One-Dimensional Reduction of Multidimensional Persistent Homology

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Abstract

A recent result on size functions is extended to higher homology modules: the persistent homology based on a multidimensional measuring function is reduced to a 1-dimensional one. This leads to a stable distance for multidimensional persistent homology. Some reflections on i-essentiality of homological critical values conclude the paper.

Keywords: Size function, measuring function, rank invariant, pattern recognition, i-essentiality.

1 Introduction

Topological Persistence started *ante litteram* in 1991 with P. Frosini, who introduced the concept of Size Function [12],[17, Sect. 8.4], a topological-geometrical tool for describing, analyzing and comparing shapes. This was actually the origin of rather large experimental research ([19, 20, 21]). Size functions were generalized by the same School in two directions: Size Homotopy Groups [14] (already in a multidimensional setting!) and Size Functor [2].

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At about the same time, Persistent Homology was independently introduced [10, 11] (see also [8, 9]). All these theories have substantially the same target: shape recognition. They are constructed on some topological features of lower level sets of a continuous real-valued function defined on the object of interest. They also share an important advantage with respect to other methods of pattern recognition: they capture qualitative aspects of shape in a formal quantitative way; so, they turn out to be particularly suited to the analysis of "natural" shapes (blood cells, signatures, gestures, melanocytic lesions, ...). Retrospectively, a size function is identifiable with the rank of a 0-th persistent homology module, while the first persistent homology module is the Abelianization of the first size homotopy group [14], and the size functor [2] is a functorial formalization of the direct sum of persistent homology modules.

The results obtained recently, involving the construction of size functions related to multidimensional measuring functions, lead us to the same generalization to persistent homology modules, which is the goal of this paper. As far as Size Theory is concerned, the main reason for such a generalization is that there are shape features, that have a multidimensional nature (such as color) and whose description can be done necessarily by a multidimensional measuring function. Moreover, there are shapes, which cannot be discriminated by n size functions related to n different real-valued measuring functions, but can be distinguished by the size function related to the n-dimensional measuring function of which those are the components (see Section 5). As mentioned in [15, Section 2.5], the study of multidimensional persistence has strong motivations, but some objective obstacles. This paper wants to pave a way out of these difficulties.

After recalling some basic notions about multidimensional size functions and 1-dimensional persistent homology in Section 2, we adapt the arguments of [1] to multidimensional persistent homology in Section 3, for proving our main result (Theorem 2). This is a reduction theorem, which takes the detection of discontinuity points back to the case of 1-dimensional persistent homology. This seems to overcome the pessimistic final considerations of [3, Section 6] on the structure of the functions $\rho_{X,i}$. In fact, although the sets, on which the functions are constant, are much more complicated than the triangles typical of the 1-dimensional case, they reduce to them when properly "sliced" by a suitable foliation. Stable distances on the leaves of the foliation define (and approximate) a global distance for rank invariants. Examples and further remarks on a different kind of reduction conclude the paper.

2 Basic notions

In the first part of this section we'll recall briefly the concept of multidimensional size functions and we'll state the theorem that gives us the tools to calculate them (Theorem 1). It asserts, indeed, that a suitable planes' foliation of a $2n$-dimensional real space makes an n-dimensional size function equal to a 1-dimensional in correspondence of each plane [1]. In the second part we shall review the definitions of persistent homology module and related concepts [5].

2.1 Multidimensional Size Functions and 1-dimensional reduction

In Multidimensional Size Theory, any pair $(X, \vec{f})$, where X is a non-empty compact and locally connected Hausdorff space, and $\vec{f} = (f_1, \dots, f_n) : X \to \mathbb{R}^n$ is a continuous function, is called a *size pair*. The function $\vec{f}$ is called an *n-dimensional measuring function*. The following relations $\preceq$ and $\prec$ are defined in $\mathbb{R}^n$: for $\vec{u} = (u_1, \dots, u_n)$ and $\vec{v} = (v_1, \dots, v_n)$, we say $\vec{u} \preceq \vec{v}$ (resp. $\vec{u} \prec \vec{v}$) if and only if $u_j \leq v_j$ (resp. $u_j < v_j$) for every index $j = 1, \dots, n$. For every n-tuple $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$, let $X\langle \vec{f} \preceq \vec{u} \rangle$ be the set $\{P \in X : f_j(P) \leq u_j, j = 1, \dots, n\}$ and let Δ^+ be the open set $\{(\vec{u}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n : \vec{u} \prec \vec{v}\}$.

Definition 1. *For every n-tuple $\vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$, we say that two points $P, Q \in X$ are* $\langle \vec{f} \preceq \vec{v} \rangle$-connected *if and only if a connected subset of $X\langle \vec{f} \preceq \vec{v} \rangle$ exists, containing P and Q.*

Definition 2. *The* (n-dimensional) size function *associated with the size pair $(X, \vec{f})$ is the function $\ell_{(X,\vec{f})} : \Delta^+ \to \mathbb{N}$, defined by setting $\ell_{(X,\vec{f})}(\vec{u}, \vec{v})$ equal to the number of equivalence classes in which the set $X\langle \vec{f} \preceq \vec{u} \rangle$ is divided by the $\langle \vec{f} \preceq \vec{v} \rangle$-connectedness relation.*

An analogous definition for multidimensional persistent homology will be given in Definition 7.

The main goal of [1] for size functions, and of the present paper for persistent homology, is to reduce computation from the multidimensional to the 1-dimensional case. This is possible through particular foliations of $\mathbb{R}^n$ by half-planes. They are determined by what are called "admissible" vector pairs.

Definition 3. *For every unit vector $\vec{l} = (l_1, \dots, l_n)$ in $\mathbb{R}^n$ such that $l_j > 0$ for $j = 1, \dots, n$, and for every vector $\vec{b} = (b_1, \dots, b_n)$ in $\mathbb{R}^n$ such that $\sum_{j=1}^{n} b_j = 0$, we shall say that the pair $(\vec{l}, \vec{b})$ is* admissible. *We shall denote the set of all admissible pairs in $\mathbb{R}^n \times \mathbb{R}^n$ by Adm_n. Given* *an admissible pair* $(\vec{l},\vec{b})$*, we define the half-plane* $\pi_{(\vec{l},\vec{b})}$ *in* $\mathbb{R}^n \times \mathbb{R}^n$ *by the following parametric equations:*

$$\begin{cases} \vec{u} = s\vec{l} + \vec{b} \\ \vec{v} = t\vec{l} + \vec{b} \end{cases}$$

for $s, t \in \mathbb{R}$*, with* $s < t$*.*

The motivation for the previous definition is the fact that for every $(\vec{u},\vec{v}) \in \Delta^+$ there exists exactly one admissible pair $(\vec{l},\vec{b})$ such that $(\vec{u},\vec{v}) \in \pi_{(\vec{l},\vec{b})}$ [1, Prop.1]. The following Lemma is substantially contained in the proof of [1, Thm. 3].

Lemma 1. *Let* $(\vec{l},\vec{b})$ *be an admissible pair and* $g : X \to \mathbb{R}$ *be defined by setting*

$$g(P) = \max_{j=1,\dots,n} \left\{ \frac{f_j(P) - b_j}{l_j} \right\}.$$

Then, for every $(\vec{u},\vec{v}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$*, the following equalities hold:*

$$X\langle \vec{f} \preceq \vec{u} \rangle = X\langle g \leq s \rangle, \qquad X\langle \vec{f} \preceq \vec{v} \rangle = X\langle g \leq t \rangle$$

Proof. For every $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$, with $u_j = sl_j + b_j, j = 1, \dots, n$, it holds that

$$\begin{aligned} X\langle \vec{f} \preceq \vec{u} \rangle &= \{P \in X : f_j(P) \leq u_j, j = 1, \dots, n\} \\ &= \{P \in X : f_j(P) \leq sl_j + b_j, j = 1, \dots, n\} \\ &= \{P \in X : \frac{f_j(P) - b_j}{l_j} \leq s, j = 1, \dots, n\} \\ &= X\langle g \leq s \rangle \end{aligned}$$

Analogously, for every $\vec{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$, with $v_j = tl_j + b_j, j = 1, \dots, n$, it holds that $X\langle \vec{f} \preceq \vec{v} \rangle = X\langle g \leq t \rangle$. □

From that, there follows the main theorem of [1]:

Theorem 1. *Let* $(\vec{l},\vec{b})$ *and* g *be defined as in Lemma 1. Then the equality*

$$\ell_{(X,\vec{f})}(\vec{u},\vec{v}) = \ell_{(X,g)}(s,t)$$

holds for every $(\vec{u},\vec{v}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l},\vec{b})}$*.*

This is indeed the theorem that we are going to extend, in Section 3, to persistent homology of all degrees. Its importance resides in the fact that essential discontinuity points ("cornerpoints" in the terminology of Size Theory) are the key to a stable distance between size functions. Unfortunately, cornerpoints do not form, in general, discrete sets in the multidimensional case. This theorem makes it possible to find them "slice by slice" with the familiar technique of dimension one. A practical use is for sampling their sets, so getting bounds for a stable distance between size functions. Our extension will produce the same opportunity for persistent homology.

2.2 1-dimensional Persistent Homology

Given a topological space X and an integer i, we denote the i-th singular homology module of X over a field k by $H_i(X)$.

Next we report two definitions of [5].

Definition 4. *Let X be a topological space and f a real function on X. A* homological critical value *of f is a real number a for which there exists an integer i such that, for all sufficiently small $\varepsilon > 0$, the map $H_i(f^{-1}(-\infty, a-\varepsilon]) \to H_i(f^{-1}(-\infty, a+\varepsilon])$ induced by inclusion is not an isomorphism.*

This is called an *i-essential critical value* in the paper [2, Def.2.6], dedicated to the *size functor*, a contemporary and not too different homological generalization of size functions.

Definition 5. *A function $f : X \to \mathbb{R}$ is* tame *if it has a finite number of homological critical values and the homology modules $H_i(f^{-1}(-\infty, a])$ are finite-dimensional for all $i \in \mathbb{Z}$ and $a \in \mathbb{R}$.*

The reader should be warned that there exist other, different meanings of "tame" in the current topological literature. Actually, "homologically tame" might be a better designation for such a type of function, but we adhere to this already current definition.

We write $F_i^u = H_i(f^{-1}(-\infty, u])$, for all $i \in \mathbb{Z}$, and for $u < v$, we let $f_i^{u,v} : F_i^u \to F_i^v$ be the map induced by inclusion of the lower level set of u in that of v, for a fixed integer i. Moreover, we indicate with $F_i^{u,v} = \mathrm{Im} f_i^{u,v}$ the image of F_i^u in F_i^v, that is called *i-th persistent homology module*.

3 Homological 1-dimensional reduction

In this section we define the i-th persistent homology module related to a continuous n-dimensional real function (substantially as in [3]). Then we show that the sets of points of $\mathbb{R}^{2n}$, where the modules change, can be obtained by computing the discontinuity points of persistent homology of a 1-dimensional function defined on particular half-planes which foliate the $2n$-space.

The first issue arises when one tries to compute the maximum between the components of a n-dimensional real function. In fact:

Remark 1. *The maximum of two tame functions is not necessarily a tame function.*

(We recall that "tame" has the meaning defined in Section 2.2.)

As an example, let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ be two tame functions defined as

$$f_1(u,v) = \begin{cases} v - u^2\sin(\frac{1}{u}) & u \neq 0 \\ v & u = 0 \end{cases} \quad , \quad f_2(u,v) = \begin{cases} -v - u^2\sin(\frac{1}{u}) & u \neq 0 \\ -v & u = 0 \end{cases}$$

and consider the function

$$f = \max(f_1, f_2).$$

Then, as we can see in Figure 3, f is not tame, since $H_0(f^{-1}(-\infty, 0])$ is an infinite-dimensional module.

Given this fault related to tame functions, the solution we propose is to introduce the following concept.

Definition 6. *Let X be a topological space and $\vec{f} : X \to \mathbb{R}^n$ a continuous function on X. We shall say that $\vec{f}$ is* max-tame *if, for every admissible pair $(\vec{l}, \vec{b})$, the function $g(P) = \max\limits_{j=1,\ldots,n} \left\{ \frac{f_j(P) - b_j}{l_j} \right\}$ is tame.*

Choosing a measuring function on X as above, let us define the multidimensional persistent modules.

Definition 7. *Let $\vec{f} : X \to \mathbb{R}^n$ be a max-tame function. For each homology degree $i \in \mathbb{Z}$ we put $F_i^{\vec{u}} = H_i(\vec{f}^{-1}(\prod\limits_{j=1}^{n}(-\infty, u_j]))$, for all $\vec{u} \in \mathbb{R}^n$. For $\vec{u} \preceq \vec{v}$ we let $f_i^{\vec{u},\vec{v}} : F_i^{\vec{u}} \to F_i^{\vec{v}}$ be the map induced by inclusion of the lower level set of $\vec{u}$ in that of $\vec{v}$, for a fixed integer i, and call $F_i^{\vec{u},\vec{v}} = \mathrm{Im}\ f_i^{\vec{u},\vec{v}}$ the i-th* multidimensional persistent homology module.

Note that the rank of $F_i^{\vec{u},\vec{v}}$ is what is called $\rho_{X,i}(\vec{u}, \vec{v})$ in [3, Def. 12].

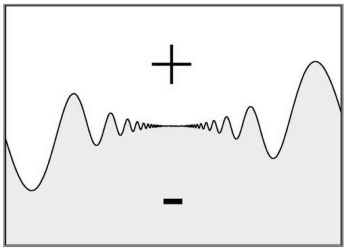


Figure 1: Lower level set of f_1 (grey area - one connected component).

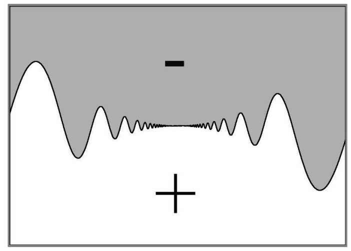


Figure 2: Lower level set of f_2 (grey area - one connected component).

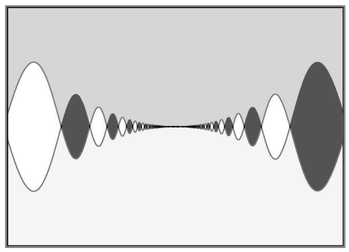


Figure 3: Lower level set of f (dark zone - infinitely many connected components).

Let $g(P) = \max\limits_{j=1,\ldots,n} \left\{ \frac{f_j(P)-b_j}{l_j} \right\}$ for a fixed $(\vec{l}, \vec{b}) \in Adm_n$, $G_i^s = H_i(g^{-1}(-\infty, s])$, for all $s \in \mathbb{R}$ and $i \in \mathbb{Z}$. For $s < t$, we let $g_i^{s,t} : G_i^s \to G_i^t$ be the map induced by inclusion of the lower level set of s in that of t, for a fixed integer i, and denote $G_i^{s,t} = \mathrm{Im}\ g_i^{s,t}$ the i-th persistent homology module.

Now we can state and prove the theorem which, in analogy with the main result of [1], enables us to reduce the computation of multidimensional persistent homology to the 1-dimensional one. This is important, not so much for finding the homology modules themselves point by point, but much more for finding points of change of the modules.

Theorem 2. *Let $(\vec{l}, \vec{b})$ be an admissible pair and $\vec{f} = (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ a max-tame function. Then, for every $(\vec{u}, \vec{v}) = (s\vec{l} + \vec{b}, t\vec{l} + \vec{b}) \in \pi_{(\vec{l}, \vec{b})}$, the following equality*

$$F_i^{\vec{u},\vec{v}} = G_i^{s,t}$$

holds for all $i \in \mathbb{Z}$ and $s, t \in \mathbb{R}$ with $s < t$.

Proof. By Lemma 1, we know that, for every $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$, with $u_j = sl_j + b_j, j =$

$1, \ldots, n$, it holds that

$$\{P \in X, f_j(P) \leq u_j, j = 1, \ldots, n\} = \{P \in X, g(P) \leq s\}$$

hence

$$\{P \in X, P \in {f_j}^{-1}(-\infty, u_j], j = 1, \ldots, n\} = \{P \in X, P \in g^{-1}(-\infty, s]\}.$$

It follows that

$$\bigcap_{j=1}^{n} {f_j}^{-1}(-\infty, u_j] = g^{-1}(-\infty, s]$$

implying

$$H_i\left(\bigcap_{j=1}^{n} {f_j}^{-1}(-\infty, u_j]\right) = H_i(g^{-1}(-\infty, s])$$

for all $i \in \mathbb{Z}$.

Analogously, for every $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$, with $v_j = tl_j + b_j, j = 1, \ldots, n$, it holds that $H_i\left(\bigcap_{j=1}^{n} {f_j}^{-1}(-\infty, v_j]\right) = H_i(g^{-1}(-\infty, t])$, for all $i \in \mathbb{Z}$. So, since $f_i^{\vec{u},\vec{v}}$ and $g_i^{s,t}$ have the same domain and codomain and they are the maps induced by inclusion, we can conclude that $f_i^{\vec{u},\vec{v}} = g_i^{s,t}$, and the claim follows. $\square$

4 Multidimensional matching distance

According to [3, Def. 12], for a given measuring function $\vec{f}' : X \to \mathbb{R}^n$, for each homology degree $i \in \mathbb{Z}$ the rank invariant $\rho'_{X,i} : \Delta^+ \to \mathbb{N}$ is defined as $\rho'_{X,i}(\vec{u}, \vec{v}) = rank(F_i^{\vec{u},\vec{v}})$.

Let $(X, \vec{f}')$, $(Y, \vec{f}'')$ be two size pairs, where $\vec{f}' : X \to \mathbb{R}^n$, $\vec{f}'' : Y \to \mathbb{R}^n$ are max-tame measuring functions, and $\overline{\rho}'_{X,i}$, $\overline{\rho}''_{Y,i}$ be the respective rank invariants. Let an admissible pair $(\vec{l}, \vec{b})$ be fixed, and let $g' : X \to \mathbb{R}$, $g'' : Y \to \mathbb{R}$ be defined by setting

$$g'(P) = \max_{j=1,\ldots,n} \left\{\frac{f'_j(P) - b_j}{l_j}\right\} \qquad g''(P) = \max_{j=1,\ldots,n} \left\{\frac{f''_j(P) - b_j}{l_j}\right\}$$

It is well-known for 1-dimensional measuring functions [18, 13, 5] that the relevant information on the rank invariants $\rho'_{X,i}$, $\rho''_{Y,i}$ of g' and g'' respectively is contained, for each degree i, in their multisets of cornerpoints, which are called "persistence diagrams". These are sets of points of the extended plane with multiplicities, augmented by adding a countable infinity of points of the diagonal $y = x$: let them be called respectively C' and C''. Each cornerpoint is determined by its coordinates $x < y \leq \infty$. The distance of two cornerpoints is

$$\delta\left((a,b),(c,d)\right) = \min\left\{\max\{|a-c|,|b-d|\},\max\left\{\frac{b-a}{2},\frac{d-c}{2}\right\}\right\}$$

It has been proved in [5] that the *matching* (or *bottleneck*) distance

$$d(\rho'_{X,i},\rho''_{Y,i}) = \min_{\sigma}\max_{P\in C'}\delta(P,\sigma(P))$$

where σ varies among all bijections from C' to C'', is stable. Mimicking [1] (and recalling that $\rho'_{X,i},\rho''_{Y,i}$ vary with $(\vec{l},\vec{b})$) we can use d to define distances between the rank invariants of the original multidimensional persistent homologies.

Definition 8. *Let $(X,\vec{f}')$, $(Y,\vec{f}'')$ be two size pairs and $\overline{\rho}'_{X,i}$, $\overline{\rho}''_{Y,i}$ be the respective rank invariants. Then the i-th* multidimensional matching distance *between rank invariants is defined as the extended distance*

$$D(\overline{\rho}'_{X,i},\overline{\rho}''_{Y,i}) = \sup_{(\vec{l},\vec{b})\in Adm_n}\ \min_{j=1,\ldots,n} l_j\cdot\ d(\rho'_{X,i},\rho''_{Y,i})$$

Note that D is by construction a global distance, i.e. not depending on $(\vec{l},\vec{b})$, but since the coefficients l_j are ≤ 1, there might be distances d, for particular admissible pairs, which take greater values. An easy corollary of our Theorem 2 is the following, which is the higher degree version of [1, Cor. 1].

Corollary 1. *For each $i\in\mathbb{Z}$ the identity $\overline{\rho}'_{X,i}\equiv\overline{\rho}''_{Y,i}$ holds if and only if $d(\rho'_{X,i},\rho''_{Y,i})=0$ for every admissible pair $(\vec{l},\vec{b})$.*

With the same argument of the analogous Proposition 4 of [1], it is easy to prove the following inequality between the multidimensional matching distance and the 1-dimensional one obtained by considering the components of the measuring functions. That this inequality can be strict, is shown in Section 5.

Proposition 1. *Let $(X,\vec{f}),(Y,\vec{h})$ be size two pairs with $\vec{f}=(f_1,\ldots,f_n)$, $\vec{h}=(h_1,\ldots,h_n)$ max-tame measuring functions. For each $i\in\mathbb{Z}$ and for each $j=1,\ldots,n$ let $\rho^{f_j}_{X,i},\rho^{h_j}_{Y,i}$ be the i-th rank invariants relative to the components f_j, h_j respectively; let then $\overline{\rho}'_{X,i}$, $\overline{\rho}''_{Y,i}$ be the rank invariants relative to $\vec{f}$, $\vec{h}$ respectively. Then it holds that*

$$d(\rho^{f_j}_{X,i},\rho^{h_j}_{Y,i})\ \leq\ D(\overline{\rho}'_{X,i},\overline{\rho}''_{Y,i})$$

The matching distance is known to be stable with respect to perturbation of 1-dimensional measuring functions [5, Section 3.1] [7, Thm. 25]. In the multidimensional setting, the stability of d with respect to an admissible pair is stated in the following proposition, whose proof is again a copy of that of [1, Prop. 2]. Here $g', g'' : X \to \mathbb{R}$ are defined in correspondence to $(\vec{l}, \vec{b})$ as at the beginning of this Section.

Proposition 2. *If $(X, \vec{f}')$, $(X, \vec{f}'')$ are size pairs, with max-tame functions $\vec{f}', \vec{f}'' : X \to \mathbb{R}^n$, and $\max_{P \in X} \|\vec{f}'(P) - \vec{f}''(P)\|_\infty \leq \epsilon$, then for every admissible pair $(\vec{l}, \vec{b})$ and for each $i \in \mathbb{Z}$ it holds that*

$$d(\rho'_{X,i}, \rho''_{X,i}) \leq \frac{\epsilon}{\min_{j=1,\dots,n} \ l_j}$$

with $\vec{l} = (l_1, \dots, l_n)$ and where $\rho'_{X,i}$, $\rho''_{X,i}$ are the rank invariants at degree i of (X, g'), (X, g'') respectively.

By the definition of D, every 1-dimensional matching distance obtained in correspondence of an admissible pair yields a lower bound for the multidimensional matching distance D; a sufficiently fine sampling by admissible pairs produces approximations of arbitrary precision of it.

Of course, $\max\limits_{i \in \mathbb{Z}} D(\overline{\rho}'_{X,i}, \overline{\rho}''_{Y,i})$ is still a meaningful distance related to the size pairs $(X, \vec{f}')$, $(Y, \vec{f}'')$. We plan to study its relation with the n-dimensional natural pseudodistance of [14, 1].

5 Examples and Remarks

We now describe a simple example, which shows that persistent homology, with respect to a multidimensional measuring function, is actually stronger than the simple collection of the persistent homologies with respect to its 1-dimensional components.

In $\mathbb{R}^3$ consider the set $\Omega = [-1, 1] \times [-1, 1] \times [-1, 1]$ and the sphere $\mathcal{S}$ of equation $u^2 + v^2 + w^2 = 1$. Let also $\vec{f} = (f_1, f_2) : \mathbb{R}^3 \to \mathbb{R}^2$ be a continuous function, defined as $\vec{f}(u, v, w) = (|u|, |v|)$. In this setting, consider the size pairs $(\mathcal{C}, \vec{\varphi})$ and $(\mathcal{S}, \vec{\psi})$, where $\mathcal{C} = \partial\Omega$ and $\vec{\varphi}$ and $\vec{\psi}$ are respectively the restrictions of $\vec{f}$ to $\mathcal{C}$ and $\mathcal{S}$.

In order to compare the persistent homology modules of $\mathcal{C}$ and $\mathcal{S}$ defined by $\vec{f}$, we are interested in studying the half-planes' foliation of $\mathbb{R}^4$, where $\vec{l} = (\cos\theta, \sin\theta)$ with $\theta \in (0, \frac{\pi}{2})$, and $\vec{b} = (a, -a)$ with $a \in \mathbb{R}$. Any such half-plane is parameterized as

$$\begin{cases} u_1 = s\cos\theta + a \\ u_2 = s\sin\theta - a \\ v_1 = t\cos\theta + a \\ v_2 = t\sin\theta - a \end{cases}$$

with $s, t \in \mathbb{R}, s < t$.

In the following, we shall always assume $0 \leq s < t$.

For example, by choosing $\theta = \frac{\pi}{4}$ and $a = 0$, i.e. $\vec{l} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $\vec{b} = (0, 0)$, we obtain that

$$g' = \sqrt{2}\max\{\varphi_1, \varphi_2\} = \sqrt{2}\max\{|u|, |v|\},$$

$$g'' = \sqrt{2}\max\{\psi_1, \psi_2\} = \sqrt{2}\max\{|u|, |v|\}.$$

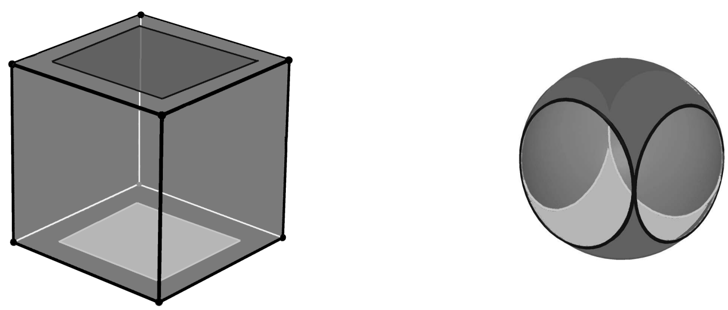
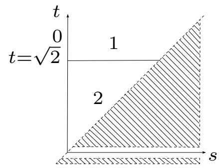


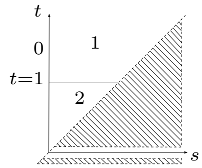
Figure 4: Lower level sets $g' \leq 1$ and $g'' \leq 1$.

Let $\rho'_{\mathcal{C},i}$ $\rho''_{\mathcal{S},i}$ be the rank invariants of the respective persistent homologies for $i \in \mathbb{Z}$. So, writing $G_i^s(\mathcal{C}) = H_i\left((g')^{-1}(-\infty, s]\right)$, $G_i^s(\mathcal{S}) = H_i\left((g'')^{-1}(-\infty, s]\right)$ and $G_i^{s,t}$ defined as above, we obtain that

$$G_0^{s,t}(\mathcal{C}) = \begin{cases} 0, & s,t < 0 \\ k^2, & 0 \le s < t < \sqrt{2} \\ k, & \text{otherwise} \end{cases}$$

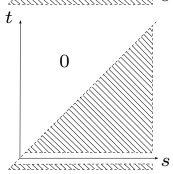


$$G_0^{s,t}(\mathcal{S}) = \begin{cases} 0, & s,t < 0 \\ k^2, & 0 \le s < t < 1 \\ k, & \text{otherwise} \end{cases}$$

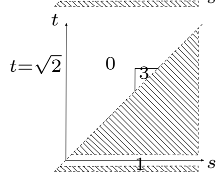


$$\Rightarrow D(\rho_{\mathcal{C},0}, \rho_{\mathcal{S},0}) \ge \tfrac{\sqrt{2}}{2} d(\rho'_{\mathcal{C},0}\ \rho''_{\mathcal{S},0}) = \tfrac{\sqrt{2}}{2}(\sqrt{2}-1)$$

$$G_1^{s,t}(\mathcal{C}) = \quad 0, \quad \text{for all}\, s,t \in \mathbb{R}$$

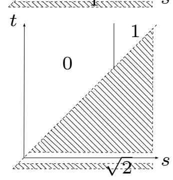


$$G_1^{s,t}(\mathcal{S}) = \begin{cases} k^3, & 1 \le s < t < \sqrt{2} \\ 0, & \text{otherwise} \end{cases}$$

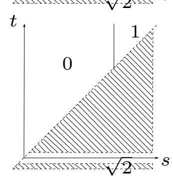


$$\Rightarrow D(\rho_{\mathcal{C},1}, \rho_{\mathcal{S},1}) \ge \tfrac{\sqrt{2}}{2} d(\rho'_{\mathcal{C},1}\ \rho''_{\mathcal{S},1}) = \tfrac{\sqrt{2}}{2}\left(\tfrac{\sqrt{2}-1}{2}\right)$$

$$G_2^{s,t}(\mathcal{C}) = \begin{cases} k, & \sqrt{2} \le s < t \\ 0, & \text{otherwise} \end{cases}$$



$$G_2^{s,t}(\mathcal{S}) = \begin{cases} k, & \sqrt{2} \le s < t \\ 0, & \text{otherwise} \end{cases}$$



$$\Rightarrow D(\rho_{\mathcal{C},2}, \rho_{\mathcal{S},2}) \ge \tfrac{\sqrt{2}}{2} d(\rho'_{\mathcal{C},2}\ \rho''_{\mathcal{S},2}) = 0$$

In other words, multidimensional persistent homology, with respect to $\vec{\varphi}$ and $\vec{\psi}$, is able to discriminate the cube and the sphere, while the 1-dimensional one, with respect to φ_1, φ_2 and ψ_1, ψ_2, cannot do that. In fact, for either manifold the lower level sets of the single components (i.e. 1-dimensional measuring functions) are homeomorphic for all values: they are topologically either circles, or annuli, or spheres.

It should be noted that the map g'' on $\mathcal{S}$ reaches the homological critical value 1 at points, at which it lacks of differentiability.

In the example above, $\vec{\varphi}$ is not a Morse function (as would be desirable, if not necessary), because $g' : \mathcal{C} \to \mathbb{R}$ has infinitely many critical points when $\max\{|u|, |v|\} = 1$; moreover, the cubic surface itself is not even $\mathcal{C}^1$. This problem can be solved by perturbing $\mathcal{C}$ so that it becomes smooth (e.g. a super-quadric [16]). In this case, the differences between homology modules of the cube and of the super–quadric are only quantitative (i.e. the levels of homological critical values are different from one another).

An even simpler example can be given on size pairs having the same support. Let X be the ellipse imbedded in $\mathbb{R}^3$ as $\begin{cases} u^2+v^2=1 \\ v=w \end{cases}$ — or parameterized as $\begin{cases} u=\cos\theta \\ v=\sin\theta \\ w=\sin\theta \end{cases}$. Let $\varphi_1,\varphi_2,\psi_1,\psi_2 : X \to \mathbb{R}$ be defined as $\varphi_1=u,\ \psi_1=v,\ \varphi_2=\psi_2=w$ and $\vec{\varphi}=(\varphi_1,\varphi_2),\ \vec{\psi}=(\psi_1,\psi_2)$. Then the persistent homology modules of (X,φ_1), (X,ψ_1), $(X,\varphi_2)=(X,\psi_2)$ are identical, while the persistent homology (in degree zero, so the size function) of $(X,\vec{\varphi})$ differs from the one of $(X,\vec{\psi})$. Indeed, while the lower level sets of $\vec{\psi}$ are always either empty or connected, the lower level sets $\vec{\varphi} \le (\overline{u},\overline{w})$, with $0<\overline{u}<1, \sqrt{1-\overline{u}^2} \le \overline{w} < 1$ consist of two connected components.

6 Reduction of i-essential critical values

The former example of the previous section suggests also some other considerations on the cooperation of measuring functions. We remind that the adjective "essential" is used here with the meaning introduced in [2] and recalled in Section 2.2 after Definition 4, so otherwise than, e.g., in [4].

A first remark is that, although the persistent homology on single components of $\vec{f}$ cannot distinguish the two spaces, the persistent homology on f_1 restricted to lower level sets of f_2 can, as can be shown as follows. Consider again the sphere $\mathbb{S}$. The value $1/\sqrt{2}$ (corresponding to the homological critical value 1 of g'') is not critical for the maps f_1, f_2 on $\mathbb{S}$ itself, but it is indeed critical for f_2 restricted to $f_1^{-1}(-\infty, 1/\sqrt{2}]$. We believe that homological critical values of the 1D reduction of multidimensional measuring functions are always clues of such phenomena.

A further speculation on the use of cooperating measuring functions — from a completely different viewpoint than the one developed in the previous sections — is the following. A problem in 1-dimensional persistent homology, as well as for the size functor, is the computation of i-essential critical values for $i>0$. A possibility is the use of several, independent measuring functions for lowering i, i.e. the degree at which the passage through the critical value causes a homology change. Lowering i is important, since 0-essential critical values are easily detected by graph-theoretical techniques [6]. The following example shows that a suitable choice of a second, auxiliary measuring function may actually take 1-essential critical values to 0-essential ones.

Let $\mathcal{T}$ be a torus of revolution around the x axis, with the innermost parallel circle of radius 2, the outermost of radius 3. On $\mathcal{T}$ define $(f_1, f_2) = (z, -z)$. Suppose we are interested in the persistent homology of the size pair $(\mathcal{T}, f_1)$. Then $(0, 0, 2)$ is a 1-essential critical point for f_1, i.e. it is a point at which 1-degree homology changes. Of course, there are computational methods (e.g. by the Euler–Poincaré characteristic) which enable us to detect it, but they will probably be tailored to the particular dimension of the manifold and to the particular homology degree.

The same point is 0-essential for its restriction to $f_2^{-1}(-\infty, 1]$, so it can be recovered by the standard graph-theoretical techniques used in degree 0, i.e. for size functions. (The two functions need not be so strictly related: f_2 could be replaced by Euclidean distance from $(0, 0, 3)$ with the same effect). We conjecture that — at least whenever torsion is not involved — one can recursively take the i-essential values of a measuring function to $(i-1)$-essential ones, down to (easily computable) 0-essential critical values by means of other (auxiliary) measuring functions, as in this example.

7 Conclusions and future work

The need of extending persistent homology to the multidimensional case is a rather widespread belief, confirmed by simple examples. The present research shows the possibility of reducing the computation of persistent homology, with respect to multidimensional measuring functions, to the 1-dimensional case, following the line of thought of an analogous extension devised for size functions in [1]. This reduction also yields a stable distance for the rank invariants of size pairs.

In the next future, we plan to characterize the multidimensional max-tame measuring functions in a way that the reduction to 1D case makes the specific features of persistent homology modules hold steady. It also would be our concern to give a rigorous definition of *multidimensional homological critical values* of a max-tame function and to relate them to the homological critical values of the maximum of its components.

Eventually, in relation to our conjecture about i-essentiality (see Section 6), we plan to build an algorithm to recursively reduce i-essential critical points of a measuring function to 0-essential ones.

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