Size functions for shape recognition in the presence of occlusions

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Abstract

In Computer Vision the ability to recognize objects in the presence of occlusions is a necessary requirement for any shape representation method. In this paper we investigate how the size function of an object shape changes when a portion of the object is occluded by another object. More precisely, considering a set $X = A \cup B$ and a measuring function φ on X, we establish a condition so that $\ell_{(X,\varphi)} = \ell_{(A,\varphi|_A)} + \ell_{(B,\varphi|_B)} - \ell_{(A\cap B,\varphi|_{A\cap B})}$. The main tool we use is the Mayer-Vietoris sequence of Čech homology groups. This result allows us to prove that size functions are able to detect partial matching between shapes by showing a common subset of cornerpoints.

Keywords: Čech homology, Mayer-Vietoris sequence, persistent homology, shape occlusion

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1 Introduction

Shape matching and retrieval are key aspects in the design of search engines based on visual, rather than keyword, information. Generally speaking, shape matching methods rely on the computation of a shape description, also called a signature, that effectively captures some essential features of the object. The ability to perform not only global matching, but also partial matching, is regarded as one of the most meaningful properties in order to evaluate the performance of a shape matching method (cf., e.g., [36]). Basically, the interest in robustness against partial occlusions is motivated by the problem of recognizing an object partially hidden by some other foreground object in the same image. However, there are also other situations in which partial matching is useful, such as when dealing with the problem of identifying similarities between different configurations of articulated objects, or when dealing with unreliable object segmentation from images. For these reasons, the ability to recognize shapes, even when they are partially occluded by another pattern, has been investigated in the Computer Vision literature by various authors, with reference to a variety of shape recognition methods (see, e.g., [8, 23, 25, 30, 34, 35]).

Size functions are a method for shape description that is suitable for any multidimensional data set that can be modelled as a topological space X, and whose shape properties can be described by a continuous function φ defined on it (e.g., a domain of $\mathbb{R}^2$ and the height function may model terrain elevations). Size functions were introduced by P. Frosini at the beginning of the 1990s (cf., e.g., [19]), and later surveyed in [2, 26]. They belong to a class of methods that are grounded in Morse theory, and are defined in terms of the number of connected components of lower level sets associated with the given space X and function φ defined on it. From the theoretical point of view, the main properties of size functions that have been studied since their introduction are the computational issues [11, 20], the robustness of size functions with respect to continuous deformations of the shape [12, 13], the conciseness of the descriptor [21, 29], the invariance of the descriptor to transformation groups [15, 37], the connections of size functions to the natural pseudo-distance in order to compare shapes [16], their algebraic topological counterparts [6, 22], and their generalization to a

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setting where many functions are used at the same time to describe the same space [3]. As far as application is concerned, the most recent papers describe the retrieval of 3D objects [4] and trademark retrieval [7].

Size functions, like most methods of their class, work on a shape as a whole. In general, it is argued that global object methods are not robust against occlusions, whereas methods based on computing local features may be more suited to this task. Our aim is to show that size functions are able to preserve local information, so that they can manage uncertainty due to the presence of occluded shapes.

We model the presence of occlusions in a shape as follows. The object of interest A is occluded by a foreground object B, so that the visible object X is given by $A \cup B$. The shapes of X, A, and B are analyzed through the size functions $\ell_{(X,\varphi)}$, $\ell_{(A,\varphi_{|A})}$, and $\ell_{(B,\varphi_{|B})}$, respectively, where $\varphi : X \to \mathbb{R}$ is the continuous function chosen to extract the shape features.

The starting point of this research is the fact that the simple topological idea behind the definition of a size function, based on the number of connected components in the lower level sets of φ, can be translated into the algebraic language, using persistent homology [6, 9]. Indeed, the size function $\ell_{(X,\varphi)}$, evaluated at a point (u,v) of $\mathbb{R}^2$, with $u < v$, is equal to the rank of the image of the homomorphism induced by inclusion between the Čech homology groups $\check{H}_0(X_u)$ and $\check{H}_0(X_v)$, where $X_u = \{P \in X : \varphi(P) \leq u\}$ and $X_v = \{P \in X : \varphi(P) \leq v\}$.

Our main result establishes a necessary and sufficient algebraic condition so that the equality

$$\ell_{(X,\varphi)}(u,v) = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \ell_{(A\cap B,\varphi_{|A\cap B})}(u,v) \tag{1}$$

holds. This is proved using the Mayer-Vietoris sequence of Čech homology groups and involves the equality between the ranks of the kernels of two homomorphisms. We illustrate the geometrical counterpart of this algebraic condition in some simple situations, when we can control the non-localisable nature of the topological quantities involved in this condition.

From the above equality (1) we can deduce that the size function of X contains features of the size functions of A and B. In particular, when size functions are represented as collections of points in the plane through their *cornerpoints* [21, 29], relation (1) allows us to prove that the set of cornerpoints for $\ell_{(X,\varphi)}$ contains a subset of cornerpoints for $\ell_{(A,\varphi_{|A})}$. These are a kind of "fingerprint" of the presence of A in X. In other words, size functions are able to detect a partial matching between two shapes by showing a common subset of cornerpoints.

The paper is organized as follows. In Section 2 we recall background notions about size functions, and give some general results concerning the link between size functions, Čech homology and persistent homology. In Section 3 we introduce the Mayer-Vietoris sequence of persistent Čech homology groups and prove our main result concerning the relationship between the size functions of A, B and $A \cup B$. Section 4 is devoted to the consequent relationship between cornerpoints for $\ell_{(A,\varphi_{|A})}$, $\ell_{(B,\varphi_{|B})}$ and $\ell_{(X,\varphi)}$ in terms of their coordinates and multiplicities. Before concluding the paper with a brief discussion of our results, we show some experimental applications in Section 5, demonstrating the potential of our approach. The reader not familiar with Čech homology and the Mayer-Vietoris sequence can find a brief survey of the subject in Appendix A. The proofs of some non-central results are demanded to Appendices B and C.

2 Background on size functions

In this section we provide the reader with the necessary mathematical background concerning size functions that will be used in the next sections. Since size functions have been recently used under the name of 0th dimensional persistence and a different terminology has been developed to deal with similar concepts, we take care of underlying the existing links between Size Theory and Persistent Homology Theory.

In this paper a pair (X,φ), where X denotes a non-empty compact and locally connected Hausdorff topological space, and $\varphi : X \to \mathbb{R}$ denotes a continuous function, is called a *size pair*. Moreover, the function φ is called a *measuring function*. For every $u \in \mathbb{R}$, we denote by X_u the lower level set $\{P \in X : \varphi(P) \leq u\}$. Moreover, we shall denote by Δ^+ the open half plane $\{(u,v) \in \mathbb{R}^2 : u < v\}$.

Definition 2.1. *The* size function *associated with the size pair* (X,φ) *is the function* $\ell_{(X,\varphi)} : \Delta^+ \to \mathbb{N}$ *such that, for every* $(u,v) \in \Delta^+$, $\ell_{(X,\varphi)}(u,v)$ *is equal to the number of connected components in* X_v *that contain at least one point of* X_u.

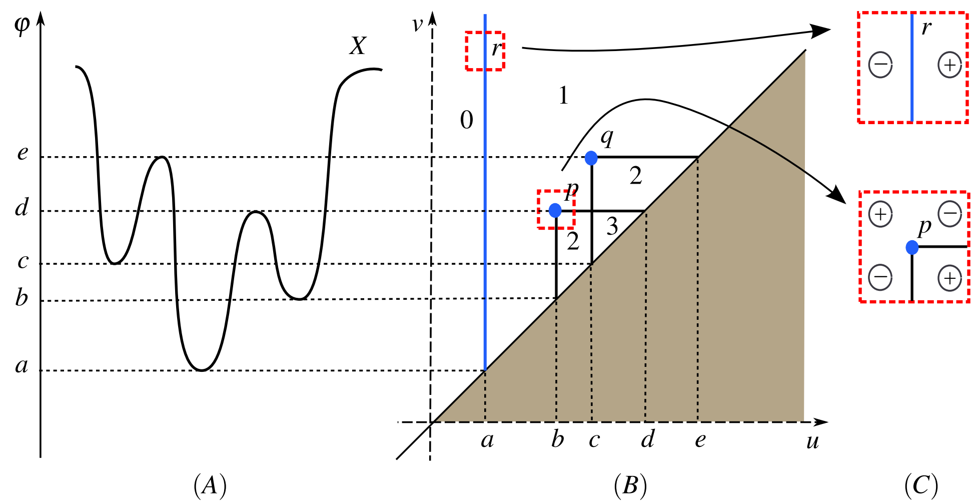

Figure 1: (A) A size pair (X,φ), where $X \subseteq \mathbb{R}^2$ is the curve represented by a continuous line, and $\varphi : X \to \mathbb{R}$ is such that $\varphi(P) = y$ for every $P = (x,y) \in X$. (B) The size function associated with (X,φ). (C) Computation of multiplicities seen through lens.

The finiteness of this number is a consequence of the compactness and local connectedness of X, and the continuity of φ.

An example of size function is displayed in Figure 1. In this example we consider the size pair (X,φ), where X is the curve of $\mathbb{R}^2$ represented by a solid line in Figure 1 (A), and $\varphi : X \to \mathbb{R}$ is the function that associates with each point $P \in X$ its ordinate in the plane. The size function associated with (X,φ) is shown in Figure 1 (B). Here, Δ^+, i.e. the domain of the size function, is divided by solid lines, representing the discontinuity points of the size function. These discontinuity points divide Δ^+ into regions where the size function is constant. The value displayed in each region is the value taken by the size function in that region. For instance, when $c \leq v < d$, X_v has three connected components. Only one of them contains at least one point of X_u, when $a \leq u < b$; two of them contain at least one point of X_u, when $b \leq u < c$; all of them contain at least one point of X_u, when $c \leq u < v < d$. Therefore, when $c \leq v < d$, $\ell_{(X,\varphi)}(u,v) = 1$ for $a \leq u < b$; $\ell_{(X,\varphi)}(u,v) = 2$ for $b \leq u < c$; $\ell_{(X,\varphi)}(u,v) = 3$ for $c \leq u < v$.

Alternatively to Definition 2.1, algebraic topology provides a homological interpretation of size functions that results to be a very powerful instrument for the analysis and the development of this shape descriptor from a theoretical point of view. It is well known that Čech homology furnishes an algebraic tool for counting connected components. Indeed, from [38, Thm. V 11.3a], it holds that the number of components of a space X is exactly the rank of the 0th Čech homology group, under the assumption that X is a compact Hausdorff space. Hence, the algebraic counterpart of a size function turns out to be a parameterized version of a Betti number, developed in the Čech setting. More precisely, given a size pair (X,φ), and $(u,v) \in \Delta^+$, denote by $\iota^{u,v}$ the inclusion of X_u into X_v. This mapping induces a homomorphism of Čech homology groups $\iota_p^{u,v} : \check{H}_p(X_u) \to \check{H}_p(X_v)$ for each integer $p \geq 0$, leading to the definition of persistent Čech homology groups.

Definition 2.2. *Given a size pair (X,φ) and a point $(u,v) \in \Delta^+$, the pth* persistent Čech homology group *$\check{H}_p^{u,v}$ is the image of the homomorphism $\iota_p^{u,v}$ between the pth Čech homology groups induced by the inclusion mapping of X_u into X_v: $\check{H}_p^{u,v}(X) = \operatorname{im} \iota_p^{u,v}$.*

Therefore, the value of a size function at a point $(u,v) \in \Delta^+$ is equal to the rank of the 0th persistent Čech homology group $\check{H}_0^{u,v}$ (also sometimes called a *persistent Betti number* [17] or a *rank invariant* [5]).

Throughout the paper the homology coefficients will be taken in a vector space over a field. In this way, from [18], we know that $\check{H}_p^{u,v}$ is a vector space over the same field. Hence, persistent homology groups are completely described by their rank.

Remark 2.3. *Persistent homology within the Čech homology setting has been previously considered in [32, 33], and the Čech method for studying spaces in persistent homology is widely illustrated in [5]. We emphasize that, from a theoretical point of view, Čech homology allows for greater generality than singular and simplicial homology, in the same way as connected components are more general than arcwise-connected components. On the other hand, in our setting, Čech homology satisfies all the ordinary homological axioms (see Appendix A), so that, it can be used in the same way as the singular and simplicial theories. Finally, the Čech approach to homology theory is currently being investigated for computational purposes [31].*

It will be useful in the sequel to consider the following link between size functions and relative Čech homology groups (see also [1]).

Corollary 2.4. *Let $(u,v) \in \Delta^+$, and let (X,φ) be a size pair such that the rank of $\check{H}_0(X_v)$ is finite. Then the value $\ell_{(X,\varphi)}(u,v)$ equals the rank of $\check{H}_0(X_v)$ minus the rank of $\check{H}_0(X_v,X_u)$.*

Proof. Let us consider the final terms of the long sequence of the pair: $\ldots \to \check{H}_0(X_u) \xrightarrow{\iota_0^{u,v}} \check{H}_0(X_v) \to \check{H}_0(X_v,X_u) \to 0$. From the exactness of this sequence, we deduce that $\ell_{(X,\varphi)}(u,v) = \operatorname{rank}\operatorname{im}\iota_0^{u,v} = \operatorname{rank}\check{H}_0(X_v) - \operatorname{rank}\check{H}_0(X_v,X_u)$. □

An equivalent representation of size functions is given by countable collections of cornerpoints (proper and at infinity), with multiplicity [21, 29]. The underlying idea is that size functions can be seen as sums of characteristic functions of (bounded or unbounded) triangular regions (see, for instance, Figure 1(B)). Formally proper cornerpoints are defined as follows.

Definition 2.5. *For every point $p=(u,v)\in\Delta^+$, the number $\mu_X(p)$ is defined as*

$$\lim_{\varepsilon\to 0^+}\left(\ell_{(X,\varphi)}(u+\varepsilon,v-\varepsilon)-\ell_{(X,\varphi)}(u-\varepsilon,v-\varepsilon)-\ell_{(X,\varphi)}(u+\varepsilon,v+\varepsilon)+\ell_{(X,\varphi)}(u-\varepsilon,v+\varepsilon)\right). \tag{2}$$

The finite number $\mu_X(p)$ is called multiplicity *of p for $\ell_{(X,\varphi)}$. Moreover, a* proper cornerpoint *for $\ell_{(X,\varphi)}$ is any point $p\in\Delta^+$ such that the number $\mu_X(p)$ is strictly positive.*

Comparing formula (2) with the definition of multiplicity given in [9], one can recognize that, in the terminology of persistence, cornerpoints are known as points of persistence diagrams. We underline that, when φ is a tame function, as in [9], cornerpoints are always isolated points. However, in the case considered here, we have to take a limit in the definition of multiplicity because cornerpoints can accumulate onto the diagonal of $\mathbb{R}^2$. An example of this phenomenon is illustrated in Figure 6.

A proper cornerpoint (u,v) encodes the level u at which a new connected component is born and the level v at which it gets merged to another connected component. The values where the lower level sets undergo a topological change are called homological critical values (see Definition B.1). This intuition about the link between cornerpoints and homological critical values needs some attention when φ is just continuous, and is treated in detail in Appendix B.

Let us now recall the definition of cornerpoints at infinity.

Definition 2.6. *For every vertical line r, with equation $u=k$ in the plane u,v, let us identify r with the pair (k,∞), and define the number $\mu_X(r)$ as $\lim_{\varepsilon\to 0^+}\left(\ell_{(X,\varphi)}(k+\varepsilon,1/\varepsilon)-\ell_{(X,\varphi)}(k-\varepsilon,1/\varepsilon)\right)$. When this finite number, called* multiplicity of *r for $\ell_{(X,\varphi)}$, is strictly positive, (k,∞) is said to be a* cornerpoint at infinity *for the size function.*

Intuitively, a cornerpoint at infinity just encodes the level u at which a new connected component of X is born.

Figure 1 (C) zooms in on some cornerpoints to explain how their multiplicity is computed. For instance, the alternating sum of the size function values at four points around p is $2-1-1+1$, giving $\mu_X(p)=1$. The alternating sum of the size function values at two points next to r is $1-0$, giving $\mu_X(r)=1$.

The importance of cornerpoints, counted with multiplicities, is revealed by the fact that they uniquely determine size functions [12], and by the possibility to translate the comparison between size functions into distances between sets of points, in a way that is robust against deformations [29].

3 The Mayer-Vietoris sequence of persistent Čech homology groups

In this section, we look for a relation expressing the size function associated with the size pair (X,φ) in terms of size functions associated with the size pairs $(A,\varphi_{|A})$ and $(B,\varphi_{|B})$, where A and B are closed locally connected subsets of X, such that $X = int_X(A) \cup int_X(B)$, and $A \cap B$ is locally connected. The notations $int_X(A)$ and $int_X(B)$ stand for the interior of the sets A and B in the topology of X, respectively. These assumptions on A, B and $A\cap B$, together with the fact that the functions $\varphi_{|A\cap B}$, $\varphi_{|A}$, and $\varphi_{|B}$ are continuous, as restrictions of the continuous function $\varphi : X \to \mathbb{R}$ to spaces endowed with the relative topology induced by the topology of X, ensure that $(A,\varphi_{|A})$, $(B,\varphi_{|B})$, and $(A\cap B,\varphi_{|A\cap B})$ are themselves size pairs. This justifies the choice of taking φ just continuous. These hypotheses on X, A, B and $A\cap B$ will be maintained throughout the paper.

We find a homological condition guaranteeing a Mayer-Vietoris formula between size functions evaluated at a point $(u,v)\in\Delta^+$, that is, $\ell_{(X,\varphi)}(u,v) = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \ell_{(A\cap B,\varphi_{|A\cap B})}(u,v)$ (see Corollary 3.6). We shall apply this relation in the next section in order to show that it is possible to match a subset of the cornerpoints for $\ell_{(X,\varphi)}$ to cornerpoints for either $\ell_{(A,\varphi_{|A})}$ or $\ell_{(B,\varphi_{|B})}$.

Our main tool is the Mayer-Vietoris sequence of the triad (X,A,B):

$$\cdots \to \check{H}_{p+1}(X) \xrightarrow{\Delta_p} \check{H}_p(A\cap B) \xrightarrow{\alpha_p} \check{H}_p(A)\oplus\check{H}_p(B) \xrightarrow{\beta_p} \check{H}_p(X) \xrightarrow{\Delta_{p-1}} \cdots \to \check{H}_0(X) \to 0,$$

where Δ_p is the homomorphism $\check{H}_{p+1}(X) \ni [z] \mapsto [\partial(z_{|A})] \in \check{H}_p(A\cap B)$, α_p is the homomorphism $\check{H}_p(A\cap B) \ni [z] \mapsto ([z],[-z]) \in \check{H}_p(A)\oplus\check{H}_p(B)$, and β_p is the homomorphism $\check{H}_p(A)\oplus\check{H}_p(B) \ni ([z],[z']) \mapsto [z+z'] \in \check{H}_p(X)$. Intuitively, the homomorphisms Δ_p, α_p, and β_p are described in Figure 2 for the case $p=0$.

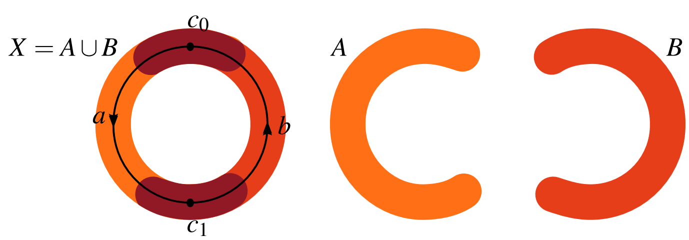

Figure 2: For this triad (X,A,B), $a+b$ is a 1-cycle in X, c_0 and c_1 are 0-cycles in $A\cap B$ (and hence also in A and B). Moreover, c_0 and c_1 are cobordant in A and B but not in $A\cap B$. We have that $\Delta_0([a+b]) = [c_1 - c_0]$, $\alpha_0([c_i]) = ([c_i],-[c_i])$, $\beta_0([c_i],[c_i]) = 2[c_i]$.

Under our assumptions, the Mayer-Vietoris sequence above is exact (see Appendix A). It is well known that the exactness of such a sequence provides a relation among the ranks of $\check{H}_p(X)$, $\check{H}_p(A)$, $\check{H}_p(B)$, $\check{H}_p(A\cap B)$, and the kernel of the homomorphisms α_p (see, e.g., [14, 18]):

$$\mathrm{rank}\check{H}_p(X) = \mathrm{rank}\check{H}_p(A) + \mathrm{rank}\check{H}_p(B) - \mathrm{rank}\check{H}_p(A\cap B) + \mathrm{rank}\ker\alpha_p + \mathrm{rank}\ker\alpha_{p-1}. \tag{3}$$

This relation can be easily checked in the example of Figure 2 for the case $p=1$.

The novelty of our approach is the study of Mayer-Vietoris sequences for different triads of lower level sets (X_u,A_u,B_u), interlacing them with long exact sequences of the pair (diagram (4)). This involves considering also a relative Mayer-Vietoris sequence (see Appendix A). Hence, exploiting the surjectivity of the map β_p, for $p=0$, we are able to generalize formula (3) to persistent homology, at least for 0th homology.

We begin by underlining some simple properties of the lower level sets of X, A, B, and $A\cap B$. Then we show that there exists a Mayer-Vietoris sequence for persistent Čech homology groups that is of order 2 (Proposition 3.4), and that, under proper assumptions, it induces a short exact sequence involving the 0th persistent Čech homology groups of X, A, B, and $A\cap B$ (Proposition 3.9).

Lemma 3.1. *Let $u\in\mathbb{R}$. Let us endow X_u with the relative topology induced by the topology of X. Then A_u and B_u are closed sets in X_u. Moreover, $X_u = int_{X_u}(A_u) \cup int_{X_u}(B_u)$ and $A_u \cap B_u = (A\cap B)_u$.*

Proof. Let us recall that the relative topology induced on X_u by the topology of X is that in which the open (closed, respectively) sets are the intersections with X_u of open (closed, respectively) sets of X [18]. Therefore, A_u is closed in X_u since $A_u = A \cap X_u$, with A closed in X. Analogously for B_u. About the second statement, the proof that $X_u \supseteq int_{X_u}(A_u) \cup int_{X_u}(B_u)$ is trivial. Let us prove that $X_u \subseteq int_{X_u}(A_u) \cup int_{X_u}(B_u)$. If $x \in X_u \subseteq X$ then $x \in int_X(A)$ or $x \in int_X(B)$. Let us suppose that $x \in int_X(A)$. Then there exists a neighborhood of x, say $U(x)$, contained in A and open in the topology of X. Clearly, $U(x) \cap X_u$ is an open neighborhood of x in the relative topology induced on X_u by the topology of X, and is contained in A_u. Hence $x \in int_{X_u}(A_u)$. The proof is analogous if $x \in int_{X_u}(B)$. Showing that $A_u \cap B_u = (A \cap B)_u$ is trivial. $\square$

Lemma 3.1 ensures that, for $(u,v) \in \Delta^+$, we can consider the following diagram:

$$
\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \check{H}_{p+1}(X_u) & \stackrel{\Delta_p^u}{\to} & \check{H}_p((A\cap B)_u) & \stackrel{\alpha_p^u}{\to} & \check{H}_p(A_u) \oplus \check{H}_p(B_u) & \stackrel{\beta_p^u}{\to} & \check{H}_p(X_u) & \to \cdots \\
 & \downarrow h_{p+1} & & \downarrow f_p & & \downarrow g_p & & \downarrow h_p & \\
\cdots \to & \check{H}_{p+1}(X_v) & \stackrel{\Delta_p^v}{\to} & \check{H}_p((A\cap B)_v) & \stackrel{\alpha_p^v}{\to} & \check{H}_p(A_v) \oplus \check{H}_p(B_v) & \stackrel{\beta_p^v}{\to} & \check{H}_p(X_v) & \to \cdots \\
 & \downarrow h'_{p+1} & & \downarrow f'_p & & \downarrow g'_p & & \downarrow h'_p & \\
\cdots \to & \check{H}_{p+1}(X_v, X_u) & \stackrel{\Delta_p^{v,u}}{\to} & \check{H}_p((A\cap B)_v, (A\cap B)_u) & \stackrel{\alpha_p^{v,u}}{\to} & \check{H}_p(A_v, A_u) \oplus \check{H}_p(B_v, B_u) & \stackrel{\beta_p^{v,u}}{\to} & \check{H}_p(X_v, X_u) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
\tag{4}
$$

where the top line belongs to the Mayer-Vietoris sequence of the triad (X_u, A_u, B_u), the second line belongs to the Mayer-Vietoris sequence of the triad (X_v, A_v, B_v), and the bottom line belongs to the relative Mayer-Vietoris sequence of the triad $((X_v, X_u), (A_v, A_u), (B_v, B_u))$. For every $p \geq 0$, the vertical maps f_p, g_p, and h_p are induced by inclusions of $(A \cap B)_u$ into $(A \cap B)_v$, (A_u, B_u) into (A_v, B_v), and X_u into X_v, respectively. Moreover, f'_p, g'_p and h'_p are induced by inclusions of $((A \cap B)_v, \emptyset)$ into $((A \cap B)_v, (A \cap B)_u)$, $((A_v, \emptyset), (B_v, \emptyset))$ into $((A_v, A_u), (B_v, B_u))$, and $(X_v, \emptyset)$ into (X_v, X_u), respectively.

Lemma 3.2. *Each vertical and horizontal line in diagram* (4) *is exact. Moreover, each square in the same diagram is commutative.*

Proof. We recall that we are assuming that X is compact and φ continuous, therefore X_u and X_v are compact, as are A_u, A_v, B_u and B_v by Lemma 3.1. Therefore, since we are also assuming that the coefficient group is a vector space over a field, it holds that the homology sequences of the pairs (X_v, X_u), $((A \cap B)_v, (A \cap B)_u)$, (A_v, A_u), (B_v, B_u) (vertical lines) are exact (cf. Theorem A.1 in Appendix A).

Analogously, the Mayer-Vietoris sequences of (X_u, A_u, B_u) and (X_v, A_v, B_v), and the relative Mayer-Vietoris sequence of $((X_v, X_u), (A_v, A_u), (B_v, B_u))$ (horizontal lines) are exact (cf. Theorems A.2 and A.4 in Appendix A).

About the commutativity of the top squares, it is sufficient to apply Theorem A.3 in Appendix A. The same conclusion can be drawn for the commutativity of the bottom squares, with X_v replaced by $(X_v, \emptyset)$, A_v by $(A_v, \emptyset)$ and B_v by $(B_v, \emptyset)$, respectively, applying Theorem A.5. $\square$

The image of the maps f_p, g_p, and h_p of diagram (4) are related to the pth persistent Čech homology groups. In particular, when $p = 0$, they are related to size functions, as the following lemma formally states.

Lemma 3.3. *For $(u,v) \in \Delta^+$, let f_p, g_p, h_p be the maps induced by inclusions of $(A\cap B)_u$ into $(A\cap B)_v$, (A_u, B_u) into (A_v, B_v), and X_u into X_v, respectively. Then $\operatorname{im} f_p = \check{H}_p^{u,v}(A\cap B)$, $\operatorname{im} g_p = \check{H}_p^{u,v}(A) \oplus \check{H}_p^{u,v}(B)$, and $\operatorname{im} h_p = \check{H}_p^{u,v}(X)$. In particular,* $\operatorname{rank}\operatorname{im} f_0 = \ell_{(A\cap B, \varphi_{|A\cap B})}(u,v)$, $\operatorname{rank}\operatorname{im} g_0 = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v)$ *and* $\operatorname{rank}\operatorname{im} h_0 = \ell_{(X,\varphi)}(u,v)$.

Proof. Trivial from Definition 2.2. □

The following proposition proves that the commutativity of squares in diagram (4) induces a Mayer-Vietoris sequence of order 2 involving the pth persistent Čech homology groups of X, A, B, and $A\cap B$, for every integer $p \geq 0$.

Proposition 3.4. *Let us consider the sequence of homomorphisms of persistent Čech homology groups*

$$\cdots \to \check{H}_{p+1}^{u,v}(X) \xrightarrow{\Delta} \check{H}_p^{u,v}(A\cap B) \xrightarrow{\alpha} \check{H}_p^{u,v}(A) \oplus \check{H}_p^{u,v}(B) \xrightarrow{\beta} \check{H}_p^{u,v}(X) \to \cdots \to \check{H}_0^{u,v}(X) \to 0$$

where $\Delta = \Delta^v_{p|\operatorname{im} h_{p+1}}$, $\alpha = \alpha^v_{p|\operatorname{im} f_p}$, and $\beta = \beta^v_{p|\operatorname{im} g_p}$. For every integer $p \geq 0$, the following statements hold:

(i) $\operatorname{im}\Delta \subseteq \ker\alpha$*;*

(ii) $\operatorname{im}\alpha \subseteq \ker\beta$*;*

(iii) $\operatorname{im}\beta \subseteq \ker\Delta$*,*

that is, the sequence is of order 2*.*

Proof. First of all, we observe that, by Lemma 3.2, $\operatorname{im}\Delta \subseteq \operatorname{im} f_p$, $\operatorname{im}\alpha \subseteq \operatorname{im} g_p$ and $\operatorname{im}\beta \subseteq \operatorname{im} h_p$. Now we prove only claim *(i)*, considering that *(ii)* and *(iii)* can be deduced analogously. Let $c \in \operatorname{im}\Delta$. Then $c \in \operatorname{im} f_p$ and $c \in \operatorname{im}\Delta^v_p = \ker\alpha^v_p$. Therefore $c \in \ker\alpha$. □

3.1 The size function of the union of two spaces

In the rest of the section we focus on the ending part of diagram (4):

$$\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \check{H}_1(X_u) & \xrightarrow{\Delta_0^u} & \check{H}_0((A\cap B)_u) & \xrightarrow{\alpha_0^u} & \check{H}_0(A_u)\oplus\check{H}_0(B_u) & \xrightarrow{\beta_0^u} & \check{H}_0(X_u) & \to 0 \\
 & \downarrow h_1 & & \downarrow f_0 & & \downarrow g_0 & & \downarrow h_0 & \\
\cdots \to & \check{H}_1(X_v) & \xrightarrow{\Delta_0^v} & \check{H}_0((A\cap B)_v) & \xrightarrow{\alpha_0^v} & \check{H}_0(A_v)\oplus\check{H}_0(B_v) & \xrightarrow{\beta_0^v} & \check{H}_0(X_v) & \to 0 \\
 & \downarrow h_1' & & \downarrow f_0' & & \downarrow g_0' & & \downarrow h_0' & \\
\cdots \to & \check{H}_1(X_v,X_u) & \xrightarrow{\Delta_0^{v,u}} & \check{H}_0((A\cap B)_v,(A\cap B)_u) & \xrightarrow{\alpha_0^{v,u}} & \check{H}_0(A_v,A_u)\oplus\check{H}_0(B_v,B_u) & \xrightarrow{\beta_0^{v,u}} & \check{H}_0(X_v,X_u) & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & 0 & & 0 & & 0 &
\end{array} \tag{5}$$

and, in the rest of the paper, the notations we use always refer to diagram (5). Moreover, hereafter it will be assumed that $\check{H}_0(X_v)$, $\check{H}_0(A_v)$, $\check{H}_0(B_v)$, and $\check{H}_0(A\cap B_v)$ are finitely generated spaces in order to apply Corollary 2.4.

Analogously to equality (3), we can deduce a relation between $\ell_{(X,\varphi)}$ and $\ell_{(A,\varphi_{|A})}$, $\ell_{(B,\varphi_{|B})}$, exploiting the surjectivity of β_0^u, β_0^v, $\beta_0^{v,u}$ as well as that of f_0', g_0', h_0' (see Theorem 3.5).

Theorem 3.5. *For every $(u,v) \in \Delta^+$, it holds that*

$$\ell_{(X,\varphi)}(u,v) = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \ell_{(A\cap B,\varphi_{|A\cap B})}(u,v) + \operatorname{rank}\ker\alpha_0^v - \operatorname{rank}\ker\alpha_0^{v,u}.$$

Proof. By the exactness of the second horizontal line of diagram (5) and by the surjectivity of the homomorphism β_0^v, repeatedly using the dimensional relation between the domain of a homomorphism, its kernel and its image, we obtain

$$\begin{aligned}\operatorname{rank}\check{H}_0(X_v) = \operatorname{rank}\operatorname{im}\beta_0^v &= \operatorname{rank}\check{H}_0(A_v)\oplus\check{H}_0(B_v) - \operatorname{rank}\ker\beta_0^v \\ &= \operatorname{rank}\check{H}_0(A_v)\oplus\check{H}_0(B_v) - \operatorname{rank}\operatorname{im}\alpha_0^v \\ &= \operatorname{rank}\check{H}_0(A_v) + \operatorname{rank}\check{H}_0(B_v) - \operatorname{rank}\check{H}_0((A\cap B)_v) + \operatorname{rank}\ker\alpha_0^v. \qquad (6)\end{aligned}$$

Similarly, by the exactness of the third horizontal line of the same diagram and by the surjectivity of $\beta_0^{v,u}$, it holds that

$$\operatorname{rank}\check{H}_0(X_v,X_u) = \operatorname{rank}\check{H}_0(A_v,A_u) + \operatorname{rank}\check{H}_0(B_v,B_u) - \operatorname{rank}\check{H}_0((A\cap B)_v,(A\cap B)_u) + \operatorname{rank}\ker\alpha_0^{v,u}. \qquad (7)$$

Now, subtracting equality (7) from equality (6), we have

$$\begin{aligned}\operatorname{rank}\check{H}_0(X_v) - \operatorname{rank}\check{H}_0(X_v,X_u) &= \operatorname{rank}\check{H}_0(A_v) - \operatorname{rank}\check{H}_0(A_v,A_u) + \operatorname{rank}\check{H}_0(B_v) - \operatorname{rank}\check{H}_0(B_v,B_u) \\ &\quad -\operatorname{rank}\check{H}_0((A\cap B)_v) + \operatorname{rank}\check{H}_0((A\cap B)_v,(A\cap B)_u) + \operatorname{rank}\ker\alpha_0^v - \operatorname{rank}\ker\alpha_0^{v,u},\end{aligned}$$

which is equivalent, in terms of size functions, to the relation claimed, because of Corollary 2.4. □

Corollary 3.6. *For every $(u,v) \in \Delta^+$, it holds that*

$$\ell_{(X,\varphi)}(u,v) = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \ell_{(A\cap B,\varphi_{|A\cap B})}(u,v)$$

if and only if $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\ker\alpha_0^{v,u}$.

Proof. Immediate from Theorem 3.5. □

Remark 3.7. *In the proof of Theorem 3.5, the key ingredient is that, for 0th homology, the maps β_0^v and $\beta_0^{v,u}$, as well as f_0', g_0', and h_0', are surjective. The surjectivity of f_0', g_0', and h_0' allows us to apply Corollary 2.4. Since surjectivity breaks in higher dimensions, as can be seen, for instance, in Figure 2, in order to generalize Theorem 3.5 to higher homology degrees one would need to add cokernels of all these maps, yielding a much more complicated relation.*

Remark 3.8. *Applying the persistent kernels construction [10], we obtain a sequence of homomorphisms* $\ldots \to \ker\alpha_0^u \to \ker\alpha_0^v \xrightarrow{\Phi} \ker\alpha_0^{v,u} \to 0$ *allowing us to rewrite the quantity* $\operatorname{rank}\ker\alpha_0^v - \operatorname{rank}\ker\alpha_0^{v,u}$ *as* $\operatorname{rank}\ker\Phi - \operatorname{rank}\operatorname{cok}\Phi$.

We now show that combining the assumption that α_0^v and $\alpha_0^{v,u}$ are both injective with Proposition 3.4, there is a short exact sequence involving the 0th persistent Čech homology groups of X, A, B, and $A\cap B$.

Proposition 3.9. *For every $(u,v) \in \Delta^+$, such that the maps α_0^v and $\alpha_0^{v,u}$ are injective, the sequence of maps*

$$0 \;\to\; \check{H}_0^{u,v}(A\cap B) \;\xrightarrow{\alpha}\; \check{H}_0^{u,v}(A)\oplus\check{H}_0^{u,v}(B) \;\xrightarrow{\beta}\; \check{H}_0^{u,v}(X) \to 0, \qquad (8)$$

where $\alpha = \alpha^v_{0|\operatorname{im} f_0}$ and $\beta = \beta^v_{0|\operatorname{im} g_0}$, is exact.

Proof. By Proposition 3.4, $\operatorname{im}\alpha \subseteq \ker\beta$, so we only have to show that β is surjective, α is injective, and $\operatorname{rank}\operatorname{im}\alpha = \operatorname{rank}\ker\beta$. We recall that $\check{H}_0^{u,v}(A\cap B) = \operatorname{im} f_0$, $\check{H}_0^{u,v}(A)\oplus\check{H}_0^{u,v}(B) = \operatorname{im} g_0$, and $\check{H}_0^{u,v}(X) = \operatorname{im} h_0$ (Lemma 3.3).

We begin by showing that β is surjective. Let $c \in \operatorname{im} h_0$. There exists $d \in \check{H}_0(X_u)$ such that $h_0(d) = c$. Since β_0^u is surjective, there exists $d' \in \check{H}_0(A_u)\oplus\check{H}_0(B_u)$ such that $h_0 \circ \beta_0^u(d') = c$. By Lemma 3.2, $\beta_0^v \circ g_0(d') = c$. Thus, taking $c' = g_0(d')$, we immediately have $\beta(c') = c$.

As for the injectivity of α, the claim is immediate because $\ker\alpha \subseteq \ker\alpha_0^v$ and we are assuming α_0^v injective.

Now we have to show that $\operatorname{rank}\operatorname{im}\alpha = \operatorname{rank}\ker\beta$. In order to do so, we observe that for every $(u,v) \in \Delta^+$ it holds that

$$\begin{aligned}\ell_{(X,\varphi)}(u,v) = \operatorname{rank}\check{H}_0^{u,v}(X) = \operatorname{rank}\operatorname{im}\beta = \operatorname{rank}H_0^{u,v}(A) \oplus \check{H}_0^{u,v}(B) - \operatorname{rank}\ker\beta \\ = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \operatorname{rank}\ker\beta.\end{aligned} \tag{9}$$

On the other hand, by Corollary 3.6, when $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\ker\alpha_0^{v,u}$ it holds that

$$\ell_{(X,\varphi)}(u,v) = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \ell_{(A\cap B,\varphi_{|A\cap B})}(u,v).$$

Hence, if $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\ker\alpha_0^{v,u}$, then $\operatorname{rank}\ker\beta = \ell_{(A\cap B,\varphi_{|A\cap B})}(u,v)$. Moreover, since $\ell_{(A\cap B,\varphi_{|A\cap B})}(u,v) = \operatorname{rank}\check{H}_0^{u,v}(A\cap B) = \operatorname{rank}\ker\alpha + \operatorname{rank}\operatorname{im}\alpha$, when $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\ker\alpha_0^{v,u}$, we have $\operatorname{rank}\ker\beta = \operatorname{rank}\ker\alpha + \operatorname{rank}\operatorname{im}\alpha$. Therefore, when $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\ker\alpha_0^{v,u}$, α is injective if and only if $\operatorname{rank}\operatorname{im}\alpha = \operatorname{rank}\ker\beta$. □

The condition $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\ker\alpha_0^{v,u} = 0$ in the previous Proposition 3.9 cannot be weakened, in fact:

Remark 3.10. *The equality* $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\ker\alpha_0^{v,u}$ *does not imply the injectivity of* α.

Indeed, Figure 3 shows an example of a topological space $X = A \cup B$ on which, taking the height function as a measuring function and $u,v \in \mathbb{R}$ as displayed, it holds that $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\ker\alpha_0^{v,u} \neq 0$, but $\operatorname{rank}\ker\alpha > 0$, making the sequence (8) not exact.

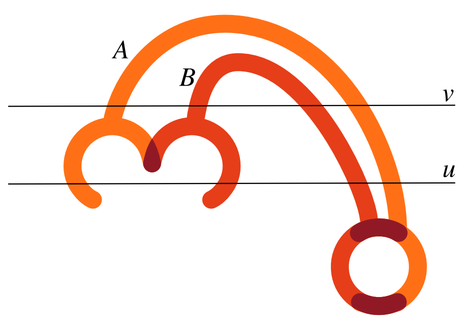

Figure 3: The sets A and B used in Remark 3.10.

To see that $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\ker\alpha_0^{v,u} \neq 0$, we note that the equalities (6) and (7) imply $\operatorname{rank}\ker\alpha_0^v = \operatorname{rank}\check{H}_0(X_v) - \operatorname{rank}\check{H}_0(A_v) - \operatorname{rank}\check{H}_0(B_v) + \operatorname{rank}\check{H}_0((A\cap B)_v) = 2-2-2+3$ and $\operatorname{rank}\ker\alpha_0^{v,u} = \operatorname{rank}\check{H}_0(X_v,X_u) - \operatorname{rank}\check{H}_0(A_v,A_u) - \operatorname{rank}\check{H}_0(B_v,B_u) + \operatorname{rank}\check{H}_0((A\cap B)_v,(A\cap B)_u) = 0-0-0+1$, respectively. To see that $\operatorname{rank}\ker\alpha = 1$, let us consider the homology sequence of the pair (X_v,X_u)

$$\cdots \to \check{H}_2(X_v,X_u) \to \check{H}_1(X_u) \xrightarrow{h_1} \check{H}_1(X_v) \xrightarrow{h_1'} \check{H}_1(X_v,X_u) \to \cdots$$

that is the first vertical line in diagram (5). In this instance, $\check{H}_2(X_v,X_u) = 0$, so it follows that h_1 is injective. Moreover, $\operatorname{rank}\check{H}_1(X_u) = \operatorname{rank}\check{H}_1(X_v) = 1$ implies the surjectivity of h_1. Recalling from Proposition 3.4 that $\Delta = \Delta^v_{0|\operatorname{im}h_1}$, we have that $\Delta = \Delta_0^v$. Then, since $\operatorname{im}\Delta \subseteq \ker\alpha \subseteq \ker\alpha^v = \operatorname{im}\Delta_0^v$ and $\operatorname{rank}\operatorname{im}\Delta = \operatorname{rank}\operatorname{im}\Delta_0^v = 1$, it follows that $\operatorname{rank}\ker\alpha = 1$.

As shown in the proof of Proposition 3.9, for every $(u,v) \in \Delta^+$, it holds that $\ell_{(X,\varphi)}(u,v) = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \operatorname{rank}\ker\beta$ (see equality (9)). So, as an immediate consequence, we observe that

Remark 3.11. $\ell_{(X,\varphi)}(u,v) \leq \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v)$ *holds for every* $(u,v) \in \Delta^+$.

3.2 Examples

In this section, we give two examples illustrating the previous results.

In both these examples, we consider a "bone" shaped object A, partially occluded by another object B, resulting in different shapes $X = A \cup B \subset \mathbb{R}^2$. The size functions $\ell_{(A,\varphi_{|A})}$, $\ell_{(B,\varphi_{|B})}$, $\ell_{(A\cap B,\varphi_{|A\cap B})}$, $\ell_{(X,\varphi)}$ are computed taking $\varphi : X \to \mathbb{R}$, $\varphi(P) = -\|P - H\|$, with H a fixed point in $\mathbb{R}^2$.

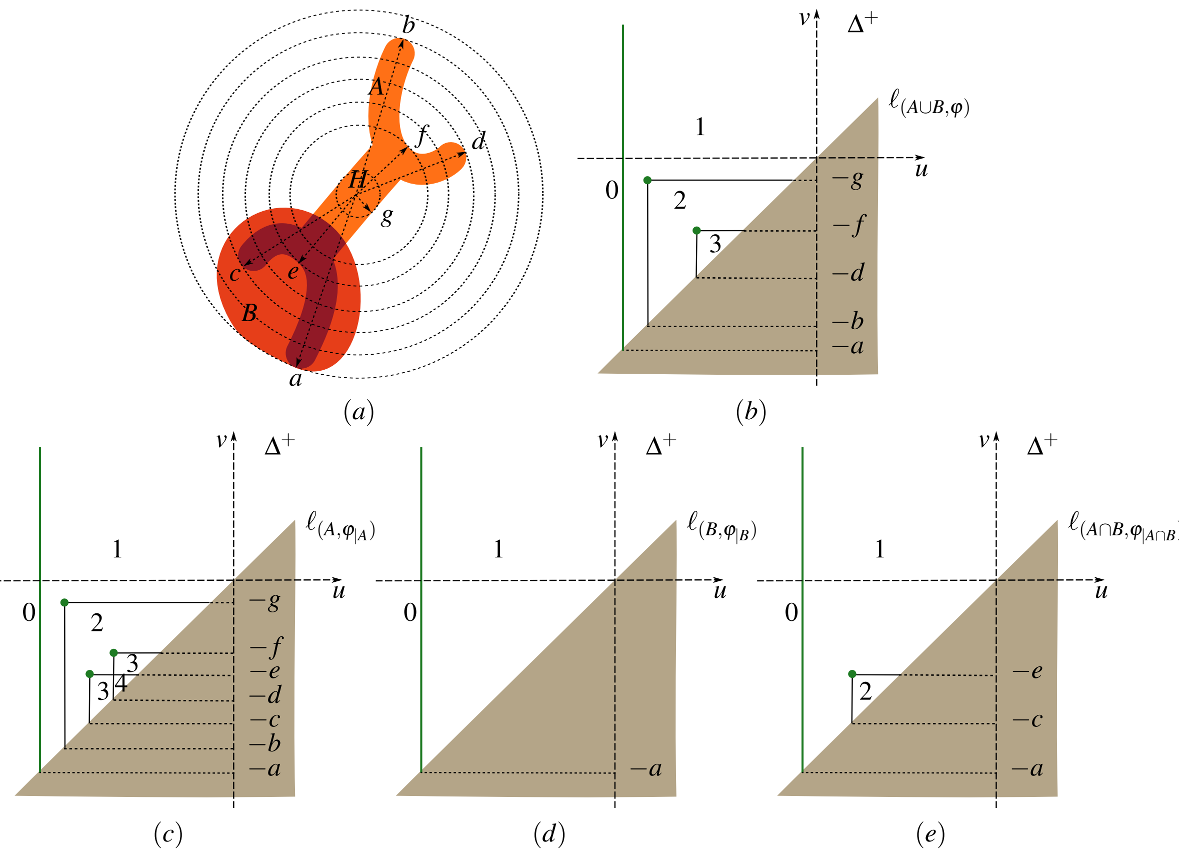

Figure 4: In (a) a "bone" shaped object A is occluded by another object B. In (b), (c), (d) and (e) we show the size functions of $(A\cup B, \varphi)$, $(A, \varphi_{|A})$, $(B, \varphi_{|B})$, and $(A \cap B, \varphi_{|A\cap B})$, respectively, computed taking $\varphi : X \to \mathbb{R}$, $\varphi(P) = -\|P - H\|$. In this example the relation $\ell_{(X,\varphi)} = \ell_{(A,\varphi_{|A})} + \ell_{(B,\varphi_{|B})} - \ell_{(A\cap B,\varphi_{|A\cap B})}$ of Corollary 3.6 holds everywhere in Δ^+.

In the first example, shown in Figure 4, the relation $\ell_{(X,\varphi)}(u,v) = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \ell_{(A\cap B,\varphi_{|A\cap B})}(u,v)$, given in Corollary 3.6, holds for every $(u,v) \in \Delta^+$.

In the second example, shown in Figure 5, the foreground object B occludes the object of interest A in such a way that the relation given in Corollary 3.6 results not valid everywhere in Δ^+. More precisely, $\operatorname{rank}\ker \alpha_0^v = 0$ for $v < -g$, while $\operatorname{rank}\ker \alpha_0^v = 1$ for $v \geq -g$. As for $\ker \alpha_0^{v,u}$, it has the same rank as $\ker \alpha_0^v$ for every $u < v$, except when $-d \leq u < -e$ and $-e \leq v < -g$, because, in that case, $\operatorname{rank}\ker \alpha_0^{v,u} = 1$, and, moreover, when $-e \leq u$ and $-g \leq v$, because, in that case, $\operatorname{rank}\ker \alpha_0^{v,u} = 0$. To simplify the visualization of the points of Δ^+ at which the equality $\ell_{(X,\varphi)}(u,v) = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \ell_{(A\cap B,\varphi_{|A\cap B})}(u,v)$ does not hold, we refer the reader to Figure 5 (c), where regions that do not verify such a relation are underlined by coloring them.

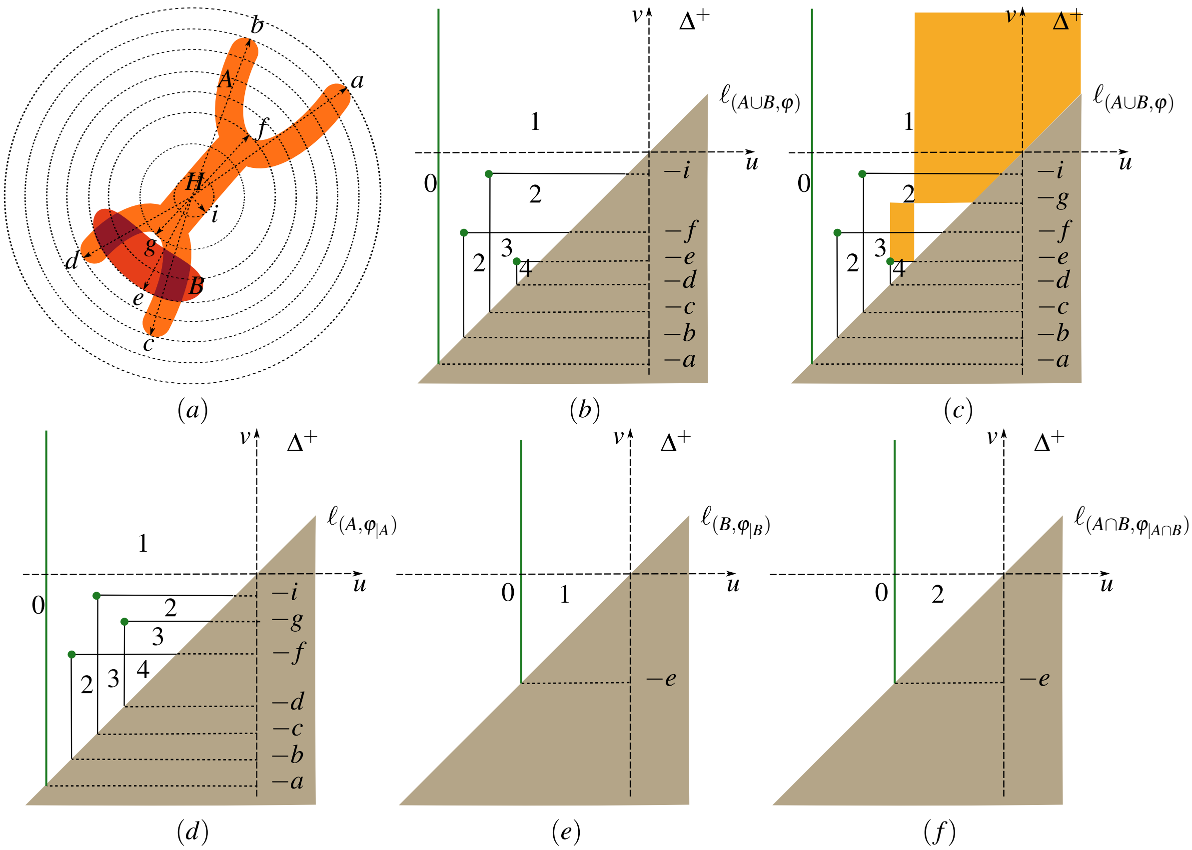

Figure 5: In (a) a "bone" shaped object A is occluded by another object B. In (b), (d), (e), (f) we display the size functions of $(A \cup B, \varphi)$, $(A, \varphi_{|A})$, $(B, \varphi_{|B})$, and $(A \cap B, \varphi_{|A\cap B})$, respectively, computed taking $\varphi : X \to \mathbb{R}$, $\varphi(P) = -\|P - H\|$. In this case the relation $\ell_{(X,\varphi)} = \ell_{(A,\varphi_{|A})} + \ell_{(B,\varphi_{|B})} - \ell_{(A\cap B,\varphi_{|A\cap B})}$ of Corollary 3.6 does not hold everywhere in Δ^+. In (c) we underline the regions of Δ^+ where the equality is not valid by coloring them.

3.3 Conditions for the exactness of $0 \to \check{H}_0^{u,v}(A\cap B) \to \check{H}_0^{u,v}(A) \oplus \check{H}_0^{u,v}(B) \to \check{H}_0^{u,v}(X) \to 0$

In this section we look for sufficient conditions in order that α_0^v and $\alpha_0^{v,u}$ are injective, so that the sequence

$$0 \quad \to \quad \check{H}_0^{u,v}(A\cap B) \quad \xrightarrow{\alpha} \quad \check{H}_0^{u,v}(A) \oplus \check{H}_0^{u,v}(B) \quad \xrightarrow{\beta} \quad \check{H}_0^{u,v}(X) \to 0$$

is exact (cf. Proposition 3.9), and the relation $\ell_{(X,\varphi)}(u,v) = \ell_{(A,\varphi_{|A})}(u,v) + \ell_{(B,\varphi_{|B})}(u,v) - \ell_{(A\cap B,\varphi_{|A\cap B})}(u,v)$ of Corollary 3.6 is satisfied.

The reason for identifying these conditions lies in the fact that they can be used as a guidance in choosing the most appropriate measuring function in order to study the shape of a partially occluded object.

The first condition we exhibit (Proposition 3.12) intuitively says that injectivity of $\alpha_0^{v,u}$ certainly holds for those values $u < v$ for which no new connected component of the lower level sets of $A\cap B$ is born between u and v. Injectivity of α_0^v certainly holds when $(A\cap B)_v$ is either empty or connected.

Proposition 3.12. *Let $(u,v) \in \Delta^+$. The following statements hold*

(i) If $\ell_{(A\cap B,\varphi_{|A\cap B})}(u,v') = \ell_{(A\cap B,\varphi_{|A\cap B})}(v,v')$ for every $(v,v') \in \Delta^+$, then $\alpha_0^{v,u} = 0$.

(ii) If $\ell_{(A\cap B,\varphi_{|A\cap B})}(v,v') \le 1$ for every $(v,v') \in \Delta^+$, then $\ker \alpha_0^v = 0$.

The proof of Proposition 3.12, being rather technical, will be demanded to Appendix C.

We observe that other sufficient conditions exist, implying that both α_0^v and $\alpha_0^{v,u}$ are injective. An example is given by the following result (Proposition 3.13). It states that it is sufficient that v ranges in the set of those values for which X_v has no holes and does not undergo topological changes.

Proposition 3.13. *If $\mathrm{rank}\check{H}_1(X_v) = 0$ and $\mathrm{rank}\check{H}_0(X_u) = \ell_{(X,\varphi)}(u,v)$, then $\ker \alpha_0^v = \ker \alpha_0^{v,u} = 0$.*

Proof. The condition $\mathrm{rank}\check{H}_1(X_v) = 0$ trivially implies $\ker \alpha_0^v = 0$. On the other hand, it implies the injectivity of the homomorphism h in the following exact sequence:

$$\cdots \quad \to \quad \check{H}_1(X_v) \quad \xrightarrow{h_1'} \quad \check{H}_1(X_v, X_u) \quad \xrightarrow{h} \quad \check{H}_0(X_u) \quad \xrightarrow{h_0} \quad \check{H}_0(X_v) \quad \xrightarrow{h_0'} \quad \check{H}_0(X_v, X_u) \to 0,$$

which is the leftmost vertical sequence in diagram (5). Therefore, by the assumption $\mathrm{rank}\check{H}_0(X_u) = \ell_{(X,\varphi)}(u,v)$, it follows that

$$\mathrm{rank}\check{H}_1(X_v, X_u) = \mathrm{rank}\,\mathrm{im}\,h = \mathrm{rank}\ker h_0 = \mathrm{rank}\check{H}_0(X_u) - \ell_{(X,\varphi)}(u,v) = 0,$$

and, consequently, the triviality of $\ker \alpha_0^{v,u}$ has been proved. □

We underline that these are only sufficient conditions, as the examples given in Section 3.2 easily show.

4 Partial matching of cornerpoints in size functions

As recalled in Section 2, in an earlier paper [21], it was shown that size functions can be concisely represented by collections of points, called cornerpoints, with multiplicities.

This representation by cornerpoints has the important property of being stable against shape continuous deformations. For this reason, in dealing with the shape comparison problem, via size functions, one actually compares the sets of cornerpoints using either the Hausdorff distance or the matching distance (see, e.g., [9, 12, 13, 29]). The Hausdorff distance and the matching distance differ in that the former does not take into account the multiplicities of cornerpoints, while the latter does.

The aim of this section is to show what happens to cornerpoints in the presence of occlusions. We prove that each cornerpoint for the size function of an occluded object X is a cornerpoint for the size function of the original object A, or the occluding pattern B, or their intersection $A\cap B$, provided that one condition holds (Corollary 4.2). Moreover, we prove that, under a finiteness condition, cornerpoints for $\ell_{(A,\varphi_{|A})}$ and $\ell_{(B,\varphi_{|B})}$ either survive in $\ell_{(X,\varphi)}$ or in $\ell_{(A\cap B,\varphi_{|A\cap B})}$ (Theorem 4.4). However, in general, it always holds that the coordinates of cornerpoints for $\ell_{(X,\varphi)}$ are also coordinates of cornerpoints for $\ell_{(A,\varphi_{|A})}$ or $\ell_{(B,\varphi_{|B})}$ or $\ell_{(A\cap B,\varphi_{|A\cap B})}$ (Theorems 4.5 and 4.6).

These facts suggest that in Size Theory the partial matching of an occluded shape with the original shape can be translated into the partial matching of cornerpoints for the corresponding size functions. This intuition will be developed in the experimental Section 5.

In the next proposition we obtain a relation involving the multiplicities of points in the size functions associated with X, A and B.

Proposition 4.1. *For every $p=(\overline{u},\overline{v})\in\Delta^+$, it holds that*

$$\mu_X(p)-\mu_A(p)-\mu_B(p)+\mu_{A\cap B}(p)=\lim_{\varepsilon\to 0^+}\left(\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}-\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}+\varepsilon}+\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}+\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}-\varepsilon}\right).$$

Proof. Applying Theorem 3.5 four times with $(u,v)=(\overline{u}+\varepsilon,\overline{v}-\varepsilon)$, $(u,v)=(\overline{u}-\varepsilon,\overline{v}-\varepsilon)$, $(u,v)=(\overline{u}+\varepsilon,\overline{v}+\varepsilon)$, $(u,v)=(\overline{u}-\varepsilon,\overline{v}+\varepsilon)$, ε being a positive real number so small that $\overline{u}+\varepsilon<\overline{v}-\varepsilon$, we get

$$\begin{aligned}
&\ell_{(X,\varphi)}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)-\ell_{(X,\varphi)}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)-\ell_{(X,\varphi)}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)+\ell_{(X,\varphi)}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)\\
={}&\ell_{(A,\varphi_{|A})}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)+\ell_{(B,\varphi_{|B})}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)-\ell_{(A\cap B,\varphi_{|A\cap B})}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)+\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}+\varepsilon}\\
&-\left(\ell_{(A,\varphi_{|A})}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)+\ell_{(B,\varphi_{|B})}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)-\ell_{(A\cap B,\varphi_{|A\cap B})}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)+\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}-\varepsilon}\right)\\
&-\left(\ell_{(A,\varphi_{|A})}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)+\ell_{(B,\varphi_{|B})}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)-\ell_{(A\cap B,\varphi_{|A\cap B})}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)+\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}+\varepsilon}\right)\\
&+\ell_{(A,\varphi_{|A})}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)+\ell_{(B,\varphi_{|B})}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)-\ell_{(A\cap B,\varphi_{|A\cap B})}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)+\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}-\varepsilon}\\
={}&\ell_{(A,\varphi)}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)-\ell_{(A,\varphi)}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)-\ell_{(A,\varphi)}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)+\ell_{(A,\varphi)}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)\\
&+\ell_{(B,\varphi)}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)-\ell_{(B,\varphi)}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)-\ell_{(B,\varphi)}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)+\ell_{(B,\varphi)}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)\\
&-\ell_{(A\cap B,\varphi)}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)+\ell_{(A\cap B,\varphi)}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)+\ell_{(A\cap B,\varphi)}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)-\ell_{(A\cap B,\varphi)}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)\\
&+\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}-\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}+\varepsilon}+\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}+\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}-\varepsilon}.
\end{aligned}$$

Hence, by definition of multiplicity of a point of Δ^+ (Definition 2.5), we have that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\left(\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}-\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}+\varepsilon}+\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}+\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}-\varepsilon}\right)\\
={}&\lim_{\varepsilon\to 0^+}\left(\ell_{(X,\varphi)}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)-\ell_{(X,\varphi)}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)-\ell_{(X,\varphi)}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)+\ell_{(X,\varphi)}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)\right)\\
&-\lim_{\varepsilon\to 0^+}\left(\ell_{(A,\varphi)}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)-\ell_{(A,\varphi)}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)-\ell_{(A,\varphi)}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)+\ell_{(A,\varphi)}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)\right)\\
&-\lim_{\varepsilon\to 0^+}\left(\ell_{(B,\varphi)}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)-\ell_{(B,\varphi)}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)-\ell_{(B,\varphi)}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)+\ell_{(B,\varphi)}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)\right)\\
&+\lim_{\varepsilon\to 0^+}\left(\ell_{(A\cap B,\varphi)}(\overline{u}+\varepsilon,\overline{v}-\varepsilon)-\ell_{(A\cap B,\varphi)}(\overline{u}-\varepsilon,\overline{v}-\varepsilon)-\ell_{(A\cap B,\varphi)}(\overline{u}+\varepsilon,\overline{v}+\varepsilon)+\ell_{(A\cap B,\varphi)}(\overline{u}-\varepsilon,\overline{v}+\varepsilon)\right)\\
={}&\mu_X(p)-\mu_A(p)-\mu_B(p)+\mu_{A\cap B}(p).
\end{aligned}$$

□

Using the previous Proposition 4.1, we find a condition ensuring that proper cornerpoints for the size function of X are also proper cornerpoints for the size function of A or B.

Corollary 4.2. *Let $p=(\overline{u},\overline{v})$ be a proper cornerpoint for $\ell_{(X,\varphi)}$ and*

$$\lim_{\varepsilon\to 0^+}\left(\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}-\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}+\varepsilon}+\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}+\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}-\varepsilon}\right)\le 0.$$

Then p is a proper cornerpoint for either $\ell_{(A,\varphi_{|A})}$ or $\ell_{(B,\varphi_{|B})}$ or both.

Proof. Let $\lim_{\varepsilon\to 0^+}\left(\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}-\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}+\varepsilon}+\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}+\varepsilon}-\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}-\varepsilon}\right)\leq 0$. From Proposition 4.1, we deduce that $\mu_X(p)\leq\mu_A(p)+\mu_B(p)-\mu_{A\cap B}(p)$. Since p is a cornerpoint for $\ell_{(X,\varphi)}$, it holds that $\mu_X(p)>0$. Since multiplicities are always non-negative, this easily implies that either $\mu_A(p)>0$ or $\mu_B(p)>0$ (or both), proving the statement. □

Remark 4.3. *If $p=(\bar{u},\bar{v})$ is a proper cornerpoint for $\ell_{(X,\varphi)}$, and Proposition 3.12 (i) applies with $u=\bar{u}\pm\varepsilon$ and $v=\bar{v}\pm\varepsilon$, for any sufficiently small $\varepsilon>0$, then p is a proper cornerpoint for either $\ell_{(A,\varphi_{|A})}$ or $\ell_{(B,\varphi_{|B})}$.*

For example, with reference to Figure 4, proper cornerpoints for $\ell_{(X,\varphi)}$ belong to a region of Δ^+ where the assumption of Proposition 3.12 *(i)* holds, and, according to Remark 4.3, they are also cornerpoints for $\ell_{(A,\varphi_{|A})}$.

The following Theorem 4.4 states that, under suitable conditions on φ, cornerpoints for $\ell_{(A,\varphi_{|A})}$ and $\ell_{(B,\varphi_{|B})}$ are cornerpoints for $\ell_{(X,\varphi)}$ or $\ell_{(A\cap B,\varphi_{|A\cap B})}$. This suggests that the shape features of the original object A either survive as cornerpoints for $\ell_{(X,\varphi)}$ or, if they are hidden by the occlusion B, can be found in the size function of $A\cap B$.

Theorem 4.4. *Let (X,φ) be a size pair with $\varphi_{|A\cap B}$ admitting at most a finite number of homological 0-critical values. Let $p=(\bar{u},\bar{v})\in\Delta^+$ be a proper cornerpoint for $\ell_{(A,\varphi_{|A})}$ or $\ell_{(B,\varphi_{|B})}$ such that neither of its coordinates is a homological 1-critical value for φ. Then p is a proper cornerpoint for either $\ell_{(X,\varphi)}$ or $\ell_{(A\cap B,\varphi_{|A\cap B})}$.*

Proof. If p is a proper cornerpoint for $\ell_{(A\cap B,\varphi_{|A\cap B})}$, then the claim is proved. If this is not the case, let us prove that p necessarily is a proper cornerpoint for $\ell_{(X,\varphi)}$. Assuming that $\varphi_{|A\cap B}$ has a finite number of homological 0-critical values, and that p is not a proper cornerpoint for $\ell_{(A\cap B,\varphi_{|A\cap B})}$, by Proposition B.3, it follows that the coordinates of p are not homological 0-critical values of $\varphi_{|A\cap B}$. Now, let us suppose that the abscissa $\bar{u}$ of p is not a homological 1-critical value of φ (the proof for $\bar{v}$ is analogous), and consider the following commutative diagram

$$\begin{array}{ccc}
\check{H}_1(X_{\bar{v}-\varepsilon},X_{\bar{u}-\varepsilon}) & \xrightarrow{\Delta_0^{\bar{v}-\varepsilon,\bar{u}-\varepsilon}} & \check{H}_0((A\cap B)_{\bar{v}-\varepsilon},(A\cap B)_{\bar{u}-\varepsilon}) \\
\downarrow & & \downarrow \\
\check{H}_1(X_{\bar{v}-\varepsilon},X_{\bar{u}+\varepsilon}) & \xrightarrow{\Delta_0^{\bar{v}-\varepsilon,\bar{u}+\varepsilon}} & \check{H}_0((A\cap B)_{\bar{v}-\varepsilon},(A\cap B)_{\bar{u}+\varepsilon})
\end{array}$$

for a sufficiently small $\varepsilon>0$, such that $\bar{u}+\varepsilon<\bar{v}-\varepsilon$. Under our hypotheses, vertical maps, that are induced by inclusions, are isomorphisms, implying that $\operatorname{rank}\operatorname{im}\Delta_0^{\bar{v}-\varepsilon,\bar{u}-\varepsilon}=\operatorname{rank}\operatorname{im}\Delta_0^{\bar{v}-\varepsilon,\bar{u}+\varepsilon}$, i.e. $\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}-\varepsilon}=\operatorname{rank}\ker\alpha_0^{\bar{v}-\varepsilon,\bar{u}+\varepsilon}$. By replacing $\bar{v}-\varepsilon$ with $\bar{v}+\varepsilon$ in the above diagram, it can also be proved that $\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}-\varepsilon}=\operatorname{rank}\ker\alpha_0^{\bar{v}+\varepsilon,\bar{u}+\varepsilon}$. Therefore, by Proposition 4.1, $\mu_X(p)-\mu_A(p)-\mu_B(p)+\mu_{A\cap B}(p)=0$. Since we are assuming that $\mu_{A\cap B}(p)=0$ and $\mu_A(p)+\mu_B(p)>0$, we have $\mu_X(p)>0$, i.e. p is a proper cornerpoint for $\ell_{(X,\varphi)}$. □

The following two theorems state that the abscissas of the cornerpoints for $\ell_{(X,\varphi)}$ are abscissas of cornerpoints for $\ell_{(A,\varphi_{|A})}$ or $\ell_{(B,\varphi_{|B})}$ or $\ell_{(A\cap B,\varphi_{|A\cap B})}$. The ordinates of the cornerpoints for $\ell_{(X,\varphi)}$ are, in general, homological 0-critical values for $(A,\varphi_{|A})$ or $(B,\varphi_{|B})$ or $(A\cap B,\varphi_{|A\cap B})$, and, under finiteness conditions, abscissas or ordinates of cornerpoints for $\ell_{(A,\varphi_{|A})}$ or $\ell_{(B,\varphi_{|B})}$ or $\ell_{(A\cap B,\varphi_{|A\cap B})}$, respectively.

These facts can easily be seen in the examples illustrated in Figures 4-5. In particular, in Figure 5, the size function $\ell_{(X,\varphi)}$ presents the proper cornerpoint $(-d,-e)$, which is a cornerpoint for neither $\ell_{(A,\varphi_{|A})}$ nor $\ell_{(B,\varphi_{|B})}$ nor $\ell_{(A\cap B,\varphi_{|A\cap B})}$. Nevertheless, its abscissa $-d$ is the abscissa of one of the cornerpoints for $\ell_{(A,\varphi_{|A})}$, while its ordinate $-e$ is the abscissa of the cornerpoint at infinity for both $\ell_{(B,\varphi_{|B})}$ and $\ell_{(A\cap B,\varphi_{|A\cap B})}$.

Theorem 4.5. *If $p=(\bar{u},\bar{v})\in\Delta^+$ is a proper cornerpoint for $\ell_{(X,\varphi)}$, then there exists at least one proper cornerpoint for $\ell_{(A,\varphi_{|A})}$ or $\ell_{(B,\varphi_{|B})}$ or $\ell_{(A\cap B,\varphi_{|A\cap B})}$ having $\bar{u}$ as abscissa. Moreover, if $(\bar{u},\infty)$ is a cornerpoint at infinity for $\ell_{(X,\varphi)}$, then it is a cornerpoint at infinity for $\ell_{(A,\varphi_{|A})}$ or $\ell_{(B,\varphi_{|B})}$.*

Proof. As for the first assertion, we proceed by contradiction.

Let us suppose that there are no proper cornerpoints for either $\ell_{(A,\varphi_{|A})}$, $\ell_{(B,\varphi_{|B})}$ or $\ell_{(A\cap B,\varphi_{|A\cap B})}$ having $\bar{u}$ as abscissa. Then, for every sufficiently small η, the following equalities hold: $\ell_{(A\cap B,\varphi_{|A\cap B})}(\bar{u}+\eta,\bar{v}\pm\eta)=\ell_{(A\cap B,\varphi_{|A\cap B})}(\bar{u}-\eta,\bar{v}\pm\eta)$, $\ell_{(A,\varphi_{|A})}(\bar{u}+\eta,\bar{v}\pm\eta)=\ell_{(A,\varphi_{|A})}(\bar{u}-\eta,\bar{v}\pm\eta)$, $\ell_{(B,\varphi_{|B})}(\bar{u}+\eta,\bar{v}\pm\eta)=\ell_{(B,\varphi_{|B})}(\bar{u}-\eta,\bar{v}\pm\eta)$.

Let us prove that $\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}+\eta,\overline{v}-\eta)=\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}-\eta,\overline{v}-\eta)$ for every η sufficiently small. The other equalities can be verified in the same way.

By [21, Prop. 6], there exists $\overline{\varepsilon}>0$ such that the open set $W_{\overline{\varepsilon}}(p)=\{(u,v)\in\Delta^+ : |u-\overline{u}|<\overline{\varepsilon}, |v-\overline{v}|<\overline{\varepsilon}, u\neq\overline{u}, v\neq\overline{v}\}$ does not contain any discontinuity point of $\ell_{(A\cap B,\varphi|_{A\cap B})}$. Therefore, for every $\eta<\overline{\varepsilon}$, $\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}+\eta,\overline{v}-\eta)=\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}+\eta,\overline{v}-\overline{\varepsilon}/2)$ and $\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}-\eta,\overline{v}-\eta)=\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}-\eta,\overline{v}-\overline{\varepsilon}/2)$. Moreover, since we are assuming that $\overline{u}$ is not the abscissa of a cornerpoint for $\ell_{(A\cap B,\varphi|_{A\cap B})}$, by [21, Lemma 3], $(\overline{u},\overline{v}-\overline{\varepsilon}/2)$ is not a discontinuity point of $\ell_{(A\cap B,\varphi|_{A\cap B})}$. It follows that $\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}+\eta,\overline{v}-\overline{\varepsilon}/2)=\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}-\eta,\overline{v}-\overline{\varepsilon}/2)$, implying the desired claim.

Now, by Corollary 2.4, $\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}+\eta,\overline{v}-\eta)=\ell_{(A\cap B,\varphi|_{A\cap B})}(\overline{u}-\eta,\overline{v}-\eta)$ implies that $\operatorname{rank}\check{H}_0((A\cap B)_{\overline{v}-\eta},(A\cap B)_{\overline{u}+\eta})=\operatorname{rank}\check{H}_0((A\cap B)_{\overline{v}-\eta},(A\cap B)_{\overline{u}-\eta})$. In a similar way we obtain $\operatorname{rank}\check{H}_0(A_{\overline{v}-\eta},A_{\overline{u}+\eta})=\operatorname{rank}\check{H}_0(A_{\overline{v}-\eta},A_{\overline{u}-\eta})$ and $\operatorname{rank}\check{H}_0(B_{\overline{v}-\eta},B_{\overline{u}+\eta})=\operatorname{rank}\check{H}_0(B_{\overline{v}-\eta},B_{\overline{u}-\eta})$. Next, let us consider the following diagram:

$$\begin{tikzcd}
\check{H}_0((A\cap B)_{\overline{v}-\eta},(A\cap B)_{\overline{u}-\eta}) \arrow[r, "\alpha_0^{\overline{v}-\eta,\overline{u}-\eta}"] \arrow[d, "j_1"] & \check{H}_0(A_{\overline{v}-\eta},A_{\overline{u}-\eta})\oplus\check{H}_0(B_{\overline{v}-\eta},B_{\overline{u}-\eta}) \arrow[d, "j_2"'] \\
\check{H}_0((A\cap B)_{\overline{v}-\eta},(A\cap B)_{\overline{u}+\eta}) \arrow[r, "\alpha_0^{\overline{v}-\eta,\overline{u}+\eta}"] & \check{H}_0(A_{\overline{v}-\eta},A_{\overline{u}+\eta})\oplus\check{H}_0(B_{\overline{v}-\eta},B_{\overline{u}+\eta}),
\end{tikzcd}$$

where the homomorphisms j_1 and j_2 are induced by inclusions. Since they are surjective and their respective domain and codomain have the same rank, we deduce that j_1 and j_2 are isomorphisms. So, we obtain that $\ker\alpha_0^{\overline{v}-\eta,\overline{u}-\eta}\simeq\ker\alpha_0^{\overline{v}-\eta,\overline{u}+\eta}$.

Proceeding analogously, we can prove that $\operatorname{rank}\check{H}_0((A\cap B)_{\overline{v}+\eta},(A\cap B)_{\overline{u}+\eta})=\operatorname{rank}\check{H}_0((A\cap B)_{\overline{v}+\eta},(A\cap B)_{\overline{u}-\eta})$, $\operatorname{rank}\check{H}_0(A_{\overline{v}+\eta},A_{\overline{u}+\eta})=\operatorname{rank}\check{H}_0(A_{\overline{v}+\eta},A_{\overline{u}-\eta})$ and $\operatorname{rank}\check{H}_0(B_{\overline{v}+\eta},B_{\overline{u}+\eta})=\operatorname{rank}\check{H}_0(B_{\overline{v}+\eta},B_{\overline{u}-\eta})$. Hence, from the diagram

$$\begin{tikzcd}
\check{H}_0((A\cap B)_{\overline{v}+\eta},(A\cap B)_{\overline{u}-\eta}) \arrow[r, "\alpha_0^{\overline{v}+\eta,\overline{u}-\eta}"] \arrow[d, "k_1"] & \check{H}_0(A_{\overline{v}+\eta},A_{\overline{u}-\eta})\oplus\check{H}_0(B_{\overline{v}+\eta},B_{\overline{u}-\eta}) \arrow[d, "k_2"'] \\
\check{H}_0((A\cap B)_{v+\eta},(A\cap B)_{\overline{u}+\eta}) \arrow[r, "\alpha_0^{\overline{v}+\eta,\overline{u}+\eta}"] & \check{H}_0(A_{\overline{v}+\eta},A_{\overline{u}+\eta})\oplus\check{H}_0(B_{\overline{v}+\eta},B_{\overline{u}+\eta}),
\end{tikzcd}$$

we can deduce that $\ker\alpha_0^{\overline{v}+\eta,\overline{u}-\eta}\simeq\ker\alpha_0^{\overline{v}+\eta,\overline{u}+\eta}$.

Since η can be chosen arbitrarily small, we have so proved that

$$\lim_{\eta\to 0^+}(\ker\alpha_0^{\overline{v}-\eta,\overline{u}-\eta}-\ker\alpha_0^{\overline{v}-\eta,\overline{u}+\eta})=0,$$
$$\lim_{\eta\to 0^+}(\ker\alpha_0^{\overline{v}+\eta,\overline{u}-\eta}-\ker\alpha_0^{\overline{v}+\eta,\overline{u}+\eta})=0.$$

Therefore, applying Proposition 4.1, it follows that $\mu_X(p)-\mu_A(p)-\mu_B(p)+\mu_{A\cap B}(p)=0$. In particular, by the assumption that $p=(\overline{u},\overline{v})$ is not a proper cornerpoint for either $\ell_{(A\cap B,\varphi|_{A\cap B})}$, $\ell_{(A,\varphi|_A)}$, or $\ell_{(B,\varphi|_B)}$, it holds that $\mu_X(p)=0$, a contradiction.

In the case of cornerpoints at infinity, we observe that, if $(\overline{u},\infty)$ is a cornerpoint at infinity for $\ell_{(X,\varphi)}$, then $\overline{u}=\min_{P\in C}\varphi(P)$, for at least one connected component C of X ([21, Prop. 9]). Furthermore, since $X=A\cup B$, it follows that $\overline{u}=\min_{P\in C\cap A}\varphi_{|A}(P)$ or $\overline{u}=\min_{P\in C\cap B}\varphi_{|B}(P)$, from which (by [21, Prop. 9]), $(\overline{u},\infty)$ is shown to be a cornerpoint at infinity for $\ell_{(A,\varphi|_A)}$ or $\ell_{(B,\varphi|_B)}$. □

Theorem 4.6. *If $p=(\overline{u},\overline{v})\in\Delta^+$ is a proper cornerpoint for $\ell_{(X,\varphi)}$, then $\overline{v}$ is a homological 0-critical value for $(A,\varphi_{|A})$ or $(B,\varphi_{|B})$ or $(A\cap B,\varphi_{|A\cap B})$. Furthermore, if there exists at most a finite number of homological 0-critical values for $(A,\varphi_{|A})$, $(B,\varphi_{|B})$, and $(A\cap B,\varphi_{|A\cap B})$, then $\overline{v}$ is the abscissa of a cornerpoint (proper or at infinity) or the ordinate of a proper cornerpoint for $\ell_{(A,\varphi|_A)}$ or $\ell_{(B,\varphi|_B)}$ or $\ell_{(A\cap B,\varphi|_{A\cap B})}$.*

Proof. Regarding the first assertion, we proceed by contradiction.

Let us suppose that $\bar{v}$ is not a homological 0-critical value for the size pairs $(A,\varphi_{|A})$, $(B,\varphi_{|B})$ and $(A\cap B,\varphi_{|A\cap B})$. Then, by Definition B.1, for every $\overline{\varepsilon}>0$, there exists ε with $0<\varepsilon<\overline{\varepsilon}$, such that the vertical homomorphisms h and k induced by inclusions in the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}_0((A\cap B)_{\bar{v}-\varepsilon}) \arrow[r] \arrow[d, "h"] & \check{H}_0(A_{\bar{v}-\varepsilon})\oplus\check{H}_0(B_{\bar{v}-\varepsilon}) \arrow[r] \arrow[d, "k"] & \check{H}_0(X_{\bar{v}-\varepsilon}) \arrow[r] \arrow[d, "\iota_0^{\bar{v}-\varepsilon,\bar{v}+\varepsilon}"] & 0 \arrow[d, "0"'] \\
\cdots \arrow[r] & \check{H}_0((A\cap B)_{\bar{v}+\varepsilon}) \arrow[r] & \check{H}_0(A_{\bar{v}+\varepsilon})\oplus\check{H}_0(B_{\bar{v}+\varepsilon}) \arrow[r] & \check{H}_0(X_{\bar{v}+\varepsilon}) \arrow[r] & 0
\end{tikzcd}$$

are isomorphisms. Therefore, extending the horizontal lines of the above diagram rightwards with two trivial homomorphisms, we can apply the Five Lemma and deduce that $\iota_0^{\bar{v}-\varepsilon,\bar{v}+\varepsilon}$ is an isomorphism. This implies that $\bar{v}$ is not a homological 0-critical value for (X,φ). Consequently, applying Proposition B.2, it holds that, for every $u<\bar{v}$, $\lim_{\varepsilon\to 0^+}\left(\ell_{(X,\varphi)}(u,\bar{v}-\varepsilon)-\ell_{(X,\varphi)}(u,\bar{v}+\varepsilon)\right)=0$. Hence, it follows that $\lim_{\varepsilon\to 0^+}\left(\ell_{(X,\varphi)}(\overline{u}-\varepsilon,\bar{v}-\varepsilon)-\ell_{(X,\varphi)}(\overline{u}-\varepsilon,\bar{v}+\varepsilon)\right)=0$, choosing $u=\overline{u}-\varepsilon$ and $\lim_{\varepsilon\to 0^+}[\ell_{(X,\varphi)}(\overline{u}+\varepsilon,\bar{v}-\varepsilon)-\ell_{(X,\varphi)}(\overline{u}+\varepsilon,\bar{v}+\varepsilon)]=0$, choosing $u=\overline{u}+\varepsilon$. Therefore, by Definition 2.5, we obtain $\mu_X(p)=0$, giving a contradiction.

Now, let us proceed with the proof of the second statement, assuming that $\bar{v}$ is a homological 0-critical value for $(A,\varphi_{|A})$. It is analogous for $(B,\varphi_{|B})$ and $(A\cap B,\varphi_{|A\cap B})$. For such a $\bar{v}$, by Definition B.1, it holds that, for every sufficiently small $\varepsilon>0$, $\iota_0^{\bar{v}-\varepsilon,\bar{v}+\varepsilon}:\check{H}_0(A_{\bar{v}-\varepsilon})\to\check{H}_0(A_{\bar{v}+\varepsilon})$ is not an isomorphism. In particular, by Proposition B.3 *(i)*, if $\iota_0^{\bar{v}-\varepsilon,\bar{v}+\varepsilon}$ is not surjective for any sufficiently small $\varepsilon>0$, then there exists $v>\bar{v}$, such that $\bar{v}$ is a discontinuity point for $\ell_{(A,\varphi_{|A})}(\cdot,v)$. This condition necessarily implies the existence of a cornerpoint (proper or at infinity) for $\ell_{(A,\varphi_{|A})}$, having $\bar{v}$ as abscissa [21, Lemma 3].

On the other hand, by Proposition B.3 *(ii)*, if $\iota_0^{\bar{v}-\varepsilon,\bar{v}+\varepsilon}$ is surjective for every sufficiently small $\varepsilon>0$, then there exists $u<\bar{v}$ such that $\bar{v}$ is a discontinuity point for $\ell_{(A,\varphi_{|A})}(u,\cdot)$. This condition necessarily implies the existence of a proper cornerpoint for $\ell_{(A,\varphi_{|A})}$, having $\bar{v}$ as ordinate [21, Lemma 3]. $\square$

5 Experimental results

In this section we present two experiments demonstrating the robustness of size functions under partial occlusions.

Psychophysical observations indicate that human and monkey perception of partially occluded shapes changes according to whether, or not, the occluding pattern is visible to the observer, and whether the occluded shape is a filled figure or an outline [28]. In particular, discrimination performance is higher for filled shapes than for outlines, and in both cases it significantly improves when shapes are occluded by a visible rather than invisible object.

In computer vision experiments, researcher usually work with invisible occluding patterns, both on outlines (see, e.g., [8, 23, 30, 34, 35]) and on filled shapes (see, e.g., [25]).

To test size function performance under occlusions, we work with 70 filled images, each chosen from a different class of the MPEG-7 dataset [39]. The two experiments differ in the visibility of the occluding pattern. Since in the first experiments the occluding pattern is visible, we aim at finding a fingerprint of the original shape in the size function of the occluded shape. In the second experiment, where the occluding pattern is invisible, we perform a direct comparison between the occluded shape and the original shape. In both experiments, the occluding pattern is a rectangular shape occluding from the top, or the left, by an area we increasingly vary from 20% to 60% of the height or width of the bounding box of the original shape. We compute size functions for both the original shapes and the occluded ones, choosing a family of eight measuring functions having only the set of black pixels as domain. They are defined as follows: four of them as the distance from the line passing through the origin (top left point of the bounding box), rotated by an angle of 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$ and $\frac{3\pi}{4}$ radians, respectively, with respect to the horizontal position; the other four as minus the distance from the same lines, respectively. This family of measuring functions is chosen only for demonstrative purposes, since the associated size functions are simple in terms of the number of cornerpoints, but, at the same time, non-trivial in terms of shape information.

The first experiment aims to show how a trace of the size function describing the shape of an object is contained in the size function related to the occluded shape when the occluding pattern is visible (see first column of Tables 2–4).

With reference to the notation used in our theoretical setting, we are considering A as the original shape, B as the black rectangle, and X as the occluded shape generated by their union.

In Table 1, for some different levels of occlusion, each 3D bar chart displays, along the z-axis, the percentage of common cornerpoints between the set of size functions associated with the 70 occluded shapes (x-axis), and the set of size functions associated with the 70 original ones (y-axis). We see that, for each occluded shape, the highest bar is always on the diagonal, that is, where the occluded object is compared with the corresponding original one.

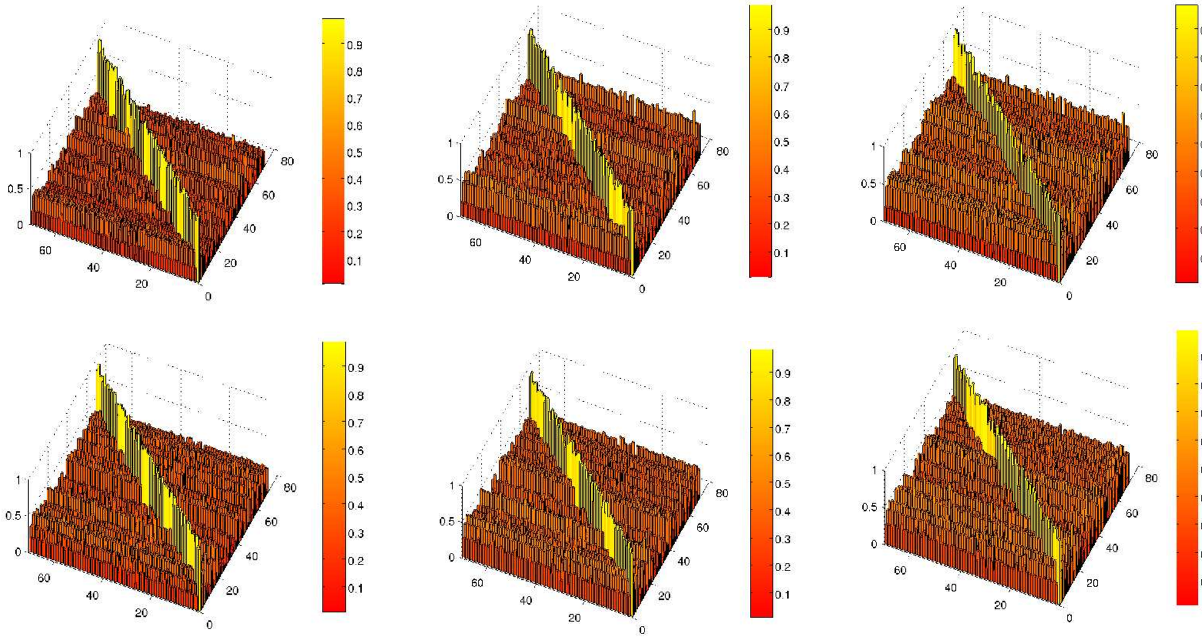

Table 1: 3D bar charts displaying the percentage of common cornerpoints (z-axis) between the 70 occluded shapes (x-axis) and the 70 original ones (y-axis) correspondingly ordered. First row: Shapes are occluded from top by 20% (column 1), by 40% (column 2), by 60% (column 3). Second row: Shapes are occluded from the left by 20% (column 1), by 40% (column 2), by 60% (column 3).

Moreover, to display the robustness of cornerpoints under occlusion, three particular instances of our dataset images are shown in Tables 2–4 (first column) with their size functions with respect to the second group of four measuring functions (the next-to-last column). The chosen images are characterized by different homological features, which will be changed in presence of occlusion. For example, the "camel" in Table 2 is a connected shape without holes, but it may happen that the occlusion makes the first homological group non-trivial (see second row, first column). On the other hand, Table 3 shows a "frog", which is a connected shape with several holes. The different percentages of occlusion can create some new holes or destroy them (see rows 3–4). Eventually, the "pocket watch", represented in Table 4, is primarily characterized by several connected components, whose number decreases as the occluding area increases. This results in a reduction of the number of cornerpoints at infinity in its size functions. In spite of these topological changes, it can easily be seen that, given a measuring function, even if the size function related to a shape and the size function related to the occluded shape are defined by different cornerpoints, because of occlusion, a common subset of these is present, making a partial matching possible between them.

The second experiment is a recognition test for occluded shapes by comparison of size functions. In this case the rectangular-shaped occlusion is not visible (see Table 5). When the original shape is disconnected by the occlusion, we retain only the connected component of greatest area. With reference to the notation used in our theoretical setting, here we are considering X as the original shape, A as the the occluded shape, and B as the invisible part of X.

By varying the amount of occluded area, we compare each occluded shape with each of the 70 original shapes.

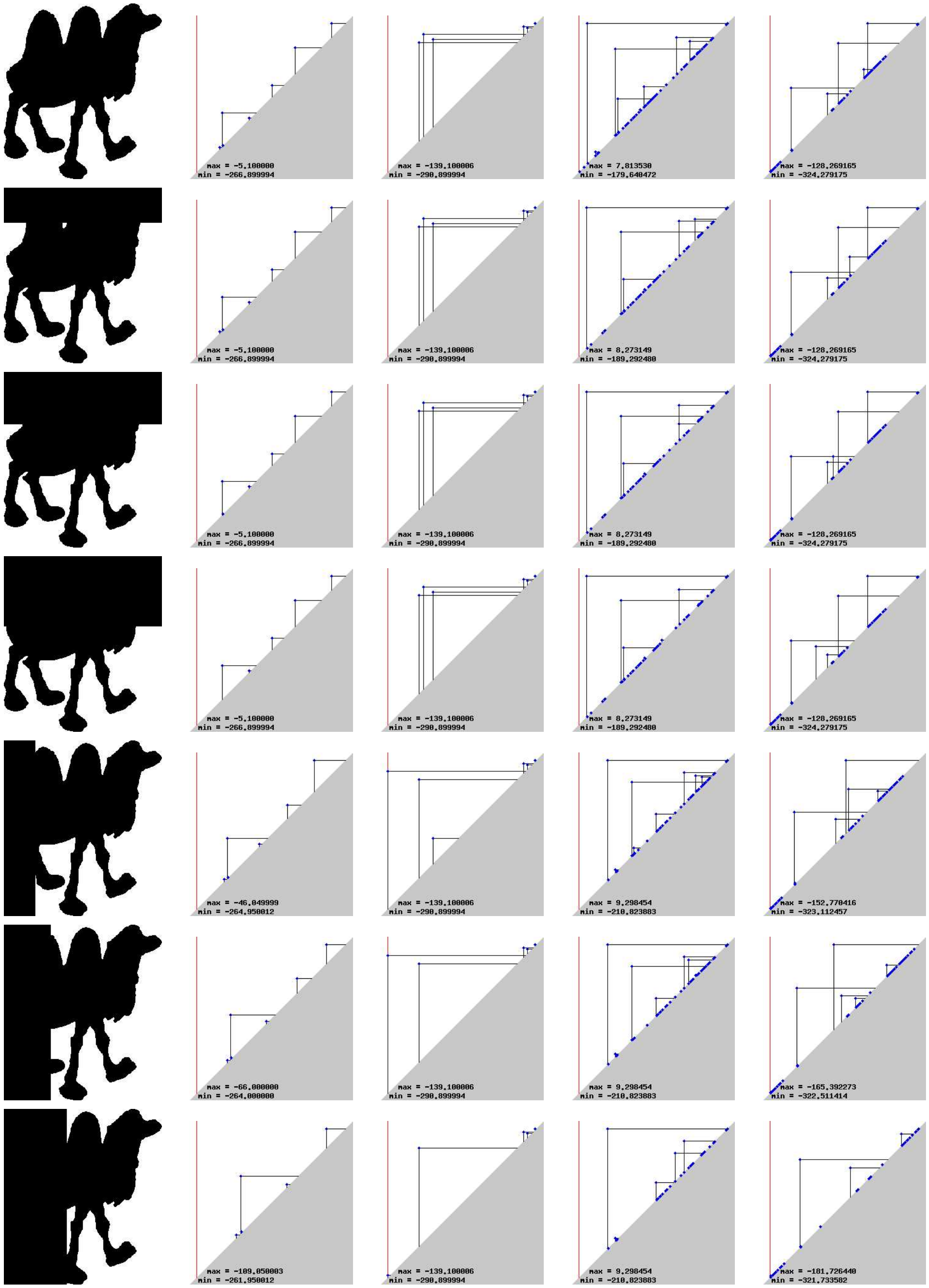

Table 2: The first column: (row 1) original "camel" shape, (rows 2–4) occluded from top by 20%, 30%, 40%, (row 5–7) occluded from left by 20%, 30%, 40%. From second column onwards: corresponding size functions related to measuring functions defined as minus distances from four lines rotated by 0, $\pi/4$, $\pi/2$, $3\pi/4$, with respect to the horizontal position.

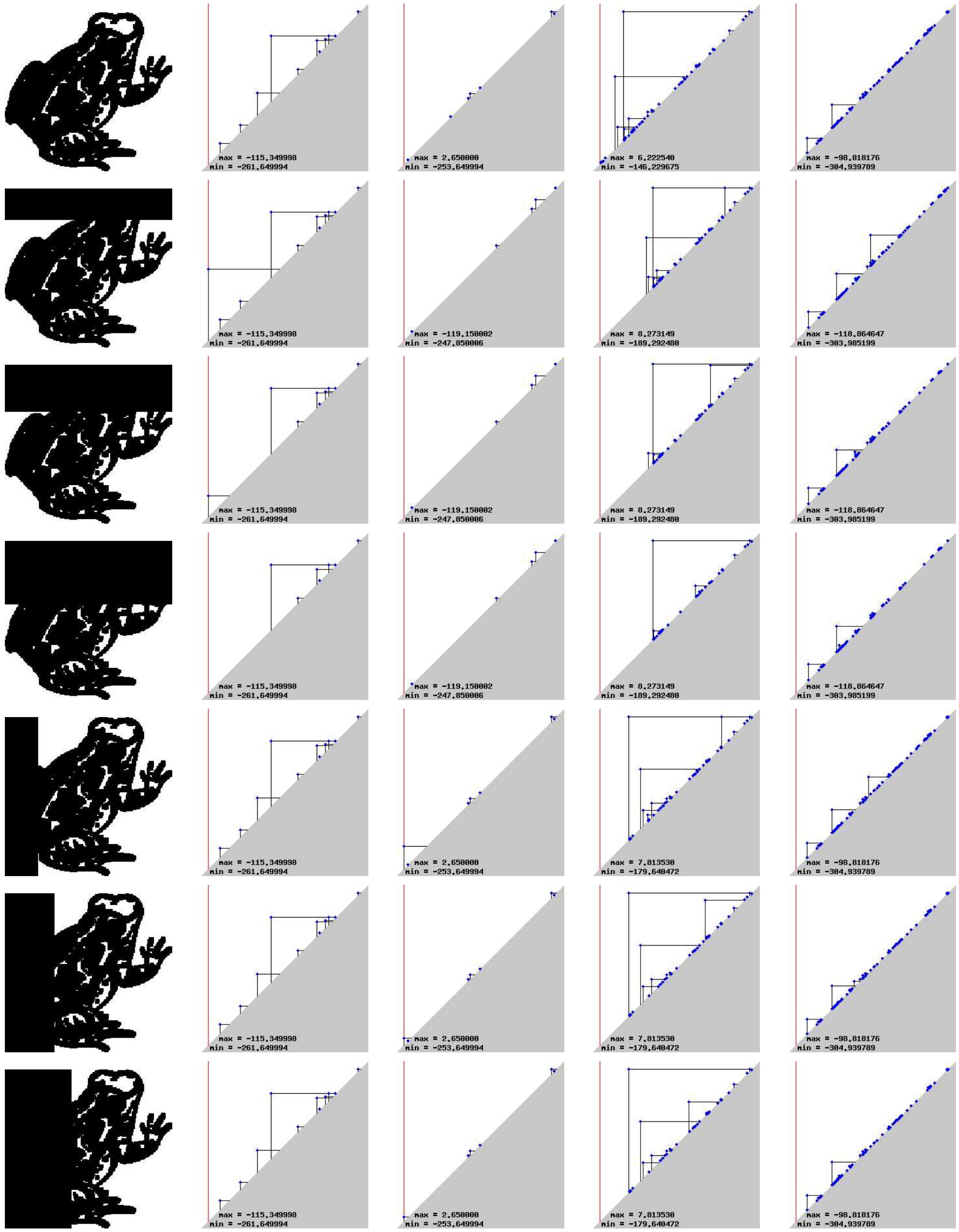

Table 3: The first column: (row 1) original "frog" shape, (rows 2–4) occluded from top by 20%, 30%, 40%, (row 5–7) occluded from left by 20%, 30%, 40%. From second column onwards: corresponding size functions related to measuring functions defined as minus distances from four lines rotated by 0, $\pi/4$, $\pi/2$, $3\pi/4$, with respect to the horizontal position.

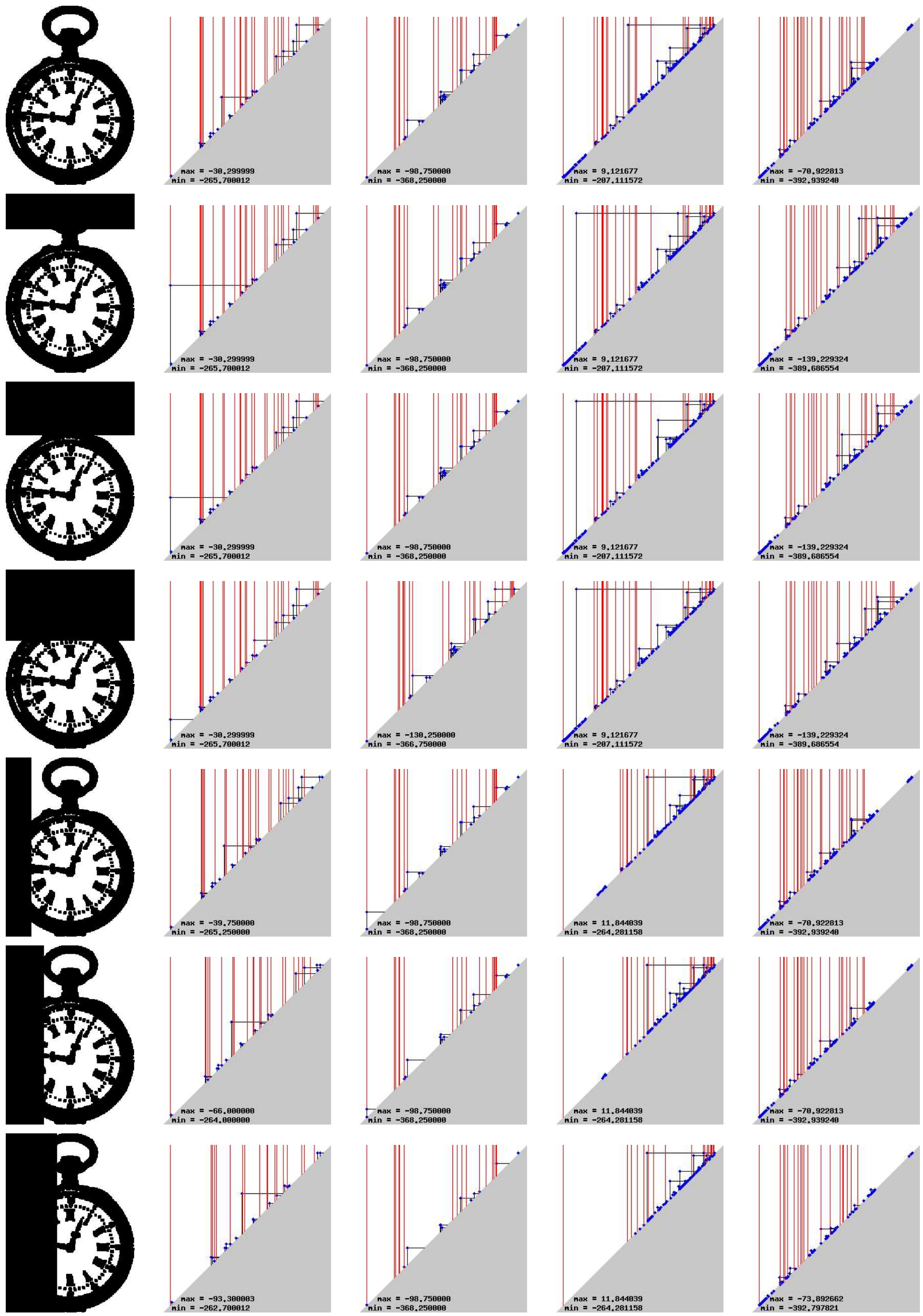

Table 4: The first column: (row 1) original "pocket watch" shape, (rows 2–4) occluded from top by 20%, 30%, 40%, (row 5–7) occluded from left by 20%, 30%, 40%. From second column onwards: corresponding size functions related to measuring functions defined as minus distances from four lines rotated by 0, $\pi/4$, $\pi/2$, $3\pi/4$, with respect to the horizontal position.

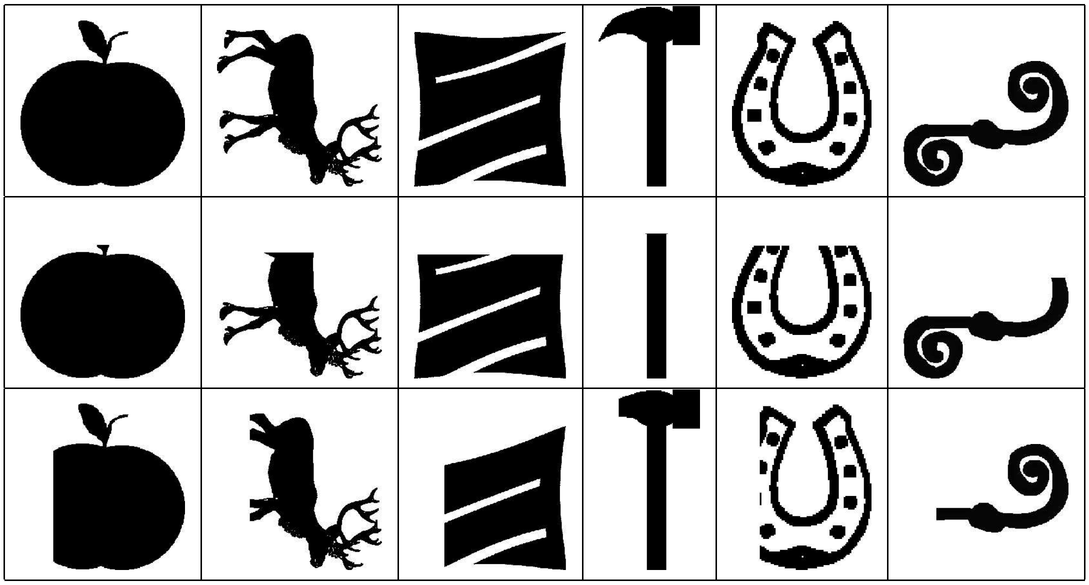

Table 5: The first row: some instances from the MPEG-7 dataset; the second and third rows: by 20% occluded from the top and from the left, respectively.

Comparison is performed by calculating the sum of the eight Hausdorff distances between the sets of cornerpoints for the size functions associated with the corresponding eight measuring functions. Then each occluded shape is assigned to the class of its nearest neighbor among the original shapes.

In Table 6, two graphs describe the rate of correct recognition in the presence of an increasing percentage of occlusion. The leftmost graph is related to the occlusion from the top, the rightmost one is related to the same occlusion from the left.

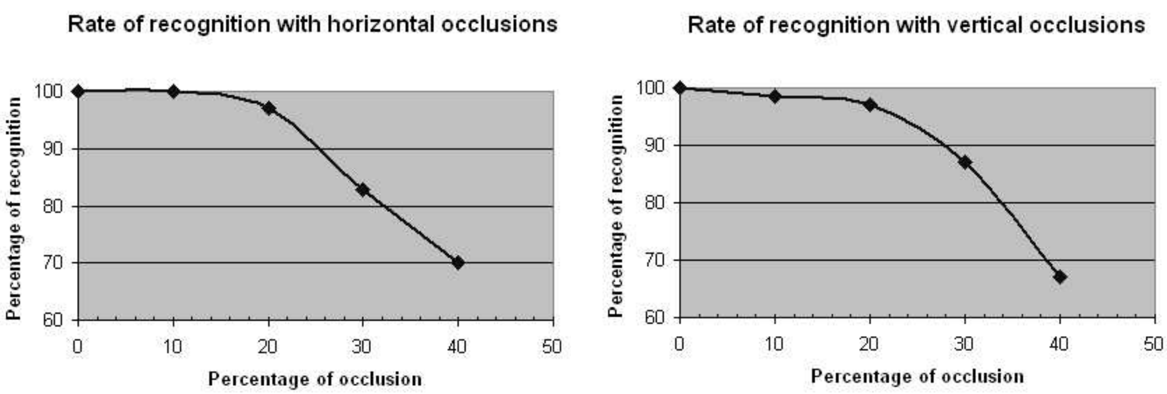

Table 6: The leftmost (rightmost, respectively) graph describes the recognition trend when the occluded area from the top (left, respectively) increases.

6 Discussion

The main contribution of this paper is the analysis of the behavior of size functions in the presence of occlusions.

Specifically we have proved that size functions assess a partial matching between shapes by showing common subsets of cornerpoints.

Therefore, using size functions, recognition of a shape that is partially occluded by a foreground shape becomes an easy task. Indeed, recognition is achieved simply by associating with the occluded shape that form whose size function presents the largest common subset of cornerpoints (as in the experiment in Table 1).

In practice, however, shapes may undergo other deformations due to perspective, articulations, or noise, for instance. As a consequence of these alterations, cornerpoints may move. Anyway, small continuous changes in shape induce small displacements in cornerpoint configuration.

It has to be expected that, when a shape is not only occluded but also deformed, it will not be possible to find a common subset of cornerpoints between the original shape and the occluded one, since the deformation has slightly changed the cornerpoint position. At the same time, however, the Hausdorff distance between the size function of the original shape and the size function of the occluded shape will not need to be small, because it takes into account the total number of cornerpoints, including, for example, those inherited from the occluding pattern (as in the experiment in Table 6).

The present work is a necessary step, in view of the more general goal of recognizing shapes in the presence of both occlusions and deformations. The development of a method to measure partial matching of cornerpoints that do not exactly overlap but are slightly shifted, would be desirable.

Acknowledgments

As pointed out to us by C. L. Schochet, there is another homology theory defined for all compact spaces and for any coefficient group G that always gives an exact Mayer-Vietoris sequence, called Steenrod homology. Under our assumptions on G, Steenrod homology coincides with Čech homology. We are grateful to C. L. Schochet for this interesting observation.

We wish to thank F. Cagliari, M. Grandis, R. Piccinini for their helpful suggestions and P. Frosini for suggesting the example in Figure 6 (a). Thanks to A. Cerri and F. Medri for their invaluable help with the software. However, the authors are solely responsible for any possible errors.

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A Appendix: Čech homology and Mayer-Vietoris sequence

The Čech approach is based on a way to generate simplicial complexes from finite covering of a space, by taking the nerve of the covering. Since the nerve of a covering is a simplicial complex, we can compute its homology groups by the usual techniques. Refining the covering one obtains an inverse system of finite triangulations that approximate a space. A Čech homology group is the inverse limit of such a system. Detailed descriptions of Čech homology theory can be found in [18]. Here we briefly survey the subject, focusing on the Mayer-Vietoris sequence of Čech homology.

Given a compact Hausdorff space X, let $\Sigma(X)$ denote the family of all finite coverings of X by open sets. The coverings in $\Sigma(X)$ will be denoted by script letters $\mathscr{U}, \mathscr{V}, \dots$ and the open sets in a covering by italic capitals $U, V, \dots$ An element $\mathscr{U}$ of $\Sigma(X)$ may be considered as a simplicial complex if we define *vertex* to mean *open set U in $\mathscr{U}$* and agree that a subcollection $U_0, \dots, U_p$ of such vertices constitutes a p-simplex if and only if the intersection $\bigcap_{i=0}^{p} U_i$ is not empty. The resulting complex is known as the *nerve of the covering $\mathscr{U}$*.

Given a covering $\mathscr{U}$ in $\Sigma(X)$, we may define the chain groups $C_p(\mathscr{U}, G)$, the cycle groups $Z_p(\mathscr{U}, G)$, the boundary groups $B_p(\mathscr{U}, G)$, and the homology groups $H_p(\mathscr{U}, G)$.

The collection $\Sigma(X)$ of finite open coverings of a space X may be partially ordered by refinement. A covering $\mathscr{V}$ refines the covering $\mathscr{U}$, and we write $\mathscr{U} < \mathscr{V}$, if every element of $\mathscr{V}$ is contained in some element of $\mathscr{U}$. It turns out that $\Sigma(X)$ is a direct set under refinement.

If $\mathscr{U} < \mathscr{V}$ in $\Sigma(X)$, then there is a simplicial mapping $\pi_{\mathscr{U}\mathscr{V}}$ of $\mathscr{V}$ into $\mathscr{U}$ called a *projection*. This is defined by taking $\pi_{\mathscr{U}\mathscr{V}}(V)$, $V \in \mathscr{V}$, to be any (fixed) element U of $\mathscr{U}$ such that V is contained in U. There may be many projections of $\mathscr{V}$ into $\mathscr{U}$. Each projection $\pi_{\mathscr{U}\mathscr{V}}$ induces a chain mapping of $C_p(\mathscr{V},G)$ into $C_p(\mathscr{U},G)$, still denoted by $\pi_{\mathscr{U}\mathscr{V}}$, and this in turn induces homomorphisms ${}_*\pi_{\mathscr{U}\mathscr{V}}$ of $H_p(\mathscr{V},G)$ into $H_p(\mathscr{U},G)$. If $\mathscr{U} < \mathscr{V}$ in $\Sigma(X)$, then it can be proved that any two projections of $\mathscr{V}$ into $\mathscr{U}$ induce the same homomorphism of $H_p(\mathscr{V},G)$ into $H_p(\mathscr{U},G)$.

Taking the limit of the inverse system $(H_p(\mathscr{V},G),{}_*\pi_{\mathscr{U}\mathscr{V}})$ one obtains the pth Čech homology group.

In general, Čech Homology Theory has all the axioms of homology theories except the exactness axiom. However, if some assumptions are made on the considered spaces and coefficients, this axiom also holds. Indeed, in [18, Chap. IX, Thm. 7.6] (see also [27]), we read the following result concerning the sequence of a pair (X,A)

$$\cdots \to \check{H}_{p+1}(X,A) \xrightarrow{\partial_p} \check{H}_p(A) \xrightarrow{i_p} \check{H}_p(X) \xrightarrow{j_p} \check{H}_p(X,A) \xrightarrow{\partial_{p-1}} \cdots \to \check{H}_0(X,A) \to 0$$

which, in general, is only of order 2 (this means that the composition of any two successive homomorphisms of the sequence is zero, i.e. im $\subseteq$ ker).

Theorem A.1. [18, Chap. IX, Thm. 7.6] *If (X,A) is compact and G is a vector space over a field, then the homology sequence of the pair (X,A) is exact.*

It follows that, if (X,A) is compact and G is a vector space over a field, Čech homology satisfies all the axioms of homology theories, and therefore all the general theorems in [18, Chap. I] also hold for Čech homology. In particular, using [18, Chap. I, Thm. 15.3], we have the exactness of the Mayer-Vietoris sequence in Čech homology:

Theorem A.2. *Let (X,A,B) be a compact proper triad and G be a vector space over a field. The Mayer-Vietoris sequence of (X,A,B) with $X = A \cup B$*

$$\cdots \to \check{H}_{p+1}(X) \xrightarrow{\Delta_p} \check{H}_p(A \cap B) \xrightarrow{\alpha_p} \check{H}_p(A) \oplus \check{H}_p(B) \xrightarrow{\beta_p} \check{H}_p(X) \xrightarrow{\Delta_{p-1}} \cdots \to \check{H}_0(X) \to 0$$

is exact.

Concerning homomorphisms between Mayer-Vietoris sequences, from [18, Chap. I, Thm. 15.4], we deduce the following result.

Theorem A.3. *If (X,A,B) and (Y,C,D) are proper triads, $X = A \cup B$, $Y = C \cup D$, and $f : (X,A,B) \to (Y,C,D)$ is a map of one proper triad into another, then f induces a homomorphism of the Mayer-Vietoris sequence of (X,A,B) into that of (Y,C,D) such that commutativity holds in the diagram*

$$\begin{array}{ccccccccc}
\cdots \to & \check{H}_{p+1}(X) & \to & \check{H}_p(A \cap B) & \to & \check{H}_p(A) \oplus \check{H}_p(B) & \to & \check{H}_p(X) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \check{H}_{p+1}(Y) & \to & \check{H}_p(C \cap D) & \to & \check{H}_p(C) \oplus \check{H}_p(D) & \to & \check{H}_p(Y) & \to \cdots
\end{array}$$

A relative form of the Mayer-Vietoris sequence, different from the one proposed in [18], is useful in the present paper. In order to obtain this sequence, we can adapt the construction explained in [24] to Čech homology and obtain the following result.

Theorem A.4. *If (X,A,B) and (Y,C,D) are compact proper triads with $X = A\cup B$, $Y = C\cup D$, $Y \subseteq X$, $C \subseteq A$, $D \subseteq B$, then there is a relative Mayer-Vietoris sequence of homology groups with coefficients in a vector space G over a field*

$$\cdots \to \check{H}_{p+1}(X,Y) \to \check{H}_p(A\cap B, C\cap D) \to \check{H}_p(A,C)\oplus \check{H}_p(B,D) \to \check{H}_p(X,Y) \to \cdots \to \check{H}_0(X,Y) \to 0$$

that is exact.

Proof. Given a covering $\mathscr{U}$ of $\Sigma(X)$, we may consider the relative simplicial homology groups $H_p(\mathscr{U},\mathscr{U}_Y)$, $H_p(\mathscr{U}_A,\mathscr{U}_C)$, $H_p(\mathscr{U}_B,\mathscr{U}_D)$, $H_p(\mathscr{U}_{A\cap B},\mathscr{U}_{C\cap D})$, for every $p \geq 0$. For these groups the relative Mayer-Vietoris sequence

$$\cdots \to H_{p+1}(\mathscr{U},\mathscr{U}_Y) \to H_p(\mathscr{U}_{A\cap B},\mathscr{U}_{C\cap D}) \to H_p(\mathscr{U}_A,\mathscr{U}_C)\oplus H_p(\mathscr{U}_B,\mathscr{U}_D) \to H_p(\mathscr{U},\mathscr{U}_Y) \to \cdots$$

is exact (cf. [24, page 152]).

We now recall that the pth Čech homology group of a pair of spaces (X,Y) over G is the inverse limit of the system of groups $\{H_p(\mathscr{U},\mathscr{U}_Y,G),_* \pi_{\mathscr{U}\mathscr{V}}\}$ defined on the direct set of all open coverings of the pair (X,Y) (cf. [18, Chap. IX, Thm. 3.2 and Def. 3.3]). The claim is proved recalling that, given an inverse system of exact lower sequences, where all the terms of the sequence belong to the category of vector spaces over a field, the limit sequence is also exact (cf. [18, Chap. VIII, Thm. 5.7] and [27]). □

The following result, concerning homomorphisms of relative Mayer-Vietoris exact sequences, holds. We omit the proof, which can be obtained in a standard way.

Theorem A.5. *If (X,A,B), (Y,C,D), (X',A',B'), (Y',C',D') are compact proper triads with $X = A\cup B$, $Y = C\cup D$, $Y \subseteq X$, $C \subseteq A$, $D \subseteq B$, and $X' = A'\cup B'$, $Y' = C'\cup D'$, $Y' \subseteq X'$, $C' \subseteq A'$, $D' \subseteq B'$, and $f : X \to X'$ is a map such that $f(Y) \subseteq Y'$, $f(A) \subseteq A'$, $f(B) \subseteq B'$, $f(C) \subseteq C'$, $f(D) \subseteq D'$, then f induces a homomorphism of the relative Mayer-Vietoris sequences such that commutativity holds in the diagram*

$$\begin{array}{ccccccccc}
\cdots \to & \check{H}_{p+1}(X,Y) & \to & \check{H}_p(A\cap B,C\cap D) & \to & \check{H}_p(A,C)\oplus\check{H}_p(B,D) & \to & \check{H}_p(X,Y) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & \check{H}_{p+1}(X',Y') & \to & \check{H}_p(A'\cap B',C'\cap D') & \to & \check{H}_p(A',C')\oplus\check{H}_p(B',D') & \to & \check{H}_p(X',Y') & \to \cdots
\end{array}$$

B Appendix: Relating homological critical values and cornerpoints

In this section we show the link between homological critical values and cornerpoints for size functions. To the best of our knowledge, this connection, which is rather intuitive, has not been proved elsewhere. Moreover, although in the case of tame functions (i.e., functions with at most a finite number of homological critical values, and whose lower level sets have finitely generated homology groups) it may be trivial, is unfortunately not so in the case considered here.

Our treatment exploits the connection between cornerpoints and discontinuities of size functions. Homological critical values have been introduced in [9], and intuitively correspond to levels where the lower level sets undergo a topological change. Discontinuity points of size functions have been thoroughly studied in [21].

In particular, we prove that the coordinates of a cornerpoint are always homological critical values (Proposition B.2), while the converse is true only assuming that the number of homological critical values is finite (Proposition B.3). Indeed, in general, there may exist homological critical values not generating discontinuities for the size function (Remark B.4).

Definition B.1. *Let (X,φ) be a size pair. A* homological p-critical value *for (X,φ) is a real number w such that, for every sufficiently small $\varepsilon > 0$, the map $\iota_p^{w-\varepsilon,w+\varepsilon} : \check{H}_p(X_{w-\varepsilon}) \to \check{H}_p(X_{w+\varepsilon})$ induced by inclusion is not an isomorphism.*

Proposition B.2. *If $w \in \mathbb{R}$ is a coordinate of a cornerpoint for $\ell_{(X,\varphi)}$, then w is a homological 0-critical value for the size pair (X,φ).*

Proof. By [21, Prop. 8], if $w \in \mathbb{R}$ is a coordinate of a cornerpoint for $\ell_{(X,\varphi)}$, then w is the coordinate of a horizontal or a vertical discontinuities of $\ell_{(X,\varphi)}$. Hence it is sufficient to prove that, if w is not a homological 0-critical value, then

(i) for every $v > w$, $\lim_{\varepsilon \to 0^+} \left(\ell_{(X,\varphi)}(w+\varepsilon, v) - \ell_{(X,\varphi)}(w-\varepsilon, v)\right) = 0$;

(ii) for every $u < w$, $\lim_{\varepsilon \to 0^+} \left(\ell_{(X,\varphi)}(u, w-\varepsilon) - \ell_{(X,\varphi)}(u, w+\varepsilon)\right) = 0$.

We begin by proving *(i)*. Let $v > w$. For every $\varepsilon > 0$ such that $v > w+\varepsilon$, we can consider the commutative diagram

$$\begin{tikzcd} \check{H}_0(X_{w-\varepsilon}) \arrow[r, "\iota_0^{w-\varepsilon,v}"] \arrow[d, "\iota_0^{w-\varepsilon,w+\varepsilon}"'] & \check{H}_0(X_v) \\ \check{H}_0(X_{w+\varepsilon}) \arrow[ur, "\iota_0^{w+\varepsilon,v}"'] & \end{tikzcd} \tag{1}$$

By the assumption that w is not a homological 0-critical value, there exist arbitrarily small real numbers $\overline{\varepsilon} > 0$ such that $\iota_0^{w-\overline{\varepsilon},w+\overline{\varepsilon}}$ is an isomorphism. So, taking $\overline{\varepsilon}$ small enough that $w+\overline{\varepsilon} < v$, we deduce that $\operatorname{rank} \operatorname{im} \iota_0^{w-\overline{\varepsilon},v} = \operatorname{rank} \operatorname{im} \iota_0^{w+\overline{\varepsilon},v}$. Consequently, $\ell_{(X,\varphi)}(w+\overline{\varepsilon}, v) = \ell_{(X,\varphi)}(w-\overline{\varepsilon}, v)$. Hence, since size functions are non-decreasing in the first variable, we deduce *(i)*.

The proof of *(ii)* is analogous, considering the following commutative diagram, for $u < w$ and for $\varepsilon > 0$ such that $u < w-\varepsilon$:

$$\begin{tikzcd} \check{H}_0(X_u) \arrow[r, "\iota_0^{u,w-\varepsilon}"] \arrow[dr, "\iota_0^{u,w+\varepsilon}"'] & \check{H}_0(X_{w-\varepsilon}) \arrow[d, "\iota_0^{w-\varepsilon,w+\varepsilon}"] \\ & \check{H}_0(X_{w+\varepsilon}) \end{tikzcd} \tag{2}$$

since size functions are non-increasing in the second variable. □

Assuming the existence of at most a finite number of homological critical values, we can say that homological critical values are always coordinates of cornerpoints.

Proposition B.3. *Let (X,φ) be a size pair with at most a finite number of homological 0-critical values. If $w \in \mathbb{R}$ is a homological 0-critical value, then there is a cornerpoint for $\ell_{(X,\varphi)}$ having w has one of its coordinates.*

Proof. By [21, Lemma 3], if w is the coordinate of a horizontal or vertical discontinuity point for $\ell_{(X,\varphi)}$, then it is the coordinate of a cornerpoint. Hence, it is sufficient to prove that w is the coordinate of either a horizontal or a vertical discontinuity point for $\ell_{(X,\varphi)}$. To this end, we show that the following statements hold:

(i) If $\iota_0^{w-\varepsilon,w+\varepsilon}$ is not surjective for any sufficiently small positive real number ε, then there exists $v > w$ such that w is a discontinuity point for $\ell_{(X,\varphi)}(\cdot, v)$;

(ii) If $\iota_0^{w-\varepsilon,w+\varepsilon}$ is surjective for every sufficiently small positive real number ε, then there exists $u < w$ such that w is a discontinuity point for $\ell_{(X,\varphi)}(u, \cdot)$.

Let us prove *(i)*. Since we are assuming the presence of at most a finite number of homological 0-critical values for (X,φ), there surely exists $v > w$ such that, for every sufficiently small $\varepsilon > 0$, $v > w+\varepsilon$ and $\iota_0^{w+\varepsilon,v} : \check{H}_0(X_{w+\varepsilon}) \to \check{H}_0(X_v)$ is an isomorphism. Hence, looking at diagram (1) in the proof of Proposition B.2, we deduce that $\operatorname{rank} \operatorname{im} \iota_0^{w-\varepsilon,v} = \operatorname{rank} \operatorname{im} \iota_0^{w-\varepsilon,w+\varepsilon} < \operatorname{rank} \check{H}_0(X_{w+\varepsilon}) = \operatorname{rank} \operatorname{im} \iota_0^{w+\varepsilon,v}$, where the first equality holds because $\iota_0^{w+\varepsilon,v}$ is an isomorphism and the diagram is commutative, the inequality holds because $\operatorname{rank} \operatorname{im} \iota_0^{w-\varepsilon,w+\varepsilon} < +\infty$ and we are assuming $\iota_0^{w-\varepsilon,w+\varepsilon}$ not surjective, and the last equality holds again because $\iota_0^{w+\varepsilon,v}$ is an isomorphism. Hence, $\ell_{(X,\varphi)}(w-\varepsilon,v) < \ell_{(X,\varphi)}(w+\varepsilon,v)$, for every $\varepsilon > 0$ such that $v > w+\varepsilon$. Therefore, $\lim_{\varepsilon\to 0^+}\big(\ell_{(X,\varphi)}(w+\varepsilon,v) - \ell_{(X,\varphi)}(w-\varepsilon,v)\big) > 0$, that is, w is a discontinuity point for $\ell_{(X,\varphi)}(\cdot,v)$.

As for *(ii)*, since we are assuming the presence of at most a finite number of homological 0-critical values for (X,φ), there surely exists $u < w$ such that, for every sufficiently small $\varepsilon > 0$, $u < w-\varepsilon$ and $\iota_0^{u,w-\varepsilon} : \check{H}_0(X_u) \to \check{H}_0(X_{w-\varepsilon})$ is an isomorphism. Moreover, since $\iota_0^{w-\varepsilon,w+\varepsilon}$ is surjective, and w is a homological 0-critical value, it necessarily follows that $\iota_0^{w-\varepsilon,w+\varepsilon}$ is not injective. Therefore, referring to diagram (2) in the proof of Proposition B.2, we have that $\operatorname{rank}\operatorname{im}\iota_0^{u,w+\varepsilon} = \operatorname{rank}\operatorname{im}\iota_0^{w-\varepsilon,w+\varepsilon} < \operatorname{rank}\check{H}_0(X_{w-\varepsilon}) = \operatorname{rank}\operatorname{im}\iota_0^{u,w-\varepsilon}$, where the first equality holds because $\iota_0^{u,w-\varepsilon}$ is an isomorphism and the diagram is commutative, the inequality holds because $\operatorname{rank}\operatorname{im}\iota_0^{w-\varepsilon,w+\varepsilon} < +\infty$ and we are assuming $\iota_0^{w-\varepsilon,w+\varepsilon}$ not injective, and the last equality holds again because $\iota_0^{u,w-\varepsilon}$ is an isomorphism. Thus, it follows that $\ell_{(X,\varphi)}(u,w-\varepsilon) > \ell_{(X,\varphi)}(u,w+\varepsilon)$ for every $\varepsilon > 0$ such that $u+\varepsilon < w$, implying $\lim_{\varepsilon\to 0^+}\big(\ell_{(X,\varphi)}(u,w-\varepsilon) - \ell_{(X,\varphi)}(u,w+\varepsilon)\big) > 0$, that is, w is a discontinuity point for $\ell_{(X,\varphi)}(u,\cdot)$. $\square$

Dropping the assumption that the number of homological 0-critical values for (X,φ) is finite, the converse of Proposition B.2 is false, as the following remark states.

Remark B.4. *From the condition that w is a homological 0-critical value, it does not follow that w is a discontinuity point for the function $\ell_{(X,\varphi)}(\cdot,v)$, $v > w$, or for the function $\ell_{(X,\varphi)}(u,\cdot)$, $u < w$.*

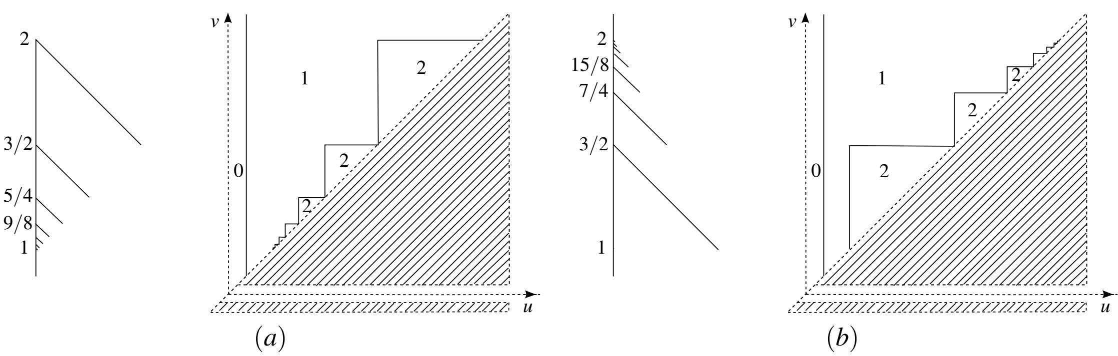

Figure 6: Two examples showing the existence of a real number w that is a homological 0-critical value for (X,φ) but not a discontinuity point for $\ell_{(X,\varphi)}(\cdot,v)$ or $\ell_{(X,\varphi)}(u,\cdot)$.

Two different examples, shown in Figure 6, support our claim.

Let us describe the first example (see Figure 6 (a)). Let (X,φ) be the size pair where X is the topological space obtained by adding an infinite number of branches to a vertical segment, each one sprouting at the height where the previous expires. These heights are chosen according to the sequence $(1+\frac{1}{2^n})_{n\in\mathbb{N}}$, converging to 1. The measuring function φ is the height function. The size function associated with (X,φ) is displayed on the right side of X. In this case, $w=1$ is a homological 0-critical value. Indeed, for $w=1$, it holds that $\operatorname{rank}\check{H}_0(X_{w-\varepsilon}) = 1$ while $\operatorname{rank}\check{H}_0(X_{w+\varepsilon}) = 2$, for every sufficiently small $\varepsilon > 0$. On the other hand, for every $v > w$, and for every small enough $\varepsilon > 0$, it holds that $\ell_{(X,\varphi)}(w+\varepsilon,v) = \ell_{(X,\varphi)}(w-\varepsilon,v) = 1$. Therefore, $\lim_{\varepsilon\to 0^+}\big(\ell_{(X,\varphi)}(w+\varepsilon,v) - \ell_{(X,\varphi)}(w-\varepsilon,v)\big) = 0$, for every $v > w$. Moreover, it is immediately verifiable that, for every $u < w$, $\lim_{\varepsilon\to 0^+}\big(\ell_{(X,\varphi)}(u,w-\varepsilon) - \ell_{(X,\varphi)}(u,w+\varepsilon)\big) = 0$.

The second example, shown in Figure 6 (b), is built in a similar way. In the chosen size pair (X,φ), φ is again the height function, and X is again obtained by adding an infinite number of branches to a vertical segment, but this time, the sequence of heights of their endpoints is $(2-\frac{1}{2^n})_{n\in\mathbb{N}}$, converging to 2. In this case, $w=2$ is a homological 0-critical value for (X,φ). Indeed, for every sufficiently small $\varepsilon > 0$, $\operatorname{rank}\check{H}_0(X_{w-\varepsilon}) = 2$ while $\operatorname{rank}\check{H}_0(X_{w+\varepsilon}) = 1$. On the other hand, for every $u < w$, and for every small enough $\varepsilon > 0$, it holds that $\ell_{(X,\varphi)}(u,w+\varepsilon) = \ell_{(X,\varphi)}(u,w-\varepsilon) = 1$ or $\ell_{(X,\varphi)}(u,w+\varepsilon) = \ell_{(X,\varphi)}(u,w-\varepsilon) = 0$. Therefore, $\lim_{\varepsilon\to 0^+}\left(\ell_{(X,\varphi)}(u,w-\varepsilon) - \ell_{(X,\varphi)}(u,w+\varepsilon)\right) = 0$, for every $u < w$, in both cases. Moreover, we can immediately verify that, for every $v > w$, $\lim_{\varepsilon\to 0^+}\left(\ell_{(X,\varphi)}(w+\varepsilon,v) - \ell_{(X,\varphi)}(w-\varepsilon,v)\right) = 0$.

C Appendix: Proof of Proposition 3.12

In this appendix we give the proof of Proposition 3.12. It requires some intermediate steps. We shall begin by showing that, for points $(u,v) \in \Delta^+$ where the size function of $A\cap B$ has no cornerpoints in the upper right region $\{(u',v') \in \Delta^+ :\ u \le u' \le v, v' > v\}$, the map $f_0 : \check{H}_0((A\cap B)_u) \to \check{H}_0((A\cap B)_v)$ induced by inclusion is necessarily surjective. Next, we prove that surjectivity of f_0 is a sufficient condition, ensuring that $\alpha_0^{v,u}$ is injective. Finally, showing that if $\ell_{(A\cap B,\varphi_{|A\cap B})}(u,v) \le 1$, then α_0^v is injective, we deduce the claim of Proposition 3.12.

The following proposition provides a condition for the surjectivity of the homomorphism between the 0th Čech homology groups induced by the inclusion map of X_u into X_v.

Proposition C.1. *Let (X,φ) be a size pair. For every $(u,v) \in \Delta^+$, $\iota_0^{u,v}$ is surjective if and only if $\ell_{(X,\varphi)}(u,v') = \ell_{(X,\varphi)}(v,v')$, for every $v' > v$.*

Proof. For every $(u,v) \in \Delta^+$, we denote by $\sim_v$ the equivalence relation on X_u such that, $P \sim_v Q$ if and only if P and Q belong to the same connected component of X_v.

For every $v' > v$, let $\frac{X_u}{\sim_{v'}}$ (respectively, $\frac{X_v}{\sim_{v'}}$) be the space obtained quotienting X_u (respectively, X_v) by the relation $\sim_{v'}$. Let us define the map $F_{v'} : \frac{X_u}{\sim_{v'}} \to \frac{X_v}{\sim_{v'}}$, such that $F_{v'}$ takes the class of P in $\frac{X_u}{\sim_{v'}}$ into the class of P in $\frac{X_v}{\sim_{v'}}$. $F_{v'}$ is well defined and injective, since $u < v < v'$. The condition that $\ell_{(X,\varphi)}(u,v') = \ell_{(X,\varphi)}(v,v')$ is equivalent to the bijectivity of $F_{v'}$.

Let $\iota_0^{u,v} : \check{H}_0(X_u) \to \check{H}_0(X_v)$ be surjective. This is equivalent to saying that, for every $P \in X_v$, there is $Q \in X_u$ with $P \sim_v Q$. Since $v < v'$, it also holds that $P \sim_{v'} Q$ and this implies $F_{v'}([Q]) = [P]$, for all $v' > v$. So, $F_{v'}$ is bijective and $\ell_{(X,\varphi)}(u,v') = \ell_{(X,\varphi)}(v,v')$, for every $v' > v$.

Conversely, let $F_{v'} : \frac{X_u}{\sim_{v'}} \to \frac{X_v}{\sim_{v'}}$ be a surjective map, for all $v' > v$. Let $P \in X_v$. Let (v_n) be a strictly decreasing sequence of real numbers converging to v. The surjectivety of F_{v_n} implies that $Q_n \in X_u$ exists, such that $F_{v_n}([Q_n]) = [P]$, for all $n \in \mathbb{N}$. Thus $P \sim_{v_n} Q_n$, for all $n \in \mathbb{N}$. Since X is compact and X_u is closed in X, there is a subsequence of (Q_n), still denoted by (Q_n), converging in X_u. Let $Q = \lim_{n\to\infty} Q_n \in X_u$. Then, necessarily, $P \sim_{v_n} Q$, for all n. In fact, let us call C_n the connected component of X_{v_n} containing P. Since (v_n) is decreasing, we have $C_n \supseteq C_{n+1}$ for every $n \in \mathbb{N}$. Let us assume that there exists $N \in \mathbb{N}$ such that $P \nsim_{v_N} Q$. Since C_N is closed, if $Q \notin C_N$, there exists an open neighborhood $U(Q)$ of Q, such that $U(Q)\cap C_N = \emptyset$. Thus, surely, there exists at least one point $Q_n \in U(Q)$, with $n > N$ and $Q_n \notin C_N$. This is a contradiction, because $Q_n \in C_n \subseteq C_N$, for all $n > N$.

Therefore, $P \sim_{v_n} Q$ for all n, and this implies that $P \sim_v Q$, because of [13, Rem. 3]. Hence, $\iota_0^{u,v} : \check{H}_0(X_u) \to \check{H}_0(X_v)$ is surjective. ☐

Remark C.2. *The condition that $\ell_{(X,\varphi)}(u,v') = \ell_{(X,\varphi)}(v,v')$, for every $v' > v$, can be restated saying that $\ell_{(X,\varphi)}$ has no points of horizontal discontinuity in the region $\{(x,y) \in \Delta^+ : u < x \le v,\ y > v\}$. In other words, the set $\{(x,y) \in \Delta^+ : u < x \le v,\ y > v\}$ does not contain any cornerpoint (either proper or at infinity) for $\ell_{(X,\varphi)}$.*

Lemma C.3. *Let $\alpha = \alpha^v_{0|\mathrm{im} f_0}$ and $\beta = \beta^v_{0|\mathrm{im} g_0}$. If f_0 is surjective, then $\mathrm{im}\,\alpha = \ker\beta$ and $\alpha_0^{v,u} = 0$.*

Proof. By Proposition 3.4*(ii)*, $\mathrm{im}\,\alpha \subseteq \ker\beta$, so we need to prove that $\ker\beta \subseteq \mathrm{im}\,\alpha$. Let $c \in \ker\beta \subseteq \ker\beta_0^v$. Since $\mathrm{im}\,\alpha_0^v = \ker\beta_0^v$, there exists $d \in \check{H}_0((A\cap B)_v)$ such that $\alpha_0^v(d) = c$. By hypothesis, f_0 is surjective, so $\check{H}_0((A\cap B)_v) = \mathrm{im} f_0$. Hence $d \in \mathrm{im} f_0$, implying $\alpha(d) = c$. Thus, $c \in \mathrm{im}\,\alpha$, and hence $\mathrm{im}\,\alpha = \ker\beta$.

Let us now show that $\alpha_0^{v,u}$ is trivial. By observing diagram (5), we see that f_0 is surjective if and only if f_0' is trivial. Since f_0' is surjective, it holds that f_0 is surjective if and only if $\check{H}_0((A\cap B)_v,(A\cap B)_u) = 0$. Therefore, if f_0 is surjective, then $\alpha_0^{v,u} = 0$. ☐

We are now ready to prove Proposition 3.12.

Proof. (of Proposition 3.12) Let us prove *(i)*. If $\ell_{(A\cap B,\varphi_{|A\cap B})}(u,v') = \ell_{(A\cap B,\varphi_{|A\cap B})}(v,v')$ for every $(v,v') \in \Delta^+$, applying Proposition C.1 with $A\cap B$ in place of X and f_0 in place of $\iota_0^{u,v}$, it follows that f_0 is surjective. Hence, by Lemma C.3, we have $\alpha_0^{v,u}$ trivial.

Let us now prove *(ii)*. From the assumption $\ell_{(A\cap B,\varphi_{|A\cap B})}(v,v') \le 1$, for every $(v,v') \in \Delta^+$, we deduce that either $(A\cap B)_v$ is empty or $(A\cap B)_v$ is non-empty and connected. If $(A\cap B)_v$ is empty, then $\check{H}_0((A\cap B)_v)$ is trivial and the claim is proved. Let us consider the case when $(A\cap B)_v$ is non-empty and connected. Let $z_0 = \{z_0(\mathscr{U}_{(A\cap B)_v})\} \in \check{H}_0((A\cap B)_v)$. If $z_0 \in \ker \alpha_0^v = \mathrm{im}\Delta_0^v$, for each $z_0(\mathscr{U}_{(A\cap B)_v}) \in H_0(\mathscr{U}_{(A\cap B)_v})$ there is a 1-chain $c_1(\mathscr{U}_{A_v})$ on A_v and a 1-chain $c_1(\mathscr{U}_{B_v})$ on B_v, such that the homology class of $\partial c_1(\mathscr{U}_{A_v}) = -\partial c_1(\mathscr{U}_{B_v})$ is equal to $z_0(\mathscr{U}_{(A\cap B)_v})$, up to homomorphisms induced by the inclusion. We now show that $\partial c_1(\mathscr{U}_{A_v})$ is a boundary on $(A\cap B)_v$. This will prove that $z_0(\mathscr{U}_{(A\cap B)_v})$ is trivial, yielding the injectivity of α_0^v. If $c_1(\mathscr{U}_{A_v}) = \sum_{i=1}^n a_i \cdot < U_i^0, U_i^1 >$, then $\partial c_1(\mathscr{U}_{A_v}) = \sum_{i=1}^n a_i \cdot U_i^1 - \sum_{i=1}^n a_i \cdot U_i^0$. From $\partial c_1(\mathscr{U}_{A_v}) = -\partial c_1(\mathscr{U}_{B_v})$, we deduce that, for $i = 1,\dots,n$, U_i^0 and U_i^1 have non-empty intersection with $(A\cap B)_v$. The connectedness of $(A\cap B)_v$ implies that there is a simple chain on $(A\cap B)_v$ connecting U_i^0 and U_i^1, for $i = 1,\dots,n$. Therefore $\partial c_1(\mathscr{U}_{A_v})$ is a boundary on $(A\cap B)_v$. □

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