On the stratification of secant varieties of Veronese varieties via symmetric rank.

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Abstract: *When considering $\sigma_r(X)$, the variety of r-secant $\mathbb{P}^{r-1}$ to a projective variety X, one question which arises is what are the possible values of the X-rank of points on $\sigma_r(X)$, apart from the generic value r? This geometric problem is of particular relevance (also for Applied Math) when X is a variety parameterizing some kind of tensors. We study here the case when X is a Veronese variety (i.e. the case of symmetric tensors). We find the complete description of the rank strata in some cases, and we give algorithms which compute the symmetric rank.*

1 Introduction

Veronese varieties and their secant varieties are geometric objects that have been studied by classical algebraic geometers, differential geometers and algebraists for a long period of time. Despite this "mathematical ancientness" it turns out that they still play a crucial role in many applications. The interesting part of the story is that when looking at the actual needs of people working in applications, many new and extremely interesting mathematical questions arise about these objects.

If we regard $\mathbb{P}^{\binom{n+d}{d}-1}$ as $\mathbb{P}(K[x_0,\ldots,x_n]_d)$, the projective space of homogeneous polynomials of degree d in $n+1$ variables on an algebraically closed field K of characteristic 0, the Veronese variety $X_{n,d} \subset \mathbb{P}^{\binom{n+d}{d}-1}$ is the variety that parameterizes those polynomial that can be written as d-th powers of linear forms (see Remark 2.14). When we view $\mathbb{P}^{\binom{n+d}{d}-1}$ as $\mathbb{P}(S^dV)$, where V is an $(n+1)$-dimensional vector space, the Veronese variety parameterizes projective classes of symmetric tensors of the type $v^{\otimes d} \in S^dV$ (see Definition 2.3).
The minimum integer r such that an element $T \in \mathbb{P}(S^dV)$ can be written as the sum of r elements in $X_{n,d}$ is called the *symmetric rank* of T (Definition 2.1). The set that parameterizes tensors in $\mathbb{P}(S^dV)$ of a given symmetric rank is not a closed variety. For many values of r, the smallest variety containing all tensors of symmetric rank r is the r-th secant variety of $X_{n,d}$, which we write $\sigma_r(X_{n,d})$ (Definition 2.5). The smallest r such that $T \in \sigma_r(X_{n,d})$ is called the *symmetric border rank* of T (Definition 2.11). This shows that, from a geometric point of view, it seems more natural to study the notion of symmetric border rank than the one of symmetric rank.

A the very classical algebraic problem, inspired by a number theory problem posed by Waring in 1770 ([**W**]), asks which is the minimum integer r such that a generic element of $K[x_0,\ldots,x_n]_d$ can be written as a sum of r d-th powers of linear forms. This problems is known as the Big Waring Problem. A geometric formulation of it asks which is the symmetric border rank of a generic symmetric tensor of S^dV. This problem was completely solved by J. Alexander and A. Hirshowithz who computed the dimensions of $\sigma_r(X_{n,d})$ for any r, n, d (see [**AH**] for the original proof and [**BO**] for a recent proof).

Although the dimensions of the $\sigma_r(X_{n,d})$'s are now all known, the same is not true for their defining equations: in general for all $\sigma_r(X_{n,d})$'s the equations coming from catalecticant matrices (Definition 3.1) are known, but they are not enough to describe their ideal; only in few cases our knowledge is complete (see for example [**K**], [**IK**], [**CGG**] and [**Ot**]). The knowledge of equations of $\sigma_r(X_{n,d})$ would give the possibility to discover the symmetric border rank of any tensor in S^dV.

Symmetric tensors show up in many applications as in Electrical Engineering (Antenna Array Processing [**ACCF**], [**DM**] and Telecommunications [**Ch**], [**dLC**]); in Statistics (cumulant tensors, see [**McC**]), or in Data Analysis (Independent Component Analysis [**Co**], [**JS**]). In most applications it turns out that it is the knowledge of the symmetric rank that is more useful, rather than knowing the symmetric border rank. Moreover the symmetric rank of a symmetric tensor extends the Singular Value Decomposition (SVD) problem for symmetric matrices ([**GVL**]).

A first efficient method to compute the symmetric rank of a symmetric tensor in $\mathbb{P}(S^dV)$ with $\dim(V) = 2$ is due to Sylvester ([**Sy**]). More than one version of such algorithm are known (see [**Sy**], [**BCMT**], [**CS**]). We present one here, in Section 3, which gives the symmetric rank of a tensor without passing through an explicit decomposition of it. The advantage of not giving an explicit decomposition is that this allows to improve very much the rapidity of the algorithm. Finding explicit decompositions is anyway a very interesting open problem (see also [**BCMT**] and [**LT**] for a study of the case $\dim(V) \geq 2$).

The aim of this paper is to explore a "projective geometry view" of the problem of finding what are the possible symmetric ranks of a tensor once its symmetric border rank is given, i.e. to determine the symmetric rank strata of the varieties $\sigma_r(X_{n,d})$. We do that for $\sigma_r(X_{1,d})$ for any r and d (see also [**BCMT**], [**CS**], [**LT**] and [**Sy**]), $\sigma_2(X_{n,d})$ and $\sigma_3(X_{n,d})$ (any n,d) (Section 4), for which we give an algorithm to compute the symmmetric rank, and for $\sigma_r(X_{2,4})$, $k = 4, 5$. Some of this results were known (see [**LT**], [**BCMT**]), with different approaches and different algorithms. In section 3 we also study the rank with respect to elliptic normal curves.

2 Preliminaries

We will always work with finite dimensional vector spaces defined on an algebraically closed field K of characteristic 0.

Definition 2.1. Let V be a vector space. The symmetric rank $s\mathrm{rk}(t)$ of a symmetric tensor $t \in S^dV$ is the minimum integer r such that there exist $v_1, \ldots, v_r \in V$ such that $t = \sum_{j=1}^r v_j^{\otimes d}$.

Notation 2.2. From now on we will indicate with T the projective class of a symmetric tensor $t \in S^dV$, i.e. if $t \in S^dV$ then $T = [t] \in \mathbb{P}(S^dV)$. We will write that an element $T \in \mathbb{P}(S^dV)$ has symmetric rank equal to r meaning that there exists a tensor $t \in S^dV$ such that $T = [t]$ and $s\mathrm{rk}(t) = r$.

Definition 2.3. Let V be a vector space of dimension $n+1$. The Veronese variety $X_{n,d} = \nu_d(\mathbb{P}(V)) \subset \mathbb{P}(S^dV) = \mathbb{P}^{\binom{n+d}{d}-1}$ is the variety parameterizing projective classes of symmetric tensors in S^dV of symmetric rank 1. I.e. $T \in X_{n,d}$ if and only if there exist $v \in V$ such that $t = v^{\otimes d}$.

Notation 2.4. If $v_1, \ldots, v_s$ belong to a vector space V, we will denote with $< v_1, \ldots, v_s >$ the subspace spanned by them. If $P_1, \ldots, P_s$ belong to a projective space $\mathbb{P}^n$ we will use the same notation $< P_1, \ldots, P_s >$ to denote the projective subspace generated by them.

Definition 2.5. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n. We define the s-th secant variety of X as follows:

$$\sigma_s(X) := \overline{\bigcup_{P_1,\ldots,P_s \in X} < P_1, \ldots, P_s >}.$$

Notation 2.6. We will indicate with $\sigma_s^0(X)$ the set $\bigcup_{P_1,\ldots,P_s \in X} < P_1, \ldots, P_s >$.

Remark 2.7. Let $X \subset \mathbb{P}^N$ be a non degenerate smooth variety. If $P \in \sigma_r^0(X) \setminus \sigma_{r-1}^0(X)$ then the minimum number of distinct points of X such that P depends linearly on them is obviously r. Let us see what happens in $\sigma_r(X)$ outside $\sigma_r^0(X)$.

Proposition 2.8. *Let $X \subset \mathbb{P}^N$ be a non degenerate smooth variety. Let H_r be the irreducible component of the Hilbert scheme of 0-dimensional schemes of degree r of X containing r distinct points, and assume that for each $y \in H_r$, the corresponding subscheme Y of X imposes independent conditions to linear forms. Then for each $P \in \sigma_r(X) \setminus \sigma_r^0(X)$ there exist a 0-dimensional scheme $Z \subset X$ of degree r such that $P \in < Z > \cong \mathbb{P}^{r-1}$.*

Conversely if there exists $Z \in H_r$ such that $P \in < Z >$, then $P \in \sigma_r(X)$.

Proof. Let us consider the map $\phi : H_r \to \mathbb{G}(r-1, \mathbb{P}^N)$, $\phi(y) = < Y >$; ϕ is well defined since $\dim < Y > = r-1$ for all $y \in H_r$ by assumption. Hence $\phi(H_r)$ is closed in $\mathbb{G}(r-1, \mathbb{P}^N)$.

Now let $\mathcal{I} \subset \mathbb{P}^N \times \mathbb{G}(r-1, \mathbb{P}^N)$ be the incidence variety, and p, q its projections on $\mathbb{P}^N$, $\mathbb{G}(r-1, \mathbb{P}^N)$ respectively.

Then, $A := pq^{-1}(\phi(H_r))$ is closed in $\mathbb{P}^N$. Moreover, A is irreducible since H_r is irreducible, so $\sigma_r^0(X)$ is dense in A. Hence $\sigma_r(X) = \overline{\sigma_r^0(X)} = A$.

□

In the following we use Proposition 2.8 when $X = X_{n,d}$, a Veronese variety, in many cases.

Remark 2.9. Let $n = 1$; in this case the Hilbert scheme of 0-dimensional schemes of degree r of $X = X_{1,d}$ is irreducible; moreover, for all y in the Hilbert scheme, Y imposes independent conditions to forms of any degree.

Also for $n = 2$ the Hilbert scheme of 0-dimensional schemes of degree r of $X = X_{2,d}$ is irreducible. Moreover, in the cases that we will study r is always small enough with respect to d to imply that all the elements in the Hilbert scheme impose independent conditions to forms of degree d.

Hence in the two cases above $P \in \sigma_r(X)$ if and only if there exists a scheme $Z \subset X$ of degree r such that $P \in < Z > \simeq \mathbb{P}^{r-1}$.

An example which shows that not always an $(r-1)$-dimensional linear space contained in $\sigma_r(X)$ is spanned by a 0-dimensional scheme of X of degree r is the following. Let $d = 6$, so that $X = \nu_6(\mathbb{P}^2) \subset \mathbb{P}^{27}$; the first r for which $\sigma_r(X)$ is the whole of $\mathbb{P}^{27}$ is 10. So if we study, for example, $\sigma_8(X)$, in $Hilb_8(\mathbb{P}^2)$ we can find a scheme Z which is the union of 8 distinct points on a line L; $\nu_6(L)$ is a rational normal curve C_6 in its $\mathbb{P}^6$, so $\dim < \nu(Z) > = 6$, hence $\nu(Z)$ does not impose independent conditions to linear forms in $\mathbb{P}^{27}$, which corresponds to the fact that Z in $\mathbb{P}^2$ imposes dependent conditions to curves of degree six. Now every linear 7-dimensional space $\Pi \subset \mathbb{P}^{27}$ containing C_6, meets X along C_6 and no other point; hence there does not exist a 0-dimensional scheme B of degree 8 on X such that $< B > \supset < \nu(Z) >$ and $< B > = \Pi$. On the other hand, consider a 1-dimensional flat family whose generic fiber Y is the union of 8 distinct points on X with $\dim < Y >= 7$ and special fiber $\nu(Z)$, and take the closure of the corresponding family of linear spaces with generic fiber $< Y >$: it still is a 1-dimensional flat family, hence it has to have a $\mathbb{P}^7$ as special fiber. Hence the closure of $\sigma_8^0(X)$ contains linear spaces of dimension 7 containing $< Z >$ which are not generated by a scheme of degree 8 on X.

Remark 2.10. A tensor $t \in S^dV$ with $\dim(V) = n+1$ has symmetric rank r if and only if $T \in \sigma_r^0(X_{n,d})$ and, for any $s < r$, we have that $T \notin \sigma_s^0(X_{n,d})$. In fact by definition of symmetric rank of an element $T \in S^dV$, there should exist at least r elements $T_1, \ldots, T_r \in X_{n,d}$ corresponding to tensors $t_1, \ldots, t_r$ of symmetric rank one such that $t = \sum_{i=1}^r t_i$. Hence $T \in \sigma_r^0(X_{n,d}) \setminus \sigma_{r-1}^0(X_{n,d})$.

Definition 2.11. If $T \in \sigma_s(X_{n,d}) \setminus \sigma_{s-1}(X_{n,d})$, we say that t has symmetric border rank s, and we write $\underline{s\mathrm{rk}}(t) = s$.

Remark 2.12. The symmetric border rank of $t \in S^dV$, with $\dim(V) = n+1$, is the smallest s such that $T \in \sigma_s(X_{n,d})$. Therefore $s\mathrm{rk}(t) \geq \underline{s\mathrm{rk}}(t)$. Moreover if $T \in \sigma_s(X_{n,d}) \setminus \sigma_s^0(X_{n,d})$ then $s\mathrm{rk}(t) > s$.

The following notation will turn out to be useful in the sequel.

Notation 2.13. We will indicate with $\sigma_{b,r}(X_{n,d}) \subset \mathbb{P}(S^dV)$ the set:

$$\sigma_{b,r}(X_{n,d}) := \{T \in \sigma_b(X_{n,d}) | s\mathrm{rk}(T) = r\},$$

i.e. the elements of $\mathbb{P}(S^dV)$ whose symmetric border rank is b and whose symmetric rank is r.

Veronese varieties can be described also as the varieties parameterizing certain kind of homogeneous polynomials.

Remark 2.14. Let V be a vector space of dimension n and let $l \in V^*$ be a linear form. Now define $\nu_d : \mathbb{P}(V^*) \to \mathbb{P}(S^dV^*)$ as $\nu_d([l]) = [l^d] \in \mathbb{P}(S^dV^*)$. The image of this map is indeed the d-uple Veronese embedding of $\mathbb{P}(V^*)$.

Remark 2.15. Remark 2.14 shows that, if V is an n-dimensional vector space, then to any symmetric tensor $t \in S^dV$ of symmetric rank r we can associate, given a basis of V, a homogeneous polynomial of degree d in $n+1$ variables that can be written as a sum of r d-th power of linear forms (see (1) below).

3 Two dimensional case

In this section we will restrict to the case that V is a 2-dimensional vector space. We first describe Sylvester algorithm which gives the symmetric rank of a symmetric tensor $t \in S^dV$ and a decomposition of t as a sum of $s\mathrm{rk}(t)$ symmetric tensors of symmetric rank one (see [**Sy**]j [**CS**], [**BCMT**]), then we give a geometric description of it and a slightly different algorithm which produces the symmetric rank of a symmetric tensor in S^dV without giving explicitly its decomposition. This algorithm makes use of a result (see Theorem 3.8) which describes the rank of tensors on the secant varieties of rational normal curves $C_d = X_{1,d}$; the Theorem has been proved in the unpublished paper [**CS**] (see also [**LT**]); we give a proof here which uses only classical projective geometry.

Moreover we extend that result to elliptic normal curves, see Theorem 3.11.

3.1 Sylvester algorithm

Let $p \in K[x_0, x_1]_d$ be a homogeneous polynomial of degree d in two variables: $p(x_0, x_1) = \sum_{k=0}^{d} a_k x_0^k x_1^{d-k}$; then p can be represented with a symmetric tensor $t = (b_{i_1,\dots,i_d})_{j=1,\dots,d;\ i_j \in \{0,1\}} \in S^d V \simeq K[x_0, x_1]_d$ where $\binom{d}{k} \cdot b_{i_1,\dots,i_d} = a_k$ for any d-uple $(i_1, \dots, i_d)$ containing exactly k zeros. This correspondence is clearly one to one:

$$\begin{array}{rcl} K[x_0, x_1]_d & \leftrightarrow & S^d V \\ \sum_{k=0}^{d} a_k x_0^k x_1^{d-k} & \leftrightarrow & (b_{i_1,\dots,i_d})_{i_j=0,1;\ j=1,\dots,d} \end{array} \tag{1}$$

with $(b_{i_1,\dots,i_d})$ as above.

Moreover, we can associate to a polynomial $p(x_0, x_1) = \sum_{k=0}^{d} a_k x_0^k x_1^{d-k}$, the so called $(d-r+1)\times(r+1)$ Catalecticant matrix (in [**BCMT**] it is called Hankel matrix) $M_{d-r,r}$ of dimension $(d-r+1)\times(r+1)$ defined as follows (for a definition of Catalecticant matrix see also [**K**]):

Definition 3.1. The Catalecticant matrix $M_{d-r,r} = M_{d-r,r}(t)$ of dimension $(d-r+1)\times(r+1)$ associated to a polynomial $p(x_0, x_1) = \sum_{k=0}^{d} a_k x_0^k x_1^{d-k} \in K[x_0, x_1]_d$, or to a tensor $t = (b_{i_1,\dots,i_d})_{i_j=0,1;\ j=1,\dots,d} \in S^d V$ with $b_{i_1,\dots,i_d} = \binom{d}{k}^{-1} a_k$ for any d-uple $(i_1, \dots, i_d)$ is the matrix whose entries are $c_{i,j} = \binom{d}{i}^{-1} a_{i+j-2}$ with $i = 1, \dots, d-r$ and $j = 1, \dots, r$.

We describe here a version of Sylvester's algorithm ([**Sy**], [**CS**], or [**BCMT**]):

Algorithm 3.2. Input: A binary form $p(x_0, x_1)$ of degree d or, equivalently, its associated symmetric tensor t.
Output: A decomposition of p as $p(x_0, x_1) = \sum_{j=1}^{k} \lambda_j l_j(x_0, x_1)^d$ with $\lambda_j \in K$ and $l_j \in K[x_0, x_1]_1$ for $j = 1, \dots, r$ with r minimal.

1. Initialize $r = 0$;
2. Increment $r \leftarrow r + 1$;
3. If the rank of the matrix $M_{d-r,r}$ is maximum, then go to step 2;
4. Else compute a basis $\{l_1, \dots, l_h\}$ of the right kernel of $M_{d-r,r}$;
5. Specialization:
   - Take a vector q in the kernel, e.g. $q = \sum_i \mu_i l_i$;
   - Compute the roots of the associated polynomial $q(x_0, x_1) = \sum_{h=0}^{r} q_h x_0^h x_1^{d-h}$. Denote them by $(\beta_j - \alpha_j)$, where $|\alpha_j|^2 + |\beta_j|^2 = 1$;
   - If the roots are not distinct in $\mathbb{P}^1$, go to step 2;
   - Else if $q(x_0, x_1)$ admits r distinct roots then compute coefficients λ_j, $1 \le j \le r$, by solving the linear system below:

$$\begin{pmatrix} \alpha_1^d & \cdots & \alpha_r^d \\ \alpha_1^{d-1}\beta_1 & \cdots & \alpha_r^{d-1}\beta_r \\ \alpha_1^{d-2}\beta_1^2 & \cdots & \alpha_r^{d-2}\beta_r^2 \\ \vdots & \vdots & \vdots \\ \beta_1^d & \cdots & \beta_r^d \end{pmatrix} \lambda = \begin{pmatrix} a_0 \\ 1/d a_1 \\ \binom{d}{2}^{-1} a_2 \\ \vdots \\ a_d \end{pmatrix};$$

6. The decomposition is $p(x_0, x_1) = \sum_{j=1}^{r} \lambda_j l_j(x_0, x_1)^d$, where $l_j(x_0, x_1) = (\alpha_j x_1 + \beta_j x_2)$.

3.2 Geometric description

If V is a two dimensional vector space, there is a well known isomorphism between $\bigwedge^{d-r+1}(S^d V)$ and $S^{d-r+1}(S^r V)$ (see [**Mu**]). Such isomorphism can be interpreted in terms of projective algebraic varieties; it allows to view the $(d-r+1)$-uple Veronese embedding of $\mathbb{P}^r$, as the set of $(r-1)$-dimensional projective subspaces of $\mathbb{P}^d$ that are r-secant to the rational normal curve. The description of this result, via coordinates, was originally given by A. Iarrobino, V. Kanev (see [**IK**]). We give here the description appeared in [**AB**] (Lemma 2.1).

Notation 3.3. With $\vec{G}(k, V)$ we denote the Grassmannian of k-dimensional subspaces of a vector space V, and with $\mathbb{G}(k-1, \mathbb{P}(V))$ we denote the $(k-1)$-dimensional projective subspaces of the projective space $\mathbb{P}(V)$.

Lemma 3.4. *Consider the map $\phi_{r,d-r+1} : \mathbb{P}(K[t_0,t_1]_r) \to \vec{G}(d-r+1, K[t_0,t_1]_d)$ that maps the class of $p_0 \in K[t_0,t_1]_r$ to the $(d-r+1)$-dimensional subspace of $K[t_0,t_1]_d$ of forms of the type $p_0 q$, with $q \in K[t_0,t_1]_{d-r}$. Then the following hold:*

(i) The image of $\phi_{r,d-r+1}$, after the Plücker embedding of $\vec{G}(d-r+1, K[t_0,t_1]_d)$, is the r-dimensional $(d-r+1)$-th Veronese variety.

(ii) Identifying $\vec{G}(d-r+1, K[t_0,t_1]_d)$ with the Grassmann variety of subspaces of dimension $r-1$ in $\mathbb{P}(K[t_0,t_1]_d^*)$, the above Veronese variety is the set of r-secant spaces to a rational normal curve $C_d \subset \mathbb{P}(K[t_0,t_1]_d^*)$.

Proof. Write $p_0 = u_0 t_0^r + u_1 t_0^{r-1} t_1 + \cdots + u_r t_1^r$. Then a basis of the subspace of $K[t_0,t_1]_d$ of forms of the type $p_0 q$ is given by:

$$\begin{cases} u_0 t_0^d + \cdots + u_r t_0^{d-r} t_1^r \\ \quad u_0 t_0^{d-1} t_1 + \cdots + u_r t_0^{d-r-1} t_1^{r+1} \\ \qquad \ddots \\ \qquad\quad u_0 t_0^r t_1^{d-r} + \cdots + u_r t_1^d. \end{cases} \tag{2}$$

The coordinates of these elements with respect to the basis $\{t_0^d, t_0^{d-1}t_1, \ldots, t_1^d\}$ of $K[t_0,t_1]_d$ are thus given by the rows of the matrix

$$\begin{pmatrix} u_0 & u_1 & \ldots & u_r & 0 & \ldots & 0 & 0 \\ 0 & u_0 & u_1 & \ldots & u_r & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & u_0 & u_1 & \ldots & u_r & 0 \\ 0 & \ldots & 0 & 0 & u_0 & \ldots & u_{r-1} & u_r \end{pmatrix}.$$

The standard Plücker coordinates of the subspace $\phi_{r,d-r+1}([p_0])$ are the maximal minors of this matrix. It is known (see for example [**AP**]), that these minors form a basis of $K[u_0,\ldots,u_r]_{d-r+1}$, so that the image of ϕ is indeed a Veronese variety, which proves (i).

To prove (ii), we still recall some standard facts from [**AP**]. Take homogeneous coordinates $z_0,\ldots,z_d$ in $\mathbb{P}(K[t_0,t_1]_d^*)$ corresponding to the dual basis of $\{t_0^d, t_0^{d-1}t_1, \ldots, t_1^d\}$. Consider $C_d \subset \mathbb{P}(K[t_0,t_1]_d^*)$ the standard rational normal curve with respect to these coordinates. Then, the image of $[p_0]$ by $\phi_{r,d-r+1}$ is precisely the r-secant space to C_d spanned by the divisor on C_d induced by the zeros of p_0. This completes the proof of (ii). $\square$

Since $\dim(V) = 2$, the Veronese variety of $\mathbb{P}(S^dV)$ is the rational normal curve $C_d \subset \mathbb{P}^d$. Hence, a symmetric tensor $t \in S^dV$ has symmetric rank r if and only if r is the minimum integer for which there exist a $\mathbb{P}^{r-1} = \mathbb{P}(W) \subset \mathbb{P}(S^dV)$ such that $T \in \mathbb{P}(W)$ and $\mathbb{P}(W)$ is r-secant to the rational normal curve $C_d \subset \mathbb{P}(S^dV)$ in r distinct points.
Consider the maps:

$$\mathbb{P}(K[t_0, t_1]_r) \overset{\phi_{r,d-r+1}}{\to} \mathbb{G}(d - r, \mathbb{P}(K[t_0, t_1]_d)) \overset{\alpha_{r,d-r+1}}{\simeq} \mathbb{G}(r - 1, \mathbb{P}(K[t_0, t_1]_d)^*). \tag{3}$$

Clearly, since $\dim(V) = 2$, we can identify $\mathbb{P}(K[t_0, t_1]_d)^*)$ with $\mathbb{P}(S^dV)$, hence the Grassmannian $\mathbb{G}(r - 1, \mathbb{P}(K[t_0, t_1]_d)^*)$ can be identified with $\mathbb{G}(r - 1, \mathbb{P}(S^dV))$.
Now, by Lemma 3.4, a projective subspace $\mathbb{P}(W)$ of $\mathbb{P}(K[t_0, t_1]_d)^* \simeq \mathbb{P}(S^dV) \simeq \mathbb{P}^d$ is r-secant to $C_d \subset \mathbb{P}(S^dV)$ in r distinct points if and only if it belongs to $\mathrm{Im}(\alpha_{r,d-r+1} \circ \phi_{r,d-r+1})$ and the preimage of $\mathbb{P}(W)$ via $\alpha_{r,d-r+1} \circ \phi_{r,d-r+1}$ is a polynomial with r distinct roots.
Therefore, a symmetric tensor $t \in S^dV$ has symmetric rank r if and only if r is the minimum integer for which:

1. T belongs to an element $\mathbb{P}(W) \in \mathrm{Im}(\alpha_{r,d-r+1} \circ \phi_{r,d-r+1}) \subset \mathbb{G}(r - 1, \mathbb{P}(S^dV))$,
2. there exist a polynomial $p_0 \in K[t_0t_1]_r$ such that $\alpha_{r,d-r+1}(\phi_{r,d-r+1}([p_0])) = \mathbb{P}(W)$ and p_0 has r distinct roots,

Fix the natural basis $\Sigma = \{t_0^d, t_0^{d-1}t_1, \ldots, t_1^d\}$ in $K[t_0, t_1]_d$. Let $\mathbb{P}(U)$ be a $(d-r)$-dimensional projective subspace of $\mathbb{P}(K[t_0, t_1]_d)$. The proof of Lemma 3.4 shows that $\mathbb{P}(U)$ belongs to the image of $\phi_{r,d-r+1}$ if and only if there exist $u_0, \ldots, u_r \in K$ such that $U =< p_1, \ldots, p_{d-r+1} >$ with $p_1 = (u_0, u_1, \ldots, u_r, 0, \ldots, 0)_\Sigma, p_2 = (0, u_0, u_1, \ldots, u_r, 0, \ldots, 0)_\Sigma, \ldots, p_{d-r+1} = (0, \ldots, 0, u_0, u_1, \ldots, u_r)_\Sigma$.
Now let $\Sigma^* = \{z_0, \ldots, z_d\}$ be the dual basis of Σ. Therefore there exist a $W \subset S^dV$ such that $\mathbb{P}(W) = \alpha_{r,d-r+1}(\mathbb{P}(U))$ if and only if $W = H_1 \cap \cdots \cap H_{d-r+1}$ and the H_i's are as follows:

$$\begin{array}{rl}
H_1 : & u_0z_0 + \cdots + u_rz_r = 0 \\
H_2 : & \quad u_0z_1 + \cdots + u_rz_{r+1} = 0 \\
 & \qquad \ddots \\
H_{d-r+1} : & \qquad u_0z_{d-r} + \cdots + u_rz_d = 0.
\end{array}$$

This is sufficient to conclude that $T \in \mathbb{P}(S^dV)$ belongs to an $(r - 1)$-dimensional projective subspace of $\mathbb{P}(S^dV)$ that is in the image of $\alpha_{r,d-r+1} \circ \phi_{r,d-r+1}$ defined in (3) if and only if there exist $H_1, \ldots, H_{d-r+1}$ hyperplanes in S^dV as above such that $T \in H_1 \cap \ldots \cap H_{d-r+1}$.
Given $t = (a_0, \ldots, a_d)_{\Sigma^*} \in S^dV$, $T \in H_1 \cap \ldots \cap H_{d-r+1}$ if and only if the following linear system admits a non trivial solution:

$$\begin{cases}
u_0a_0 + \cdots + u_ra_r = 0 \\
u_0a_1 + \cdots + u_ra_{r+1} = 0 \\
\vdots \\
u_0a_{d-r} + \cdots + u_ra_d = 0.
\end{cases}$$

If $d - r + 1 < r + 1$ this system admits an infinite number of solutions.
If $r \leq d/2$, it admits a non trivial solution if and only if all the maximal $(r + 1)$-minors of the following

$(d-r+1)\times(r+1)$ catalecticant matrix, defined in Definition 3.1, vanish :

$$M_{d-r,r}=\begin{pmatrix} a_0 & \cdots & a_r \\ a_1 & \cdots & a_{r+1} \\ \vdots & & \vdots \\ a_{d-r} & \cdots & a_d \end{pmatrix}.$$

The following three remarks contain results on rational normal curves and their secant varieties that are classically known and that we will need in our description.

Remark 3.5. The dimension of $\sigma_r(C_d)$ is the minimum between $2r-1$ and d. Actually $\sigma_r(C_d)\subsetneq\mathbb{P}^d$ if and only if $1\le r<\lceil\frac{d+1}{2}\rceil$.

Remark 3.6. An element $T\in\mathbb{P}^d$ belongs to $\sigma_r(C_d)$ for $1\le r<\lceil\frac{d+1}{2}\rceil$ if and only if the catalecticant matrix $M_{r,d-r}$ defined in Definition 3.1 does not have maximal rank.

Remark 3.7. Any divisor $D\subset C_d$ is such that $\dim <D>=\deg D-1$.

The following result has been proved by G. Comas and M. Seiguer in the unpublished paper [**CS**] (see also [**LT**]), and it describes the structure of the stratification by symmetric rank of symmetric tensors in S^dV with $\dim(V)=2$. The proof we give here is a strictly "projective geometry" one.

Theorem 3.8. *Let $X_{1,d}=C_d\subset\mathbb{P}(S_dV)$, $\dim(V)=2$, be the rational normal curve, parameterizing decomposable symmetric tensors ($C_d=\{T\in\mathbb{P}(S^dV)\,|\,\mathrm{srk}(T)=1\}$), i.e. homogeneous polynomials in $K[t_0,t_1]_d$ which are d-th powers of linear forms. Then:*

$$\forall\, r,\ 2\le r\le\left\lceil\frac{d+1}{2}\right\rceil: \qquad \sigma_r(C_d)\setminus\sigma_{r-1}(C_d)=\sigma_{r,r}(C_d)\cup\sigma_{r,d-r+2}(C_d)$$

where $\sigma_{r,r}(C_d)$ and $\sigma_{r,d-r+2}(C_d)$ are defined as in Notation 2.13.

Proof. Of course, for all $t\in S^dV$, if $\mathrm{srk}(t)=r$, with $r\le\lceil\frac{d+1}{2}\rceil$, we have $T\in\sigma_r(C_d)\setminus\sigma_{r-1}(C_d)$. Thus we have to consider the case $\mathrm{srk}(t)>\lceil\frac{d+1}{2}\rceil$.

If a point in $K[t_0,t_1]_d^*$ represents a tensor t with $\mathrm{srk}(t)>\lceil\frac{d+1}{2}\rceil$, then we want to show that $\mathrm{srk}(t)=d-r+2$, where r is the minimum such that $T\in\sigma_r(C_d)$, $r\le\lceil\frac{d+1}{2}\rceil$.

Let us consider the case $r=2$ first: Let $T\in\sigma_2(C_d)\setminus C_d$. If $\mathrm{srk}(t)>2$, it means that T lies on a line t_P, tangent to C_d at a point P (since T has to lie on a $\mathbb{P}^1$ which is the image of a non-reduced form of degree 2: $p_0=l^2$ with $l\in K[x_0,x_1]_1$, otherwise $\mathrm{srk}(t)=2$). We want to show that $\mathrm{srk}(t)=d$; in fact, if $\mathrm{srk}(t)=r<d$, there would exist points $P_1,\dots,P_{d-1}\in C_d$, such that $T\in<P_1,\dots,P_{d-1}>$; in this case the hyperplane $H=<P_1,\dots,P_{d-1},P>$ would be such that $t_P\subset H$, a contradiction, since $H\cap C_d=2P+P_1+\cdots+P_{d-1}$, which has degree $d+1$.

Notice that $\mathrm{srk}(t)=d$ is possible, since obviously there is a $(d-1)$-space (i.e. a hyperplane) through T cutting d distinct points on C_d (any generic hyperplane through T will do). This also shows that d is the maximum possible rank.

Now let us generalize the procedure above; let $T\in\sigma_r(C_d)\setminus\sigma_{r-1}(C_d)$, $r\le\lceil\frac{d+1}{2}\rceil$; we want to prove that if $\mathrm{srk}(t)\neq r$, then $\mathrm{srk}(t)=d-r+2$. Since $\mathrm{srk}(t)>r$, we know that T must lie on a $\mathbb{P}^{r-1}$ which cuts a non-reduced divisor $Z\in C_d$ with $\deg(Z)=r$; therefore there is a point $P\in C_d$ such that $2P\in Z$. If we had

$s\mathrm{rk}(t) \leq d-r+1$, then T would be on a $\mathbb{P}^{d-r}$ which cuts C_d in distinct points $P_1,\dots,P_{d-r+1}$; if that were true the space $< P_1,\dots,P_{d-r+1}, Z-P >$ would be $(d-1-\deg(Z-2P)\cap\{P_1,\dots,P_{d-r+1}\})$-dimensional and cut $P_1+\cdots+P_{d-r+1}+Z-(Z-2P)\cap\{P_1,\dots,P_{d-r+1}\}$ on C_d, which is impossible.

So we got $s\mathrm{rk}(t) \geq d-r+2$; now we have to show that the rank is actually $d-r+2$. Let's consider the divisor $Z-2P$ on C_d; we have $\deg(Z-2P) = r-2$, and the space $\Gamma =< Z-2P, T >$ which is $(r-2)$-dimensional since $< Z-2P >$ does not contain T (otherwise $T \in \sigma_{r-3}(C_d)$). Consider the linear series cut on C_d by the hyperplanes containing Γ: we will be finished if we show that its generic divisor is reduced.

If it is not, there should be a fixed non-reduced part of the series, i.e. at least a divisor of type $2Q$. If this is the case, each hyperplane through Γ would contain $2Q$, hence $2Q \subset \Gamma$, which is impossible, since we would have $\deg(\Gamma\cap C_d) = r$, while $\dim\Gamma = r-2$.

Thus $s\mathrm{rk}(t) = d-r+2$, as required. □

Remark 3.9. In the proof above we have seen that if t is a symmetric tensor such that $T \in \sigma_r(C_d) \setminus \sigma_{r-1}(C_d)$, and $T \notin \sigma_r^0(C_d)$, then there exists a non reduced 0-dimensional scheme $Z \subset \mathbb{P}^2$, which is a divisor of degree r on C_d, such that $T \in< Z >$. Let $Z = m_1P_1+\dots m_sP_s$, with $P_1,\dots,P_s$ distinct points on the curve, and $m_1+\cdots+m_s = r$ and at least for one value of i we have $m_i \geq 2$. Then t^* can be written as

$$t^* = l_1^{d-m_1+1}f_1+\cdots+l_s^{d-m_s+1}f_s$$

where $l_1,\dots,l_s$ are homogeneous linear forms in two variables and each f_i is a homogeneous form of degree m_i-1 for $i=1,\dots,s$.

In the theorem above it is implicitly proved that each form of this type has symmetric rank $d-r+2$. In particular, every monomial of type $x^{d-s}y^s$ is such that

$$s\mathrm{rk}(x^{d-s}y^s) = \max\{d-s+1, s+1\}.$$

3.3 A result on elliptic normal curves.

We can use the same kind of construction to prove the following result on elliptic normal curves.

Notation 3.10. If $\Gamma_{d+1} \subset \mathbb{P}^d$, with $d \geq 3$, is an elliptic normal curve, and $T \in \mathbb{P}^d$, we say that T has rank r with respect to Γ_{d+1} and we write $\mathrm{rk}_{\Gamma_{d+1}}(T)$ if r is the minimum number of points of Γ_{d+1} such that T depends linearly on them

Theorem 3.11. *Let $\Gamma_{d+1} \subset \mathbb{P}^d$, $d \geq 3$, be an elliptic normal curve. Let $d \geq 4$, then:*

- *For all $2 \leq r < \lceil\frac{d+1}{2}\rceil$:* $\qquad \sigma_r(\Gamma_{d+1}) \setminus \sigma_{r-1}(\Gamma_{d+1}) = \sigma_{r,r}(\Gamma_{d+1}) \cup \sigma_{r,d-r+1}(\Gamma_{d+1})$.
- *For $r = \lceil\frac{d+1}{2}\rceil$:* $\qquad \sigma_r(\Gamma_{d+1}) \setminus \sigma_{r-1}(\Gamma_{d+1}) = \sigma_{r,r}(\Gamma_{d+1}) \quad$ *(here $\sigma_r(\Gamma_{d+1}) = \mathbb{P}^d$).*

When $d = 3$, we have : $\qquad \sigma_2(\Gamma_4) \setminus \Gamma_4 = \sigma_{2,2}(\Gamma_4) \cup \sigma_{2,3}(\Gamma_4)$; *(here $\sigma_2(\Gamma_4) = \mathbb{P}^3$).*
Here the $\sigma_{i,j}(\Gamma_{d+1})$'s are defined as in Notation 2.13, but with respect to Γ_{d+1}, i.e. $\sigma_{i,j}(\Gamma_{d+1}) = \{t \in \mathbb{P}^d | \mathrm{rk}_{\Gamma_{d+1}}(t) = j, t \in \sigma_i(\Gamma_{d+1})\}$.

Proof. Of course, for all $T \in \mathbb{P}^d$, if $\mathrm{rk}_{\Gamma_{d+1}}(T) = r$, with $r \leq \lceil\frac{d+1}{2}\rceil$, we have $T \in \sigma_r(\Gamma_{d+1}) \setminus \sigma_{r-1}(\Gamma_{d+1})$. Thus we have to consider the case $\mathrm{rk}_{\Gamma_{d+1}}(T) > \lceil\frac{d+1}{2}\rceil$.

First let $d \geq 4$; if a point $T \in \mathbb{P}^d$ has $\mathrm{rk}_{\Gamma_{d+1}}(T) > \lceil \frac{d+1}{2} \rceil$, then we want to show that $\mathrm{rk}_{\Gamma_{d+1}}(T) = d - r + 1$, where r is the minimum such that $T \in \sigma_r(\Gamma_{d+1})$, $r < \lceil \frac{d+1}{2} \rceil$.

Let us consider the case $r = 2$ first: Let $T \in \sigma_2(\Gamma_{d+1}) \setminus \Gamma_{d+1}$. If $\mathrm{rk}_{\Gamma_{d+1}}(T) > 2$, it means that T lies on a line t_P, tangent to Γ_{d+1} at a point P. We want to show that $\mathrm{rk}_{\Gamma_{d+1}}(T) = d - 1$. Let us check that we cannot have $\mathrm{rk}_{\Gamma_{d+1}}(T) = r < d-1$, first. In fact, in that case there would exist points $P_1, \dots, P_{d-2} \in \Gamma_{d+1}$, such that $T \in < P_1, \dots, P_{d-2} >$; in this case the space $< P_1, \dots, P_{d-2}, P >$ would be $(d-2)$-dimensional, and such that $< P_1, \dots, P_{d-2}, 2P > = < P_1, \dots, P_{d-2}, P >$, since T is on $< P_1, \dots, P_{d-2} >$, so the line $< 2P > = t_P$ is in $< P_1, \dots, P_{d-2}, P >$ already. But this is a contradiction, since $< P_1, \dots, P_{d-2}, 2P >$ has to be $(d-1)$-dimensional (on Γ_{d+1} every divisor of degree $< d+1$ imposes independent conditions to hyperplanes).

Now we want to check that $\mathrm{rk}_{\Gamma_{d+1}}(T) \leq d - 1$. We have to show that there exist $d-1$ distinct points $P_1, \dots, P_{d-1}$ on Γ_{d+1}, such that $T \in < P_1, \dots, P_{d-2} >$. Consider the hyperplanes in $\mathbb{P}^d$ containing the line t_P; they cut a g^{d-2}_{d+1} on Γ_{d+1}, which is made of the fixed divisor $2P$, plus a complete linear series g^{d-2}_{d-1} which is of course very ample; among the divisors of this linear series, the ones which span a $\mathbb{P}^{d-2}$ containing T form a sub-series g^{d-3}_{d-1}, whose generic element is smooth by Bertini's theorem, hence it is made of $d-1$ distinct points whose span contains T, as required.

Now let us generalize the procedure above; let $T \in \sigma_r(\Gamma_{d+1}) \setminus \sigma_{r-1}(\Gamma_{d+1})$, $r < \lceil \frac{d+1}{2} \rceil$; we want to prove that if $\mathrm{rk}_{\Gamma_{d+1}}(T) \neq r$, then it is actually $= d - r + 1$. All works exactly as in the case $r = 2$; if $\mathrm{rk}_{\Gamma_{d+1}}(T) > r$, we know that T must lie on a $\mathbb{P}^{r-1}$ which cuts a non-reduced divisor $Z \in \Gamma_{d+1}$ with $\deg(Z) = r$; therefore there is a point $P \in \Gamma_{d+1}$ such that $2P \in Z$. If we had $\mathrm{rk}_{\Gamma_{d+1}}(T) \leq d - r$, then T would be on a $\mathbb{P}^{d-r-1}$ which cuts Γ_{d+1} in distinct points $P_1, \dots, P_{d-r}$; if that were true the space $< P_1, \dots, P_{d-r}, Z - P >$ would be $(d-2)$-dimensional, and coincide with the space $< P_1, \dots, P_{d-r}, Z >$, which has to be $(d-1)$-dimensional, a contradiction.

In order to show that $\mathrm{rk}_{\Gamma_{d+1}}(T) \leq d - r + 1$, we consider the hyperplanes containing $< Z >$, which cut a g^{d-r}_{d+1-r} on Γ_{d+1}, outside Z; the divisors of that linear series passing through T form a g^{d-r-1}_{d+1-r} which is very ample, hence a generic element of it is made of $(d+1-r)$ distinct points, as required.

Now let us consider the case $T \in \sigma_r(\Gamma_{d+1})$, $r = \lceil \frac{d+1}{2} \rceil$; if $T \in < Z >$, where $Z \subset \Gamma_{d+1}$ is a non-reduced subscheme of degree r, let us consider the two cases:

Let d be odd, so $d = 2r - 1$. The hyperplanes through Z cut a complete g^{r-1}_r on Γ_{d+1}, and $r \geq 3$, so the linear series is very ample; its divisors D are such that $< D > \cap < Z >$ is a point ($< D > + < Z >$ is an hyperplane in $\mathbb{P}^{2r-1}$); the divisors D such that $T \in < D >$ form a subseries g^{r-2}_r whose generic element is reduced (again by Bertini), hence $\mathrm{rk}_{\Gamma_{d+1}}(T) = r$, as required.

If d is even, then $d = 2r - 2$. Consider the complete g^{r-1}_r on Γ_{d+1} defined by Z itself; since $r \geq 3$, so the linear series is very ample; its divisors D are such that $< D > \cap < Z >$ is a point (they are two $\mathbb{P}^{r-1}$'s in $\mathbb{P}^{2r-2}$); the divisor D such that $T \in < D >$ form a series g^{r-2}_r whose generic element is reduced, hence $\mathrm{rk}_{\Gamma_{d+1}}(T) = r$, as required.

Eventually, let $d = 3$; obviously $\sigma_2(\Gamma_4) = \mathbb{P}^3$; if a point $T \in (\sigma_2(\Gamma_4)\sigma_r\Gamma_4)$ is on a tangent line t_P of the curve, consider the planes through t_P: they cut a g^1_2 on Γ_4 outside $2P$; each divisor D of such g^1_2 spans a line which meets t_P in a point ($< D > + < 2P >$ is a plane in $\mathbb{P}^3$), so the g^1_2 defines a $2:1$ map $\Gamma_4 \to t_P$ which, by Hurwitz theorem, has two ramification points. Hence for a generic point of t_P there is a secant line through it (i.e. it lies on $\sigma_{2,2}(\Gamma_4)$), but for those special points no such line exists (namely, for the points in which two tangent lines at Γ_4 meet), hence those points have $\mathrm{rk}_{\Gamma_4} = 3$ (a generic hyperplane through one point cuts 4 distinct points on Γ_4, and three of them span it). □

Remark 3.12. Let $T \in \mathbb{P}^d$ and $C \subset \mathbb{P}^d$ be a smooth curve not contained in a hyperplane. In [**LT**]

(Corollary 5.3) it is proved that $\mathrm{rk}_C(T) \leq d$. This value of the rank with respect to a smooth curve is attained by a tensor T if C is the rational normal curve (precisely if T belongs to a tangent line to C, see Theorem 3.8). Actually Theorem 3.11 shows that, if $d = 3$, then there are tensors of $\mathbb{P}^3$ whose rank with respect to an elliptic normal curve $\Gamma_4 \subset \mathbb{P}^3$ is precisely 3. To our knowledge $\Gamma_4 \subset \mathbb{P}^3$ is the only example apart from the rational normal curve where this value of the rank with respect to a smooth curve not contained in a hyperplane is attained.

3.4 Simplified version of Sylvester's Algorithm

Theorem 3.2 allows to get a simplified version of Sylvester algorithm (see also [**CS**]), which computes only the symmetric rank of a symmetric tensor, without computing the actual decomposition.

Algorithm 3.13. Sylvester Symmetric Rank Algorithm:

Input: A symmetric tensor $t \in S^d V$ with $\dim(V) = 2$
Output: $s\mathrm{rk}(t)$.

1. Initialize $r = 0$;
2. Increment $r \leftarrow r + 1$;
3. Compute $M_{d-r,r}(t)$'s $(r+1) \times (r+1)$-minors; if they are not all equal to zero go to step 2.; else, $T \in \sigma_r(C_d) \setminus \sigma_{r-1}(C_d)$ (notice that this happens for $r \leq \lceil \frac{d+1}{2} \rceil$).
4. Choose a solution $(\overline{u}_0, \ldots, \overline{u}_d)$ of the system $M_{d-r,r} \cdot (u_0, \ldots, u_r)^t = 0$. If the polynomial $\overline{u}_0 t_0^d + \overline{u}_1 t_0^{d-1} t_1 + \cdots + \overline{u}_r t_1^r$ has distinct roots, then $s\mathrm{rk}(t) = r$, otherwise $s\mathrm{rk}(t) = d - r + 2$.

4 Beyond dimension two

The sequence in (3) has to be reconsidered when working on $\mathbb{P}^n$, $n \geq 2$, and with secant varieties to the Veronese variety $X_{n,d} \subset \mathbb{P}^N$, $N = \binom{d+n}{n} - 1$. Here a polynomial in $K[x_0, \ldots, x_n]_r$ gives a divisor, which is not a 0-dimensional scheme, hence via the previous construction we would not obtain $(r-1)$-spaces which are r-secant to the Veronese variety.

Actually in this case, when following the construction in (3), we associate to a polynomial $f \in K[x_0, \ldots, x_n]_r$, the vector space $(f)_d \subset K[x_0, \ldots, x_n]_d$, which is $\binom{d-r+n}{n}$-dimensional. Then, working by duality as before, we get a linear space in $\mathbb{P}^N$ which has dimension $\binom{d+n}{n} - \binom{d-r+n}{n} - 1$ and it is the intersection of the hyperplanes containing the image $\nu_d(F) \subset \nu_d(\mathbb{P}^n)$ of the divisor $F = \{f = 0\}$ where ν_d is the Veronese map defined in Notation 2.14.

Since the condition for a point in $\mathbb{P}^N$ to belong to such space a is given by the annihilation of the maximal minors of the catalecticant matrix $M_{d-r,r(n)}$, this shows that such minors define in $\mathbb{P}^N$ a variety which is the union of the linear spaces spanned by the images of the divisors (hypersurfaces in $\mathbb{P}^n$) of degree r on the Veronese $X_{n,d}$ (see [**Gh**]).

In order to consider linear spaces which are r-secant to $X_{n,d}$, we will change our approach by considering $Hilb_r(\mathbb{P}^n)$ instead of $K[x_0, \ldots, x_n]_r$:

$$\begin{aligned} Hilb_r(\mathbb{P}^n) \overset{\phi}{\dashrightarrow} \ \vec{G}\left(\binom{d+n}{n} - r, K[x_0, \ldots, x_n]_d\right) \cong ... \\ ... \cong \mathbb{G}\left(\binom{d+n}{n} - r - 1, \mathbb{P}(K[x_0, \ldots, x_n]_d)\right) \to \mathbb{G}(r-1, \mathbb{P}(K[x_0, \ldots, x_n]_d)^*). \end{aligned} \tag{4}$$

The map ϕ in (4) sends a scheme Z (0-dimensional with $\deg(Z) = r$) to the vector space $(I_Z)_r$; it is defined in the open set of the Z's which imposes independent conditions to forms of degree d.

As in the case $n = 1$, the final image in the above sequence gives the $(r-1)$-spaces which are r-secant to the Veronese variety in $\mathbb{P}^N \cong \mathbb{P}(K[x_0, \ldots, x_n]_d)^*$; moreover each such space cuts the image of Z on the Veronese.

Notation 4.1. From now on we will always use the notation Π_Z to indicate the projective linear subspace of dimension $r-1$ in $\mathbb{P}(S^d V)$, with $\dim(V) = n+1$, generated by the image of a 0-dimensional scheme $Z \subset \mathbb{P}^n$ of degree r via Veronese embedding.

4.1 The chordal varieties to Veronese varieties

Here we describe $\sigma_r(X_{n,d})$ for $r = 2$ and $n, d \geq 1$. More precisely we give a stratification of $\sigma_r(X_{n,d})$ in terms of the symmetric rank of its elements. We will end with an algorithm that allows to determine if an element belongs to $\sigma_2(X_{n,d})$ and, if this is the case, to compute $s\mathrm{rk}(t)$.

We premit a remark that will be useful in the sequel.

Remark 4.2. We recall (e.g. see [**LS**], [**LT**]) that for any form $f \in K[x_0, \ldots, x_n]$, the symmetric rank of its corresponding symmetric tensor with respect to $X_{n,d}$ is the same as the one with respect to $X_{m,d}$, $m < n$, when f can be written using less variables, i.e. $f \in K[l_0, \ldots, l_m]$, for $l_j \in K[x_0, \ldots, x_n]_1$. In particular, when a tensor is such that $T \in \sigma_r(X_{n,d}) \subset \mathbb{P}(S^d V)$, $\dim(V) = n+1$, then, if $r < n+1$, there is a subspace $W \subset V$ with $\dim(W) = r$ such that $T \in \mathbb{P}(S^d W)$; i.e. the form corresponding to T can be written with respect to r variables.

Theorem 4.3. *Any $T \in \sigma_2(X_{n,d}) \subset \mathbb{P}(V)$, with $\dim(V) = n+1$, can only have symmetric rank equal to 1, 2 or d. More precisely:*

$$\sigma_2(X_{n,d})\sigma_r X_{n,d} = \sigma_{2,2}(X_{n,d}) \cup \sigma_{2,d}(X_{n,d})$$

where $\sigma_{2,2}(X_{n,d})$ and $\sigma_{2,d}(X_{n,d})$ are defined in Notation 2.13 and the locus of tensors $T \in \mathbb{P}(S^d W)$ of symmetric rank d is the tangential variety to $X_{n,d}$.

Proof. Since $r = 2$, every $Z \in Hilb_2(\mathbb{P}^n)$ is the complete intersection of a line and a quadric, hence the structure of I_Z is well known: $I_Z = (l_1, \ldots, l_{n-1}, q)$, where $l_i \in R_1$, linearly independent, and $q \in R_2 - (l_1, \ldots, l_{n-1})_2$.

If $T \in \sigma_2(\nu_d(\mathbb{P}^n))$ we have two possibilities; either $s\mathrm{rk}(T) = 2$ (i.e. $T \in \sigma_2^0(\nu_2(\mathbb{P}^n))$), or $s\mathrm{rk}(T) > 2$ i.e. T lies on a tangent line to the Veronese, defined by the image of Z via the maps (4). In this case Π_Z is that tangent line. We can view T in the projective linear space $H \cong \mathbb{P}^d$ in $\mathbb{P}(S_d V)$ generated by the rational normal curve $C_d \subset X_{n,d}$, which is the image of the line L defined by the ideal $(l_1, \ldots, l_{n-1})$ in $\mathbb{P}^n$ with $l_1, \ldots, l_{n-1} \in V^*$; hence we can apply Theorem 3.8 in order to get that $s\mathrm{rk}(T) \leq d$.

Moreover, by Remark 4.2, we have $s\mathrm{rk}(T) = d$. □

Remark 4.4. Let us check that it is the annihilation of the (3×3)-minors of the first two catalecticant matrices, $M_{d-1,1}$ and $M_{d-2,2}$ which determines $\sigma_2(\nu_d(\mathbb{P}^n))$ (actually such minors are the generators of $I_{\sigma_2(\nu_d(\mathbb{P}^n))}$, see [**K**]).

Following the construction before Theorem 3.3, we can notice that the linear spaces defined by the forms $l_i \in V^*$ in the ideal I_Z, are such that their coefficients are the solutions of a linear system whose matrix is given by the catalecticant matrix $M_{d-1,1}$ defined in Definition 3.1 (where the a_i's are the coefficients of the polynomial defined by t); since the space of solutions has dimension $n-1$, we get $\mathrm{rk}(M_{d-1,1}) = 2$. When we consider the quadric q in I_Z, instead, the analogous construction gives that its coefficients are the solutions of a linear systems defined by the catalecticant matrix $M_{d-2,2}$, and the space of solutions has to give q and all the quadrics in $(l_1, \ldots, l_{n-1})_2$, which are $\binom{n}{2} + 2n - 1$, hence $\mathrm{rk}(M_{d-2,2}) = \binom{n+2}{2} - (\binom{n}{2} + 2n) = 2$.

We can therefore write down an algorithm to test if an element $T \in \sigma_2(X_{n,d})$ has symmetric rank 2 or d.

Algorithm 4.5. Algorithm for the symmetric rank of an element of $\sigma_2(X_{n,d})$

Input: The projective class of a symmetric tensor $T \in \mathbb{P}(S^dV)$, with $\dim(V) = n+1$;
Output: $T \notin \sigma_2(X_{n,d})$, or $T \in \sigma_{2,2}(X_{n,d})$, or $T \in \sigma_{2,d}(X_{n,d})$, or $T \in X_{n,d}$.

1. Rewrite T with the minimum number of variables possible (methods are described in [**Ca**] or [**Ol**]), if this is 1 then $T \in X_{n,d}$; if it is > 2 then $T \notin \sigma_2(X_{n,d})$, otherwise T can be viewed as a point in $\mathbb{P}(S^dW) \cong \mathbb{P}^d \subset \mathbb{P}(S^dV)$, and $\dim(W) = 2$, and go to step 2.

2. Apply the Algorithm 3.13 to conclude.

4.2 Varieties of secant planes to Veronese varieties

In this section we give a stratification of $\sigma_3(X_{n,d}) \subset \mathbb{P}(S^dV)$ with $\dim(V) = n+1$ via the symmetric rank of its elements. We will denote by X_d the Veronese surface $X_{2,d} \subset \mathbb{P}(S^2U)$ where U is a 3-dimensional vector space.

Lemma 4.6. *Let $Z \subset \mathbb{P}^n$, $n \geq 2$, be a 0-dimensional scheme, with $\deg(Z) \leq 2d+1$. A necessary and sufficient condition for Z to impose independent conditions to hypersurfaces of degree d is that no line $L \subset \mathbb{P}^n$ is such that $\deg(Z \cap L) \geq d+2$.*

Proof. The statement is probably classically known, we prove it here for lack of a precise reference. Let us work by induction on n and d; if $d = 1$ the statement is trivial; so let us suppose that $d \geq 2$ and now let's work by induction on n; let us consider the case $n = 2$ first. If there is a line L which intersects Z with multiplicity $\geq d+2$, then trivially Z cannot impose independent condition to curves of degree d, since the fixed line gives $d+1$ conditions, hence we already have missed one. So, suppose that no such line exist, and let L be a line such that $Z \cap L$ is as big as possible (but $Z \cap L \leq d+1$). Let $Tr_L Z$, the *Trace* of Z on L, be the schematic intersection $Z \cap L$ and $Res_L Z$, the *Residue* of Z with respect to L, be the scheme defined by $(I_Z : I_L)$. We have the following exact sequence of ideal sheaves:

$$0 \to \mathcal{I}_{Res_L Z}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Tr_L Z}(d) \to 0.$$

Since no line can intersect $Res_L Z$ with multiplicity $\geq d+1$ (because $\deg(Z) \leq 2d+1$), we have $h^1(\mathcal{I}_{Res_L Z}(d-1)) = 0$, by induction on d; on the other hand, we have $h^1(\mathcal{I}_{Tr_L Z}(d)) = h^1(\mathcal{O}_{\mathbb{P}^1}(d-$

$\deg(Tr_L Z))) = 0$, hence also $h^1(\mathcal{I}_Z(d)) = 0$, i.e. Z imposes independent conditions to curves of degree d, since the condition $\deg(Z) \leq 2d+1$ imposes $h^0(\mathcal{I}_Z(d)) > 0$.

With the case $n = 2$ done, let us finish by induction on n; let $n \geq 3$ now; again, if there is a line L which intersects Z with multiplicity $\geq d+2$, we can conclude that Z does not impose independent conditions to forms of degree d, as in the case $n = 2$. Otherwise, consider a hyperplane H, with maximum multiplicity of intersection with Z, and consider the exact sequence:

$$0 \to \mathcal{I}_{Res_H Z}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Tr_H Z}(d) \to 0.$$

We have $h^1(\mathcal{I}_{Res_H Z}(d-1)) = 0$, by induction on d, and $h^1(\mathcal{I}_{Tr_H Z}(d)) = 0$, by induction on n, so we conclude again that $h^1(\mathcal{I}_Z(d)) = 0$, and we are done. □

Remark 4.7. Notice that if $\deg L \cap Z$ is exactly $d+2$, then the dimension of the space of curves of degree d through them increases exactly by one.

We will need this definition in the sequel.

Definition 4.8. *A t-jet is a 0-dimensional scheme $J \subset \mathbb{P}^n$ of degree t with support at a point $P \in \mathbb{P}^n$ and contained in a line L; namely the ideal of J is of type: $I_P^t + I_L$, where $L \subset \mathbb{P}^n$ is a line containing P. We will say that $J_1, \dots, J_s$ are generic t-jets in $\mathbb{P}^n$, if the points $P_1, \dots, P_s$ are generic in $\mathbb{P}^n$ and $L_1, \dots, L_s$ are generic lines through $P_1, \dots, P_s$.*

Theorem 4.9. *Let $d \geq 3$, $X_{n,d} \subset \mathbb{P}^(V)$. Then:*

$$\sigma_3(X_{n,d}) \setminus \sigma_2(X_{n,d}) = \sigma_{3,3}(X_{n,d}) \cup \sigma_{3,d-1}(X_{n,d}) \cup \sigma_{3,d+1}(X_{n,d}) \cup \sigma_{3,2d-1}(X_{n,d}), \text{ if } d > 3,$$

$$\sigma_3(X_{n,3}) \setminus \sigma_2(X_{n,3}) = \sigma_{3,3}(X_{n,3}) \cup \sigma_{3,4}(X_{n,d}) \cup \sigma_{3,7}(X_{n,d}) \text{ if } d = 3,$$

where $\sigma_{3,3}(X_{n,d}), \sigma_{3,d-1}(X_{n,d}), \sigma_{3,d+1}(X_{n,d})$ and $\sigma_{3,2d-1}(X_{n,d})$ are as in Notation 2.13.

Proof. For any scheme $Z \in Hilb_3(\mathbb{P}(V))$ there exist a subspace $U \subset V$ of dimension 3 such that $Z \subset \mathbb{P}(U)$. Hence, when we make the construction in (4) we get that Π_Z is always a $\mathbb{P}^2$ contained in $\mathbb{P}(S^d U)$ and $\nu_d(\mathbb{P}(U))$ is a Veronese surface $X_d \subset \mathbb{P}(S^d U) \subset \mathbb{P}(S^d V)$. Therefore, by Remark 4.2, it is sufficient to prove the statement for $X_d \subset \mathbb{P}(S^d U)$.

We will consider first the case when there is a line L such that $Z \subset L$. In this case, let $C_d = \nu_d(L)$, where ν_d is defined in Remark 2.14; we get that $T \in \sigma_3(C_d)$, hence either $T \in \sigma_{3,3}(C_d)$ (hence $T \in \sigma_{3,3}(X_d)$), or (only when $d \geq 4$) $T \in \sigma_{3,d-1}(C_d)$, hence $s\mathrm{rk}(T) \leq d-1$. It is actually $d-1$ by Remark 4.2.

Now we let Z not to be on a line; the scheme $Z \in Hilb_3(\mathbb{P}^n)$ can have support on 3 , 2 distinct points or on one point.

If $Supp(Z)$ is the union of 3 distinct points then clearly Π_Z, that is the image of Z via (4), intersects X_d in 3 different points and hence any $T \in \Pi_Z$ has symmetric rank precisely 3, so $T \in \sigma_{3,3}(X_d)$.

If $Supp(Z) = \{P, Q\}$ with $P \neq Q$, then the scheme Z is the union of a simple point, Q, and of a 2-jet J (see Definition 4.8) at P. The structure of 2-jet on P implies that there exist a line $L \subset \mathbb{P}^n$ whose intersection with Z is a 0-dimensional scheme of degree 2. Hence $\Pi_Z =< T_{\nu_d(P)}(C_d), \nu_d(Q) >$ where $T_{\nu_d(P)}(C_d)$ is the projective tangent line at $\nu_d(P)$ on $C_d = \nu_d(L)$. Since $T \in \Pi_Z$, the line $< T, \nu_d(Q) >$ intersects $T_{\nu_d(P)}(C_d)$ in a point $Q' \in \sigma_2(C_d)$. From Theorem 3.8 we know that $s\mathrm{rk}(Q') = d$. We may assume that $T \neq Q'$ because otherwise T should belong to $\sigma_2(X_d)$.

We have $Q \notin L$ because Z is not in a line, so T can be written as a combination of a tensor of symmetric rank d and a tensor of symmetric rank 1, hence $s\mathrm{rk}(t) \leq d+1$. Now suppose that $s\mathrm{rk}(t) = d$, hence there should exist $Q_1,\dots,Q_d\in X_d$ such that $T\in<Q_1,\dots,Q_d>$; notice that $Q_1,\dots,Q_d$ are not all on C_d, otherwise $T\in\sigma_2(X_d)$. Let $P_1,\dots,P_d$ be the pre-image via ν_d of $Q_1,\dots,Q_d$; then $P_1,\dots,P_d$ together with J and Q should not impose independent conditions to curves of degree d, so, by Lemma 4.6, either $P_1,\dots,P_d,J$ are on L, or $P_1,\dots,P_d,P,Q$ are on a line L'. The first case is not possible, since $Q_1,\dots,Q_d$ are not on C_d. In the other case notice that, by Lemma 4.6 and the Remark 4.7, should have that $<Q_1,\dots,Q_d,T_{\nu_d(P)}(C_d),\nu_d(Q)>\cong\mathbb{P}^d$, but since $<Q_1,\dots,Q_d>$ and $<T_{\nu_d(P)}(C_d),\nu_d(Q)>$ have $T,\nu_d(P)$ and $\nu_d(Q)$ in common, they generate a $(d-1)$-dimensional space, a contradiction. Hence $s\mathrm{rk}(t)=d+1$.

This construction shows also that $T\in\sigma_{3,d+1}(X_d)$, and that there exist $W\subset V$ with $\dim(W)=2$ and $l_1,\dots,l_d\in W^*$ and $l_{d+1}\in V^*$ such that $t=l_1^d+\cdots+l_d^d+l_{d+1}^d$ and $t=[T]$.

If $Supp(Z)$ is only one point $P\in\mathbb{P}^2$, then Z can only be one of the following: either Z is 2-fat point, or there exists a smooth conic containing Z.
If Z is a double fat point then Π_Z is the tangent space to X_d at $\nu_d(P)$, hence if $T\in\Pi_Z$, then the line $<\nu_d(P),T>$ turns out to be a tangent line to some rational normal curve of degree d contained in X_d, hence in this case $T\in\sigma_2(X_d)$.
If there exists a smooth conic $C\subset\mathbb{P}^2$ containing Z, write $Z=3P$ and consider $C_{2d}=\nu_d(C)$, hence $T\in\sigma_3(C_{2d})$, therefore by Theorem 3.8 clearly $s\mathrm{rk}(t)\le 2d-1$. Suppose that $s\mathrm{rk}(t)\le 2d-2$, hence there exist $P_1,\dots,P_{2d-2}\in\mathbb{P}^2$ distinct points that are neither on a line nor on a conic containing $3P$, such that $T\in\Pi_{Z'}$ with $Z'=P_1+\cdots+P_{2d-2}$ and $Z+Z'=3P+P_1+\cdots+P_{2d-2}$ doesn't impose independent conditions to the planes curves of degree d. Now, by Lemma 4.6 we get that $3P+P_1+\cdots+P_{2d-2}$ doesn't impose independent conditions to the plane curves of degree d if and only if there exists a line $L\subset\mathbb{P}^2$ such that $\deg((Z+Z')\cap L)\ge d+2$. Observe that Z' cannot have support contained in a line because otherwise $T\in\sigma_2(X_d)$. Moreover $Z+Z'$ cannot have support on a conic $C\subset\mathbb{P}^2$ because in that case T would have symmetric rank $2d-1$ with respect to $\nu_d(C)=C_{2d}$.
We have to check the following cases:

1. There exist $P_1,\dots,P_{d+2}\in Z'$ on a line $L\subset\mathbb{P}^2$;
2. There exist $P_1,\dots,P_{d+1}\in Z'$ such that together with $P=Supp(Z)$ they are on the same line $L\subset\mathbb{P}^2$;
3. There exist $P_1,\dots,P_d\in Z'$ such that together with the 2-jet $2P$ they are on the same line $L\subset\mathbb{P}^2$.

Case 1. Let $P_1,\dots,P_{d+2}\in L\subset\mathbb{P}^2$, then $\nu_d(L)=C_d\subset\mathbb{P}^d\subset\mathbb{P}^N$ with $N=\binom{d+2}{2}-1$. Clearly $T\in\Pi_Z\cap\Pi_{Z'}$, then $\dim(\Pi_Z+\Pi_{Z'})\le\dim(\Pi_Z)+\dim(\Pi_{Z'})$, moreover $\Pi_{Z'}$ doesn't have dimension $2d-3$ as expected because $\nu_d(P_1),\dots,\nu_d(P_{d+2})\in C_d\subset\mathbb{P}^d$, hence $\dim(\Pi_{Z'})\le 2d-4$ and $\dim(\Pi_Z+\Pi_{Z'})\le 2d-2$. But this is not possible because $Z+Z'$ imposes to the plane curves of degree d only one condition less then the expected, hence $\dim(I_{Z+Z'}(d))=\binom{d+1}{2}-d+1$ and then $\dim(\Pi_Z+\Pi_{Z'})=2d-1$, that is a contradiction.

Case 2. Let $P_1,\dots,P_{d+1},P\in L\subset\mathbb{P}^2$, then $\nu_d(P_1),\dots,\nu_d(P_{d+1}),\nu_d(P)\in\nu_d(L)=C_d$. Now $\Pi_Z\cap\Pi_{Z'}\supset\{\nu_d(P),T\}$, then again $\dim(\Pi_Z+\Pi_{Z'})\le 2d-2$.

Case 3. Let $P_1,\dots,P_d,2P\in L\subset\mathbb{P}^2$, as previously $\nu_d(P_1),\dots,\nu_d(P_{d+1}),\nu_d(2P)\in\nu_d(L)=C_d$, then now $T_{\nu_d(P)}(C_d)$ is contained in $<C_d>\cap\Pi_Z$. Since $<\nu_d(P_1,\dots,\nu_d(P_d)>)$ is an hyperplane in $<C_d>=\mathbb{P}^d$, it will intersect $T_{\nu_d(P)}(C_d)$ in a point Q different form $\nu_d(P)$. Again $\dim(\Pi_Z\cap\Pi_{Z'})\ge 1$ and then $\dim(\Pi_Z+\Pi_{Z'})\le 2d-2$.

$\square$

Now we are almost ready to present an algorithm which allows to indicate if a projective class of a symmetric tensor in $\mathbb{P}^{\binom{n+d}{d}-1}$ belongs to $\sigma_3(X_{n,d})$, and in this case to determine its rank. Before giving the algorithm we need to recall a result about $\sigma_3(X_3)$:

Remark 4.10. The secant variety $\sigma_3(X_3) \subset \mathbb{P}^9$ is a hypersurface and its defining equation it is the "Aronhold (or Clebsch) invariant" (for an explicit expression see e.g. [**Ot**]).

Notice that there is a very direct and well known way of getting the equations for the secant variety $\sigma_s(X_{n,d})$, which we describe in the next remark. The problem with this method is that it is computationally very inefficient, and it can be worked out only in simple cases.

Remark 4.11. Let $T = \left[z_0, \ldots, z_{\binom{n+d}{d}}\right] \in \mathbb{P}(S^d(V))$, where V is an $(n+1)$-dimensional vector space. T is an element of $\sigma_s(X_{n,d})$ if there exist $P_i = [x_{0,i}, \ldots, x_{n,i}] \in \mathbb{P}^n = \mathbb{P}(V)$, $i = 1, \ldots, s$, and $\lambda_1, \ldots, \lambda_s \in K$, such that $T = \lambda_1 Q_1 + \cdots + \lambda_s Q_s$, where $Q_i = \nu_d(P_i) \subset \mathbb{P}^{\binom{n+d}{d}-1} = \mathbb{P}(S^d V)$, $i = 1, \ldots, s$ (i.e. $Q_i = [x_{0,i}^d, x_{0,i}^{d-1}x_1, \ldots, x_{n,i}^d]$).

This can be expressed via the following system of equations:

$$\begin{cases} z_0 = \lambda_1 x_{0,1}^d + \cdots + \lambda_s x_{0,s}^d \\ z_1 = \lambda_1 x_{0,1}^{d-1} x_{1,1} + \cdots + \lambda_s x_{0,s}^{d-1} x_{1,s} \\ \vdots \\ z_{\binom{n+d}{d}-1} = \lambda_1 x_{n,1}^d + \cdots + \lambda_s x_{s,s}^d \end{cases}.$$

Now consider the ideal $I_{s,n,d}$ defined by the above polynomials in the weighted coordinate ring

$$R = K\left[x_{0,1}, \ldots, x_{n,1}; \ldots; x_{0,s}, \ldots, x_{n,s}; \lambda_1, \ldots, \lambda_s; z_0, \ldots, z_{\binom{n+d}{d}-1}\right]$$

where the z_i's have degree $d+1$:

$$I_{s,n,d} = (z_0 - \lambda_1 x_{0,1}^d + \cdots + \lambda_s x_{0,s}^d, z_1 - \lambda_1 x_{0,1}^{d-1} x_{1,1} + \cdots + \lambda_s x_{0,s}^{d-1} x_{1,s}, \ldots, z_{\binom{n+d}{d}-1} - \lambda_1 x_{n,1}^d + \cdots + \lambda_s x_{s,s}^d).$$

Now eliminate from $I_{s,n,d}$ the variables λ_i's and $x_{j,i}$'s, $i = 1, \ldots, s$ and $j = 0, \ldots, n$. The elimination ideal $J_{s,n,d} \subset K\left[z_0, \ldots, z_{\binom{n+d}{d}-1}\right]$ that we get from this process is an ideal of $\sigma_s(X_{n,d})$.

Obviously $J_{s,n,d}$ contains all the $(s+1) \times (s+1)$ minors of the catalecticant matrix of order $r \times (d-r)$ (if they exist).

Algorithm 4.12. Algorithm for the symmetric rank of an element of $\sigma_3(X_{n,d})$

Input: A symmetric tensor $t \in S^d V$, with $\dim(V) = n+1$;
Output: $T \notin \sigma_3(X_{n,d})$ or $T \in \sigma_2(X_{n,d})$ or $T \in \sigma_{3,3}(X_{n,d})$ or $T \in \sigma_{3d-1}(X_{n,d})$ or $T \in \sigma_{3,d+1}(X_{n,d})$ or $T \in \sigma_{3,2d-1}$.

1. Run the first step of Algorithm 4.5; If T can be written in one variable, then $T \in X_{n,d}$; if the two variables are needed, then use Algorithm 4.5 to determine $\mathrm{srk}T$. If output is > 3, then $T \notin \sigma_3(X_d)$. Otherwise (three variables) rewrite t as a polynomial in three variables. From now on consider $t \in S^d(V)$, with $\dim(V) = 3$; go to next step;

2. Evaluate the Aronhold invariant (see 4.10) on T, if is zero on T then go to step 3. Otherwise $T \notin \sigma_3(X_d)$;

3. Consider the space S of the solutions of the system $M_{2,d-2} \cdot (b_{0,0}, \ldots, b_{2,2})^T = 0$. Choose three generators of S and let F_1, F_2, F_3 the conics in $K[x_0, x_1, x_2]$ associated to them;

4. Compute the radical ideal I of the ideal (F_1, F_2, F_3);

5. If F_1, F_2, F_3 (i.e. the output of step 3.) have a common factor then go to step 6., otherwise go to step 7.;

6. Compute the cardinality of the support of the ideal I computed in step 4;
   If it is 1 or 2 then $T \in \sigma_{3,d-1}(X_{n,d})$;
   if it is 3 then $T \in \sigma_{3,3}(X_{n,d})$.

7. Consider the generators of I. If there are two linear forms among them, then $T \in \sigma_{3,2d-1}(X_{n,d})$, if there is only one linear form then $T \in \sigma_{3,d+1}(X_{n,d})$, if there are no linear forms then $T \in \sigma_{3,3}(X_{n,d})$.

4.3 Secant varieties of X_3

In this section we describe all possible symmetric ranks that can occur in $\sigma_s(X_3)$ for any $s \geq 1$.

Theorem 4.13. *Let U be a 3-dimensional vector space. The stratification of the cubic forms of $\mathbb{P}(S^3U^*)$ via the symmetric rank is the following:*

- $X_3 = \{T \in \mathbb{P}(S^3U) \mid \mathrm{srk}(T) = 1\}$;
- $\sigma_2(X_3) \setminus X_3 = \sigma_{2,2}(X_3) \cup \sigma_{2,3}(X_3)$;
- $\sigma_3(X_3) \setminus \sigma_2(X_3) = \sigma_{3,3}(X_3) \cup \sigma_{3,4}(X_3) \cup \sigma_{3,5}(X_3)$;
- $\mathbb{P}^9 \setminus \sigma_3(X_3) = \sigma_{4,4}(X_3)$;

where $\sigma_{s,m}(X_3)$ is defined as in Notation 2.13.

Proof. We only need to prove that $\mathbb{P}^9 \setminus \sigma_3(X_3) = \sigma_{4,4}(X_3) \setminus \sigma_{3,4}(X_3)$ because X_3 is by definition the set of symmetric tensors of symmetric rank 1 and the cases of $\sigma_2(X_3)$ and $\sigma_3(X_3)$ are consequences of Theorem 4.3 and Theorem 4.9 respectively.

First of all we show that all symmetric tensors in $\mathbb{P}^9 \setminus \sigma_3(X_3)$ are of symmetric rank 4. Clearly, since they do not belong to $\sigma_3(X_3)$, they have symmetric rank ≥ 4; hence we need to show that their symmetric rank is actually less or equal than 4.
Let $T \in \mathbb{P}^9 \setminus \sigma_3(X_3)$ and consider the system $M_{2,1} \cdot (b_{0,0}, \ldots, b_{2,2})^T = 0$. The space of solutions of this system gives a vector space of conics which has dimension 3; moreover it is not the degree 2 part of any ideal representing a 0-dimensional scheme of degree 3, hence the generic solution of that system is a smooth conic. Therefore in the space of the cubics through T, there is a subspace given by $< C \cdot x_0, C \cdot x_1, C \cdot x_2 >$ where C is indeed a smooth conic given by the previous system. Hence, if C_6 is the image of C via the Veronese embedding ν_3, we have that $T \in < C_6 >$, in particular $T \in \sigma_4(C_6) \setminus \sigma_3(C_6)$, therefore $\mathrm{srk}(t) \leq 6 - 4 + 2 = 4$. □

4.4 Secant varieties of X_4

We recall that the k-th osculating variety to $X_{n,d}$, denoted by $\mathcal{O}_{k,n,d}$, is the union of the k-osculating planes to the Veronese variety $X_{n,d}$, where the k-osculating plane $\mathcal{O}_{k,n,d,P}$ at the point $P \in X_{n,d}$ is the linear space generated by the k-th infinitesimal neighborhood $(k+1)P$ of P on $X_{n,d}$ (see for example [**BCGI**] 2.1, 2.2). Hence for example the first osculating variety is the tangential variety.

Lemma 4.14. *The second osculating variety $\mathcal{O}_{2,2,4}$ of X_4 is contained in $\sigma_4(X_4)$*

Proof. Let T be a generic element of $\mathcal{O}_{2,2,4} \subset \mathbb{P}(S^4V)$ with $\dim(V) = 3$. Hence $T = l^2\mathcal{C}$ where l and $\mathcal{C}$ are a linear and a quadratic generic forms respectively of $\mathbb{P}(S^4V)$ regarded as a projectivization of the homogeneous polynomials of degree 4 in 3 variables, i.e. $K[x,y,z]_4$ (see [**BCGI**]). We can always assume that $l = x$ and $\mathcal{C} = a_{0,0}x^2 + a_{0,1}xy + a_{0,2}xz + a_{1,1}y^2 + a_{1,2}yz + a_{2,2}z^2$. The catalecticant matrix $M_{2,2}$ (defined in general in Definition 3.1) for a plane quartic $a_{0000}x^4 + a_{0001}x^3y + \cdots + a_{2222}z^4$ is the following:

$$M_{2,2} = \begin{pmatrix} a_{0000} & a_{0001} & a_{0002} & a_{0011} & a_{0012} & a_{0022} \\ a_{0001} & a_{0011} & a_{0012} & a_{0111} & a_{0112} & a_{0122} \\ a_{0002} & a_{0012} & a_{0022} & a_{0112} & a_{0122} & a_{0222} \\ a_{0011} & a_{0111} & a_{0112} & a_{1111} & a_{1112} & a_{1122} \\ a_{0012} & a_{0112} & a_{0122} & a_{1112} & a_{1122} & a_{1222} \\ a_{0022} & a_{0122} & a_{0222} & a_{1122} & a_{1222} & a_{2222} \end{pmatrix}$$

hence in the specific case of the quartic above $l^2\mathcal{C} = x^2(a_{0,0}x^2 + a_{0,1}xy + a_{0,2}xz + a_{1,1}y^2 + a_{1,2}yz + a_{2,2}z^2)$ it becomes:

$$M_{2,2}(T) = \begin{pmatrix} a_{0000} & a_{0001} & a_{0002} & a_{0011} & a_{0012} & a_{0022} \\ a_{0001} & a_{0011} & a_{0012} & 0 & 0 & 0 \\ a_{0002} & a_{0012} & a_{0022} & 0 & 0 & 0 \\ a_{0011} & 0 & 0 & 0 & 0 & 0 \\ a_{0012} & 0 & 0 & 0 & 0 & 0 \\ a_{0022} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that clearly has rank less or equal than 4, hence $O^2(X_4) \subset \sigma_4(X_4)$. □

Lemma 4.15. *If $Z \in Hilb_4(\mathbb{P}^2)$ and Z is contained in a line, then $r = s\mathrm{rk}(T) \leq 4$ for any $T \in \Pi_Z$, where Π_Z is defined in Notation 4.1, and T belongs either to $\sigma_2(X_4)$ or to $\sigma_3(X_4)$. Moreover there exists W of dimension 2 and $l_1, \ldots, l_r \in S^1W^*$ such that $t = l_1^4 + \cdots + l_r^4$ with $r \leq 4$.*

Proof. If there exist a 2-dimensional subspace $W \subset V$ with $\dim(V) = 3$ such that $Supp(Z) \subset \mathbb{P}(W)$ then any $T \in \Pi_Z \subset \mathbb{P}(S^4V)$ belongs to $\sigma_4(\nu_4(\mathbb{P}(W))) \simeq \mathbb{P}^4$, therefore $s\mathrm{rk}(T) \leq 4$. If $s\mathrm{rk}(T) = 2, 4$ then $T \in \sigma_2(X_4)$, otherwise $T \in \sigma_3(X_4)$. □

Lemma 4.16. *If $Z \subset Hilb_4(\mathbb{P}^2)$ and there exist a smooth conic $C \subset \mathbb{P}^2$ such that $Z \subset C$, then any $T \in \Pi_Z$ and $T \notin \sigma_3(X_4)$ is of symmetric rank 4 or 6.*

Proof. Clearly $T \in \sigma_4(\nu_4(C))$ and $\nu_4(C)$ is a rational normal curve of degree 8, then $s\mathrm{rk}(T) \leq 6$. If $\sharp\{Supp(Z)\} = 4$ then $s\mathrm{rk}(T) = 4$. Otherwise $s\mathrm{rk}(T)$ cannot be less or equal than 5 because there would exists a 0-dimensional scheme $Z' \subset \mathbb{P}^2$ made of 5 distinct points such that $T \in \Pi_{Z'}$, then $Z + Z'$ should not impose independent conditions to plane curves of degree 4. In fact by Lemma 4.6 the scheme $Z + Z'$ doesn't impose independent conditions to the plane quartic if and only if there exists a line $M \subset \mathbb{P}^2$ such that $\deg((Z+Z')\cap M)\geq 6$. If $\deg((Z')\cap M)\geq 5$ then $T\in\sigma_2(X_4)$ or $T\in\sigma_3(X_4)$. Hence assume that $\deg((Z+Z')\cap M)\geq 6$ and $\deg((Z')\cap M)<5$. Consider first the case $\deg((Z+Z')\cap M)=6$. Then $\deg((Z')\cap M)=4$ and $\deg((Z)\cap M)=2$. We have that $\Pi_{Z+Z'}$ should be a $\mathbb{P}^7$ but actually it is at most a $\mathbb{P}^6$ in fact $\Pi_{(Z+Z')\cap M}=\mathbb{P}^4$ because $<\nu_4(M)>=\mathbb{P}^4$, moreover $T\in\Pi_Z\cap\Pi_{Z'}$ hence $\Pi_{Z+Z'}$ is at most a $\mathbb{P}^6$. Analogously if $\deg((Z+Z')\cap M)=7$ (it cannot be more) one can see that $\Pi_{Z+Z'}$ should have dimension 6 but it must have dimension strictly less than 6. □

Theorem 4.17. *The s-th secant varieties to X_4 up to $s=4$ are described in terms of symmetric ranks as follows:*

- $X_4=\{T\in S^4V \mid s\mathrm{rk}(T)=1\}$;
- $\sigma_2(X_4)\setminus X_4=\sigma_{2,2}(X_4)\cup\sigma_{2,4}(X_4)$;
- $\sigma_3(X_4)\setminus\sigma_2(X_4)=\sigma_{3,3}(X_4)\cup\sigma_{3,5}(X_4)\cup\sigma_{3,7}(X_4)$;
- $\sigma_4(X_4)\setminus\sigma_3(X_4)=\sigma_{4,4}(X_4)\cup\sigma_{4,6}(X_4)\cup\sigma_{4,7}(X_4)$;
- $\sigma_5(X_4)\setminus\sigma_4(X_4)=\sigma_{5,5}(X_4)\cup\sigma_{5,6}(X_4)\cup\sigma_{5,7}(X_4)$.

Proof. By definition of $X_{n,d}$ we have that X_4 is the variety parameterizing symmetric tensors of S^4V having symmetric rank 1 and the cases of $\sigma_2(X_4)$ and $\sigma_3(X_4)$ are consequences of Theorem 4.3 and Theorem 4.9 respectively.

Now we study $\sigma_4(X_4)\setminus\sigma_3(X_4)$. Let $Z\in Hilb_4(\mathbb{P}^2)$ and $T\in\Pi_Z$ be defined as in Notation 4.1.

- Let Z be contained in a line L; then by Lemma 4.15 we have that T belongs either to $\sigma_2(X_4)$ or to $\sigma_3(X_4)$.
- Let $Z\subset C$, with C a smooth conic. Then by Lemma 4.16, $T\in\sigma_{4,4}(X_4)$ or $T\in\sigma_{4,6}(X_4)$.
- If there are no smooth conics containing Z then either there is a line L such that $\deg(Z\cap L)=3$, or I_Z can be written as (x^2,y^2). We study separately those two cases.

  1. In the first case the ideal of Z in degree 2 can be written either as $<x^2,xy>$ or $<xy,xz>$.

     If $(I_Z)_2=<x^2,xy>$ then it can be seen that the catalecticant matrix of T is

$$M_{2,2}(T)=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&a_{0222}\\ 0&0&0&a_{1111}&a_{1112}&a_{1122}\\ 0&0&0&a_{1112}&a_{1122}&a_{1222}\\ 0&0&a_{0222}&a_{1122}&a_{1222}&a_{2222}\end{pmatrix}.$$

     Hence, for a generic such T, we have that $T\notin\sigma_3(X_4)$ since the rank of $M_{2,2}(T)$ is 4, while it has to be 3 for points in $\sigma_3(X_4)$. In this case if Z has support in a point then I_Z can be written as (x^2,xy,y^3) and the catalecticant matrix defined in Definition 3.1 evaluated in T turns out to be:

$$M_{2,2}(T)=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&a_{0222}\\ 0&0&0&0&0&a_{1122}\\ 0&0&0&0&a_{1122}&a_{1222}\\ 0&0&a_{0222}&a_{1122}&a_{1222}&a_{2222}\end{pmatrix}$$

that clearly has rank less or equal then 3. Hence $T\in\sigma_3(X_4)$.
Otherwise Z is either made of two 2-jets or one 2-jet and two simple points. In both cases denote by R the line $y=0$. We have $\deg(Z\cap R)=2$. Thus Π_Z is the sum of the linear space $\Pi_{Z\cap L}\simeq\mathbb{P}^2$ and $\Pi_{Z\cap R}\simeq\mathbb{P}^1$. Hence $T=Q+Q'$ for suitable $Q\in\Pi_{Z\cap L}$ and $Q'\in\Pi_{Z\cap R}$. Since $Q\in\sigma_3(\nu_4(L))$ and Q' is in a tangent line to $\nu_4(R)$ we have that $s\mathrm{rk}(T)\leq 7$. Working as in Lemma 4.16 we can prove that $s\mathrm{rk}(T)=7$.

Eventually if $(I_Z)_2$ can be written as (xy,xz) then Z is made of a subscheme of degree 3 on the line L and a simple disjoint point. In this case $s\mathrm{rk}(T)=4$ (T can be viewed as the sum of a point in $\sigma_3(\nu_4(L))$ and a simple point in X_4).

2. In the last case we have that $I_Z$can be written as (x^2,y^2). If we write the catalecticant matrix defined in Definition 3.1 evaluated in T we get the following matrix:

$$M_{2,2}(T)=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&a_{0122}\\ 0&0&0&0&a_{0122}&a_{0222}\\ 0&0&0&0&0&0\\ 0&0&a_{0122}&0&0&a_{1222}\\ 0&a_{0122}&a_{0222}&0&a_{1222}&a_{2222}\end{pmatrix}.$$

Clearly if $a_{0122}=0$ the rank of $M_{2,2}(T)$ is three, hence such a T belongs to $\sigma_3(X_4)$, otherwise we can make a change of coordinates (that corresponds to do a Gauss elimination on $M_{2,2}(T)$) that allows to write the above matrix as follows:

$$M_{2,2}(T)=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&a_{0122}\\ 0&0&0&0&a_{0122}&0\\ 0&0&0&0&0&0\\ 0&0&a_{0122}&0&0&0\\ 0&a_{0122}&0&0&0&0\end{pmatrix}.$$

This matrix is associated to a tensor $t\in S^4V$, with $\dim(V)=3$, that can be written as the polynomial $t(x_0,x_1,x_2)=x_0x_1x_2^2$. Now $s\mathrm{rk}(t)=6$ (see [**LT**], Proposition 11.9).

We now study $\sigma_5(X_4)\setminus\sigma_4(X_4)$,so in the following we assume $T\notin\sigma_4(X_4)$, which implies $s\mathrm{rk}(T)\geq 5$. We have to study the cases with $\deg(Z)=5$, i.e., $Z\in Hilb_5(\mathbb{P}^2)$. The scheme Z is hence always contained in a conic, which can be a smooth conic, the union of 2 lines or a double line. In the last two cases, Z might be contained in a line; we now distinguish the various cases according to these possibilities.

- Z is contained in a line L: $\Pi_Z \cong \mathbb{P}^4$ is spanned by the rational curve $\nu(L) = C_4$, hence $s\mathrm{rk}(T) \leq 4$, against assumptions.

- Z is contained in a smooth conic C. Hence Π_Z is spanned by the subscheme $\nu(Z)$ of the rational curve $\nu(C) = C_8$, so that $T \in \sigma_5(C_8)$ and by Theorem 3.8 $s\mathrm{rk}(T) = 5$.

- Z is contained in the union of two lines L and R. We say that Z is of type (i, j) if $\deg(Z \cap L) = i$ and $\deg(Z \cap R) = j$ and for any other couple of lines in the ideal of Z the degree of the intersections is not smaller. Four different cases can occur: Z is of type $(3, 2)$, in which case $Z \cap L \cap R = \emptyset$, Z is of type $(3, 3)$ or $(4, 2)$, and in these two cases Z, L and R meet in a point P, Z is of type $(4, 1)$, in which case R is not unique. We set $C_4 = \nu(L)$, $C'_4 = \nu(R)$, $O = \nu(P)$, $\Pi_L =< \nu(Z \cap L) >$ and $\Pi_R =< \nu(Z \cap R) >$.

  - Z is of type $(4, 1)$. Hence Π_Z is sum of the linear space $\Pi_L \subseteq \sigma_4(C_4)$ and the point $Q = \Pi_R \in X_4$, so that $T = Q' + Q$ for a suitable $Q' \in \sigma_4(C_4)$, and since $s\mathrm{rk}(Q') \leq 4$ by Theorem 3.8, we get $s\mathrm{rk}(Q') \leq 5$.
  - Z is of type $(3, 2)$. Hence Π_Z is sum of the linear spaces $\Pi_L \cong \mathbb{P}^2$ and the line Π_R, so that $T = Q' + Q$ for suitable $Q \in \Pi_L \subseteq \sigma_3(C_4)$ and $Q' \in \Pi_R \subseteq \sigma_2(C'_4)$. Since $s\mathrm{rk}(Q) \leq 3$ and $s\mathrm{rk}(Q') \leq 4$, we get $s\mathrm{rk}(Q) \leq 7$.
  - Z is of type $(3, 3)$. Hence Π_Z is sum of the linear spaces $\Pi_L \cong \mathbb{P}^2$ and $\Pi_R \cong \mathbb{P}^2$ meeting at one point, so that $T = Q' + Q$ for suitable $Q \in \Pi_L \subseteq \sigma_3(C_4)$ and $Q' \in \Pi_R \subseteq \sigma_3(C'_4)$. Since $s\mathrm{rk}(Q) \leq 3$ and $s\mathrm{rk}(Q') \leq 3$, we get $s\mathrm{rk}(T) \leq 6$. Moreover if Z has support on 4 points, we see that $s\mathrm{rk}(T) = 6$, using the same kind of argument as in Lemma 4.16.
  - Z is of type $(4, 2)$. In this case $(I_Z)_2$ can be written as $< xy, x^2 >$, then working as above we can see that the catalecticant matrix $M_{2,2}(T)$ has rank 4. Since at least set theoretically $I(\sigma_4(X_4))$ is generated by the 5×5 minors of $M_{2,2}$, we conclude that such T belongs to $\sigma_4(X_4)$.

- Z is contained in a double line. We distinguish the following cases:

  - The support of Z is a point P, i.e. the ideal of Z is either of type (x^3, x^2y, y^2) or, in affine coordinates, $(x - y^2, y^4) \cap (x^2, y)$. In the first case Z is contained in the 3-fat point supported on P, so that Π_Z is contained in in the second osculating variety and by Lemma 4.14 $T \in \sigma_4(X_4)$. In the second case it easy to see that the homogeneous ideal contains x^2, xy^2 and y^4 and this fact forces the catalecticant matrix $M_{2,2}(T)$ to have rank smaller or equal to 4. Hence $T \in \sigma_4(X_4)$.
  - The support of Z consists of two poihnts, i.e. the ideal of Z is of type $(x^2, y^2) \cap (x - 1, y)$ or $(x^2, xy, y^2) \cap (x - 1, y^2)$.

    In the first case Z is union of a scheme Y of degree 4 and of a point P, hence Π_Z is sum of the linear spaces Π_Y and Π_P, so that $T = Q + \nu(P)$ for suitable $Q \in \Pi_Y$. The above description of the case corresponding to I_Z of the type (x^2, y^2) shows that either $Q \in \sigma_3(X_4)$ or $s\mathrm{rk}(Q) = 6$. Now if $Q \in \sigma_3(X_4)$ then clearly $T \in \sigma_4(X_4)$, if $s\mathrm{rk}(Q) = 6$ then $s\mathrm{rk}(T) = 7$.

    In the second case Z is union of a jet and of a 2-fat point, hence Π_Z is sum of two linear spaces, both contained in the tangent spaces of X_4 at two different points, so that $T = Q + Q'$ with Q, Q' contained in the tangential variety; then both Q and Q' belongs to $\sigma_2(X_4)$ hence $T \in \sigma_4(X_4)$.

– The support of Z consists of three points, i.e. the ideal of Z is of type $(x, y) \cap ((x^2 - 1), y^2)$. Let P_1, P_2, P_3 be the points supporting Z, with η_1, η_2 jets such that $Z = \eta_1 \cup \eta_2 \cup P_3$. There exists a smooth conic C containing $\eta_1 \cup \eta_2$, and $\nu(C)$ is a C_8. Then Π_Z is the sum of $\nu(P_3)$ and of the linear space $< \nu(\eta_1), \nu(\eta_2) >$, so that $T = Q + \nu(P_3)$ for a suitable $Q \in \sigma_4(C_8)$, so that $s\mathrm{rk}(Q) \leq 6$ and we get $s\mathrm{rk}(T) \leq 7$.

$\square$

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