 = a_1 + \varphi_{z_0}(z_1)a_0$, we have

$$\begin{aligned} b_{2,1}\xi_0 + b_{2,2}\xi_1 &= [\varphi_{z_0}(z_2) - \varphi_{z_0}(z_1)\varphi_{z_1}(z_2)]a_0 + \varphi_{z_1}(z_2)[a_1 + \varphi_{z_0}(z_1)a_0] \\ &= \varphi_{z_0}(z_2)a_0 + \varphi_{z_1}(z_2)a_1 \\ &= \omega_2, \end{aligned}$$

which proves (2.16) for $k = 2$. We also have (2.18) for $1 \leq j \leq i = 2$ since the bound

$$|b_{2,2}| \leq 1 + \lambda(z_1)$$

is obvious from (2.6), and the bound for $b_{2,1}$ follows from (2.6), (2.19) and Lemma 1:

$$\begin{aligned}|b_{2,1}| &\leq |\varphi_{z_0}(z_2) - \varphi_{z_0}(z_1)| + |\varphi_{z_0}(z_1)| \, |1 - \varphi_{z_1}(z_2)| \\ &\leq \left|\varphi'_{z_0}(\zeta_0)\right| |z_2 - z_1| + (1 + \lambda(z_0))(1 + \lambda(z_1)),\end{aligned}$$

and since

$$\begin{aligned}\left|\varphi'_{z_0}(\zeta_0)\right| |z_2 - z_1| &\leq \left(1 - |z_0|^2\right)^{-\alpha} |z_2 - z_1| \\ &\leq \left(1 - |z_0|^2\right)^{-\alpha} \left(1 - |z_1|^2\right)^{\beta} \\ &\leq \left(1 - |z_0|^2\right)^{\beta\eta - \alpha},\end{aligned}$$

we obtain

$$|b_{2,1}| \leq \left(1 - |z_0|^2\right)^{\beta\eta - \alpha} + e^{\lambda(z_0) + \lambda(z_1)}. \tag{2.20}$$

We now define functions $b_{2,1}(z)$ and $b_{2,2}(z)$ by

$$\begin{aligned}b_{2,1}(z) &= \varphi_{z_0}(z) - b_{1,1}\varphi_{z_1}(z), \\ b_{2,2}(z) &= \varphi_{z_1}(z),\end{aligned}$$

i.e. we replace z_2 by z throughout the formulas for $b_{2,1}$ and $b_{2,2}$. If we then set

$$\begin{aligned}b^*_{2,1} &= b_{2,1}(z_3), \\ b^*_{2,2} &= b_{2,2}(z_3),\end{aligned}$$

we obtain as above that

$$\begin{aligned}b_{3,1} &= b^*_{2,1} - b_{2,1}\beta_{3,3} = [\varphi_{z_0}(z_3) - b_{1,1}\varphi_{z_1}(z_3)] - [\varphi_{z_0}(z_2) - b_{1,1}\varphi_{z_1}(z_2)] \varphi_{z_2}(z_3), \\ b_{3,2} &= b^*_{2,2} - b_{2,2}\beta_{3,3} = \varphi_{z_1}(z_3) - \varphi_{z_1}(z_2) \varphi_{z_2}(z_3), \\ b_{3,3} &= \beta_{3,3} = \varphi_{z_2}(z_3),\end{aligned}$$

which proves (2.16) for $k = 3$. Moreover, we again have (2.18) for $1 \leq j \leq i = 3$. Indeed,

$$|b_{3,3}| \leq 1 + \lambda(z_2),$$

and the arguments used above to obtain (2.20) show that

$$|b_{3,2}| \leq \left(1 - |z_1|^2\right)^{\beta\eta - \alpha} + e^{\lambda(z_1) + \lambda(z_2)}.$$

Finally,

$$\begin{aligned}|b_{3,1}| &\leq \left|b^*_{2,1} - b_{2,1}\right| + |b_{2,1}| \, |1 - \beta_{3,3}| \\ &\leq \left\{\left|\varphi'_{z_0}(\zeta_0)\right| + |b_{1,1}| \left|\varphi'_{z_1}(\zeta_1)\right|\right\} |z_2 - z_3| \\ &\quad + \left(1 - |z_0|^2\right)^{\beta\eta - \alpha} + e^{\lambda(z_0) + \lambda(z_1) + \lambda(z_2)},\end{aligned}$$

and since

$$\begin{aligned}
&\left\{\left|\varphi_{z_0}'(\zeta_0)\right| + |b_{1,1}|\left|\varphi_{z_1}'(\zeta_1)\right|\right\}|z_2 - z_3| \\
&\quad\leq \left\{\left(1-|z_0|^2\right)^{-\alpha} + (1+\lambda(z_0))\left(1-|z_1|^2\right)^{-\alpha}\right\}|z_2-z_3| \\
&\quad\leq \left\{\left(1-|z_1|^2\right)^{-\frac{\alpha}{\eta}} + (1+\lambda(z_0))\left(1-|z_1|^2\right)^{-\alpha}\right\}\left(1-|z_2|^2\right)^{\beta} \\
&\quad\leq \left\{(1+A\lambda(z_0))\left(1-|z_1|^2\right)^{-\alpha}\right\}\left(1-|z_1|^2\right)^{\beta\eta}, \\
&\quad\leq (1+A\lambda(z_0))\left(1-|z_1|^2\right)^{\beta\eta-\alpha},
\end{aligned}$$

for a large constant A, we have

$$|b_{3,1}| \leq e^{A\lambda(z_0)}\left(1-|z_1|^2\right)^{\beta\eta-\alpha} + e^{\lambda(z_0)+\lambda(z_1)+\lambda(z_2)}.$$

Continuing in this way with

$$\begin{aligned}
b_{i,j}(z) &= b_{i-1,j}(z) - b_{i-1,j}\varphi_{z_{i-1}}(z), \\
b_{i,j} &= b_{i,j}(z_i), \\
b_{i,j}^* &= b_{i,j}(z_{i+1}),
\end{aligned} \tag{2.21}$$

we can prove (2.16) and (2.18) by induction on k and i (see below). The bound C in (2.18) will use the fact that

$$\lambda(z_0) + \lambda(z_1) + \lambda(z_2) + ... \leq C\lambda(z_0). \tag{2.22}$$

To see (2.22) we use Lemma 1.

Now if $\mathcal{G}_\ell = [k_0, k_1, ..., k_{m-1}, k_m]$, then by applying (2.3) repeatedly, we obtain

$$\left(1-|z_{k_i}|^2\right) \leq \left(1-|z_{k_0}|^2\right)^{\eta^i},$$

and so combining these estimates we have

$$\begin{aligned}
\lambda(z_0) + \lambda(z_1) + \lambda(z_2) + ... &\leq C\sum_{i\in\mathcal{G}_\ell\setminus\{k_0\}}\left(\log\frac{1}{1-\left|z_{P(i)}\right|^2}\right)^{-1} \\
&\leq C\left(\sum_{j=0}^{m-1}\eta^{-j}\right)\left(\log\frac{1}{1-|z_{k_0}|^2}\right)^{-1} \\
&\leq C_\eta\left(\log\frac{1}{1-|z_{k_0}|^2}\right)^{-1} = C_\eta\lambda(z_0)
\end{aligned} \tag{2.23}$$

since $\eta > 1$, which yields (2.22).

We now give the induction details for proving (2.16) and (2.18). The proof of (2.16) is straightforward by induction on k, so we concentrate on proving (2.18) by induction on i. If we denote the i^{th} row

$$\left[\begin{array}{ccccccc} b_{i,1} & b_{i,2} & \cdots & b_{i,i} & 0 & \cdots & 0 \end{array}\right]$$

of the matrix in (2.16) by $\mathbf{B}_i$, the corresponding row of starred components

$$\left[\begin{array}{ccccccc} b_{i,1}^* & b_{i,2}^* & \cdots & b_{i,i}^* & 0 & \cdots & 0 \end{array}\right]$$

by $\mathbf{B}_i^*$, and the row having all zeroes except a one in the i^{th} place by $\mathbf{E}_i$, then we have the recursion formula

$$\begin{aligned}\mathbf{B}_i &= \mathbf{B}_{i-1}^* - \mathbf{B}_{i-1} + (1-\beta_{i,i})\,\mathbf{B}_{i-1} + \beta_{i,i}\mathbf{E}_i \\ &= \{(1-\beta_{i,i})\,\mathbf{B}_{i-1} + \beta_{i,i}\mathbf{E}_i\} - \left(\mathbf{B}_{i-1} - \mathbf{B}_{i-1}^*\right)\end{aligned} \tag{2.24}$$

which expresses $\mathbf{B}_i$ as a "convex combination" of the previous row and the unit row $\mathbf{E}_i$, minus the difference of the previous row and its starred counterpart. In terms of the components of the rows, we have

$$b_{i,j} = \left[b_{i-1,j}^* - b_{i-1,j}\right] + (1-\beta_{i,i})\,b_{i-1,j} + \beta_{i,i}\delta_{i,j}. \tag{2.25}$$

For a large constant A that will be chosen later so that the induction step works, we prove the following estimate by induction on i:

$$|b_{i,j}| \leq e^{A\{\lambda(z_{j-1})+\ldots+\lambda(z_{i-1})\}}, \qquad i \geq j. \tag{2.26}$$

The initial case $i = j$ follows from

$$|b_{j,j}| = \left|\varphi_{z_{j-1}}(z_j)\right| \leq 1 + \lambda(z_{j-1}) \leq e^{\lambda(z_{j-1})}.$$

Now (2.21) yields

$$b_{i,j}'(z) = b_{i-1,j}'(z) - b_{i-1,j}\varphi_{z_{i-1}}'(z),$$

and so by the induction assumption for indices smaller than i, we have from (2.19) that

(2.27)

$$\begin{aligned}\left\|b_{i,j}'\right\|_{L^\infty} &\leq |b_{i-1,j}|\left(1-|z_{i-1}|^2\right)^{-\alpha} + \left\|b_{i-1,j}'\right\|_{L^\infty} \\ &\leq |b_{i-1,j}|\left(1-|z_{i-1}|^2\right)^{-\alpha} + |b_{i-2,j}|\left(1-|z_{i-2}|^2\right)^{-\alpha} + \left\|b_{i-2,j}'\right\|_{L^\infty} \\ &\vdots \\ &\leq \left\{\sup_{j\leq k\leq i-1}|b_{k,j}|\right\}\left[\left(1-|z_{i-1}|^2\right)^{-\alpha} + \ldots + \left(1-|z_j|^2\right)^{-\alpha}\right] + \left(1-|z_{j-1}|^2\right)^{-\alpha} \\ &\leq e^{A\{\lambda(z_{j-1})+\ldots+\lambda(z_{i-2})\}}\left[\left(1-|z_{i-1}|^2\right)^{-\alpha} + \ldots + \left(1-|z_{j-1}|^2\right)^{-\alpha}\right] \\ &\leq e^{A\{\lambda(z_{j-1})+\ldots+\lambda(z_{i-2})\}}\left[\left(1-|z_{i-1}|^2\right)^{-\alpha} + \ldots + \left(1-|z_{i-1}|^2\right)^{-\frac{\alpha}{\eta^{i-j}}}\right],\end{aligned}$$

where the last line uses (2.3). Thus we have from (2.27), (2.25) and (2.21),

(2.28)

$$\begin{aligned}|b_{i,j}| &\leq \left|b_{i-1,j}^* - b_{i-1,j}\right| + |(1-\beta_{i,i})\,b_{i-1,j}| \\ &\leq |b_{i-1,j}(z_{i+1}) - b_{i-1,j}(z_i)| + |1-\beta_{i,i}|\,|b_{i-1,j}| \\ &\leq \left\|b_{i,j}'\right\|_{L^\infty}|z_{i+1}-z_i| + (1+\lambda(z_{i-1}))\,|b_{i-1,j}| \\ &\leq e^{A\{\lambda(z_{j-1})+\ldots+\lambda(z_{i-2})\}}\times \\ &\left\{1+\lambda(z_{i-1}) + \left[\left(1-|z_{i-1}|^2\right)^{-\alpha} + \ldots + \left(1-|z_{i-1}|^2\right)^{-\frac{\alpha}{\eta^{i-j}}}\right]\left(1-|z_{i-1}|^2\right)^{\beta\eta}\right\},\end{aligned}$$

upon using the inequality $|z_{i+1}-z_i| \leq \left(1-|z_i|^2\right)^{\beta} \leq \left(1-|z_{i-1}|^2\right)^{\beta\eta}$, which follows from Lemma 1.

Finally we use the inequality (see below for a proof)

$$\left(1-|z_{i-1}|^2\right)^{-\alpha} + ... + \left(1-|z_{i-1}|^2\right)^{-\frac{\alpha}{\eta^{i-j}}} \leq \left\{C_\eta \left(1-|z_{i-1}|^2\right)^{-\alpha} + 2\,(i-j+1)\right\} \tag{2.29}$$

to obtain that

$$\begin{aligned} &\left[\left(1-|z_{i-1}|^2\right)^{-\alpha} + ... + \left(1-|z_{i-1}|^2\right)^{-\frac{\alpha}{\eta^{i-j}}}\right]\left(1-|z_{i-1}|^2\right)^{\beta\eta} \\ &\quad \leq 2\left\{C_\eta\left(1-|z_{i-1}|^2\right)^{\beta\eta-\alpha} + 2i\left(1-|z_{i-1}|^2\right)^{\beta\eta}\right\} \\ &\quad \leq (A-1)\left(\log\frac{1}{1-|z_{i-1}|^2}\right)^{-1} \\ &\quad = (A-1)\,\lambda\,(z_{i-1})\,, \end{aligned} \tag{2.30}$$

for all i if A is chosen large enough. With such a choice of A, (2.28) yields

$$\begin{aligned} |b_{i,j}| &\leq e^{A\{\lambda(z_{j-1})+...+\lambda(z_{i-2})\}}\left\{1+\lambda\,(z_{i-1}) + (A-1)\,\lambda\,(z_{i-1})\right\} \\ &\leq e^{A\{\lambda(z_{j-1})+...+\lambda(z_{i-2})+\lambda(z_{i-1})\}}, \end{aligned}$$

which proves (2.26), and hence (2.18) by (2.22). To see (2.29), we rewrite it as

$$\sum_{\ell=0}^{N} R^{\eta^{-\ell}} \leq C_\eta R + 2N + 2,$$

and to prove this, note that for $R^{\eta^{-\ell}} > 2$ the ratio of the consecutive terms $R^{\eta^{-(\ell+1)}}$ and $R^{\eta^{-\ell}}$ is $\left(R^{\eta^{-\ell}}\right)^{1-\eta} < 2^{1-\eta}$, i.e. this portion of the series is supergeometric. Thus we have

$$\begin{aligned} \sum_{\ell=0}^{N} R^{\eta^{-\ell}} &\leq \sum_{\ell\geq 0:R^{\eta^{-\ell}}>2} R^{\eta^{-\ell}} + \sum_{\ell\leq N:R^{\eta^{-\ell}}\leq 2} R^{\eta^{-\ell}} \\ &\leq R\sum_{j=0}^{\infty}\left(2^{1-\eta}\right)^j + 2\,(N+1)\,. \end{aligned}$$

We now claim the following crucial property. Recall that $P\,(m) = m-1$. If $\sigma > 0$ and $\gamma_{m-1}\,(z_m) > \alpha+\sigma$, i.e. $z_m \in V_{z_{m-1}}^{\alpha+\sigma}$, then

$$|b_{i,j}| \leq C\left(1-|z_{m-1}|^2\right)^{\sigma} \text{ for all } j < m \leq i. \tag{2.31}$$

We first note that from (2.19), we have for $z_m \in V_{z_{m-1}}^{\alpha+\sigma}$,

$$\begin{aligned}\beta_{m,m} &= \varphi_{z_{m-1}}(z_m) = \varphi_{z_{m-1}}(z_{m-1}) + \left[\varphi_{z_{m-1}}(z_m) - \varphi_{z_{m-1}}(z_{m-1})\right] \\ &= 1 + O\left(\left(1-|z_{m-1}|^2\right)^{-\alpha}|z_m - z_{m-1}|\right) \\ &= 1 + O\left(\left(1-|z_{m-1}|^2\right)^{-\alpha}\left(1-|z_{m-1}|^2\right)^{\alpha+\sigma}\right) \\ &= 1 + O\left(\left(1-|z_{m-1}|^2\right)^{\sigma}\right).\end{aligned} \tag{2.32}$$

From (2.24) we then obtain

$$\|\mathbf{B}_m - \beta_{m,m}\mathbf{E}_m\|_\infty \le \left\|\mathbf{B}_{m-1}^* - \mathbf{B}_{m-1}\right\|_\infty + O\left(\left(1-|z_{m-1}|^2\right)^{\sigma}\right)\|\mathbf{B}_{m-1}\|_\infty .$$

Next, the estimate

$$\left\|\mathbf{B}_{m-1}^* - \mathbf{B}_{m-1}\right\|_\infty \le C\left(1-|z_{m-1}|^2\right)^{\sigma},$$

follows from (2.21), (2.27) and (2.30) with $\beta\eta$ replaced with $\alpha+\sigma$:

$$\begin{aligned}\left|b_{m-1,j}^* - b_{m-1,j}\right| &= |b_{m-1,j}(z_m) - b_{m-1,j}(z_{m-1})| \\ &\le \left\|b_{m-1,j}'\right\|_{L^\infty}|z_m - z_{m-1}| \\ &\le \sum_{k=1}^{m}\left\{C_\eta\left(1-|z_{k-1}|^2\right)^{-\alpha} + 2k\right\}\left(1-|z_{m-1}|^2\right)^{\alpha+\sigma} \\ &\le C_\sigma\left(1-|z_{m-1}|^2\right)^{\sigma}.\end{aligned}$$

Thus altogether we have proved that the top row of the rectangle $\mathbf{R}_m = [b_{i,j}]_{j<m\le i}$ satisfies (2.31), i.e. $b_{m,j} \le C\left(1-|z_{m-1}|^2\right)^{\sigma}$ for $j<m$. The proof for the remaining rows is similar using (2.25).

For convenience in notation we now define

$$\Gamma = \left\{m : z_m \in V_{z_{m-1}}^{\alpha+\sigma}\right\}.$$

If we take $0 < \sigma \le (\eta-1)\alpha$ and iterate the proof of (2.31) and use (2.18), we obtain the improved estimate

$$|b_{i,j}| \le C\prod_{m\in\Gamma: j<m\le i}\left(1-|z_{m-1}|^2\right)^{\sigma}, \qquad i>j. \tag{2.33}$$

To see this we first look at the simplest case when $2,3\in\Gamma$ and establish the corresponding inequality

$$|b_{3,1}| \le C\left(1-|z_1|^2\right)^{\sigma}\left(1-|z_2|^2\right)^{\sigma}. \tag{2.34}$$

We have from (2.25) that

$$b_{3,1} = \left[b_{2,1}^* - b_{2,1}\right] + (1-\beta_{3,3})\,b_{2,1}.$$

From (2.32) and (2.31) we have

$$|b_{2,1}|\,|1-\beta_{3,3}| \le C\left(1-|z_1|^2\right)^{\sigma}\left(1-|z_2|^2\right)^{\sigma}.$$

From (2.21) we have

$$\begin{aligned}\left|b_{2,1}-b_{2,1}^{*}\right| &= \left|b_{2,1}\left(z_{2}\right)-b_{2,1}\left(z_{3}\right)\right| \\ &= \left|\left[b_{1,1}\left(z_{2}\right)-b_{1,1}\varphi_{z_{1}}\left(z_{2}\right)\right]-\left[b_{1,1}\left(z_{3}\right)-b_{1,1}\varphi_{z_{1}}\left(z_{3}\right)\right]\right| \\ &\leq \left|b_{1,1}\left(z_{2}\right)-b_{1,1}\left(z_{3}\right)\right|+\left|b_{1,1}\right|\left|\varphi_{z_{1}}\left(z_{2}\right)-\varphi_{z_{1}}\left(z_{3}\right)\right| \\ &\leq \left\|b_{1,1}'\right\|_{\infty}\left|z_{2}-z_{3}\right|+\left|b_{1,1}\right|\left\|\varphi_{z_{1}}'\right\|_{\infty}\left|z_{2}-z_{3}\right|\end{aligned}$$

where

$$\begin{aligned}\left\|b_{1,1}'\right\|_{\infty}\left|z_{2}-z_{3}\right| &\leq C\left(1-\left|z_{0}\right|^{2}\right)^{-\alpha}\left(1-\left|z_{2}\right|^{2}\right)^{\alpha+\sigma} \\ &\leq C\left(1-\left|z_{1}\right|^{2}\right)^{-\frac{\alpha}{\eta}}\left(1-\left|z_{1}\right|^{2}\right)^{\eta\alpha}\left(1-\left|z_{2}\right|^{2}\right)^{\sigma}\end{aligned}$$

and

$$\begin{aligned}\left|b_{1,1}\right|\left\|\varphi_{z_{1}}'\right\|_{\infty}\left|z_{2}-z_{3}\right| &\leq C\left(1-\left|z_{1}\right|^{2}\right)^{-\alpha}\left(1-\left|z_{2}\right|^{2}\right)^{\alpha+\sigma} \\ &\leq C\left(1-\left|z_{1}\right|^{2}\right)^{(\eta-1)\alpha}\left(1-\left|z_{2}\right|^{2}\right)^{\sigma}\end{aligned}$$

are both dominated by $C\left(1-\left|z_{1}\right|^{2}\right)^{\sigma}\left(1-\left|z_{2}\right|^{2}\right)^{\sigma}$ if $0<\sigma\leq(\eta-1)\alpha$. Altogether we have proved (2.34).

Now we suppose that $4\in\Gamma$ as well and prove the estimate

$$\left|b_{4,1}\right|\leq C\left(1-\left|z_{1}\right|^{2}\right)^{\sigma}\left(1-\left|z_{2}\right|^{2}\right)^{\sigma}\left(1-\left|z_{3}\right|^{2}\right)^{\sigma}. \tag{2.35}$$

Again we have from (2.25) that

$$b_{4,1}=\left[b_{3,1}^{*}-b_{3,1}\right]+\left(1-\beta_{4,4}\right)b_{3,1},$$

and from (2.32) and (2.34) we have

$$\left|b_{3,1}\right|\left|1-\beta_{4,4}\right|\leq C\left(1-\left|z_{1}\right|^{2}\right)^{\sigma}\left(1-\left|z_{2}\right|^{2}\right)^{\sigma}\left(1-\left|z_{3}\right|^{2}\right)^{\sigma}.$$

From (2.21) we have

$$\begin{aligned}\left|b_{3,1}-b_{3,1}^{*}\right| &= \left|b_{3,1}\left(z_{3}\right)-b_{3,1}\left(z_{4}\right)\right| \\ &= \left|\left[b_{2,1}\left(z_{3}\right)-b_{2,1}\varphi_{z_{2}}\left(z_{3}\right)\right]-\left[b_{2,1}\left(z_{4}\right)-b_{2,1}\varphi_{z_{2}}\left(z_{4}\right)\right]\right| \\ &\leq \left|b_{2,1}\left(z_{3}\right)-b_{2,1}\left(z_{4}\right)\right|+\left|b_{2,1}\right|\left|\varphi_{z_{2}}\left(z_{3}\right)-\varphi_{z_{2}}\left(z_{4}\right)\right| \\ &\leq \left\|b_{2,1}'\right\|_{\infty}\left|z_{3}-z_{4}\right|+\left|b_{2,1}\right|\left\|\varphi_{z_{2}}'\right\|_{\infty}\left|z_{3}-z_{4}\right|,\end{aligned}$$

where

$$\begin{aligned}\left\|b_{2,1}'\right\|_{\infty}\left|z_{3}-z_{4}\right| &\leq C\left(1-\left|z_{1}\right|^{2}\right)^{-\alpha}\left(1-\left|z_{3}\right|^{2}\right)^{\alpha+\sigma} \\ &\leq C\left(1-\left|z_{1}\right|^{2}\right)^{-\alpha}\left(1-\left|z_{2}\right|^{2}\right)^{\eta\alpha}\left(1-\left|z_{3}\right|^{2}\right)^{\sigma} \\ &\leq C\left(1-\left|z_{1}\right|^{2}\right)^{(\eta-1)\alpha}\left(1-\left|z_{2}\right|^{2}\right)^{(\eta-1)\alpha}\left(1-\left|z_{3}\right|^{2}\right)^{\sigma}\end{aligned}$$

and

$$\begin{aligned}\left|b_{2,1}\right|\left\|\varphi_{z_{2}}'\right\|_{\infty}\left|z_{3}-z_{4}\right| &\leq C\left(1-\left|z_{1}\right|^{2}\right)^{\sigma}\left(1-\left|z_{2}\right|^{2}\right)^{-\alpha}\left(1-\left|z_{3}\right|^{2}\right)^{\alpha+\sigma} \\ &\leq C\left(1-\left|z_{1}\right|^{2}\right)^{\sigma}\left(1-\left|z_{2}\right|^{2}\right)^{(\eta-1)\alpha}\left(1-\left|z_{3}\right|^{2}\right)^{\sigma}\end{aligned}$$

are both dominated by $C\left(1-|z_1|^2\right)^{\sigma}\left(1-|z_2|^2\right)^{\sigma}\left(1-|z_3|^2\right)^{\sigma}$ if $0<\sigma\leq(\eta-1)\alpha$. This completes the proof of (2.35), and the general case is similar.

The consequence we need from (2.33) is that if $m_1<m_2<...<m_N\leq k$ is an enumeration of the $m\in\Gamma$ such that $m\leq k$, then

(2.36)

$$\begin{aligned}|a_k| &\leq |\xi_k-\omega_k| \\ &\leq |\xi_k|+|\omega_k| \\ &\leq |\xi_k|+C\sum_{i=1}^{N}\left\{\prod_{i\leq\ell\leq N}\left(1-|z_{m_\ell-1}|^2\right)^{\sigma}\right\}\sum_{m_{i-1}\leq j\leq m_i}|\xi_{j-1}|+C\sum_{m_N\leq j\leq k}|\xi_{j-1}|\end{aligned}$$

for $0\leq k\leq\ell$.

We now return our attention to the tree $\mathcal{Y}$. For each $\alpha\in\mathcal{Y}$, with corresponding index $j\in\{j\}_{j=1}^{J}$, there are values $a(\alpha)=a_j$, $\xi(\alpha)=\xi_j$ and $m(\alpha)=z_{m(j)}\in\mathcal{Y}$. Define functions $f(\alpha)=|a(\alpha)|$ and $g(\alpha)=|\xi(\alpha)|$ on the tree $\mathcal{Y}$. Note that we are simply relabelling the indices $\{j\}_{j=1}^{J}$ as $\alpha\in\mathcal{Y}$ to emphasize the tree structure of $\mathcal{Y}$ when convenient. If we define operators

$$\begin{aligned}J_k g(\alpha) &= \sum_{m_{k-1}(\alpha)\leq\beta\leq m_k(\alpha)} g(A\beta), \\ J_\infty g(\alpha) &= g(\alpha)+\sum_{m_{N(\alpha)}(\alpha)\leq\beta\leq\alpha} g(A\beta)\end{aligned}$$

on the tree $\mathcal{Y}$, then inequality (2.36) implies in particular that

(2.37)
$$f(\alpha)\leq C\left(J_\infty g(\alpha)+\sum_{k=1}^{N(\alpha)}\left\{\prod_{k\leq\ell\leq N(\alpha)}\left(1-|z_{m_\ell-1}|^2\right)^{\sigma}\right\}J_k g(\alpha)\right),\qquad \alpha\in\mathcal{Y}.$$

Recall that we are assuming that the measure $d\mu=\sum_{\alpha\in\mathcal{Y}}\left(\log\frac{1}{1-|z_\alpha|^2}\right)^{-1}$, where $z_\alpha=z_j\in\mathbb{D}$ if α corresponds to j, satisfies the weak simple condition,

$$\beta(0,t)\sum_{j:z_j\in S(t)\text{ is minimal}}\mu(j)\leq C,\qquad t\in\mathcal{T}. \tag{2.38}$$

Note that this last inequality refers to the tree $\mathcal{T}$ rather than to $\mathcal{Y}$. Using the fact that $\beta(0,\alpha)\approx\log\frac{1}{1-|z_\alpha|^2}$, we obtain from this weak simple condition that if $S(t)\approx V_{z_k}$, i.e. $t\approx\left[1-\left(1-|z_k|^{\beta}\right)\right]z_k$, then

$$\begin{aligned}\sum_{j:z_j\in S(t)\text{ is minimal}}\mu(j) &\leq C\beta(0,t)^{-1}\approx C\left(\log\frac{1}{\left(1-|z_k|^2\right)^{\beta}}\right)^{-1} \\ &\approx C\left(\log\frac{1}{1-|z_k|^2}\right)^{-1}=C\mu(z_k),\end{aligned}$$

by the definition of the region V_{z_k}. To utilize this inequality on the tree $\mathcal{Y}$ we need the following crucial property of the sequence Z: if $[\alpha,\beta]$ is a geodesic in $\mathcal{Y}$ such that $\gamma \notin \Gamma$ for all $\alpha < \gamma \leq \beta$, then the geodesic $[\alpha, \beta]$, considered as a set of points in the tree $\mathcal{T}$, is *scattered in* $\mathcal{T}$ in the sense that no two distinct points $\gamma, \gamma' \in [\alpha, \beta]$ are comparable in $\mathcal{T}$, i.e. neither $\gamma \leq \gamma'$ nor $\gamma' \leq \gamma$ in $\mathcal{T}$. With this observation we obtain that on the tree $\mathcal{Y}$, the adjoint J_k^* of J_k satisfies

$$J_k^* \mu(\alpha) \leq C \mu(\alpha), \quad \alpha \in \mathcal{Y}. \tag{2.39}$$

Now (2.13) will follow from (2.37) together with the inequality

$$\sum_{\alpha \in \mathcal{Y}} J_k g(\alpha)^2 \mu(\alpha) \leq C \sum_{\alpha \in \mathcal{Y}} g(\alpha)^2 \mu(\alpha), \quad g \geq 0, \tag{2.40}$$

uniformly in k, and thus it suffices to show the equivalence of (2.40) and (2.39).

To see this we first claim that the inequality

$$\sum_{\alpha \in \mathcal{Y}} Ig(\alpha)^2 \mu(\alpha) \leq C \sum_{\alpha \in \mathcal{Y}} g(\alpha)^2 \mu(\alpha), \tag{2.41}$$

is equivalent to

$$I^* \mu(\alpha) \leq C \mu(\alpha), \quad \alpha \in \mathcal{Y}. \tag{2.42}$$

Indeed, (2.42) is obviously necessary for (2.41). To see the converse, we use our more general tree theorem for the tree $\mathcal{Y}$:

$$\sum_{\alpha \in \mathcal{Y}} Ig(\alpha)^2 w(\alpha) \leq C \sum_{\alpha \in \mathcal{Y}} g(\alpha)^2 v(\alpha), \quad g \geq 0,$$

if and only if

$$\sum_{\beta \geq \alpha} I^* w(\beta)^2 v(\beta)^{-1} \leq C I^* w(\alpha) < \infty, \quad \alpha \in \mathcal{Y}. \tag{2.43}$$

With $w = v = \mu$, (2.42) yields condition (2.43) as follows:

$$\sum_{\beta \geq \alpha} I^* \mu(\beta)^2 \mu(\beta)^{-1} \leq C \sum_{\beta \geq \alpha} \mu(\beta)^2 \mu(\beta)^{-1} = C \sum_{\beta \geq \alpha} \mu(\beta) = C I^* \mu(\alpha),$$

and this completes the proof of the claim.

In general, condition (2.39), a consequence of the weak simple condition, does not imply the simple condition (2.42). However, we can again exploit the crucial property of the sequence Z mentioned above - namely that if $[\alpha, \beta]$ is a geodesic in $\mathcal{Y}$ with $(\alpha, \beta] \cap \Gamma = \phi$, then $[\alpha, \beta]$, considered as a set of points in the tree $\mathcal{T}$, is *scattered in* $\mathcal{T}$. Now decompose the tree $\mathcal{Y}$ into a family of pairwise disjoint forests $\mathcal{Y}_\ell$ as follows. Let $\mathcal{Y}_1$ consist of the root o of $\mathcal{Y}$ together with all points $\beta > o$ having $\gamma \notin \Gamma$ for $o < \gamma \leq \beta$. Then let $\mathcal{Y}_2$ consist of each minimal point α in $\mathcal{Y} \setminus \mathcal{Y}_1$ together with all points $\beta > \alpha$ having $\gamma \notin \Gamma$ for $\alpha < \gamma \leq \beta$, then let $\mathcal{Y}_3$ consist of each minimal point α in $\mathcal{Y} \setminus (\mathcal{Y}_1 \cup \mathcal{Y}_2)$ together with all points $\beta > \alpha$ having $\gamma \notin \Gamma$ for $\alpha < \gamma \leq \beta$, etc.

A key property of this decomposition is that on $\mathcal{Y}_\ell$ the operator J_k sees only the values of g on $\mathcal{Y}_\ell$ itself. A second key property is that since the geodesics in $\mathcal{Y}_\ell$ are scattered, we see that the restriction μ_ℓ of μ to the forest $\mathcal{Y}_\ell$ satisfies the *simple* condition, rather than just the *weak simple* condition. As a consequence, upon decomposing each forest $\mathcal{Y}_\ell$ into trees and applying the above claim with μ_ℓ in place of μ, i.e. (2.41) holds if and only if (2.42) holds, we conclude that

$$\sum_{\alpha \in \mathcal{Y}_\ell} J_k g(\alpha)^2 \mu(\alpha) \leq C \sum_{\alpha \in \mathcal{Y}_\ell} g(\alpha)^2 \mu(\alpha), \quad g \geq 0,$$

uniformly in k for each $\ell \geq 1$. Summing in ℓ and using the finite overlap, we obtain the sufficency of (2.39) for (2.40).

Finally, to see that (2.13) now follows from (2.37), we use that

$$\sum_{i=1}^{N}\left\{\prod_{i\leq \ell\leq N}\left(1-\left|z_{m_\ell -1}\right|^2\right)^{\sigma}\right\}\leq C$$

in (2.37) to obtain (2.13). This completes the proof of Lemma 3.

Now we prove that the function $\varphi = \sum_{i=1}^{J} a_i\varphi_i$ constructed above comes close to interpolating the data $\{\xi_j\}_{j=1}^{J}$ provided we choose $\varepsilon > 0$ sufficiently small in (2.9).

Lemma 4. *Suppose $s > -1$, that $\{\xi_j\}_{j=1}^{J}$ is a sequence of complex numbers, and let $0 < \delta < 1$. Let φ_j, g_j and γ_j correspond to z_j as in Lemma 2 and with the same s. Then for $\varepsilon > 0$ sufficiently small in (2.9), there is $\{a_i\}_{i=1}^{J}$ such that $\varphi = \sum_{i=1}^{J} a_i\varphi_i$ satisfies*

$$\left\|\{\xi_j - \varphi(z_j)\}_{j=1}^{J}\right\|_{\ell^2(\mu)} < \delta\left\|\{\xi_j\}_{j=1}^{J}\right\|_{\ell^2(\mu)} \tag{2.44}$$

and

$$\|\varphi\|_{B_2(\mathbb{D})} \leq C\left\|\{a_j\}_{j=1}^{J}\right\|_{\ell^2(\mu)}. \tag{2.45}$$

Remark 2. *The proof below will show that the series $\sum_{i=1}^{J} a_i\varphi_i$ in Lemma 4 satisfies the estimate (using $\#\mathcal{G}_\ell \leq C\beta(0, z_\ell)$)*

$$\sum_{i=1}^{\infty}|\varphi_i(z)| \leq C\left(1+\log\frac{1}{1-|z|^2}\right), \qquad z\in\mathbb{D},$$

with a constant independent of J.

Remark 3. *The construction in the proof below shows that both the sequence $\{a_i\}_{i=1}^{J}$ and the function φ depend linearly on the data $\{\xi_j\}_{j=1}^{J}$.*

Proof: We now show that both (2.44) and (2.45) hold for the function $\varphi = \sum_{i=1}^{J} a_i\varphi_i$ constructed above. Fix an index $\ell \in \mathbb{N}$, and with notation as above, let $\mathcal{F}_\ell = \mathbb{N}\setminus\mathcal{G}_\ell$ and write using (2.12),

(2.46)

$$\begin{aligned}
\varphi(z_\ell) - \xi_\ell &= \sum_{i=1}^{\infty} a_i\varphi_i(z_\ell) - \xi_\ell \\
&= \left(\sum_{i\in\mathcal{G}_{P(\ell)}} a_i\varphi_i(z_\ell) + a_\ell\varphi_\ell(z_\ell) + \sum_{i\in\mathcal{F}_\ell} a_i\varphi_i(z_\ell)\right) - \left(a_\ell + \sum_{i\in\mathcal{G}_\ell\setminus\{0\}}\beta_{\ell,i}a_{P(i)}\right) \\
&= \sum_{i\in\mathcal{G}_\ell\setminus\{0\}} a_{P(i)}\left(\varphi_{P(i)}(z_\ell) - \beta_{\ell,i}\right) + \sum_{i\in\mathcal{F}_\ell} a_i\varphi_i(z_\ell) \\
&= \sum_{i\in\mathcal{F}_\ell} a_i\varphi_i(z_\ell) \equiv B_\ell,
\end{aligned}$$

since $\varphi_\ell(z_\ell) = 1$ and $\varphi_{P(i)}(z_\ell) = \beta_{\ell,i}$.

We now claim that

$$|B_\ell| \leq C \sum_{i=1}^{J} |a_i| \, \mu(z_i) \, . \tag{2.47}$$

We first note that if $z_\ell \notin V_{z_i}$, then

$$|\varphi_i(z_\ell)| \leq C \left(1 - |z_i|^2\right)^\sigma , \qquad \sigma > 0, \tag{2.48}$$

by the third line in (2.6). On the other hand, if $z_\ell \in V_{z_i}$, then $|z_i| < |z_\ell|$, and if $\mathcal{G}_\ell = [k_0, k_1, ..., k_{m-1}, k_m]$, then either $|z_i| < |z_{k_0}|$ or there is j such that $\left|z_{k_{j-1}}\right| < |z_i| \leq \left|z_{k_j}\right|$. Note however that equality cannot hold here by Lemma 1, and so we actually have $\left|z_{k_{j-1}}\right| < |z_i| < \left|z_{k_j}\right|$. From (2.8) we obtain that no index $m \in (k_{j-1}, k_j)$ satisfies $V_{z_{k_j}} \subset V_{z_m}$. Since $i \notin \mathcal{G}_\ell$, we have $i \in (k_{j-1}, k_j)$ and thus we have both

$$V_{z_{k_j}} \subsetneq V_{z_i} \text{ and } \left|z_{k_j}\right| > |z_i| \, .$$

Now using Lemma 1 and $\beta\eta > 1$, we obtain

$$\left(1 - \left|z_{k_j}\right|^2\right)^\beta \leq \left(1 - |z_i|^2\right)^{\beta\eta} \ll \left(1 - |z_i|^2\right) .$$

If we choose $w \in V_{z_{k_j}} \setminus V_{z_i}$, then $w, z_\ell \in V_{z_{k_j}}$ implies $|z_\ell - w| \leq C \left(1 - \left|z_{k_j}\right|^2\right)^\beta$ by definition, and $w \notin V_{z_i}$ implies $|1 - \overline{w} \cdot P z_i| \geq c \left(1 - |z_i|^2\right)^\beta$. Together with the reverse triangle inequality we thus have

$$\begin{aligned} |1 - \overline{z_\ell} \cdot P z_i| &\geq |1 - \overline{w} \cdot P z_i| - |\overline{z_\ell} \cdot P z_i - \overline{w} \cdot P z_i| \\ &\geq c \left(1 - |z_i|^2\right)^\beta - C \left(1 - |z_i|^2\right)^{\beta\eta} \\ &\geq (1 - |z_i|)^{\beta_1} , \end{aligned}$$

for some $\beta_1 \in (\beta, \rho)$ (again provided the $|z_i|$ are large enough). Thus in the case $z_\ell \in V_{z_i}$, estimate (2.48) again follows from the third line in (2.6). Finally, the estimate $\left(1 - |z_i|^2\right)^\sigma \leq C\mu(z_i)$ is trivial and this yields (2.47).

Combining (2.47) and (2.9) we then have for the sequence $\{\xi_j - \varphi(z_j)\}_{j=1}^J$,

$$\begin{aligned} \left\| \{\xi_j - \varphi(z_j)\}_{j=1}^J \right\|_{\ell^2(d\mu)} &\leq C \left\| \{B_j\}_{j=1}^J \right\|_{\ell^2(d\mu)} \\ &\leq C \sum_{i=1}^J |a_i| \, \mu(z_i) \left\{ \sum_{j=1}^J \mu(z_j) \right\}^{\frac{1}{2}} \\ &\leq C \left\{ \sum_{i=1}^J |a_i|^2 \mu(z_i) \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^J \mu(z_i) \right\}^{\frac{1}{2}} \left\{ \sum_{j=1}^J \mu(z_j) \right\}^{\frac{1}{2}} \\ &= C \|\mu\| \left\| \{a_j\}_{j=1}^J \right\|_{\ell^2(\mu)} \\ &< C\varepsilon \left\| \{a_j\}_{j=1}^J \right\|_{\ell^2(\mu)} . \end{aligned}$$

This completes the proof of (2.44).

We now prove the estimate $\|\varphi\|_{B_2(\mathbb{D})} \leq C$ in (2.45) whenever $\left\|\{a_j\}_{j=1}^J\right\|_{\ell^2(d\mu)} = 1$, independent of $J \geq 1$. Thus we must show that

$$\int_{\mathbb{D}} |\nabla \varphi(z)|^2 \, dz \leq C,$$

independent of $J \geq 1$. Now

$$\varphi = \sum_{i=1}^J a_i \varphi_i = \sum_{i=1}^J a_i \Gamma_s g_i = \Gamma_s g$$

where $g = \sum_{i=1}^J a_i g_i$ with $\left\|\{a_i\}_{i=1}^J\right\|_{\ell^2(\mu)} = 1$. Moreover,

$$|\nabla \Gamma_s g(z)| \leq C_s' \widehat{T_s} |g|(z)$$

where the operator $\widehat{T_s}$ is given by

$$\widehat{T_s} f(z) = c_s \int_{\mathbb{D}} \frac{f(w)\left(1 - |w|^2\right)^s}{|1 - \overline{w} \cdot z|^{1+s}} dw.$$

Thus we must estimate $\int_{\mathbb{D}} \left|\widehat{T_s}|g|(z)\right|^2 dz$.Now by Theorem 2.10 in [Zhu], $\widehat{T_s}$ is bounded on L^2 if and only if $s > -\frac{1}{2}$. Thus choosing $s - \frac{1}{2}$, we have

$$\begin{aligned}
\int_{\mathbb{D}} |\nabla^n \varphi(z)|^2 \, dz &\leq C \int_{\mathbb{D}} |g(z)|^2 \, dz \\
&= C \int_{\mathbb{D}} |g(z)|^2 \, dz.
\end{aligned}$$

Since the supports of the g_i are pairwise disjoint by the separation condition, we obtain from (2.7) that $g = \sum_{i=1}^J a_i g_i$ satisfies

$$\begin{aligned}
\int_{\mathbb{D}} |g(z)|^2 \, dz &= \sum_{j=1}^J |a_i|^2 \int_{\mathbb{D}} |g_i(z)|^2 \, dz \\
&\leq C \sum_{j=1}^J |a_i|^2 \left(\log \frac{1}{1 - |z_j|^2}\right)^{-1} \\
&= C \left\|\{a_j\}_{j=1}^J\right\|_{\ell^2(d\mu)} = C.
\end{aligned}$$

This completes the proof of Lemma 4.

Now we finish proving the sufficiency portion of Theorem 3. Fix $s > -1$, $0 < \delta < 1$ and $\{\xi_j\}_{j=1}^J$ with $\left\|\{\xi_j\}_{j=1}^J\right\|_{\ell^2(\mu)} = 1$. Then by Lemma 4 there is $f_1 = \sum_{i=1}^J a_i^1 \varphi_i \in B_2(\mathbb{D})$ such that $\left\|\{\xi_j - f_1(z_j)\}_{j=1}^J\right\|_{\ell^2(\mu)} < \delta$ and using Lemma 3 as well, $\left\|\{a_i^1\}_{i=1}^J\right\|_{\ell^2(\mu)}, \|f_1\|_{B_2(\mathbb{D})} \leq C$ where C is the product of the constants in (2.45) and (2.13). Now apply Lemma 4 to the sequence $\{\xi_j - f_1(z_j)\}_{j=1}^J$ to obtain the existence of $f_2 = \sum_{i=1}^\infty a_i^2 \varphi_i \in B_2(\mathbb{D})$ such that $\left\|\{\xi_j - f_1(z_j) - f_2(z_j)\}_{j=1}^J\right\|_{\ell^2(\mu)} < \delta^2$ and again using Lemma 3 as well, $\left\|\{a_i^2\}_{i=1}^J\right\|_{\ell^2(\mu)}, \|f_2\|_{B_2(\mathbb{D})} \leq C\delta$ where C is again the product of the constants in (2.45) and (2.13). Continuing inductively, we obtain $f_m = \sum_{i=1}^{J} a_i^m \varphi_i \in B_2(\mathbb{D})$ such that

$$\left\| \left\{ \xi_j - \sum_{i=1}^{m} f_i(z_j) \right\}_{j=1}^{J} \right\|_{\ell^\infty} < \delta^m,$$
$$\left\| \{a_i^m\}_{i=1}^{J} \right\|_{\ell^2(\mu)}, \|f_m\|_{B_2(\mathbb{D})} \le C\delta^{m-1}.$$

If we now take

$$\varphi = \sum_{m=1}^{\infty} f_m = \sum_{m=1}^{\infty} \left\{ \sum_{i=1}^{J} a_i^m \varphi_i \right\} = \sum_{i=1}^{J} a_i \varphi_i,$$

we have

$$\xi_j = \varphi(z_j), \quad 1 \le j \le J, \tag{2.49}$$
$$\left\| \{a_i\}_{i=1}^{J} \right\|_{\ell^2(\mu)} \le C,$$
$$\|\varphi\|_{B_2(\mathbb{D})} \le C,$$

if $\varepsilon > 0$ is chosen small enough in (2.9). A limiting argument using $J \to \infty$ now completes the sufficiency proof of Theorem 3.

2.1.2. *Riesz bases of Böe functions.* The proof that the weak simple condition (1.6) is necessary for onto interpolation for the Böe space $B_{2,Z}(\mathbb{D})$ requires additional tools, including the fact that the Böe functions $\{\varphi_{z_j}\}_{j=1}^{\infty}$ corresponding to a separated sequence $Z = \{z_j\}_{j=1}^{\infty}$ in the disk $\mathbb{D}$ form a Riesz basis for the Böe space $B_{2,Z}(\mathbb{D})$, at least in the presence of a mild summability condition on Z. It is interesting to note that for a separated sequence Z in $\mathbb{D}$, the set of Dirichlet reproducing kernels $\{k_{z_j}\}_{j=1}^{\infty}$ form a Riesz basis if and only if μ_Z is B_2-Carleson ([Bo]), a condition much stronger than the mild summability used for the Böe functions. This points to an essential advantage of the set of Böe functions $\{\varphi_{z_j}\}_{j=1}^{\infty}$ over the set of corresponding normalized reproducing kernels $\left\{k_{z_j}(z_j)^{-1} k_{z_j}\right\}_{j=1}^{\infty}$. The feature of Böe functions responsible for this advantage is the fact that the supports of the functions g_{z_i} are pairwise disjoint.

Proposition 1. *Let $Z = \{z_j\}_{j=1}^{\infty} \subset \mathbb{D}$ satisfy the separation condition (1.3) and the mild summability condition $\sum_{j=1}^{\infty}(1-|z_j|^2)^{\sigma} < \infty$ for all $\sigma > 0$. Then there is a finite subset S of Z such that $\{\varphi_{z_j}\}_{z_j \in Z\setminus S}$ is a Riesz basis for the closed linear span $B_{2,Z}(\mathbb{D})$ of $\{\varphi_{z_j}\}_{j=1}^{\infty}$ in the Dirichlet space $B_2(\mathbb{D})$.*

Proof: A sequence of Böe functions $\{\varphi_{z_j}\}_{j=1}^{\infty}$ is a Riesz basis if

$$C^{-1} \left\| \{a_j\}_{j=1}^{\infty} \right\|_{\ell^2(\mu)}^2 \le \left\| \sum_{j=1}^{\infty} a_j \varphi_{z_j} \right\|_{B_2}^2 \le C \left\| \{a_j\}_{j=1}^{\infty} \right\|_{\ell^2(\mu)}^2 \tag{2.50}$$

holds for all sequences $\{a_j\}_{j=1}^{\infty}$ with a positive constant C independent of $\{a_j\}_{j=1}^{\infty}$. Here $\mu = \sum_{j=1}^{\infty} \left\|\varphi_{z_j}\right\|_{B_2}^{-2} \delta_{z_j}$ and $\mu(z_j) = \left\|\varphi_{z_j}\right\|_{B_2}^{-2} \approx d(z_j)^{-1}$. The inequality on the right follows from (2.7) and the disjoint supports of the g_{z_i} - see the argument use to prove (2.45) above - so we concentrate on proving the leftmost inequality in (2.50) for an appropriate set of Böe functions. We begin with

$$\begin{aligned}\left\|\sum_{j=1}^{\infty} a_j \varphi_{z_j}\right\|_{B_2}^2 &= \int_{\mathbb{D}} \left|\sum_{j=1}^{\infty} a_j \varphi'_{z_j}(z)\right|^2 dz \\ &= \sum_{j,k=1}^{\infty} a_j \overline{a_k} \int_{\mathbb{D}} \varphi'_{z_j}(z) \overline{\varphi'_{z_k}(z)} dz \\ &= \sum_{j=1}^{\infty} |a_j|^2 \mu(z_j) + \sum_{j\neq k} a_j \overline{a_k} \int_{\mathbb{D}} \varphi'_{z_j}(z) \overline{\varphi'_{z_k}(z)} dz.\end{aligned}$$

We now claim that by discarding finitely many points of Z, we have

$$\left|\sum_{j\neq k} a_j \overline{a_k} \int_{\mathbb{D}} \varphi'_{z_j}(z) \overline{\varphi'_{z_k}(z)} dz\right| < \frac{1}{2} \sum_{j=1}^{\infty} |a_j|^2 \mu(z_j). \tag{2.51}$$

Indeed, we will estimate (2.51) using the following derivative estimates for Böe functions in the unit disk.

Lemma 5. *Let $\varphi_w(z)$ be as in Lemma 2. Then we have*

$$\begin{cases} |\varphi'_w(z)| \leq C\left(1-|w|^2\right)^{-\alpha}, & z \in V_w^\alpha \\ |\varphi'_w(z)| \leq C|z-w^*|^{-1} \leq \left(1-|w|^2\right)^{-\alpha}, & z \in V_w^\rho \setminus V_w^\alpha \\ |\varphi'_w(z)| \leq C\frac{\left(1-|w|^2\right)^{\rho(1+s)}}{|z-w^*|^{2+s}} \leq \left(1-|w|^2\right)^{-\rho}, & z \notin V_w^\rho \end{cases},$$

where $V_w^\beta = \{z \in \mathbb{D} : \gamma_w(z) \geq \beta\}$ and $\gamma_w(z)$ is given by

$$|z-w^*| = \left(1-|w|^2\right)^{\gamma_w(z)}.$$

Proof. This follows readily from the formula

$$\varphi_w(z) = \Gamma_s g_w(z) = \int_{\mathbb{D}} \frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}z\right)^{1+s}} d\zeta,$$

together with the estimate in [ArRoSa2],

$$|g_w(\zeta)| \leq C\left(\log\frac{1}{1-|w|^2}\right)^{-1} |\zeta-w^*|^{-1}, \qquad \zeta \in \mathbb{D},$$

and the fact that the support of g_w lives in the annular sector $\mathcal{S}$ centred at w^* given as the intersection of the annulus

$$\mathcal{A} = \mathcal{A}_w = \left\{\zeta \in \mathbb{D} : \left(1-|w|^2\right)^\alpha \leq |\zeta-w^*| \leq \left(1-|w|^2\right)^\rho\right\}$$

and the 45° angle cone $\mathcal{C}_w$ with vertex at w^*. Note that the cone $\mathcal{C}_w$ corresponds to the geodesic in the Bergman tree $\mathcal{T}$ joining the root to the "boundary point" w^*.

The estimate we will prove is, for $j \neq k$,

$$\begin{aligned} \left|\left\langle \varphi_{z_j}, \varphi_{z_k} \right\rangle\right| &= \left| \int_{\mathbb{D}} \varphi'_{z_j}(z) \overline{\varphi'_{z_k}(z)} dz \right| \\ &\leq C \left(1 - |z_j|^2\right)^{\sigma} \left(1 - |z_k|^2\right)^{\sigma} \mu(z_j) \mu(z_k), \end{aligned} \tag{2.52}$$

for some $\sigma > 0$. We may assume that $1 - |z_j|^2 \leq 1 - |z_k|^2$ and write

$$\begin{aligned} \left| \int_{\mathbb{D}} \varphi'_{z_j}(z) \overline{\varphi'_{z_k}(z)} dz \right| &\leq \int_{\mathbb{D}} \left|\varphi'_{z_j}(z)\right| \left|\varphi'_{z_k}(z)\right| dz \\ &= \left\{ \int_{V_{z_j}} + \int_{\mathbb{D} \setminus V_{z_j}} + \right\} \left|\varphi'_{z_j}(z)\right| \left|\varphi'_{z_k}(z)\right| dz \\ &= I + II. \end{aligned}$$

To estimate II we use Lemma 5 to obtain

$$\begin{aligned} II &\leq C \int_{\mathbb{D} \setminus V_{z_j}} \frac{\left(1 - |z_j|^2\right)^{\rho(1+s)}}{|1 - \overline{z} z_j|^{2+s}} \left(1 - |z_k|^2\right)^{-\alpha} dz \\ &= C \left(1 - |z_k|^2\right)^{-\alpha} \left(1 - |z_j|^2\right)^{\rho(1+s)} \int_{\mathbb{D} \setminus V_{z_j}} \frac{dz}{|1 - \overline{z} z_j|^{2+s}} \\ &\leq C \left(1 - |z_k|^2\right)^{-\alpha} \left(1 - |z_j|^2\right)^{\rho(1+s)} \left(1 - |z_j|^2\right)^{-\beta s} \\ &\leq C \left(1 - |z_j|^2\right)^{\rho(1+s) - \beta s - \alpha}. \end{aligned}$$

Using (2.1) we see that the exponent $\rho(1+s) - \beta s - \alpha$ is positive, and using $1 - |z_j|^2 \leq 1 - |z_k|^2$ we easily obtain (2.52).

To estimate I we consider two cases. In the case that $V_{z_j} \cap V_{z_k} \neq \phi$, we have from Lemma 5 and the estimate $\left|V_{z_j}\right| \leq C \left(1 - |z_j|^2\right)^{2\beta}$ that

$$\begin{aligned} I &\leq C \sup_{\mathbb{D}} \left|\varphi'_{z_j}\right| \sup_{\mathbb{D}} \left|\varphi'_{z_k}\right| \left|V_{z_j}\right| \\ &\leq C \left(1 - |z_j|^2\right)^{-\alpha} \left(1 - |z_k|^2\right)^{-\alpha} \left(1 - |z_j|^2\right)^{2\beta} \\ &\leq C \left(1 - |z_j|^2\right)^{2\beta - \alpha} \left(1 - |z_k|^2\right)^{-\alpha}, \end{aligned}$$

and now Lemma 1 yields

$$I \leq C \left(1 - |z_j|^2\right)^{2\beta - \alpha} \left(1 - |z_j|^2\right)^{-\frac{\alpha}{\eta}} = C \left(1 - |z_j|^2\right)^{2\beta - \alpha - \frac{\alpha}{\eta}}.$$

Now using (2.1) we see that the exponent $2\beta-\alpha-\frac{\alpha}{\eta}$ is positive and we again obtain (2.52). On the other hand, if $V_{z_j}\cap V_{z_k}=\phi$, then we use

$$\begin{aligned}
I &\leq C\sup_{\mathbb{D}}\left|\varphi'_{z_j}\right|\sup_{V_{z_j}}\left|\varphi'_{z_k}\right|\left|V_{z_j}\right| \\
&\leq C\left(1-|z_j|^2\right)^{-\alpha}\frac{\left(1-|z_k|^2\right)^{\rho(1+s)}}{\left|1-\overline{z_j}\cdot z_k\right|^{2+s}}\left(1-|z_j|^2\right)^{2\beta} \\
&\leq C\left(1-|z_j|^2\right)^{2\beta-\alpha}\left(1-|z_k|^2\right)^{\rho(1+s)-\beta(2+s)} \\
&\leq C\left(1-|z_j|^2\right)^{\varepsilon}\left(1-|z_k|^2\right)^{\rho(1+s)-\beta(2+s)+2\beta-\alpha-\varepsilon},
\end{aligned}$$

upon using $\left(1-|z_j|^2\right)^{2\beta-\alpha-\varepsilon}\leq\left(1-|z_k|^2\right)^{2\beta-\alpha-\varepsilon}$ in the last line. Now choosing

$$s>\frac{\alpha-\rho+\varepsilon}{\rho-\beta},$$

the exponent $\rho(1+s)-\beta(2+s)+2\beta-\alpha-\varepsilon$ is positive, and once more we obtain (2.52).

Now we can estimate the left side of (2.51) by (2.52) and (2.4) to obtain

$$\begin{aligned}
\left|\sum_{j\neq k}a_j\overline{a_k}\int_{\mathbb{D}}\varphi'_{z_j}(z)\overline{\varphi'_{z_k}(z)}dz\right| &\leq C\sum_{j\neq k}|a_ja_k|\left(1-|z_j|^2\right)^{\sigma}\left(1-|z_k|^2\right)^{\sigma}\mu(z_j)\mu(z_k) \\
&\leq C\left\{\sum_k\left(1-|z_k|^2\right)^{\sigma}\mu(z_k)\right\}\sum_{j=1}^{\infty}|a_j|^2\mu(z_j) \\
&<\frac{1}{2}\sum_{j=1}^{\infty}|a_j|^2\mu(z_j)
\end{aligned}$$

if $\sum_k\left(1-|z_k|^2\right)^{\sigma}\mu(z_k)$ is sufficiently small, which can be achieved by discarding a sufficiently large finite subset F from Z. This shows that $\left\{\varphi_{z_j}\right\}_{z_j\in Z\setminus F}$ is a Riesz basis. However, if $w\in F$ is *not* in the closed linear span of the Riesz basis $\left\{\varphi_{z_j}\right\}_{z_j\in Z\smallsetminus F}$, then it is immediate that $\left\{\varphi_{z_j}\right\}_{z_j\in Z\smallsetminus F}\cup\{\varphi_w\}$ is also a Riesz basis. We can continue adding Böe functions φ_w with $w\in G\subset F$ so that $\left\{\varphi_{z_j}\right\}_{z_j\in Z\setminus F}\cup\{\varphi_w\}_{w\in G}$ is a Riesz basis, and such that *all* of the remaining Boe functions φ_w with $w\in F\smallsetminus G$ lie in the closed linear span of the Riesz basis $\left\{\varphi_{z_j}\right\}_{z_j\in Z\setminus F}\cup\{\varphi_w\}_{w\in G}$. This completes the proof of Proposition 1 with $S=F\smallsetminus G$.

2.1.3. *Necessity of separation and the weak simple condition.* Now we consider the necessity of the two conditions (1.3) and (1.6) in Theorem 3. First we observe that by the closed graph theorem, there is a bounded right inverse $\mathcal{S}:\ell^2(\mu_Z)\to B_{2,Z}(\mathbb{D})$ to the restriction map $\mathcal{U}$. In particular Z is then onto interpolating for $B_2(\mathbb{D})$ and so (1.3) is necessary. To see that (1.6) is necessary, we note that by Proposition 1 above (the summability hypothesis there is a consequence of $\|\mu\|<\infty$), we can remove a finite subset S from Z so that $B_{2,Z\smallsetminus S}(\mathbb{D})=B_{2,Z}(\mathbb{D})$ and $\left\{\varphi_{z_j}\right\}_{z_j\in Z\smallsetminus S}$ is a Riesz basis. We can obviously add finitely many points to a sequence satisfying the weak simple condition and obtain a new sequence satisfying the weak simple condition. Thus we may assume that (2.2) holds for Z.

Now let e_j be the function on Z that is 1 at z_j and vanishes on the rest of Z. Denote the collection of all children of z_j in the forest structure $\mathcal{F}$ by $\mathcal{C}(z_j)$, and let $\mu = \mu_Z$. We now claim that for j sufficiently large,

$$\mathcal{S}e_j = \varphi_{z_j} - \sum_{z_i \in \mathcal{C}(z_j)} \varphi_{z_j}(z_i)\, \varphi_{z_i} + f_j, \tag{2.53}$$

where $f_j \in B_{2,Z}(\mathbb{D})$ has the form

$$f_j = \sum_{i=1}^{\infty} a_i \varphi_{z_i}$$

with $\{a_i\}_{i=1}^{\infty} \in \ell^2(\mu)$ and

$$\begin{aligned} |a_j| &< \frac{1}{2}, \\ |a_i| &< \frac{1}{2}, \qquad z_i \in \mathcal{C}(z_j). \end{aligned}$$

Indeed, by (2.2) we have

$$\mathcal{S}e_j = \sum_{i=1}^{\infty} b_i \varphi_{z_i} \tag{2.54}$$

with $\{b_i\}_{i=1}^{\infty} \in \ell^2(\mu)$ and $\left\|\{b_i\}_{i=1}^{\infty}\right\|_{\ell^2(\mu)}^2 \approx \mu(z_j)$.

Now let $\mathcal{Y}$ be the Böe tree containing j and

$$\mathcal{G}_j = [j_0, j] = \{j_0, j_1, ..., j_{m-1}, j_m = j\}$$

be the geodesic $\mathcal{G}_j$ in $\mathcal{Y}$ joining j_0 to j. If we evaluate both sides of (2.54) at z_{j_ℓ} where $0 \le \ell < m$, we have

$$0 = \mathcal{S}e_j(z_{j_\ell}) = \sum_{k=0}^{\ell} b_{j_k} \varphi_{z_{j_k}}(z_{j_\ell}) + \sum_{i \notin \{j_0, j_1, \ldots, j_\ell\}} b_i \varphi_{z_i}(z_{j_\ell}). \tag{2.55}$$

Subtracting the cases ℓ and $\ell+1$ in (2.55) we obtain

$$\begin{aligned} 0 &= \mathcal{S}e_j\left(z_{j_{\ell+1}}\right) - \mathcal{S}e_j(z_{j_\ell}) \\ &= \sum_{k=0}^{\ell-1} b_{j_k}\left[\varphi_{z_{j_k}}\left(z_{j_{\ell+1}}\right) - \varphi_{z_{j_k}}(z_{j_\ell})\right] + b_{j_\ell}\left(\varphi_{z_{j_\ell}}\left(z_{j_{\ell+1}}\right) - 1\right) \\ &\quad + b_{j_{\ell+1}} + \sum_{i \notin \{j_0, j_1, \ldots, j_{\ell+1}\}} b_i \varphi_{z_i}\left(z_{j_{\ell+1}}\right) - \sum_{i \notin \{j_0, j_1, \ldots, j_\ell\}} b_i \varphi_{z_i}(z_{j_\ell}). \end{aligned}$$

From Hölder's inequality and the third estimate in (2.6) we obtain

(2.56)

$$\begin{aligned}\left|\sum_{i\notin\{j_0,j_1,...,j_\ell\}} b_i\varphi_{z_i}\left(z_{j_\ell}\right)\right| &\le \left\{\sum_i \left|b_i\right|^2\mu\left(z_i\right)\right\}^{\frac{1}{2}}\left\{\sum_{i\notin\{j_0,j_1,...,j_\ell\}}\left|\varphi_{z_i}\left(z_{j_0}\right)\right|^2\mu\left(z_i\right)^{-1}\right\}^{\frac{1}{2}}\\ &\le C\mu\left(z_j\right)^{\frac{1}{2}}\left\{\sum_{i\neq j_0}\left|d\left(z_i\right)^{-1}\left(1-\left|z_i\right|^2\right)^{\sigma}\right|^2 d\left(z_i\right)\right\}^{\frac{1}{2}}\\ &\le C_0\mu\left(z_j\right)^{\frac{1}{2}},\end{aligned}$$

where the final term in braces is bounded by hypothesis. We also have from (2.6)

$$\left|\varphi_{z_{j_\ell}}\left(z_{j_{\ell+1}}\right)-1\right|\le\left(1+C\mu\left(z_{j_\ell}\right)\right)$$

and

$$\begin{aligned}&\left|\sum_{k=0}^{\ell-1} b_{j_k}\left[\varphi_{z_{j_k}}\left(z_{j_{\ell+1}}\right)-\varphi_{z_{j_k}}\left(z_{j_\ell}\right)\right]\right|\\ &\le C\sum_{k=0}^{\ell-1}\left|b_{j_k}\right|\mu\left(z_{j_k}\right)\le C\left\|\mu\right\|^{\frac{1}{2}}\left\{\sum_{k=0}^{\ell-1}\left|b_{j_k}\right|^2\mu\left(z_{j_k}\right)\right\}^{\frac{1}{2}}.\end{aligned}$$

Altogether then we have

$$\begin{aligned}\left|b_{j_{\ell+1}}\right| &\le \left|b_{j_\ell}\right|\left(1+C\mu\left(z_{j_\ell}\right)\right)+C\left\|\mu\right\|^{\frac{1}{2}}\left\{\sum_{k=0}^{\ell-1}\left|b_{j_k}\right|^2\mu\left(z_{j_k}\right)\right\}^{\frac{1}{2}}+2C_0\mu\left(z_j\right)^{\frac{1}{2}}\\ &\le \left|b_{j_\ell}\right|\left(1+C\mu\left(z_{j_\ell}\right)\right)+C_1\mu\left(z_j\right)^{\frac{1}{2}}.\end{aligned}$$

Now the case $\ell=0$ of (2.55) together with (2.56) yields

$$\left|b_{j_0}\right|=\left|\sum_{i\notin\{j_0,j_1,...,j_\ell\}} b_i\varphi_{z_i}\left(z_{j_\ell}\right)\right|\le C_0\mu\left(z_j\right)^{\frac{1}{2}},$$

and now by induction on ℓ we obtain that for $0\le\ell\le m-1$, $\left|b_{j_\ell}\right|$ is dominated by

$$C_0\mu\left(z_j\right)^{\frac{1}{2}}\left\{\prod_{k=0}^{\ell-1}\left(1+Cd\left(z_{j_k}\right)^{-1}\right)+\prod_{k=1}^{\ell-1}\left(1+Cd\left(z_{j_k}\right)^{-1}\right)+...+\left(1+Cd\left(z_{j_{\ell-1}}\right)^{-1}\right)\right\}.$$

In particular,

$$\left|b_{j_\ell}\right|\le C_0\mu\left(z_j\right)^{\frac{1}{2}}\ell\exp\left(C\sum_{k=0}^{\ell-1}d\left(z_{j_k}\right)^{-1}\right) \tag{2.57}$$

for $0\le\ell\le m-1$.

Now evaluate both sides of (2.54) at $z_j=z_{j_m}$ to obtain

$$1=b_j+\sum_{k=0}^{m-1} b_{j_k}\varphi_{z_{j_k}}\left(z_j\right)+\sum_{i\notin\{j_0,j_1,...,j_m\}} b_i\varphi_{z_i}\left(z_j\right),$$

which by the argument above yields

$$|b_j - 1| \leq C_0 \mu(z_j)^{\frac{1}{2}} \, m \exp\left(C \sum_{k=0}^{m} d(z_{j_k})^{-1} \right).$$

Similarly, for $z_i \in \mathcal{C}(z_j)$ we obtain

$$\left| b_i - b_j \varphi_{z_j}(z_i) \right| \leq C_0 \mu(z_j)^{\frac{1}{2}} (m+1) \exp\left(C d(z_i)^{-1} + C \sum_{k=0}^{m} d(z_{j_k})^{-1} \right).$$

Now the separation condition (1.3) yields $d(z_{j_k}) \geq (1+c) \, d\left(z_{j_{k-1}}\right)$ for $1 \leq k \leq m$ and it follows that

$$\sum_{k=0}^{m} d(z_{j_k})^{-1} \leq C \tag{2.58}$$

independent of $\dot{j}$. Thus we see that

$$\left| b_i - b_j \varphi_{z_j}(z_i) \right| \leq C (m+1) \mu(z_j)^{\frac{1}{2}}, \qquad z_i \in \mathcal{C}(z_j),$$

with a constant C independent of $\dot{j}$. If we take j_0 large enough, then since $d(z_j) = d(z_{j_m}) \geq (1+c)^m d(z_{j_0})$, we have

$$|b_j - 1| \leq C m \mu(z_j)^{\frac{1}{2}} = C m d(z_j)^{-\frac{1}{2}} \leq C \frac{m}{(1+c)^{\frac{m}{2}}} d(z_{j_0})^{-\frac{1}{2}} < \frac{1}{2}.$$

It follows that

$$\left| b_i - \varphi_{z_j}(z_i) \right| \leq \left| b_i - b_j \varphi_{z_j}(z_i) \right| + |b_j - 1| \left| \varphi_{z_j}(z_i) \right| < \frac{1}{2}, \qquad z_i \in \mathcal{C}(z_j),$$

which proves (2.53).

By (2.2) we then have using (2.53) and the fact that $\varphi_{z_j}(z_i) = 1$ for $z_i \in \mathcal{C}(z_j) \cap V_{z_j}^{\alpha}$:

$$\|\mathcal{S} e_j\|_{B_{2,Z}(\mathbb{D})} \approx \left\{ \sum_i |b_i|^2 \mu(z_i) \right\}^{\frac{1}{2}} \geq \frac{1}{2} \left\{ \sum_{z_i \in \mathcal{C}(z_j) \cap V_{z_j}^{\alpha}} \mu(z_i) \right\}^{\frac{1}{2}}.$$

It follows that

$$\mu(z_j) = \|e_j\|_{\ell^2(\mu)}^2 \geq c^2 \|\mathcal{S} e_j\|_{B_{2,Z}(\mathbb{D})}^2 \geq c' \sum_{z_i \in \mathcal{C}(z_j) \cap V_{z_j}^{\alpha}} \mu(z_i),$$

which yields (1.6) for $\alpha = z_j \in Z$ with j large, and hence for all j with a worse constant.

Now we suppose that $\alpha \in \mathcal{T} \setminus Z$. We claim that with either $z_0 = \alpha$ or $z_0 = A^M \alpha$, where $M = \left[\frac{c}{10} d(\alpha) \right]$ and c is as in (1.3), the set $Z' = Z \cup \{z_0\}$ is separated with separation constant in (1.3) at least $\frac{c}{100}$. Indeed, if $Z \cup \{\alpha\}$ fails to satisfy (1.3) with separation constant $\frac{c}{100}$, then there is some w in Z such that

$$\beta(\alpha, w) < \frac{c}{50} (1 + \beta(o, w)).$$

From this we obtain that

$$\begin{aligned}\beta\left(A^M\alpha,w\right) &\geq \beta\left(A^M\alpha,\alpha\right)-\beta\left(\alpha,w\right)\\ &> \frac{c}{10}\left(1+\beta\left(o,w\right)\right)-\frac{c}{50}\left(1+\beta\left(o,w\right)\right)\\ &> \frac{c}{20}\left(1+\beta\left(o,w\right)\right),\end{aligned}$$

and then for any $z\in Z\setminus\{w\}$,

$$\begin{aligned}\beta\left(A^M\alpha,z\right) &\geq \beta\left(w,z\right)-\beta\left(A^M\alpha,w\right)\\ &> \beta\left(w,z\right)-\left\{\beta\left(A^M\alpha,\alpha\right)+\beta\left(\alpha,w\right)\right\}\\ &> c\left(1+\beta\left(o,w\right)\right)-\left\{\frac{c}{10}\left(1+\beta\left(o,w\right)\right)+\frac{c}{50}\left(1+\beta\left(o,w\right)\right)\right\}\\ &> \frac{c}{2}\left(1+\beta\left(o,w\right)\right),\end{aligned}$$

which shows that $Z\cup\left\{A^M\alpha\right\}$ satisfies (1.3) with separation constant $\frac{c}{20}$. Now we associate a Böe function φ_{z_0} with z_0, but take the parameters $\beta,\beta_1,\rho,\alpha$ so close to 1 for this additional function φ_{z_0} that the extended set of Böe functions $\{\varphi_z\}_{z\in Z'}=\{\varphi_z\}_{z\in Z}\cup\{\varphi_{z_0}\}$ satisfy the property that the supports of the associated functions g_z are pairwise disjoint for $z\in Z'$.

Now we define a bounded linear operator S' from $\ell^2\left(\mu_{Z'}\right)$ into $B_{2,Z'}\left(\mathcal{T}\right)$ by

$$S'\left[\xi'\right]=S\xi+\left(\xi_0-S\xi\left(z_0\right)\right)\left\{\varphi_{z_0}-S\left[\varphi_{z_0}\mid_Z\right]\right\},$$

where $\xi'=\left(\xi_0,\xi\right)=\left(\xi_0,\xi_1,...\right)$. For $j\geq 1$ we have

$$\begin{aligned}S'\left[\xi'\right]\left(z_j\right) &= \xi_j+\left(\xi_0-S\xi\left(z_0\right)\right)\left\{\varphi_{z_0}\left(z_j\right)-S\left[\varphi_{z_0}\mid_Z\right]\left(z_j\right)\right\}\\ &= \xi_j+\left(\xi_0-S\xi\left(z_0\right)\right)\left\{0\right\}=\xi_j,\end{aligned}$$

and for $j=0$,

$$\begin{aligned}S'\left[\xi'\right]\left(z_0\right) &= S\xi\left(z_0\right)+\left(\xi_0-S\xi\left(z_0\right)\right)\left\{1-S\left[\varphi_{z_0}\mid_Z\right]\left(z_0\right)\right\}\\ &= \xi_0-S\left[\varphi_{z_0}\mid_Z\right]\left(z_0\right)\left(\xi_0-S\xi\left(z_0\right)\right).\end{aligned}$$

Now $S\left[\varphi_{z_0}\mid_Z\right]\left(z_0\right)$ is small by the argument used to prove (2.53) above, and in fact (2.57) and (2.58) of that argument yield

$$\left|S\left[\varphi_{z_0}\mid_Z\right]\left(z_0\right)\right|\leq C\mu\left(z_0\right)^{\frac{1}{2}}.$$

At this point we may assume that $C\mu\left(z_0\right)^{\frac{1}{2}}<\varepsilon$ since there are only finitely many (depending on $\varepsilon>0$) points α in the tree $\mathcal{T}$ having such a point z_0 that fails this condition. Thus S' is an *approximate* bounded right inverse to the restriction map $\mathcal{U}$, and in fact,

$$\mathcal{U}S'\xi'-\xi'=S\left[\varphi_{z_0}\mid_Z\right]\left(z_0\right)\left(\xi_0-S\xi\left(z_0\right)\right)e_{z_0},$$

so that

$$\left\|\mathcal{U}S'\xi'-\xi'\right\|_{\ell^2(\mu)}\leq\varepsilon C\left\|\xi'\right\|_{\ell^2(\mu)}<\frac{1}{2}\left\|\xi'\right\|_{\ell^2(\mu)}$$

if $\varepsilon>0$ is small enough. Then $\mathcal{U}S'$ is invertible on $\ell^2\left(\mu\right)$, and so the operator $S''=S'\left(\mathcal{U}S'\right)^{-1}$ is an exact bounded right inverse to the restriction map $\mathcal{U}$ since $\mathcal{U}S''=\mathcal{U}S'\left(\mathcal{U}S'\right)^{-1}=\mathbb{I}_{\ell^2(\mu)}$. Then the result proved in the previous paragraph shows that the weak simple condition (1.6) holds at z_0 with a controlled constant, and thus also at α with a controlled constant. This completes the proof of Theorem 3.

2.1.4. *The simple condition.* Suppose that $Z \subset \mathbb{D}$ satisfies the separation condition (1.3) and that the associated measure μ is finite. Here we show that $R\left(B_{2,Z}\left(\mathbb{D}\right)\right) \subset \ell^2\left(\mu\right)$ if the simple condition (1.4) holds, and conversely that if R maps $B_{2,Z}\left(\mathbb{D}\right)$ into $\ell^2\left(\mu\right)$, then a weaker version (2.61) of condition (1.4) holds. To see this we fix $f = \sum_{i=1}^{\infty} a_i \varphi_{z_i} \in B_{2,Z}\left(\mathbb{D}\right)$, $z_j \in Z$, and as in the previous subsection, we let $\mathcal{Y}$ be the Böe tree containing j and

$$\mathcal{G}_j = [j_0, j] = \{j_0, j_1, ..., j_{m-1}, j_m = j\}$$

be the geodesic $\mathcal{G}_j$ in $\mathcal{Y}$ joining j_0 to j. Then we have

$$f\left(z_j\right) = \sum_{k=0}^{m} a_{j_k} \varphi_{z_{j_k}}\left(z_j\right) + \sum_{i \notin \{j_0, j_1, \ldots, j_m\}} a_i \varphi_{z_i}\left(z_j\right).$$

From Hölder's inequality, (2.2) (which follows from Proposition 1) and the third estimate in (2.6) we obtain

$$\begin{aligned}
\left|\sum_{i \notin \{j_0, j_1, \ldots, j_m\}} a_i \varphi_{z_i}\left(z_j\right)\right| &\leq C \left\{\sum_i \left|a_i\right|^2 \mu\left(z_i\right)\right\}^{\frac{1}{2}} \left\{\sum_{i \notin \{j_0, j_1, \ldots, j_\ell\}} \left|\varphi_{z_i}\left(z_j\right)\right|^2 \mu\left(z_i\right)^{-1}\right\}^{\frac{1}{2}} \\
&\leq C \left\|f\right\|_{B_{2,Z}(\mathbb{D})} \left\{\sum_{i \neq j_0} \left|d\left(z_i\right)^{-1} \left(1 - \left|z_i\right|^2\right)^{\sigma}\right|^2 d\left(z_i\right)\right\}^{\frac{1}{2}} \\
&\leq C \left\|f\right\|_{B_{2,Z}(\mathbb{D})}.
\end{aligned}$$

We also have

$$\left|\sum_{k=0}^{m} a_{j_k} \varphi_{z_{j_k}}\left(z_j\right)\right| \leq C \sum_{k=0}^{m} \left|a_{j_k}\right| = C I a\left(z_j\right)$$

where I denotes the summation operator on the Böe tree $\mathcal{Y}$. Thus we have

$$\left\|Rf\right\|_{\ell^2(\mu)} \leq C \left\|Ia\right\|_{\ell^2(\mu)} + C \left\|f\right\|_{B_{2,Z}(\mathbb{D})} \left\|\mu\right\|^{\frac{1}{2}}. \tag{2.59}$$

By [ArRoSa] I is bounded on $\ell^2\left(\mu\right)$ if and only if

$$\sum_{\beta \geq \alpha} \frac{I^*\mu\left(\beta\right)^2}{\mu\left(\beta\right)} \leq C I^*\mu\left(\alpha\right), \qquad \alpha \in \mathcal{Y}. \tag{2.60}$$

Now if μ satisfies the simple condition (1.4) then $I^*\mu\left(\beta\right) \leq C\mu\left(\beta\right)$ for $\beta \in \mathcal{Y}$, and we see that (2.60) holds. Thus $\left\|Ia\right\|_{\ell^2(\mu)} \leq C \left\|a\right\|_{\ell^2(\mu)} \approx \left\|f\right\|_{B_{2,Z}(\mathbb{D})}$ and this combined with (2.59) completes the proof that R maps $B_{2,Z}\left(\mathbb{D}\right)$ boundedly into $\ell^2\left(\mu\right)$.

Conversely, if R is bounded from $B_{2,Z}\left(\mathbb{D}\right)$ to $\ell^2\left(\mu\right)$, then we have

$$\sum_{z_k \in V_{z_j}^{\alpha}} \mu\left(z_k\right) \leq \left\|R\varphi_{z_j}\right\|_{\ell^2(\mu)}^2 \leq C \left\|\varphi_{z_j}\right\|_{B_{2,Z}(\mathbb{D})}^2 = C\mu\left(z_j\right) \tag{2.61}$$

for all $z_j \in Z$, a weaker version of the simple condition (1.4).

2.2. **Example of an onto interpolating sequence with $\|\mu\| = \infty$.** Here we prove the existence of an onto interpolating sequence $Z = \{z_j\}_{j=1}^{\infty}$ for the Dirichlet space with *infinite* mass, thus answering a question of Bishop [Bi]. The key is to let Z be a subtree of $\mathcal{T}$, so that if $z \in Z$ is a Böe child of $w \in Z$, then z actually lies in the Bergman successor set $S(w)$ of w, and hence the value of $c_{\rho,\alpha}(\gamma_w(z))$ in Lemma 2 is 1, which is exploited in (2.65) below. The advantage when assuming (1.7) is that we may dispense with the complicated inductive definition of the coefficients a_k in (2.12) for the holomorphic function $\mathcal{S}\xi$ in (2.10) approximating ξ on Z, and instead use the elementary construction in (2.63) below of a holomorphic function $M\xi$ approximating the *integrated* sequence $(I\xi)_j = \sum_{z_i \leq z_j} \xi_i$ on Z. This permits us to interpolate the *difference* sequence $\triangle\xi$ using the operator $\triangle M$, whose kernel is better localized. Of course in the absence of (1.7), the values $c_{\rho,\alpha}(\gamma_w(z))$ may lie in $[0,1)$ and then $M\xi$ will *not* be a good approximation to $I\xi$ on Z.

Theorem 4. *Suppose $Z = \{z_j\}_{j=1}^{\infty} \subset \mathbb{D}$ is a subtree of $\mathcal{T}$ that satisfies the separation condition (1.3), and if c is the constant in (1.3), that there is $\beta \in \left(1 - \frac{c}{2}, 1\right)$ satisfying (1.7). Then Z is onto interpolating for the Böe space $B_{2,Z}(\mathbb{D})$ if and only if the weak simple condition (1.6) holds.*

There are subtrees Z of $\mathcal{T}$ that satisfy (1.3), (1.6) and (1.7) and yet $\|\mu_Z\| = \infty$, thus yielding an onto interpolating sequence for the Dirichlet space $B_2(\mathbb{D})$ with infinite mass. See Subsubsection 3.1 below.

Proof: (of Theorem 4) To see the necessity of (1.6) when Z is onto interpolating for the Böe space $B_{2,Z}(\mathbb{D})$, we note that a subtree of a dyadic tree has branching number at most 2, and it follows easily from the separation condition that

$$\sum_{j=1}^{\infty} \left(1 - |z_j|^2\right)^{\sigma} < \infty$$

for all $\sigma > 0$. Thus Proposition 1 can be applied together with the argument used above to prove necessity of (1.6) in the case $\|\mu_Z\| < \infty$.

To establish sufficiency, fix $\{\xi_j\}_{j=1}^{\infty}$ with

$$\left\| \left\{ \frac{\xi_j}{\left\|k_{z_j}\right\|_{B_2}} \right\}_{j=1}^{\infty} \right\|_{\ell^2} = 1.$$

Recall that $\left\|k_{z_j}\right\|_{B_2} \approx \left(\log \frac{1}{1-|z_j|^2}\right)^{\frac{1}{2}}$ and that we may suppose $Z \subset \mathcal{T}$. We note that (2.4) holds here - in fact the proof is simpler using the separation condition (1.3) and the assumption that Z is a subtree of $\mathcal{T}$ (and hence has branching number at most 2). We can always add the origin to Z, and so in particular we obtain that $\sum_{j=1}^{\infty} (1 - |z_j|)^{\sigma} < \infty$. Thus given any $\sigma > 0$, we can use Remark 1 to discard all points from Z that lie in some ball $B(0,R)$, $R < 1$, and reorder the remaining points so that

$$\left(\log \frac{1}{1-R^2}\right)^{-1},\ 1 - R^2,\ \sum_{j=1}^{\infty} (1 - |z_j|)^{\sigma} < \varepsilon. \tag{2.62}$$

We now suppose in addition that the sequence $Z = \{z_j\}_{j=1}^{J}$ is finite, and obtain an appropriate estimate independent of $J \geq 1$. Given a sequence of complex numbers $\xi = \{\xi_j\}_{j=1}^J$ we define a holomorphic function $M\xi$ on the ball by

$$M\xi(z) = \sum_{j=1}^J \xi_j \varphi_{z_j}(z), \qquad z \in \mathbb{D}, \tag{2.63}$$

where $\varphi_w(z)$ is as in Lemma 2. View μ as the measure assigning mass $\left(\log \frac{1}{1-|z_j|^2}\right)^{-1}$ to the point $j \in \{0, 1, 2, ..., J\}$. We have

$$\left\| \{\xi_j\}_{j=1}^J \right\|_{\ell^2(d\mu)} \approx \left\| \left\{ \frac{\xi_j}{\left\| k_{z_j} \right\|_{B_2}} \right\}_{j=1}^J \right\|_{\ell^2},$$

for any complex sequence $\{\xi_j\}_{j=1}^J$. We will use another useful consequence of Lemma 1:

$$1 - |Az_j|^2 \le \left(1 - |z_\ell|^2\right)^\eta, \qquad \text{for } z_j \in V_{z_\ell} \setminus \mathcal{C}(z_\ell). \tag{2.64}$$

Indeed, if $z_j \in V_{z_\ell} \setminus \mathcal{C}(z_\ell)$, then $Az_j \neq z_\ell$ and $|Az_j| \ge |z_\ell|$ by the construction in (2.8). Then $V_{z_\ell} \cap V_{Az_j}$ contains z_j and is thus nonempty, and Lemma 1 now shows that $1 - |Az_j|^2 \le \left(1 - |z_\ell|^2\right)^\eta$.

Now define a linear map T from $\ell^2(d\mu)$ to $\ell^2(d\mu)$ by

$$T\xi = \triangle(M\xi)|_Z = \{M\xi(z_k) - M\xi(Az_k)\}_{j=1}^J = \left\{ \sum_{j=1}^J \xi_j \left[\varphi_{z_j}(z_k) - \varphi_{z_j}(Az_k)\right] \right\}_{j=1}^J,$$

where Az_j denotes the predessor of z_j in the forest structure on Z defined in (2.8) above (we identify z_k with k here). Let $\mathcal{R}$ denote the set of all roots of maximal trees in the forest. In the event that $z_k \in \mathcal{R}$, then Az_k isn't defined and our convention is to define $\varphi_{z_j}(Az_k) = 0$. We claim that T is a bounded invertible map on $\ell^2(d\mu)$ with norms independent of $J \ge 1$. To see this it is enough to prove that $\mathbb{I} - T$ has small norm on $\ell^2(d\mu)$ where $\mathbb{I}$ denotes the identity operator. We have

$$\begin{aligned}(\mathbb{I} - T)\xi &= \left\{ \xi_k - \sum_{j=1}^J \xi_j \left[\varphi_{z_j}(z_k) - \varphi_{z_j}(Az_k)\right] \right\}_{k=1}^J \\ &= \{\xi_k \varphi_{z_k}(Az_k)\}_{k=1}^J - \left\{ \sum_{j:j\neq k} \xi_j \left[\varphi_{z_j}(z_k) - \varphi_{z_j}(Az_k)\right] \right\}_{k=1}^J\end{aligned}$$

since $\varphi_{z_k}(z_k) = 1$.

Now we estimate the kernel $K(k, j)$ of the operator $\mathbb{I} - T$. We have on the diagonal,

$$|K(k,k)| = \begin{cases} |\varphi_{z_k}(Az_k)| \quad \le \quad \left(1 - |z_k|^2\right)^{(\rho - \beta_1)(1+s)} & \text{if } z_k \notin \mathcal{R} \\ 0 & \text{if } z_k \in \mathcal{R} \end{cases},$$

by the third estimate in (2.6).

Suppose now that $z_k \notin \mathcal{R}$ and $j \neq k$. Lemma 5 shows that $\left|\varphi'_{z_j}(\zeta_k)\right| \le \left(1 - |z_j|^2\right)^{-\alpha}$ and the definition of V_{z_j} shows that $|z_k - Az_k| \le \left(1 - |Az_k|^2\right)^\beta$.

Thus if $1-|Az_k|^2 \leq \left(1-|z_j|^2\right)^{\eta}$, then

$$\begin{aligned}|K(k,j)| &= \left|\varphi_{z_j}(z_k)-\varphi_{z_j}(Az_k)\right| \leq \left|\varphi'_{z_j}(\zeta_k)\right| |z_k - Az_k| \\ &\leq C\left(1-|z_j|^2\right)^{-\alpha}\left(1-|Az_k|^2\right)^{\beta} \\ &\leq C\left(1-|z_j|^2\right)^{\eta(\beta-\delta)-\alpha}\left(1-|Az_k|^2\right)^{\delta},\end{aligned}$$

where the exponent $\eta(\beta-\delta)-\alpha$ is positive if we choose δ small enough, since $\alpha<1<\beta\eta$ by Lemma 1.

Suppose instead that $1-|Az_k|^2 > \left(1-|z_j|^2\right)^{\eta}$. Then $Az_k \notin V_{z_j}$ by Lemma 1. If $z_k \notin \mathcal{C}(z_j)$, then $z_k \notin V_{z_j}$ by (2.64), and this time we use the third estimate in (2.6) to obtain

$$\begin{aligned}|K(k,j)| &= \left|\varphi_{z_j}(z_k)-\varphi_{z_j}(Az_k)\right| \leq \left|\varphi_{z_j}(z_k)\right| + \left|\varphi_{z_j}(Az_k)\right| \\ &\leq C\left(1-|z_j|^2\right)^{(\rho-\beta_1)(1+s)} \\ &\leq C\left(1-|z_j|^2\right)^{(\rho-\beta_1)(1+s)-\delta}\left(1-|Az_k|^2\right)^{\frac{\delta}{\rho}}.\end{aligned}$$

On the other hand, if $z_k \in \mathcal{C}(z_j)$, then $|z_k| \geq |z_j|$ and our hypothesis (1.7) implies that $z_k \in S(z_j)$. Then we have

$$|K(k,j)| = \left|\varphi_{z_j}(z_k)-\varphi_{z_j}(z_j)\right| \leq \left(\log\frac{1}{1-|z_j|^2}\right)^{-1} \tag{2.65}$$

by the first two estimates in (2.6) since $c_{\rho,\alpha}\left(\gamma_{z_j}(z_k)\right)=1$ in Lemma 2 if $z_k \in S(z_j)$.

Finally, we consider the case when $z_k \in \mathcal{R}$ and $j \neq k$. The third estimate in (2.6) shows that

$$|K(k,j)| = \left|\varphi_{z_j}(z_k)\right| \leq C\left(1-|z_j|^2\right)^{(\rho-\beta_1)(1+s)},$$

where the exponent $(\rho-\beta_1)(1+s)$ can be made as large as we wish by taking s sufficently large. Combining all cases we have in particular the following estimate for some $\sigma_1, \sigma_2 > 0$:

$$|K(k,j)| \leq C \begin{cases} \left(1-|z_j|^2\right)^{\sigma_1}\left(1-|Az_k|^2\right)^{\sigma_2}, & \text{if } z_k \notin \mathcal{R} \text{ and } z_k \notin \mathcal{C}(z_j) \\ \left(\log\frac{1}{1-|z_j|^2}\right)^{-1} & \text{if } z_k \notin \mathcal{R} \text{ and } z_k \in \mathcal{C}(z_j) \\ \left(1-|z_j|^2\right)^{3}, & \text{if } z_k \in \mathcal{R} \end{cases}.$$

Now we obtain the boundedness of $\mathbb{I}-T$ on $\ell^2(d\mu)$ with small norm by Schur's test. It is here that we use the assumption that μ satisfies the weak simple condition

(1.6). With $\xi \in \ell^2(d\mu)$ and $\eta \in \ell^2(d\mu)$, we have

$$\begin{aligned}\left|\langle(\mathbb{I}-T)\xi,\eta\rangle_\mu\right| &= \left|\sum_k\left(\sum_j K(k,j)\xi_j\right)\overline{\eta_k}\mu(k)\right| \\ &\leq C\sum_j\sum_{k\notin\mathcal{R},z_k\notin\mathcal{C}(z_j)}\left(1-|z_j|^2\right)^{\sigma_1}\left(1-|Az_k|^2\right)^{\sigma_2}|\xi_j||\eta_k|\mu(k) \\ &\quad+C\sum_j\sum_{k\notin\mathcal{R},z_k\in\mathcal{C}(z_j)}\left(\log\frac{1}{1-|z_j|^2}\right)^{-1}|\xi_j||\eta_k|\mu(k) \\ &\quad+C\sum_j\sum_{k\in\mathcal{R}}\left(1-|z_j|^2\right)^3|\xi_j||\eta_k|\mu(k),\end{aligned}$$

and since $\mu(j) \leq \left(1-|z_j|^2\right)^\varepsilon$, we have with $\sigma_1' = \sigma_1 - \varepsilon$,

$$\begin{aligned}\left|\langle(\mathbb{I}-T)\xi,\eta\rangle_\mu\right| &\leq C\sum_j\sum_{k\notin\mathcal{R},z_k\notin\mathcal{C}(z_j)}\left(1-|z_j|^2\right)^{\sigma_1'}\left(1-|Az_k|^2\right)^{\sigma_2}|\xi_j|\mu(j)|\eta_k|\mu(k) \\ &\quad+C\sum_j\sum_{k\notin\mathcal{R},z_k\in\mathcal{C}(z_j)}|\xi_j|\mu(j)|\eta_k|\mu(k) \\ &\quad+C\sum_j\sum_{k\in\mathcal{R}}\left(1-|z_j|^2\right)^2|\xi_j|\mu(j)|\eta_k|\mu(k).\end{aligned}$$

By Schur's test it suffices to show

$$\mu(Ak)+\sum_{j=1}^J\left(1-|z_j|^2\right)^{\sigma_1'}\mu(j) < C\varepsilon < 1, \tag{2.66}$$

$$\sum_{k:z_k\in\mathcal{C}(z_j)}\mu(k)+\sum_{k\notin\mathcal{R}}\left(1-|Az_k|^2\right)^{\sigma_2}\mu(k) < C\varepsilon < 1,$$

$$\sum_{k\in\mathcal{R}}\left(1-|z_j|^2\right)^3\mu(k) < C\varepsilon < 1.$$

Now (2.62) yields

$$\sum_{j=1}^J\left(1-|z_j|^2\right)^{\sigma_1'}\mu(j) \leq C\sum_j\left(1-|z_j|^2\right)^{\sigma_1''} < C\varepsilon,$$

and combined with the weak simple condition (1.6), we have

$$\begin{aligned}\sum_{k=1}^J\left(1-|Az_k|^2\right)^{\sigma_2}\mu(k) &= \sum_\ell\left(1-|z_\ell|^2\right)^{\sigma_2}\left(\sum_{z_k\in\mathcal{C}(z_\ell)}\mu(k)\right) \\ &\leq C\sum_\ell\left(1-|z_\ell|^2\right)^{\sigma_2}\mu(\ell) \\ &\leq C\sum_\ell\left(1-|z_\ell|^2\right)^{\sigma_2'} < C\varepsilon.\end{aligned}$$

Finally we write the annulus $B(0,1) \setminus B(0,R)$ as a pairwise disjoint union $\cup_{i=1}^{N} B_i$ of Carleson boxes of "size" R where $N \approx \left(1-R^2\right)^{-1}$. Then

$$\sum_{z_k \in B_i : k \in \mathcal{R}} \mu(k) \leq C \left(1 + \log \frac{1}{1-R^2}\right)^{-1} \leq C$$

by the weak simple condition (1.6), and thus the left side of the final estimate in (2.66) satisfies

$$\begin{aligned}
\sum_{k \in \mathcal{R}} \left(1 - |z_j|^2\right)^2 \mu(k) &\leq \left(1-R^2\right)^2 \sum_{i=1}^{N} \sum_{z_k \in B_i : k \in \mathcal{R}} \mu(k) \\
&\leq C \left(1-R^2\right)^2 N \\
&\leq C \left(1-R^2\right) < C\varepsilon,
\end{aligned}$$

by (2.62) as required.

Thus T^{-1} exists uniformly in J. Now we take $\xi \in \ell^2(d\mu)$ and set $\eta = \triangle \xi$. Here we use the convention that $\xi(A\alpha) = 0$ if α is a root of a tree in the forest Z. By the weak simple condition we have the estimate

$$\begin{aligned}
\|\eta\|_{\ell^2(d\mu)}^2 &= \sum_j |\eta_j|^2 \mu(j) = \sum_j |\xi_j - \xi_{Aj}|^2 \mu(j) \\
&\leq C \sum_j |\xi_j|^2 \mu(j) + C \sum_\ell |\xi_\ell|^2 \left(\sum_{z_j \in \mathcal{C}(z_\ell)} \mu(j) \right) \\
&\leq C \sum_j |\xi_j|^2 \mu(j) + C \sum_\ell |\xi_\ell|^2 \mu(\ell) \\
&\leq C \|\xi\|_{\ell^2(d\mu)}^2 .
\end{aligned} \tag{2.67}$$

Then let $h = M\left(T^{-1}\eta\right)$ so that

$$\triangle h \mid_Z = \triangle \left(MT^{-1}\eta\right) \mid_Z = TT^{-1}\eta = \eta = \triangle \xi.$$

Thus the holomorphic function h satisfies

$$h \mid_Z = \xi.$$

Finally, from (2.7) and then (2.67) we have the Besov space estimate ([ArRoSa2]),

$$\begin{aligned}
\|h\|_{B_2(\mathbb{D})}^2 &\leq C \sum_{j=1}^{J} \left|\left(T^{-1}\eta\right)_j\right|^2 \int_{\mathbb{D}} \left|\left(1-|\zeta|^2\right) g_w(\zeta)\right|^2 d\lambda_1(\zeta)\, d\zeta \\
&\leq C \sum_{j=1}^{J} \left|\left(T^{-1}\eta\right)_j\right|^2 \left(\log \frac{1}{1-|w|^2}\right)^{-1} \\
&\leq C \left\|T^{-1}\eta\right\|_{\ell^2(d\mu)}^2 \leq C \|\eta\|_{\ell^2(d\mu)}^2 \leq C \|\xi\|_{\ell^2(d\mu)}^2 .
\end{aligned}$$

Since all of this is uniform in J we may let $J \to \infty$ and use a normal families argument to complete the proof of Theorem 4. Indeed, if $h_J \in B_2(\mathbb{D})$ satisfies

$$\begin{aligned}
\|h_J\|_{B_2(\mathbb{D})} &\leq C, \quad 1 \leq j \leq J, \\
h_J(z_j) &= \xi_j, \quad 1 \leq j \leq J,
\end{aligned}$$

then $|h_J(z)| \leq C\left(1+\log\frac{1}{1-|z|^2}\right)^{\frac{1}{2}} \|h_J\|_{B_2(\mathbb{D})}$ shows that $\{h_J\}_{J=1}^{\infty}$ is a normal family on the ball $\mathbb{D}$. If the subsequence $\{h_{J_k}\}_{k=1}^{\infty}$ converges uniformly on compact subsets of $\mathbb{D}$, then the limit $h=\lim_{k\to\infty} h_{J_k}$ satisfies

$$\int_{\mathbb{D}} \left|\left(1-|z|^2\right) f'(z)\right|^2 d\lambda_1(z) < \infty,$$

$$\begin{aligned}
\|h\|_{B_2(\mathbb{D})} &= \left(\int_{\mathbb{D}} \left|\left(1-|z|^2\right) h'(z)\right|^2 d\lambda_1(z)\right)^{\frac{1}{2}} \\
&\leq \liminf_{k\to\infty} \left(\int_{\mathbb{D}} \left|\left(1-|z|^2\right) h'_{J_k}(z)\right|^2 d\lambda_1(z)\right)^{\frac{1}{2}} \\
&\leq C, \\
h_J(z_j) &= \xi_j, \quad 1 \leq j < \infty.
\end{aligned}$$

3. Examples

3.1. **An onto interpolating subtree with infinite measure.** Let $a, b > 1$ satisfy $\left[a^{k+1}b\right] \geq \left[a^k b\right] + 1$ for all $k \geq 0$ (in particular this will hold if $(a-1)\,b \geq 2$), and define a Cantor-like sequence Z

$$Z = Z_{a,b} = \cup_{k=0}^{\infty} \left\{z_j^k\right\}_{j=1}^{2^k} \subset \mathcal{T}$$

as follows. Pick a point $z_1^0 = 0$ of $\mathcal{T}$ satisfying $d\left(z_1^0\right) = [b]$. Then choose 2^1 points $\left\{z_1^1, z_2^1\right\} \subset \mathcal{T}$ that are successors to distinct children of z_1^0 and having $d\left(z_j^1\right) = [ab]$, $1 \leq j \leq 2^1$, and $\beta\left(z_i^1, z_j^1\right) \gtrapprox [ab]$ for $i \neq j$. Then choose 2^2 points $\left\{z_1^2, z_2^2, z_3^2, z_4^2\right\} \subset \mathcal{T}$ that are successors to distinct children of the points in $\left\{z_1^1, z_2^1\right\}$ and having $d\left(z_j^2\right) = \left[a^2 b\right]$, $1 \leq j \leq 2^2$, and $\beta\left(z_i^2, z_j^2\right) \gtrapprox \left[a^2 b\right]$ for $i \neq j$. Having constructed 2^k points $\left\{z_j^k\right\}_{j=1}^{2^k} \subset \mathcal{T}$ in this way, we then choose 2^{k+1} points $\left\{z_j^{k+1}\right\}_{j=1}^{2^{k+1}} \subset \mathcal{T}$ that are successors to distinct children of the points in $\left\{z_j^k\right\}_{j=1}^{2^k}$ and having $d\left(z_j^{k+1}\right) = \left[a^{k+1}b\right]$, $1 \leq j \leq 2^k$, and $\beta\left(z_i^{k+1}, z_j^{k+1}\right) \gtrapprox \left[a^{k+1}b\right]$ for $i \neq j$. Note that the condition $\left[a^{k+1}b\right] \geq \left[a^k b\right] + 1$ allows for the existence of such points. Then $Z = \cup_{k=0}^{\infty} \left\{z_j^k\right\}_{j=1}^{2^k}$ satisfies the separation condition (1.3) with constant roughly $a-1$ and condition (1.7) with β close to 1, and the associated measure μ_Z satisfies the weak simple condition (1.6) with constant 2. Thus Theorem 4 applies to show that Z is onto interpolating for $B_2(\mathbb{D})$. Yet the total mass of the measure μ_Z satisfies

$$\|\mu_Z\| = \sum_{k=0}^{\infty} 2^k \left[a^k b\right]^{-1} \approx \frac{1}{b} \sum_{k=0}^{\infty} \left(\frac{2}{a}\right)^k = \infty$$

if $a \leq 2$.

3.2. **A separated sequence with finite measure satisfying weak simple but not simple.** We now use the above example to construct a separated sequence W in the disk with *finite* measure $\mu = \mu_W$ satisfying the weak simple condition but not the simple condition. This yields an example of a sequence which fails the simple condition, but to which Theorem 1 applies.

With notation as in the previous subsubsection, we choose $a = 2$ for convenience, let $b, N \in \mathbb{N}$ be large integers, and replace the sequence $Z_{2,b}$ above with the truncated sequence $Z_{2,b,N} = \cup_{k=0}^{N} \left\{ z_j^k \right\}_{j=1}^{2^k}$. Then $Z_{2,b,N}$ satisfies the separation condition (1.3) with constant roughly 1, the associated measure $\mu_{Z_{2,b,N}}$ satisfies the weak simple condition (1.6) with constant 2, and the total mass of $\mu_{Z_{2,b,N}}$ is about $\frac{N}{b}$:

$$\left\| \mu_{Z_{2,b,N}} \right\| = \sum_{k=0}^{N} 2^k \left[2^k b \right]^{-1} \approx \frac{1}{b} \sum_{k=0}^{N} \left(\frac{2}{2} \right)^k \approx \frac{N}{b}.$$

On the other hand the constant $C\left(\mu_{Z_{2,b,N}}\right)$ in the simple condition (1.4) for $\mu_{Z_{2,b,N}}$ satisfies

$$C\left(\mu_{Z_{2,b,N}}\right) \gtrsim N, \tag{3.1}$$

since

$$\frac{N}{b} \approx \left\| \mu_{Z_{2,b}^*} \right\| = \sum_{\alpha \geq z_0} \mu_{Z_{2,b}^*} (\alpha) \leq C \frac{1}{d(z_0)} = \frac{C}{b}.$$

It is now an easy exercise to choose sequences of parameters $\{b(n)\}_{n=1}^{\infty}$ and $\{N(n)\}_{n=1}^{\infty}$, and initial points $\left\{ z_1^0 (n) \right\}_{n=1}^{\infty}$ so that the corresponding sequences $Z_{2,b(n),N(n)} = \cup_{k=0}^{N(n)} \left\{ z_j^k (n) \right\}_{j=1}^{2^k}$ satisfy

$$\left\| \mu_{Z_{2,b(n),N(n)}} \right\| \approx \frac{N(n)}{b(n)} \leq 2^{-n} \tag{3.2}$$

and

$$\lim_{n \to \infty} N(n) = \infty, \tag{3.3}$$

along with the nested property

$$z_1^0 (n+1) \geq z_1^{b(n)} (n), \qquad n \geq 1. \tag{3.4}$$

Then the union $W = \cup_{n=1}^{\infty} Z_{2,b(n),N(n)}$ satisfies the separation condition and the associated measure μ_W is finite by (3.2), satisfies the weak simple condition by (3.4), yet fails the simple condition by (3.1) and (3.3).

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