UNIQUENESS IN THE CAUCHY PROBLEM FOR A CLASS OF
HYPOELLIPTIC ULTRAPARABOLIC OPERATORS

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Abstract. We consider a class of hypoelliptic ultraparabolic operators in the form

\[ \mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t, \]

under the assumption that the vector fields \( X_1, \ldots, X_m \) and \( X_0 - \partial_t \) are invariant with respect to a suitable homogeneous Lie group \( G \). We show that if \( u, v \) are two solutions of \( \mathcal{L}u = 0 \) on \( \mathbb{R}^N \times [0, T] \) and \( u(\cdot, 0) = \varphi \), then each of the following conditions: \( |u(x, t) - v(x, t)| \) can be bounded by \( M \exp(c|x|_G^2) \), or both \( u \) and \( v \) are non-negative, implies \( u \equiv v \).

We use a technique which relies on a pointwise estimate of the fundamental solution of \( \mathcal{L} \).

1. Introduction and Main Results

We consider a class of linear second order operators in \( \mathbb{R}^{N+1} \) of the form

\[ (1.1) \quad \mathcal{L} = \sum_{j=1}^{m} X_j^2 + X_0 - \partial_t, \]

where \( 1 \leq m \leq N \). In (1.1) the \( X_j \)'s are smooth vector fields on \( \mathbb{R}^N \), i.e. denoting \( z = (x, t) \) the point in \( \mathbb{R}^{N+1} \),

\[ X_j(x) = \sum_{k=1}^{N} a_{j,k}(x) \partial_{x_k}, \quad j = 0, \ldots, m, \]

and any \( a_{j,k} \) is a \( C^\infty \) function. In the sequel we also consider the \( X_j \)'s as vector fields in \( \mathbb{R}^{N+1} \) and denote

\[ Y = X_0 - \partial_t. \]

Our main assumption on the operator \( \mathcal{L} \) is the invariance with respect to a homogeneous Lie group structure, and a controllability condition:

[H.1] there exists a homogeneous Lie group \( G = (\mathbb{R}^{N+1}, \circ, \delta_\lambda) \) such that

i) \( X_1, \ldots, X_m, Y \) are left translation invariant on \( C \);

ii) \( X_1, \ldots, X_m \) are \( \delta_\lambda \)-homogeneous of degree one and \( Y \) is \( \delta_\lambda \)-homogeneous of degree two;

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for every \((x,t),(y,s)\) \(\in \mathbb{R}^{N+1}\) with \(t > s\), there exists an absolutely continuous path \(\gamma: [0, t - s] \to \mathbb{R}^{N+1}\) such that

\[
\gamma(0) = (x,t), \quad \gamma(t - s) = (y,s), \quad \gamma'(\tau) = \sum_{k=1}^{m} \omega_k(\tau)X_k(\gamma(\tau)) + Y(\gamma(\tau)), \quad \text{a.e. in } [0, t - s],
\]

(1.2)

with \(\omega_1, \ldots, \omega_m \in L^\infty([0, t - s])\). We say that this curve is an \(\mathcal{L}\)-admissible path connecting \((x,t)\) with \((y,s)\), and in the sequel we will denote it by \(\gamma((x,t),(y,s),\omega)\).

It is known that \([H.1]\) implies that the coefficients \(a_j^k\) of the \(X_j\)'s are polynomial functions, hence \([H.2]\) yields

\[(1.3) \quad \text{rank Lie}\{X_1, \ldots, X_m, Y\}(z) = N + 1, \quad \text{for every } z \in \mathbb{R}^{N+1},\]

(see, for instance, [8] or [16, Chap. II, Sec. 8]). This is the well known Hörmander condition for the hypoellipticity of \(\mathcal{L}\) (see [11]).

Operators of the form (1.1), verifying assumptions \([H.1]-[H.2]\), have been studied by Kogoj and Lanconelli in [12], [13] and [14]. Note that in [12] and [14] the \(\mathcal{L}\)-admissible path \(\gamma\) in \([H.2]\) is supposed to satisfy \(\gamma'(\tau) = \sum_{k=1}^{m} \omega_k(\tau)X_k(\gamma(\tau)) + \mu(\tau)Y(\gamma(\tau))\) for piecewise constant real functions \(\omega_1, \ldots, \omega_m, \mu\), with \(\mu \geq 0\). However, even if this definition is slightly different from our one, the main results stated in these papers hold true also with our assumption. In [12] it is proved that \(\mathcal{L}\) has a fundamental solution \(\Gamma(\cdot, \zeta)\) which shares several properties of the fundamental solution of the heat equation (see Section 2 for the details).

In this paper we prove some uniqueness results for the Cauchy problem related to \(\mathcal{L}\):

\[
\begin{aligned}
\mathcal{L}u &= 0 \quad \text{in } \mathbb{R}^N \times ]0,T[; \\
u(\cdot, 0) &= \varphi \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]

(1.4)

where \(\varphi \in C(\mathbb{R}^N)\). Our main achievements are the following ones:

**Theorem 1.1.** Assume that \(\mathcal{L}\) satisfies conditions \([H.1]-[H.2]\), and let \(u, v \in C(\mathbb{R}^N \times [0,T])\) be two solutions of the Cauchy problem (1.4). If there exists a positive constant \(c\) such that

\[
\int_0^T \int_{\mathbb{R}^N} e^{-\left|\cdot\right|_\mathcal{G}^2} |u(x,t) - v(x,t)| \, dx \, dt < \infty
\]

then \(u \equiv v\).

(See Section 2 for the definition of the norm \(|\cdot|_\mathcal{G}\)).

**Theorem 1.2.** Assume that \(\mathcal{L}\) satisfies conditions \([H.1]-[H.2]\), and let \(u, v \in C(\mathbb{R}^N \times [0,T])\) be two solutions of the Cauchy problem (1.4). If both \(u\) and \(v\) are non negative, then \(u \equiv v\).

Our theorems 1.1 and 1.2 extend some classical uniqueness result for parabolic operators. We first quote the paper by Tychoff [19], where it is shown that the Cauchy problem for the heat equation has an unique solution satisfying \(u(x,t) \leq M e^{c|t|^2}\) for every \((x,t) \in \mathbb{R}^N \times [0,T]\). On the other hand, Widder in [20] proved that the unique non-negative solution of the heat equation in \(\mathbb{R} \times [0,T]\) such that \(u(\cdot,0) = 0\) is the null function.

With regard to more general parabolic operators, Krzyżański [15] showed that the Tychonoff condition ensures uniqueness for parabolic operators in non-divergence form with
bounded and continuous coefficients. Serrin in [18] extended Widder’s result to solutions of equations in the form \(u_t = a(x)u_{xx,x} + b(x)u_x + c(x)u\) with H"older continuous and uniformly bounded coefficients. Aronson and Besala in [1] proved that, if \(u\) is a solution of a divergence form parabolic equation with measurable coefficients satisfying certain growth condition at infinity, the uniqueness of the homogeneous Cauchy problem is guaranteed by the following integral condition:

\[
\int_0^T \int_{\mathbb{R}^N} e^{-\epsilon|\xi|^2} u^2(x,t) \, dx \, dt < \infty.
\]

Moreover, the same authors in [2] proved that an hypothesis analogous to (1.5) yields the uniqueness for a class of uniformly parabolic operators

\[
\sum_{i,j} a_{i,j}(x,t) \partial_{x_i} x_j + \sum_j b_j(x,t) \partial_{x_j} - \partial_t
\]

with locally H"older continuous coefficients which grow at most linearly at infinity.

Concerning Kolmogorov-type operators, some results like our Theorem 1.1 and 1.2 have been obtained by Polidoro [17], Di Francesco and Pascucci [9], Di Francesco and Polidoro [10]. In [17] it is showed that there is only one solution which is in the Tychonoff class or non-negative to the operator \(L = \text{div}(A(z)D) + \langle x, BD \rangle - \partial_t\); here \(L\) is homogeneous with respect to a suitable Lie group structure, and the \(a_{i,j}(z)\)'s are uniformly H"older continuous with respect to the geometry of \(L\). This results have been improved respectively in [9] and [10] for Kolmogorov equations in non-divergence form, assuming that the coefficients and their derivatives are bounded and H"older continuous (in a certain sense), and removing the homogeneity assumption. In all these papers, the authors relied on pointwise estimates for the fundamental solution of the operator considered.

Finally, we quote the paper of Bonfiglioli, Lanconelli and Uguzzoni [3], where Tychonoff-type and Widder-type uniqueness theorems are extended for the heat operator \(H = L - \partial_t\) related to the sub-Laplacian \(L\) on a stratified Carnot group.

We recall that Tychonoff constructed in [19] a non trivial solution \(u\) to the Cauchy problem for the heat equation such that \(u(\cdot,0) = 0\) and \(u(x,t) \leq Me^{c|x|^2/2+\epsilon}\) in \(\mathbb{R}^N \times [0,T]\). Since our results apply to heat operators, this example shows that the growth condition allowed in Theorem 1.1 cannot be improved by increasing the exponent of \(|x|\). Nevertheless, it seem possible to sharpen hypothesis (1.5) by using the value function \(V\) (see Remark 2.4 in Section 2).

This paper is organized as follows: in Section 2 we collect some preliminaries, and in Section 3 we give the proof of our main results, Theorem 1.1 and 1.2.

2. Notations and Preliminary Results

We say that a Lie group \(G = (\mathbb{R}^{N+1}, \circ)\) is homogeneous if on \(G\) there exists a family of dilations \(\{\delta_\lambda\}_{\lambda > 0}\) which is an automorphism of the group:

\[
\delta_\lambda(z \circ \zeta) = (\delta_\lambda z) \circ (\delta_\lambda \zeta), \quad \text{for all } z, \zeta \in \mathbb{R}^{N+1} \text{ and } \lambda > 0.
\]

As we stated in the Introduction, [H.1] and [H.2] imply the H"ormander condition (1.3). From [H.1] and [H.2] it follows also that the composition law \(\circ\) is euclidean in the “time” variable, i.e.

\[
(x,t) \circ (\xi,\tau) = (S(x,t,\xi,\tau), t + \tau)
\]

for a suitable \(C^\infty\) function \(S\) with value in \(\mathbb{R}^N\). Moreover the dilation of the group induces a direct sum decomposition on \(\mathbb{R}^N\)

\[
\mathbb{R}^N = V_1 \oplus \cdots \oplus V_k,
\]
as follows. If \( x = x^{(1)} + x^{(2)} + \cdots + x^{(k)} \) with \( x^{(j)} \in V_j \), then \( \delta_\lambda(x, t) = (D(\lambda)x, \lambda^2 t) \), where
\[
D(\lambda)(x^{(1)} + x^{(2)} + \cdots + x^{(k)}) = (\lambda x^{(1)} + \lambda^2 x^{(2)} + \cdots + \lambda^k x^{(k)}).
\]
The natural number
\[
Q = 2 + \sum_{j=1}^k j \dim V_j
\]
is usually called the \textit{homogeneous dimension} of \( G \) with respect to \( \delta_\lambda \). We set
\[
|x|_G = \left( \sum_{j=1}^k \sum_{i=1}^{\dim V_j} (x^{(j)}_i)^{\frac{2k_i}{\lambda}} \right)^{\frac{1}{\dim V_j}}, \quad \|(x, t)\|_G = \left( |x|_G^{\dim V_j} + |t|^{\dim V_j} \right)^{\frac{1}{\dim V_j}},
\]
and we observe that the above functions are \( \delta_\lambda \)-homogeneous of degree 1, respectively on \( \mathbb{R}^N \) and \( \mathbb{R}^{N+1} \):
\[
\left| (\lambda x^{(1)} + \cdots + \lambda^k x^{(k)}) \right|_G = \lambda |x|_G, \quad \left| \delta_\lambda(x, t) \right|_G = \lambda \|(x, t)\|_G,
\]
for every \((x, t) \in \mathbb{R}^{N+1}\) and for any \( \lambda > 0 \). We define the quasi-distance in \( G \) as
\[
d(z, \zeta) := \|\zeta^{-1} \circ z\|_G, \quad \text{for all } z, \zeta \in \mathbb{R}^{N+1},
\]
and we recall that, for a positive constant \( c \),
\[i) \ d(z, \zeta) \leq c d(\zeta, z) \quad \text{for all } z, \zeta \in \mathbb{R}^{N+1};
\]
\[ii) \ d(z, \zeta) \leq c (d(z, z_1) + d(z_1, \zeta)) \quad \text{for all } z, z_1, \zeta \in \mathbb{R}^{N+1}.
\]
Moreover,
\[
d(\delta_\lambda z; \delta_\lambda \zeta) = \lambda d(z, \zeta), \quad \text{for all } z, \zeta \in \mathbb{R}^{N+1}, \lambda > 0.
\]
Throughout the paper we shall write \( d(z) \) instead of \( d(0, z) = \|z^{-1}\|_G \). Obviously, from \( i) \), we have
\[
c^{-1} \|z\|_G \leq d(z) \leq c \|z\|_G.
\]
We also use the following notation for the \( |\cdot|_G \)-ball of radius \( r > 0 \) centered at the origin:
\[
B_r = \{ x \in \mathbb{R}^N \mid |x|_G < r \}.
\]
We recall the following weak maximum principle on strips (see e.g. [6, Theorem 4.1] for an elementary proof based on suitable mean-value representation formulas).

**Proposition 2.1.** Let \( u \in C^2(\mathbb{R}^N \times [0, T]) \). If \( Lu \geq 0 \), \( \lim\sup u \leq 0 \) both in \( \mathbb{R}^N \times \{0\} \) and at infinity, then \( u \leq 0 \) in the whole strip.

We next collect some useful facts on the fundamental solution of the hypoelliptic operator \( \mathcal{L} \). If \( \Gamma(\zeta, \cdot) \) is the fundamental solution of \( \mathcal{L} \) with pole in \( \zeta \in \mathbb{R}^{N+1} \), then \( \Gamma \) is smooth out of the pole and has the following properties (see [12]):
\[i) \ \text{for any } z \in \mathbb{R}^{N+1}, \Gamma(\cdot, z) \text{ and } \Gamma(z, \cdot) \text{ belong to } L^1_{\text{loc}}(\mathbb{R}^{N+1});
\]
\[ii) \ \Gamma(z, \zeta) \geq 0, \text{ and } \Gamma(x, t, \xi, \tau) > 0 \text{ if, and only if, } t > \tau;
\]
\[iii) \ \text{for every } \varphi \in C^\infty_c(\mathbb{R}^{N+1}) \text{ and } x \in \mathbb{R}^N \text{ we have}
\]
\[
\lim_{t \to \tau^+} \int_{\mathbb{R}^N} \Gamma(x, t, \xi, \tau) \varphi(\xi) \, d\xi = \varphi(x);
\]
Let \( \varphi \in C^\infty_0(\mathbb{R}^{N+1}) \) and \( z \in \mathbb{R}^{N+1} \) we have
\[
\mathcal{L} \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) \varphi(\zeta) \, d\zeta = \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) \mathcal{L}\varphi(\zeta) \, d\zeta = -\varphi(z);
\]
\( \mathcal{L} \Gamma(\cdot, \cdot) = -\delta_\zeta \), where \( \delta_\zeta \) denotes the Dirac measure supported at \( \{\zeta\} \); \( \mathcal{L} \Gamma(\cdot, \cdot) = -\delta_\zeta \), where \( \delta_\zeta \) denotes the Dirac measure supported at \( \{\zeta\} \); \( \mathcal{L} \Gamma(\cdot, \cdot) = -\delta_\zeta \), where \( \delta_\zeta \) denotes the Dirac measure supported at \( \{\zeta\} \); \( \mathcal{L} \Gamma(\cdot, \cdot) = -\delta_\zeta \), where \( \delta_\zeta \) denotes the Dirac measure supported at \( \{\zeta\} \);
\[
\Gamma(x, \xi, \tau) = \int_{\mathbb{R}^N} \Gamma(x, t, \xi, \tau) \, dx = 1;
\]
\[
\Gamma(x, t, \xi, \tau) = \int_{\mathbb{R}^N} \Gamma(x, t, y, s) \varphi(y, \xi, \tau) \, dy;
\]
\[
\Gamma^*(z, \zeta) := \Gamma(\zeta, z) is the fundamental solution of the adjoint \( \mathcal{L}^* \) of \( \mathcal{L} \).
\]

Moreover, \( \Gamma \) is invariant with respect to the group operation and \( \delta_\lambda \)-homogeneous of degree \( 2 - Q \):
\[
\Gamma(z, \eta) = \Gamma(\zeta^{-1} \circ z, 0) =: \Gamma(\zeta^{-1} \circ z), \quad z, \zeta \in \mathbb{R}^{N+1}, z \neq \zeta;
\]
\[
\Gamma(\delta_\lambda(z)) = \lambda^{2-Q} \Gamma(z), \quad z \in \mathbb{R}^{N+1} \setminus \{0\}, \ \lambda > 0.
\]

Theorem 2.2. Let \( \varphi \in C(\mathbb{R}^N) \) satisfying \( \varphi(x) = O(|x|^n) \) as \( |x| \to \infty \), for some \( n \in \mathbb{N} \). Then the function
\[
u(x, t) = \int_{\mathbb{R}^N} \Gamma(x, t, \xi, \tau) \varphi(\xi) \, d\xi, \quad x \in \mathbb{R}^N, t > \tau
\]
is well defined and is a solution to the Cauchy problem
\[
\begin{cases}
\mathcal{L}u = 0 & \text{in } \mathbb{R}^N \times ]\tau, \infty[, \\
\lim_{(x,t) \to (y,\tau)} u(x, t) = \varphi(y), & \text{for every } y \in \mathbb{R}^N.
\end{cases}
\]

The main tools we shall employ in the proof of our results are the following pointwise estimates of \( \Gamma \) and its derivatives, proved by Kogoj and Lanconelli. There exists a positive constant \( C \) such that
\[
\Gamma(x, t, \xi, \tau) \leq \frac{C}{(t - \tau)^{\frac{Q}{2}}} \exp \left( - \frac{d^2((\xi, \tau)^{-1} \circ (x, t))}{C(t - \tau)} \right),
\]
for every \( x, \xi \in \mathbb{R}^N \) and \( t > \tau \) (see [12, (5.1)] and also [14, (2)]). Moreover, as a consequence of a Harnack-type inequality for non-negative solution to \( \mathcal{L}u = 0 \), for any \( j = 1, \ldots, m \) there exists \( c_j > 0 \) with
\[
|X_j \Gamma(x, t, \xi, \tau)| \leq c_j (t - \tau)^{-\frac{1}{2}} \Gamma \left( (\xi, \tau)^{-1} \circ (x, t) \circ \left( 0, -\frac{t - \tau}{2} \right) \right)^{-1},
\]
for all \( x, \xi \in \mathbb{R}^N, t > \tau \) (see (7) in [14]). We point out that the upper bound in (2.5) is not the best possible. Indeed, in [7] it is given a more precise asymptotic behavior of the exponent in terms of the value function \( V \) of the following optimal control problem related to the ordinary differential equation in (1.2):
Definition 2.3. Let \((x,t),(y,s) \in \mathbb{R}^{N+1}\), with \(t > s\), and let \(\gamma((x,t),(y,s),\omega)\) be any \(L\)-admissible path connecting \((x,t)\) with \((y,s)\):

\[
\gamma'(\tau) = \sum_{k=1}^{m} \omega_k(\tau)X_k(\gamma(\tau)) + Y(\gamma(\tau)), \quad \gamma(0) = (x,t), \quad \gamma(t-s) = (y,s).
\]

We consider the set of functions \(\omega_1, \ldots, \omega_m\) as the control of the path \(\gamma\), and the integral

\[
\Phi(\omega) = \int_0^{t-s} (\omega_1^2(\tau) + \cdots + \omega_m^2(\tau)) \, d\tau
\]

as its cost. We then define the value function

\[
V(x,t,y,s) = \inf \{ \Phi(\omega) | \gamma((x,t),(y,s),\omega) \text{ is an } L\text{-admissible path} \}.
\]

Hence, if \(V\) is locally Lipschitz continuous, for every \(\varepsilon > 0\) there exists a positive constant \(C_\varepsilon\), only depending on the vector fields \(X_1, \ldots, X_m, Y\) and on \(\varepsilon\), such that

\[
\Gamma(x,t,0,0) \leq \frac{C_\varepsilon}{t^{\frac{2}{2^*}}} \exp \left( -\frac{1}{32} V((0,\varepsilon t) \circ (x,t) \circ (0,\varepsilon t),0,0) \right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}^+.
\]

(see [7, Theorem 1.6]). On the other hand, Theorem 1.2 of [5] provides a lower bound of \(\Gamma\), also stated in terms of \(V\): there exists two constants \(C > 0\) and \(\theta \in ]0,1[\) only depending on \(L\), such that

\[
\Gamma(x,t,0,0) \geq \frac{1}{C t^{\frac{2}{2^*}}} \exp \left( -CV(x,\theta^2 t,0,0) \right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}^+.
\]

Remark 2.4. With these more careful Gaussian estimates at hands, maybe it is possible to improve Theorem 1.1. Indeed, by comparing (2.7) with (2.5), we deduce that

\[
V(x,\theta^2 t,0,0) \geq c_0 \frac{|x|^2}{t}, \quad x \in \mathbb{R}^N, \quad t > 0,
\]

for a positive constant \(c_0\). Thus, a growth condition on \(u - v\) in terms of \(V\) of this kind:

\[
\int_0^T \int_{\mathbb{R}^N} e^{-CV(x,\tau,0,0)} |u(x,t) - v(x,t)| \, dx \, dt < \infty, \quad \tau \in \mathbb{R}^+,
\]

is sharper than (1.5). However, working with \(V\) yields some technical problems, mainly due to the fact that we do not always know explicitly the expression of \(V\). We plan to check on this in a forthcoming paper.

3. Proof of the main results

The purpose of this section is the proof of Theorems 1.1 and 1.2, our main results. We start with a lemma.

Lemma 3.1. Let \(u,v \in C(\mathbb{R}^N \times [0,T])\) be two solutions of the Cauchy problem (1.4). If we set \(w = u - v\), then there exists a positive constant \(C\) such that

\[
|w(x,t)| \leq C \int_0^t \int_{\mathbb{R}^N \setminus B_R} |w(\xi,\tau)| \left( |\Gamma(x,t,\xi,\tau)| + \sum_{j=1}^m |X_j \Gamma(x,t,\xi,\tau)| \right) \, d\xi \, d\tau,
\]

for every \((x,t) \in \mathbb{R}^N \times ]0,T[\) and \(R > |x|_G\).
Proof. Since the vector fields \( X_j = \sum_{k=1}^{N} a^j_k \partial_{x_k} \) for \( j = 0, \ldots, m \) are \( \delta \lambda \)-homogeneous of a positive degree, the coefficient \( a^j_k(x) \) does not depend of \( x_k \), for any \( k = 1, \ldots, N \). As a consequence, the \( X_j \)'s are divergence free, \( X_j^2 = -X_j \) and

\[
\psi L w - w L^* \psi = \sum_{h,k=1}^{N} \partial_{x_k} (a_{h,k}(\psi \partial_{x_k} w - w \partial_{x_k} \psi)) + \sum_{k=1}^{N} \partial_{x_k} (a^0_k w \psi) - \partial_t (w \psi),
\]

for any \( w, \psi \in C^\infty_0(\mathbb{R}^{N+1}) \).

Now, let \( (x, t) \in \mathbb{R} \times ]0, T[ \) be fixed. For any \( R > |x|_G \) we consider \( h_R \in C^\infty_0(B_{R+1}) \), \( 0 \leq h_R \leq 1 \), such that \( h_R \equiv 1 \) on \( B_{R} \) and with first and second order derivatives bounded uniformly w.r.t. \( R \). We integrate the Green’s identity (3.3) with \( w = u - v \) and \( \psi(x, \xi, \tau) = h_R(\xi) \Gamma(x, t, \xi, \tau) \) over the domain \( \{ \xi \in \mathbb{R}^{N+1} | \xi \in B_{R+1}, 0 < \tau < t - \delta \} \), for some \( \delta > 0 \). Recalling that \( Lw = 0 \) and using the divergence theorem, we get

By hypothesis, the last three terms in the above equation are null. Hence, as \( \delta \) tends to \( 0^+ \), by using the property \( vi \) of \( \Gamma \) and (2.5), we obtain

\[
\psi L w = \lim_{\delta \to 0^+} \int_{B_{R+1}} \psi L w = \int_{B_{R+1}} \psi L^* \psi \d \xi \d \tau.
\]

Being \( L^* \Gamma(x, t, \xi, \tau) = 0 \) and \( \text{supp} (\partial_{\xi_k} h_R) \subset B_{R+1} \setminus B_R \), we obtain

\[
w(x, t) = \int_{B_{R+1}} w(x, t) (\Gamma(x, t, \xi, \tau) L^* h_R(\xi) + 2A(\xi) \nabla \Gamma(x, t, \xi, \tau), \nabla h_R(\xi)) \d \xi \d \tau.
\]
Now (3.1) directly follows from the above equation, by using the Cauchy-Schwartz inequality and (3.2). The assertion is proved.

As a simple corollary of the previous lemma, we have the following

**Proposition 3.2.** Let \( u, v \in C(\mathbb{R}^N \times [0, T]) \) be two solutions of the Cauchy problem (1.4). If

\[
(3.4) \quad \int_0^t \int_{\mathbb{R}^N} |u(\xi, \tau) - v(\xi, \tau)|(\Gamma(x, t, \xi, \tau) + \sum_{j=1}^m |X_j \Gamma(x, t, \xi, \tau)|) \, d\xi \, d\tau < \infty
\]

for every \((x, t) \in \mathbb{R}^N \times [0, T]\), then \( u \equiv v \).

**Proof.** Condition (3.4) implies

\[
\lim_{R \to \infty} \int_0^t \int_{\mathbb{R}^N \setminus B_R} |u(\xi, \tau) - v(\xi, \tau)|(\Gamma(x, t, \xi, \tau) + \sum_{j=1}^m |X_j \Gamma(x, t, \xi, \tau)|) \, d\xi \, d\tau = 0,
\]

then, by (3.1), \( u \equiv v \) in the strip \( \mathbb{R} \times [0, T] \). This ends the proof.

As we will see, the hypothesis of Theorem 1.1 is another condition which, like (3.4), together with Lemma 3.1 implies the uniqueness of the solution of the Cauchy problem (1.4).

**Proof of Theorem 1.1.** Let \( u, v \in C(\mathbb{R}^N \times [0, T]) \) be two solutions to the Cauchy problem (1.4). We first prove that \( u \equiv v \) in a thin strip \( \mathbb{R}^N \times ]0, \epsilon[ \), where \( \epsilon \in ]0, \min\{1, T\} \) will be suitably chosen later.

Let \((x, t) \in \mathbb{R}^N \times ]0, \epsilon[ \) be fixed. Aiming to use (3.1), we estimate the fundamental solution \( \Gamma \) with pole in \((\xi, \tau) \in \mathbb{R}^N \times ]0, t[ \) valued in \((x, t) \), and every \( X_j \Gamma(x, t, \xi, \tau) \)'s.

By using the pseudo-triangular inequality for \( d \), we get

\[
d((\xi, \tau)^{-1} \circ (x, t)) \geq c^{-1} d((\xi, \tau)^{-1} \circ (x, t), (\xi, \tau)^{-1})
\]

\[
= c^{-1} \| (\xi, \tau) \|_G - \| (x, t) \|_G \geq c^{-1} |\xi|_G - \| (x, t) \|_G \geq (2c)^{-1} |\xi|_G,
\]

if we take \( |\xi|_G \geq 2c \sup_{0 < t < T} \| (x, t) \|_G =: R_1(x) \). Hence, by the Gaussian estimate (2.5),

\[
(3.5) \quad \Gamma(x, t, \xi, \tau) \leq \frac{C}{(t - \tau)^{\frac{N+1}{2}} \exp \left(-\frac{|\xi|_G^2}{4c^2C(t - \tau)}\right)}, \quad \text{if} \quad |\xi|_G \geq R_1(x).
\]

On the other hand, (2.6) and (2.5) imply that for any \( j = 1, \ldots, m \) there exists a positive constant \( c_j \) such that

\[
|X_j \Gamma(x, t, \xi, \tau)| \leq \frac{c_j C}{(t - \tau)^{\frac{N+1}{2}}} \exp \left(-\frac{d^2((\xi, \tau)^{-1} \circ (x, t) \circ (0, -\frac{t-\tau}{2})^{-1})}{C(t - \tau)}\right);
\]

see also (2) in [14]. With the same argument as above, we have

\[
d((\xi, \tau)^{-1} \circ (x, t) \circ (0, -\frac{t-\tau}{2})^{-1}) \geq c^{-1} d((0, \frac{t-\tau}{2})^{-1} \circ (x, t) \circ (0, -\frac{t-\tau}{2})^{-1}) \geq (\sqrt{2} c)^{-2} |\xi|_G,
\]

\[
\geq \delta \left(\sqrt{(t-\tau)^2/2}((0, -1)^{t-1})\right)_G \geq \sqrt{2} \delta c |\xi|_G - d((0, 1))^2 \geq |\xi|_G - d((0, 1))^2 \geq |\xi|_G - d((0, 1))^2 \geq (\sqrt{2} c)^{-2} |\xi|_G.
\]
for $|\xi|_G \geq R_1(x) + c^2 d((0, -1)) \sqrt{2T} =: R(x)$. It follows that, for every $j = 1, \ldots, m$,

$$
|X_j \Gamma(x, t, \xi, \tau)| \leq \frac{c_j C}{(t - \tau)^{\frac{Q-1}{2}}} \exp \left(- \frac{|\xi|^2_{V}}{\sqrt{2} c C(t - \tau)}\right), \quad \text{if } |\xi|_G \geq R(x).
$$

As a consequence, from (3.1), (3.5) and (3.6),

$$
|u(x, t) - v(x, t)| \leq C \int_0^t \int_{\mathbb{R}^N \setminus B_R} |u(\xi, \tau) - v(\xi, \tau)| \left(\Gamma(x, t, \xi, \tau) + \sum_{j=1}^{m} |X_j \Gamma(x, t, \xi, \tau)|\right) d\xi d\tau
$$

$$
\leq C_1 \int_0^t \int_{\mathbb{R}^N \setminus B_R} |u(\xi, \tau) - v(\xi, \tau)| \frac{1}{(t - \tau)^{\frac{Q-1}{2}}} \exp \left(- \frac{C_2|\xi|^2_{V}}{2(t - \tau)}\right) d\xi d\tau,
$$

for every $R > \max\{|x|_G, R(x)\}$, where $C_1, C_2$ are two positive constants only dependent on $c$ and on the operator $L$. Now set $\varepsilon = \min \left\{ \frac{C_2}{2c^2}, 1, T \right\}$, where $c > 0$ is the constant in (1.5). Since the function $(\xi, \tau) \mapsto (t - \tau)^{-\frac{Q-1}{2}} \exp \left(- \frac{C_2|\xi|^2_{V}}{2(t - \tau)}\right)$ is bounded on $(\mathbb{R}^N \setminus B_R) \times ]0, T[$, by the choice of $\varepsilon$ we get

$$
|u(x, t) - v(x, t)| \leq C_3 \int_0^t \int_{\mathbb{R}^N \setminus B_R} e^{-c|\xi|^2_{V}} |u(\xi, \tau) - v(\xi, \tau)| d\xi d\tau, \quad \forall R > \max\{|x|_G, R(x)\}.
$$

On the other hand, hypothesis (1.5) implies that

$$
\lim_{R \to \infty} \int_0^t \int_{\mathbb{R}^N \setminus B_R} e^{-c|\xi|^2_{V}} |u(\xi, \tau) - v(\xi, \tau)| d\xi d\tau = 0,
$$

whence $u(x, t) = v(x, t)$ for every $(x, t) \in \mathbb{R}^N \setminus ]0, \varepsilon[$. The thesis follows by repeating the previous argument finitely many times (note that $\varepsilon$ depends only on $c$, on the constant $c$ in (1.5) and on the operator $L$).

In order to prove Theorem 1.2, we need a preliminary result.

**Lemma 3.3.** Let $u \in C(\mathbb{R}^N \times ]0, T[)$ be a non-negative solution of $Lu = 0$ in $\mathbb{R}^N \times ]0, T[$. Then

$$
(3.7) \quad u(x, t) \geq \int_{\mathbb{R}^N} \Gamma(x, t, \xi, \tau) u(\xi, \tau) \, d\xi,
$$

for every $(x, t) \in \mathbb{R}^N \times ]0, T[$ and $0 < \tau < t$.

**Proof.** Fix $\tau \in ]0, T[$. For every $n \in \mathbb{N}$ and $(x, t) \in \mathbb{R}^N \times ]\tau, T[$, set

$$
u_n(x, t, \tau) := \int_{\mathbb{R}^N} \Gamma(x, t, \xi, \tau) h \left(\frac{|\xi|_G}{n}\right) u(\xi, \tau) \, d\xi,
$$

where $h \in C^\infty(\mathbb{R})$ is a fixed non-increasing cut-off function such that $h(s) = 1$ if $s \leq 1$ and $h(s) = 0$ if $s \geq 2$. By Theorem 2.2, we know that $u_n$ is a solution to the Cauchy problem

$$
\begin{cases}
Lu_n(\cdot, \tau) = 0 & \text{in } \mathbb{R}^N \setminus \{\tau\}, T], \\
\lim_{(x, t) \to (y, \tau)} u_n(x, t, \tau) = h\left(\frac{|y|_G}{n}\right) u(y, \tau), & \text{for every } y \in \mathbb{R}^N.
\end{cases}
$$

...
Furthermore, by the estimate (2.5),

\[
0 \leq u_n(x, t, \tau) \leq \frac{C}{(t - \tau)^{\frac{n-2}{2}}} \int_{\mathbb{R}^N} \exp \left(-\frac{d^2((\xi, \tau)^{-1} \circ (x, t))}{C(t - \tau)}\right) h\left(\frac{\|\xi\|}{n}\right) u(\xi, \tau) \, d\xi
\]

(3.8)

\[
\leq \frac{C}{(t - \tau)^{\frac{n-2}{2}}} \int_{B_{2n}} \exp \left(-\frac{d^2((\xi, \tau)^{-1} \circ (x, t))}{C(t - \tau)}\right) u(\xi, \tau) \, d\xi.
\]

Recalling the properties of the quasi-distance \(d\), we obtain

\[
d((\xi, \tau)^{-1} \circ (x, t)) \geq c^{-1}d((\xi, \tau)^{-1} \circ (x, t), 0) \geq c^{-2}\|x\|_G - c^{-1}(\xi, \tau)\|G\|
\]

\[
\geq c^{-2}\|x\|_G - c^{-1}(2n)^{2k_1} + T^{k_1}) \frac{1}{n^{2k_1}},
\]

so that, by (3.8),

\[
0 \leq u_n(x, t, \tau) \leq \max_{B_{2n}} u(\cdot, \tau) \frac{C \text{meas}(B_{2n})}{(t - \tau)^{\frac{n-2}{2}}} \exp \left(-\frac{(c^{-1}\|x\|_G - (2n)^{2k_1} + T^{k_1})}{c^2C(t - \tau)}\right) \rightarrow 0
\]

as \(\|x\|_G \rightarrow \infty\). We now apply the weak maximum principle to the \(L\)-harmonic function \(v_n = u_n(\cdot, \tau) - u\) in the strip \(\mathbb{R}^N \times [\tau, T]\). Indeed, we have \(\lim_{(x, t) \to (y, \tau)} v_n(x, t) \leq 0\) for every \(y \in \mathbb{R}^N\) being \(h \leq 1\), and \(\limsup v_n \leq 0\) at infinity in the strip, recalling that \(u \geq 0\). The maximum principle of Proposition 2.1 then gives

\[
0 \leq u_n(x, t, \tau) \leq u(x, t), \quad \text{for every } (x, t) \in \mathbb{R}^N \times [\tau, T].
\]

Letting \(n\) go to infinity, from the above inequality we obtain (3.7), since

\[
u_n(x, t, \tau) \not\to \int_{\mathbb{R}^N} \Gamma(x, t, \xi, \tau) u(\xi, \tau) \, d\xi \quad \text{as } n \to \infty
\]

by monotone convergence. This accomplishes the proof. \(\square\)

**Proof of Theorem 1.2.** We first show that, if \(u \in C(\mathbb{R}^N \times [0, T])\) is a non-negative solution of \(Lu = 0\) on the strip \(\mathbb{R}^N \times [0, T]\), then

\[
\int_0^t \int_{\mathbb{R}^N} u(\xi, \tau) \left(\Gamma(x, t, \xi, \tau) + \sum_{j=1}^m |X_j \Gamma(x, t, \xi, \tau)|\right) \, d\xi \, d\tau < \infty
\]

(3.9)

for every \((x, t) \in \mathbb{R}^N \times ]0, \frac{2}{3} T[.\) Indeed, by integrating the inequality (3.7) with respect to \(\tau \in ]0, t[,\) we obtain

\[
\int_0^t \int_{\mathbb{R}^N} u(\xi, \tau) \Gamma(x, t, \xi, \tau) \, d\xi \, d\tau \leq u(x, t) t < \infty, \quad \text{for every } (x, t) \in \mathbb{R}^N \times [0, T].
\]

On the other hand, from (2.6) it follows that

\[
\int_0^t \int_{\mathbb{R}^N} u(\xi, \tau) \sum_{j=1}^m |X_j \Gamma(x, t, \xi, \tau)| \, d\xi \, d\tau
\]

\[
\leq C \int_0^t \frac{1}{\sqrt{t - \tau}} \left(\int_{\mathbb{R}^N} u(\xi, \tau) \Gamma((x, t) \circ (0, -\frac{t - \tau}{2})^{-1}, \xi, \tau) \, d\xi\right) \, d\tau \quad \text{(by (3.7))}
\]

\[
\leq C \int_0^t \frac{1}{\sqrt{t - \tau}} u((x, t) \circ (0, -\frac{t - \tau}{2})^{-1}) \, d\tau \leq C C_{x, t} 2 \sqrt{t} < \infty,
\]
for every \((x, t) \in \mathbb{R}^N \times \left[0, \frac{3}{2} T \right]\). In the last but one inequality we have used that, since \(u\) is a continuous function on \(\mathbb{R}^N \times [0, T]\), there exists a constant \(C_{x, t} > 0\) such that

\[
u((x, t) \circ (0, -\frac{t - \tau}{2} - 1) \leq \sup_{\tau \in [0, t]} u(S(x, t, \tau), 3\frac{t - \tau}{2}) =: C_{x, t} < \infty.
\]

Hence, (3.9) is proved.

We next conclude the proof of the theorem. Let \(u, v \in C(\mathbb{R}^N \times [0, T])\) be two non-negative solutions to the Cauchy problem (1.4). As \(|u - v| \leq u + v\), the first part of the proof yields condition (3.4) for every \((x, t) \in \mathbb{R}^N \times \left[0, \frac{3}{2} T \right]\), whence \(u \equiv v\) in \(\mathbb{R}^N \times \left[0, \frac{3}{2} T \right]\) by Proposition 3.2. We repeat the above argument, and we find, at the \(n\)-th step, \(u \equiv v\) in \(\mathbb{R}^N \times \left[0, (1 - \frac{1}{3^n}) T \right]\).

This proves the theorem.

\[\square\]

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