Regularity near the Initial State in the Obstacle Problem for a class of Hypoelliptic Ultraparabolic Operators

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Abstract

This paper is devoted to a proof of regularity, near the initial state, for solutions to the Cauchy-Dirichlet and obstacle problem for a class of second order differential operators of Kolmogorov type. The approach used here is general enough to allow us to consider smooth obstacles as well as non-smooth obstacles.

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1 Introduction

In this paper we continue the study in [10] concerning regularity of solutions to the obstacle problem for a class of second order differential operators of Kolmogorov type of the form

$$L = \sum_{i,j=1}^{m} a_{ij}(x,t)\partial_{x_i x_j} + \sum_{i=1}^{m} b_i(x,t)\partial_{x_i} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t \tag{1.1}$$

where $(x,t) \in \mathbb{R}^{N+1}$, m is a positive integer satisfying $m \le N$, the functions $\{a_{ij}(\cdot,\cdot)\}$ and $\{b_i(\cdot,\cdot)\}$ are continuous and bounded and $B = \{b_{ij}\}$ is a matrix of constant real numbers. Let $\Omega \subset \mathbb{R}^{N+1}$ be an open subset, let $\partial_P \Omega$ denote the *parabolic* boundary of Ω, let $g, f, \psi : \bar{\Omega} \to \mathbb{R}$ be such that $g \ge \psi$ on $\bar{\Omega}$ and assume that g, f, ψ are continuous and bounded on $\bar{\Omega}$. We consider the following obstacle problem for the operator L,

$$\begin{cases} \max\{Lu(x,t) - f(x,t), \psi(x,t) - u(x,t)\} = 0, & \text{in } \Omega, \\ u(x,t) = g(x,t), & \text{on } \partial_P \Omega. \end{cases} \tag{1.2}$$

The structural assumptions imposed on the operator L, which will imply that L is a hypoelliptic ultraparabolic operator of Kolmogorov type, as well as the regularity assumptions on a_{ij}, b_i, f, ψ and g will be given and discussed below. We note that in case $m = N$ the assumptions we impose imply that the operator L is uniformly parabolic, while if $m < N$ the operator L is strongly degenerate. We are mainly interested in the case $m < N$.

To motivate our study we note that the problem (1.2) occurs in mathematical finance and in particular in the pricing of options of American type. More precisely, consider a financial model where the dynamics of the state variables is described by a N-dimensional diffusion process $X = \left(X_t^{x_0,t_0}\right)$ which is a solution to the stochastic differential equation

$$dX_t^{x_0,t_0} = BX_t^{x_0,t_0} + \sigma(X_t^{x_0,t_0}, t)dW_t, \qquad X_{t_0}^{x_0,t_0} = x_0, \tag{1.3}$$

where $(x_0, t_0) \in \mathbb{R}^N \times [0,T]$ and $W = \{W_t\}$ denotes a m-dimensional Brownian motion, $m \le N$, on a filtered space. An American option with pay-off ψ is a contract which gives the holder the right to receive a payment equal to $\psi(X_\tau)$ assuming that the holder choose to exercise the option at $\tau \in [0,T]$. By the classical arbitrage theory (see, for instance, [2]) the fair price of the American option, assuming that the risk-free interest rate is zero, is given by the solution to the optimal stopping problem

$$U(x,t) = \sup_{t \le \tau \le T} \mathbb{E}[\psi(X_\tau^{x,t})], \tag{1.4}$$

where the supremum is taken with respect to all stopping times $\tau \in [t,T]$. The main result in [16] states that if u is a solution to a problem in the form (1.2), with $f \equiv 0$, $g \equiv \psi$ and $\Omega = \mathbb{R}^N \times [0,T]$, then $u(x,t) = U(x, T-t)$. In this case the operator L is the Kolmogorov operator associated to X, that is

$$L = \frac{1}{2}\sum_{i,j=1}^{m} (\sigma\sigma^T)_{ij}\partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t. \tag{1.5}$$

There are several significant classes of American contracts, commonly traded in financial markets, whose corresponding diffusion process X is associated with Kolmogorov type operators which are not uniformly parabolic, i.e. $m < N$. Some examples are provided by American Asian style options, see [1], and by American options priced in the stochastic volatility introduced in [12], see also [6] and [9]. Obstacle problems for degenerate diffusions also arise in the study of pension plans, see [11], and have recently been considered in connection with stock loans, see [5]. In this framework the two regions

$$\begin{aligned}\mathscr{E} &= \{(x,t) \in \mathbb{R}^N \times [0,T] : U(x,t) = \psi(x)\},\\ \mathscr{C} &= \{(x,t) \in \mathbb{R}^N \times [0,T] : U(x,t) > \psi(x)\}\end{aligned}$$

are usually referred to as the *coincidence* and *continuation sets* respectively. The boundary $\mathscr{F}$ of $\mathscr{E}$ is called *associated free boundary* or *optimal exercise boundary.* To clarify the distinction between the results in this paper and the results established in [10] we note that the results in [10] apply in a neighborhood of any interior point $(x_0, t_0) \in \mathscr{F}$, $t_0 < T$, while in this paper we focus on the regularity of the solution at points $(x_0, t_0) \in \mathscr{F} \cap \{t = T\}$. In particular, we focus on the regularity of the solution up to the maturity and we establish quite general results which apply in many problems where operators of Kolmogorov type occur.

In this paper we impose the same assumptions concerning the operator L and the problem in (1.2) as in [10]. In particular, we assume that

H1 the coefficients $a_{ij} = a_{ji}$ are bounded continuous functions for $i, j = 1, \dots, m$. Moreover, there exists a positive constant λ such that

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j \le \lambda|\xi|^2, \qquad \xi \in \mathbb{R}^m,\ (x,t) \in \mathbb{R}^{N+1};$$

H2 the operator

$$Ku := \sum_{i=1}^{m} \partial_{x_i x_i} u + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u - \partial_t u \tag{1.6}$$

is hypoelliptic, i.e. every distributional solution of $Ku = f$ is a smooth function, whenever f is smooth;

H3 the coefficients a_{ij}, b_i, for $i, j = 1, \dots, m$, and f, g belong to the space $C_K^{0,\alpha}$ of Hölder continuous functions defined in (2.9), for some $\alpha \in]0,1]$.

We set

$$Y = \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t$$

and we recall that **H2** can be stated in terms of the well-known Hörmander condition [13]

$$\text{rank Lie}(\partial_{x_1}, \dots, \partial_{x_m}, Y) = N + 1, \tag{1.7}$$

where $\text{Lie}(\partial_{x_1}, \dots, \partial_{x_m}, Y)$ denotes the Lie algebra generated by the vector fields $\partial_{x_1}, \dots, \partial_{x_m}, Y$. To simplify our presentation, we also assume the following technical condition:

H4 the operator K is δ_r-homogeneous of degree two with respect to the dilations group $(\delta_r)_{r>0}$ in (2.3) below.

We recall that existence and uniqueness of a strong solution to (1.2) have been proved in [7] and [16]. We say that $u \in S^1_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ is a strong solution to problem (1.2) if the differential inequality is satisfied a.e. in Ω and the boundary datum is attained at any point of $\partial_P\Omega$. We refer to Section 2 for the definition of the Hölder spaces $C^{n,\alpha}_K$ and Sobolev-Stein spaces S^p. In [10] we proved the following internal estimates.

Theorem 1.1 *Assume hypotheses* **H1-4**. *Let $\alpha \in]0,1]$ and let Ω, Ω' be domains of $\mathbb{R}^{N+1}$ such that $\Omega' \subset\subset \Omega$. Let u be a solution to problem* (1.2)*:*

i) if $\psi \in C^{0,\alpha}_K(\Omega)$ then $u \in C^{0,\alpha}_K(\Omega')$ and

$$\|u\|_{C^{0,\alpha}_K(\Omega')} \le c\left(\alpha, \Omega, \Omega', L, \|f\|_{C^{0,\alpha}_K(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|\psi\|_{C^{0,\alpha}_K(\Omega)}\right);$$

ii) if $\psi \in C^{1,\alpha}_K(\Omega)$ then $u \in C^{1,\alpha}_K(\Omega')$ and

$$\|u\|_{C^{1,\alpha}_K(\Omega')} \le c\left(\alpha, \Omega, \Omega', L, \|f\|_{C^{0,\alpha}_K(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|\psi\|_{C^{1,\alpha}_K(\Omega)}\right);$$

iii) if $\psi \in C^{2,\alpha}_K(\Omega)$ then $u \in S^\infty(\Omega')$ and

$$\|u\|_{S^\infty(\Omega')} \le c\left(\alpha, \Omega, \Omega', L, \|f\|_{C^{0,\alpha}_K(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|\psi\|_{C^{2,\alpha}_K(\Omega)}\right).$$

Hereafter when we say that a constant depends on the operator L, we mean that it depends on the dimension N, the parabolicity constant λ and the Hölder norms of its coefficients. The aim of this paper is to extend the above estimates to the initial state. In particular we consider problem (1.2) on the domain

$$\Omega_{t_0} := \Omega \cap \{t > t_0\} \tag{1.8}$$

and prove Hölder estimates on $\Omega'_{t_0} = \Omega' \cap \{t > t_0\}$ for every $\Omega' \subset\subset \Omega$. We explicitly remark that Ω'_{t_0} is not a compact subset of Ω_{t_0}. Our main result is the following

Theorem 1.2 *Assume hypotheses* **H1-4**. *Let $\alpha \in]0,1]$ and let Ω, Ω' be domains of $\mathbb{R}^{N+1}$ such that $\Omega' \subset\subset \Omega$. Let u be a solution to problem* (1.2) *in the domain Ω_{t_0}, $t_0 \in \mathbb{R}$, defined in* (1.8)*:*

i) if $g, \psi \in C^{0,\alpha}_K(\Omega_{t_0})$ then $u \in C^{0,\alpha}_K(\Omega'_{t_0})$ and

$$\|u\|_{C^{0,\alpha}_K(\Omega'_{t_0})} \le c\left(\alpha, \Omega, \Omega', L, \|f\|_{C^{0,\alpha}_K(\Omega_{t_0})}, \|g\|_{C^{0,\alpha}_K(\Omega_{t_0})}, \|\psi\|_{C^{0,\alpha}_K(\Omega_{t_0})}\right);$$

ii) if $g, \psi \in C^{1,\alpha}_K(\Omega_{t_0})$ then $u \in C^{1,\alpha}_K(\Omega'_{t_0})$ and

$$\|u\|_{C^{1,\alpha}_K(\Omega'_{t_0})} \le c\left(\alpha, \Omega, \Omega', L, \|f\|_{C^{0,\alpha}_K(\Omega_{t_0})}, \|g\|_{C^{1,\alpha}_K(\Omega_{t_0})}, \|\psi\|_{C^{1,\alpha}_K(\Omega_{t_0})}\right);$$

iii) if $g, \psi \in C^{2,\alpha}_K(\Omega_{t_0})$ then $u \in S^\infty(\Omega'_{t_0})$ and

$$\|u\|_{S^\infty(\Omega'_{t_0})} \le c\left(\alpha, \Omega, \Omega', L, \|f\|_{C^{0,\alpha}_K(\Omega_{t_0})}, \|g\|_{C^{2,\alpha}_K(\Omega_{t_0})}, \|\psi\|_{C^{2,\alpha}_K(\Omega_{t_0})}\right).$$

We note that Theorem 1.2 concerns the optimal interior regularity for the solution u to the obstacle problem under different assumption on the regularity of the obstacle ψ and the datum g. As a preliminary result in the proof of Theorem 1.2, we also give new results concerning the regularity at the initial state of solutions to the Cauchy-Dirichlet problem

$$\begin{cases} Lu(x,t) = f(x,t), & \text{in } \Omega, \\ u(x,t) = g(x,t), & \text{on } \partial_P \Omega. \end{cases} \tag{1.9}$$

These results are of independent interest and read as follows

Theorem 1.3 *Assume hypotheses* **H1-4**. *Let* $\alpha \in]0,1]$ *and let* Ω, Ω' *be domains of* $\mathbb{R}^{N+1}$ *such that* $\Omega' \subset\subset \Omega$. *Let* u *be a solution to problem* (1.9) *in the domain* Ω_{t_0}, $t_0 \in \mathbb{R}$, *defined in* (1.8)*:*

i) if $g \in C_K^{0,\alpha}(\Omega_{t_0})$ *then* $u \in C_K^{0,\alpha}(\Omega'_{t_0})$ *and*

$$\|u\|_{C_K^{0,\alpha}(\Omega'_{t_0})} \leq c\left(\alpha, \Omega, \Omega', L, \|f\|_{C_K^{0,\alpha}(\Omega_{t_0})}, \|g\|_{C_K^{0,\alpha}(\Omega_{t_0})}\right);$$

ii) if $g \in C_K^{1,\alpha}(\Omega_{t_0})$ *then* $u \in C_K^{1,\alpha}(\Omega'_{t_0})$ *and*

$$\|u\|_{C_K^{1,\alpha}(\Omega'_{t_0})} \leq c\left(\alpha, \Omega, \Omega', L, \|f\|_{C_K^{0,\alpha}(\Omega_{t_0})}, \|g\|_{C_K^{1,\alpha}(\Omega_{t_0})}\right);$$

iii) if $g \in C_K^{2,\alpha}(\Omega_{t_0})$ *then* $u \in S^\infty(\Omega'_{t_0})$ *and*

$$\|u\|_{S^\infty(\Omega'_{t_0})} \leq c\left(\alpha, \Omega, \Omega', L, \|f\|_{C_K^{0,\alpha}(\Omega_{t_0})}, \|g\|_{C_K^{2,\alpha}(\Omega_{t_0})}\right).$$

Concerning Theorem 1.2, we note that even in the uniformly elliptic-parabolic case, $m = N$, there is a very limited theory of the regularity up to the initial state. In fact we are only aware of the results by Nyström [15], Shahgholian [18] (see also Petrosyan and Shahgholian [17]). While the arguments in [18] allow for certain fully non-linear parabolic equations, in [15] the techniques was conveyed in context of pricing of multi-dimensional American options in a financial market driven by a general multi-dimensional Ito diffusion. In [15] the machinery and techniques were developed and described, in the case $m = N$, assuming more regularity on the operator and the obstacle than needed and in the standard context of American options. However, the results in [15] and [18] do not apply in the setting of Asian options or the Hobson-Rogers model for stochastic volatility [12].

Note that our results also apply to uniformly parabolic equations ($m = N$). In this case we slightly improve Theorem 4.3 in [17] (see also Theorems 1.2 and 1.3 in [18]) since we get the Hölder regularity of the solution with the optimal exponent.

The techniques used in this paper are structurally similar to those in [10] and were introduced by Caffarelli, Karp and Shahgholian in [4] in the stationary case and by Caffarelli, Petrosyan and Shahgholian [3] in the study of the heat equation. In this paper we build the core part of the argument on the function

$$S_k^+(u) = \sup_{Q_{2^{-k}}^+} |u| \tag{1.10}$$

where u is a solution to the obstacle problem in Q^+ and Q_r^+ is defined in (2.5). In particular, as an important step in the proof of Theorem 1.2, we prove that there exists a positive constant c such that, for all $k \in \mathbb{N}$,

$$S_{k+1}^+(u-F) \leq \max\left(c\, 2^{-(k+1)\gamma}, \frac{S_k^+(u-F)}{2^\gamma}, \frac{S_{k-1}^+(u-F)}{2^{2\gamma}}, \dots, \frac{S_0^+(u-F)}{2^{(k+1)\gamma}} \right) \tag{1.11}$$

assuming that (u, g, f, ψ) belongs to certain function classes defined in the bulk of the paper. Moreover given ψ, in this construction we let F and γ be determined as follows:

$$\begin{aligned} \text{Theorem 1.2-}i)\text{:} \qquad & F = P_0^{(0,0)} g = g(0,0),\ \gamma = \alpha, \\ \text{Theorem 1.2-}ii)\text{:} \qquad & F = P_1^{(0,0)} g,\ \gamma = 1+\alpha, \\ \text{Theorem 1.2-}iii)\text{:} \qquad & F = P_2^{(0,0)} g,\ \gamma = 2, \end{aligned}$$

where $P_n^{(0,0)}$ is the intrinsic Taylor expansion defined in Remark 2.1. In either case the proof of (1.11) is based on an argument by contradiction and this argument differs at key points compared to the corresponding proof in [10] due to the presence of the boundary at $t = 0$.

The rest of this paper is organized as follows. In Section 2 we collect a number of important facts concerning operators of Kolmogorov type. In Section 3 we prove Theorem 1.3. In Section 4 we develop the bulk of the estimates needed in the proof of Theorem 1.2. Then we conclude the proof of Theorem 1.2 in Section 5.

2 Preliminaries on operators of Kolmogorov type

In this section we collect a number of results concerning operators of Kolmogorov type to be used in the proof of Theorems 1.2 and 1.3.

We recall that the natural setting for operators satisfying a Hörmander condition is that of the analysis on Lie groups. In particular, as shown in [14] the relevant Lie group related to the operator K in (1.6) is defined using the group law

$$(x,t) \circ (y,s) = (y + E(s)x, t+s), \qquad E(s) = \exp(-sB^T), \qquad (x,t),(y,s) \in \mathbb{R}^{N+1}, \tag{2.1}$$

where B^T denotes the transpose of the matrix B. It is known that a condition equivalent to our assumption **H2** is that there exists a basis for $\mathbb{R}^N$ such that the matrix B takes the form

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_\kappa \\ * & * & * & \cdots & * \end{pmatrix} \tag{2.2}$$

where B_j, for $j \in \{1,..,\kappa\}$, is a $m_{j-1} \times m_j$ matrix of rank m_j, $1 \leq m_\kappa \leq ... \leq m_1 \leq m$ and $m + m_1 + ... + m_\kappa = N$, while $*$ represents arbitrary matrices with constant entries. Moreover, if the matrices denoted by $*$ in (2.2) are null then there is a natural family of dilations

$$D_r = \operatorname{diag}(rI_m, r^3 I_{m_1}, .., r^{2\kappa+1} I_{m_\kappa}), \qquad \delta_r = \operatorname{diag}(D_r, r^2),\ r > 0, \tag{2.3}$$

associated to the Lie group. In (1.14) I_k, $k \in \mathbb{N}$, is the k-dimensional identity matrix.

For $x \in \mathbb{R}^N$ and $r > 0$ we let $B_r(x)$ denote the open ball in $\mathbb{R}^N$ with center x and radius r. We let $\mathbf{e}_1$ be the unit vector pointing in the x_1-direction in the canonical basis for $\mathbb{R}^N$. We let

$$\begin{aligned} Q &= \left(B_1(\tfrac{1}{2}\mathbf{e}_1) \cap B_1(-\tfrac{1}{2}\mathbf{e}_1)\right) \times]-1,1[, \\ Q^+ &= \left(B_1(\tfrac{1}{2}\mathbf{e}_1) \cap B_1(-\tfrac{1}{2}\mathbf{e}_1)\right) \times]0,1[, \\ Q^- &= \left(B_1(\tfrac{1}{2}\mathbf{e}_1) \cap B_1(-\tfrac{1}{2}\mathbf{e}_1)\right) \times]-1,0[. \end{aligned} \tag{2.4}$$

Then Q is a space-time cylinder, Q^+ will be referred to as the upper half-cylinder and Q^- will be referred to as the lower half-cylinder. We also let, whenever $(x,t) \in \mathbb{R}^{N+1}$, $r > 0$,

$$Q_r = \delta_r(Q), \quad Q_r(x,t) = (x,t) \circ Q_r, \quad Q_r^{\pm} = \delta_r(Q^{\pm}), \quad Q_r^{\pm}(x,t) = (x,t) \circ Q_r^{\pm}. \tag{2.5}$$

Then $Q_r(x,t)$ is the cylinder Q scaled to size r and translated to the point (x,t). As outlined in [10] the main reason we work with these cylinders is that these domains are regular for the Dirichlet problem for the operators considered in this paper.

We define a quasi-distance and a quasi-norm on $\mathbb{R}^{N+1}$ by setting

$$d_K((x,t),(\xi,\tau)) = \inf\{r > 0 \mid (x,t) \in Q_r(\xi,\tau)\}, \qquad \|(x,t)\|_K = d_K((x,t),(0,0)). \tag{2.6}$$

We recall (cf. Remark 1.3 in [10]) that $\|\delta_r(x,t)\|_K = r\|(x,t)\|_K$ and the following triangular inequality (cf. [8]): for any compact subset H of $\mathbb{R}^{N+1}$, there exists a positive constant c such that

$$\|z^{-1}\|_K \le c\|z\|_K, \qquad \|z \circ w\|_K \le c\left(\|z\|_K + \|w\|_K\right), \qquad z, w \in H. \tag{2.7}$$

By (2.7), for any $r_0 > 0$ there exists a positive constant c such that:

i) if $(x,t) \in Q_r(\xi,\tau)$ then $(\xi,\tau) \in Q_{cr}(x,t)$ for $r \in]0, r_0[$;

ii) if $(x,t) \in Q_r(\xi,\tau)$ then $Q_\rho(x,t) \subseteq Q_{c(r+\rho)}(\xi,\tau)$ for $r, \rho \in]0, r_0[$.

We also note that as a consequence we have that if $(x,t) \in Q_r(\xi,\tau)$ then

$$Q_r(\xi,\tau) \subseteq Q_{C_1 r}(x,t) \qquad r \in]0, r_0[, \tag{2.8}$$

for some positive constant C_1.

We next introduce the functional setting (Hölder and Sobolev spaces) for Kolmogorov equations. Let $\alpha \in (0,1]$ and let Ω be a domain of $\mathbb{R}^{N+1}$. We denote by $C_K^{0,\alpha}(\Omega)$, $C_K^{1,\alpha}(\Omega)$ and $C_K^{2,\alpha}(\Omega)$ the Hölder spaces defined by the following norms:

$$\begin{aligned} \|u\|_{C_K^{0,\alpha}(\Omega)} &= \sup_{\Omega} |u| + \sup_{\substack{z,\zeta \in \Omega \\ z \neq \zeta}} \frac{|u(z) - u(\zeta)|}{d_K(z,\zeta)^\alpha}, \\ \|u\|_{C_K^{1,\alpha}(\Omega)} &= \|u\|_{C_K^{0,\alpha}(\Omega)} + \sum_{j=1}^m \|\partial_{x_j} u\|_{C_K^{0,\alpha}(\Omega)} + \sup_{\substack{z,\zeta \in \Omega \\ z \neq \zeta}} \frac{|u(z) - u(\zeta) - \sum_{j=1}^m (z_j - \zeta_j)\partial_{x_j} u(\zeta)|}{d_K(z,\zeta)^{1+\alpha}}, \\ \|u\|_{C_K^{2,\alpha}(\Omega)} &= \|u\|_{C_K^{1,\alpha}(\Omega)} + \sum_{i,j=1}^m \|\partial_{x_i x_j} u\|_{C_K^{0,\alpha}(\Omega)} + \|Yu\|_{C_K^{0,\alpha}(\Omega)}. \end{aligned} \tag{2.9}$$

Remark 2.1 *Denote*

$$
\begin{aligned}
&P_0^{(\xi,\tau)}u(x,t) = u(\xi,\tau),\\
&P_1^{(\xi,\tau)}u(x,t) = u(\xi,\tau) + \sum_{j=1}^{m}(x_j-\xi_j)\partial_{x_j}u(\xi,\tau),\\
&P_2^{(\xi,\tau)}u(x,t) = u(\xi,\tau) + \sum_{j=1}^{m}(x_j-\xi_j)\partial_{x_j}u(\xi,\tau)\\
&+\sum_{i,j=1}^{m}(x_i-\xi_i)(x_j-\xi_j)\partial_{x_ix_j}u(\xi,\tau) - (t-\tau)Yu(\xi,\tau).
\end{aligned}
$$

If $u \in C_K^{n,\alpha}$ *(with* $n = 0,1,2$*) then we have*

$$
\left|u(x,t) - P_n^{(\xi,\tau)}u(x,t)\right| \le \|u\|_{C_K^{n,\alpha}}\, d_K\left((x,t),(\xi,\tau)\right)^{n+\alpha}.
$$

Let $n \in \{0,1,2\}$, $\alpha \in (0,1]$. If $u \in C_K^{n,\alpha}(\Omega')$ for every compact subset Ω' of Ω then we write $u \in C_{K,\mathrm{loc}}^{n,\alpha}(\Omega)$. Furthermore, for $p \in [1,\infty]$ we define the Sobolev-Stein spaces

$$
S^p(\Omega) = \{u \in L^p(\Omega) :\ \partial_{x_i}u, \partial_{x_ix_j}u, Yu \in L^p(\Omega),\ i,j = 1,...,m\}
$$

and we let

$$
\|u\|_{S^p(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{i=1}^{m}\|\partial_{x_i}u\|_{L^p(\Omega)} + \sum_{i,j=1}^{m}\|\partial_{x_ix_j}u\|_{L^p(\Omega)} + \|Yu\|_{L^p(\Omega)}.
$$

If $u \in S^p(\Omega')$ for every compact subset Ω' of Ω then we write $u \in S^p_{\mathrm{loc}}(\Omega)$.

We end this section by stating a version of some technical lemmas established in [10]. We first need to introduce some additional notations. For any positive T, R, and $(x_0,t_0) \in \mathbb{R}^{N+1}$ we put $Q^+(T) = \left(B_1(\frac{1}{2}\mathbf{e}_1) \cap B_1(-\frac{1}{2}\mathbf{e}_1)\right) \times (0,T]$, and $Q_R^+(x_0,t_0,T) = (x_0,t_0) \circ \delta_R\left(Q^+(R^{-2}T)\right)$. Note that, from (2.3) it follows that T is the height of $Q_R^+(x_0,t_0,T)$. The following lemmas correspond respectively to Lemma 2.7 and Corollary 2.6 in [10]. The function Γ in Lemma 2.3 is the fundamental solution of L.

Lemma 2.2 *Assume* **H1-4**. *Let* $R > 0$ *be given. Then there exist constants* $R_0, C_0, C_1 > 0$, $R_0 \ge 2R$, *such that*

$$
\sup_{Q_R^+}|v| \le C_0 e^{-C_1\tilde{R}^2} \sup_{\partial_P Q_{\tilde{R}}^+ \cap \{(x,t):\ 0<t\le R^2\}} |v|
$$

for any $\tilde{R} \ge R_0$ *and for every* v *solution of* $Lv = 0$ *in* $Q_{\tilde{R}}^+(0,0,R^2)$ *such that* $v(\cdot,0) = 0$.

Lemma 2.3 *Define, for* $\gamma > 0$, *the function*

$$
u(x,t) = \int_{\mathbb{R}^N}\Gamma(x,t,y,0)\|(y,0)\|_K^{\gamma}dy, \qquad x \in \mathbb{R}^N,\ t > 0.
$$

There exists a positive constant c_γ *such that*

$$
u(x,t) \le c_\gamma\|(x,t)\|_K^{\gamma}.
$$

3 Estimates for the Cauchy-Dirichlet problem

In this section we prove some preliminary estimates for the Cauchy-Dirichlet problem at the initial state.

Definition 3.1 *Let L be an operator of the form* (1.1) *satisfying hypotheses* **H1-4**, $\Omega \subset \mathbb{R}^{N+1}$ *be a given domain,* $n \in \{0,1,2\}$, $\alpha \in (0,1]$ *and* M_1, M_2, M_3 *be three positive constants. Then we say that* (u,f,g) *belongs to the class* $\mathscr{D}_n(L,\Omega,\alpha,M_1,M_2,M_3)$ *if* u *is a solution to problem* (1.9) *with* $f \in C_K^{0,\alpha}(\Omega)$, $g \in C_K^{n,\alpha}(\overline{\Omega})$ *and*

$$\|u\|_{L^\infty(\Omega)} \le M_1, \quad \|f\|_{C_K^{0,\alpha}(\Omega)} \le M_2, \quad \|g\|_{C_K^{n,\alpha}(\Omega)} \le M_3.$$

The main result of this section is the following

Lemma 3.2 *Let* $R, \alpha \in]0,1]$, $n = 0,1,2$, $(x_0,t_0) \in \mathbb{R}^{N+1}$ *and let* M_1, M_2, M_3 *be positive constants. Assume that*

$$(u,f,g) \in \mathscr{D}_n(L, Q_R^+(x_0,t_0), \alpha, M_1, M_2, M_3).$$

Then there exists $C_\alpha = C(L,\alpha,M_1,M_2,M_3)$ *such that*

$$\sup_{Q_r^+(x_0,t_0)} |u-g| \le C_\alpha r^{n+\alpha}, \qquad r \in]0,R[, \quad \text{for } n = 0,1$$

and

$$\sup_{Q_r^+(x_0,t_0)} |u-g| \le C_\alpha r^2, \qquad r \in]0,R[, \quad \text{for } n = 2.$$

Proof. By the invariance properties of L under translation and scaling (cf. Remark 4.2), it is not restrictive to assume $(x_0,t_0) = (0,0)$ and $R = 1$. Moreover by the triangle inequality and Remark 2.1, it suffices to prove it

$$\sup_{Q_r^+(0,0)} \left| u - P_n^{(0,0)} g \right| \le c r^\gamma, \qquad r \in]0,1[,$$

where $\gamma = \alpha + n$ if $n = 0,1$ and $\gamma = 2$ if $n = 2$. We also remark that the function $v_n = u - P_n^{(0,0)} g$ satisfies the equation

$$L v_n = f - L P_n^{(0,0)} g =: f_n.$$

Since $f_n \in C_K^{0,\alpha}$, without restriction we may assume $P_n^{(0,0)} g = 0$.

After these preliminary reductions, we first consider the case $n = 0$ and we denote by v_1, v_2, v_3 the solutions of the following boundary value problems:

$$\begin{cases} Lv_1 = 0 & \text{in } Q^+, \\ v_1 = 0 & \text{on } \partial_P^+ Q^+, \\ v_1 = g & \text{on } \partial_P^- Q^+, \end{cases} \qquad \begin{cases} Lv_2 = 0 & \text{in } Q^+, \\ v_2 = g & \text{on } \partial_P^+ Q^+, \\ v_2 = 0 & \text{on } \partial_P^- Q^+, \end{cases} \qquad \begin{cases} Lv_3 = -\|f\|_{C_K^{0,\alpha}(Q^+)} & \text{in } Q^+, \\ v_3 = 0 & \text{on } \partial_P Q^+, \end{cases}$$

where

$$\partial_P^+ Q^+ = \partial_P Q^+ \cap \{t > 0\}, \qquad \partial_P^- Q^+ = \partial_P Q^+ \cap \{t = 0\}.$$

Then, by the maximum principle we have

$$v_1 + v_2 - v_3 \le u \le v_1 + v_2 + v_3 \qquad \text{in } Q^+, \tag{3.1}$$

so that we only have to prove that

$$\sup_{Q_r^+} \left(|v_1| + |v_2| + |v_3|\right) \le c r^\alpha, \tag{3.2}$$

for r suitably small.

Since $\|g\|_{C_K^{0,\alpha}(Q^+)} \le M_3$, by the maximum principle we have

$$|v_1(x,t)| \le M_3 \int_{\mathbb{R}^N} \Gamma(x,t,y,0) \|(y,0)\|_K^\alpha dy$$

so that by Lemma 2.3 we get

$$|v_1(x,t)| \le M_3 c_\alpha \|(x,t)\|_K^\alpha.$$

We next apply Lemma 2.2 with $R = 1$. We have

$$\sup_{Q_r^+} |v_2| \le C_0 \exp\left(-\frac{C_1}{r^2}\right) \sup_{\partial_P^+ Q^+} |v_2|$$

for any $r \le \frac{1}{R_0}$ where R_0, C_0, C_1 are as in Lemma 2.2. Since $|v_2|$ agrees with $|u|$ on $\partial_P^+ Q^+$ we can conclude that

$$\sup_{Q_r^+} |v_2| \le C_0\, M_1 \exp\left(-\frac{C_1}{r^2}\right) \le c_2 r^2, \qquad \text{for every} \quad r \in \left]0, \frac{1}{R_0}\right].$$

Finally, we have

$$|v_3(x,t)| \le \|f\|_{C_K^{0,\alpha}(Q^+)} \int_0^t \int_{\mathbb{R}^N} \Gamma(x,t,y,s) dy ds \le t\, \|f\|_{C_K^{0,\alpha}(Q^+)} \le M_2 \|(x,t)\|_K^2.$$

This proves (3.2) and the claim plainly follows as $n = 0$.

For $n = 1, 2$ we can use the same argument. In particular, we now apply Lemma 2.3 with $\gamma = n + \alpha$ and we find

$$|v_1(x,t)| \le M_3 c_{n+\alpha} \|(x,t)\|_K^{n+\alpha}.$$

□

4 Estimates for the obstacle problem at the initial state

In this section we prove estimates at the initial state for the solution of the obstacle problem. The main result is Lemma 4.3 below.

Definition 4.1 *Let L be an operator of the form (1.1) satisfying hypotheses* **H1-4**, $\Omega \subset \mathbb{R}^{N+1}$ *be a given domain,* $\alpha \in (0,1]$ *and* M_1, M_2, M_3, M_4 *be four positive constants. Then, for* $n \in \{0,1,2\}$ *we say that* (u,f,g,ψ) *belongs to the class* $\mathscr{P}_n(L,\Omega,\alpha,M_1,M_2,M_3,M_4)$ *if* u *is a strong solution to problem (1.2) with* $f \in C_K^{0,\alpha}(\Omega)$, $\psi, g \in C_K^{n,\alpha}(\overline{\Omega})$, $g \geq \psi$ *on* $\partial_P\Omega$ *and*

$$\|u\|_{L^\infty(\Omega)} \leq M_1, \quad \|f\|_{C_K^{0,\alpha}(\Omega)} \leq M_2, \quad \|g\|_{C_K^{n,\alpha}(\Omega)} \leq M_3, \quad \|\psi\|_{C_K^{n,\alpha}(\Omega)} \leq M_4.$$

The proof of Lemma 4.3 is based on certain blow-up arguments. In particular we define the blow-up of a function $v \in C(\Omega)$ as

$$v^r(x,t) := v\left(\delta_r(x,t)\right), \qquad r > 0, \tag{4.1}$$

whenever $\delta_r(x,t) \in \Omega$. A direct computation shows that

$$Lv = f \text{ in } \Omega \quad \text{if and only if} \quad L_r v^r = r^2 f^r \text{ in } \delta_{1/r}\Omega, \tag{4.2}$$

where

$$L_r = \sum_{i,j=1}^m a_{ij}^r \partial_{x_i x_j} + \sum_{i=1}^m r b_i^r \partial_{x_i} + \sum_{i,j=1}^N b_{ij} x_i \partial_{x_j} - \partial_t. \tag{4.3}$$

Remark 4.2 *Given* $r \in]0,1[$ *and* $(x_0,t_0) \in \mathbb{R}^{N+1}$*, we also set*

$$u^{r,(x_0,t_0)}(x,t) = u((x_0,t_0) \circ \delta_r(x,t)). \tag{4.4}$$

We remark that $u \in C_K^{n,\alpha}$ *if and only if* $u^{r,(x_0,t_0)} \in C_K^{n,\alpha}$ *and*

$$\|u^{r,(x_0,t_0)}\|_{C_K^{n,\alpha}} \leq \|u\|_{C_K^{n,\alpha}}.$$

Indeed in the case $n = 0$ *we have*

$$\|u^{r,(x_0,t_0)}\|_{C_K^{0,\alpha}(\Omega)} = \sup_\Omega |u| + r^\alpha \sup_{\substack{z,\zeta \in \Omega \\ z \neq \zeta}} \frac{|u(z) - u(\zeta)|}{\|\zeta^{-1} \circ z\|_K^\alpha} \leq \|u\|_{C_K^{0,\alpha}(\Omega)}.$$

Moreover

$$Lu = f \ \textit{in}\ (x_0,t_0) \circ \delta_r(\Omega) \quad \textit{if and only if} \quad L^{r,(x_0,t_0)} u^{r,(x_0,t_0)} = r^2 f^{r,(x_0,t_0)} \ \textit{in}\ \Omega,$$

where

$$L_r^{(x_0,t_0)} = \sum_{i,j=1}^m a_{ij}^{r,(x_0,t_0)}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m r b_i^{r,(x_0,t_0)}(x,t) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^N b_{ij} x_i \frac{\partial}{\partial x_j} - \frac{\partial}{\partial t}. \tag{4.5}$$

Therefore

$$u \in \mathscr{P}_n(L,(x_0,t_0) \circ \delta_r(\Omega),\alpha,M_1,M_2,M_3,M_4) \implies u^{r,(x_0,t_0)} \in \mathscr{P}_n(L_r^{(x_0,t_0)},\Omega,\alpha,M_1,M_2,M_3,M_4).$$

Lemma 4.3 *Let $R, \alpha \in]0,1]$, $n = 0,1,2$, $(x_0,t_0) \in \mathbb{R}^{N+1}$ and let M_1, M_2, M_3, M_4 be positive constants. Assume that*

$$(u, f, g, \psi) \in \mathscr{P}_n(L, Q_R^+(x_0,t_0), \alpha, M_1, M_2, M_3, M_4).$$

Then there exists $c = c(L, \alpha, M_1, M_2, M_3, M_4)$ such that

$$\sup_{Q_r^+(x_0,t_0)} |u - g| \le c r^{n+\alpha}, \qquad r \in]0, R[, \quad \textit{for } n = 0, 1$$

and

$$\sup_{Q_r^+(x_0,t_0)} |u - g| \le c r^2, \qquad r \in]0, R[, \quad \textit{for } n = 2.$$

Proof. We first prove that there exists $C_\alpha = C(L, \alpha, M_1, M_2, M_3)$ such that

$$\inf_{Q_r^+(x_0,t_0)} u - g \ge -C_\alpha r^{n+\alpha}, \qquad r \in]0, R[, \quad \text{for } n = 0, 1 \tag{4.6}$$

and

$$\inf_{Q_r^+(x_0,t_0)} u - g \ge -C_\alpha r^2, \qquad r \in]0, R[, \quad \text{for } n = 2. \tag{4.7}$$

In fact, consider v solution to the Dirichlet problem (1.9) in the domain $\Omega = Q_r^+(x_0, t_0)$. Then by the comparison principle we have $u \ge v$ and (4.6)-(4.7) are a direct consequence of Lemma 3.2 since

$$(v, f, g) \in \mathscr{D}_n(L, Q_R^+(x_0,t_0), \alpha, M_1, M_2, M_3).$$

Armed with (4.6)-(4.7) we next proceed with the proof of Lemma 4.3. We start with some preliminary problem reduction steps. To start with we first note that, by Remark 4.2, we can assume, without loss of generality, that $(x_0, t_0) = (0, 0)$ and $R = 1$. Moreover, as in the proof of Lemma 3.2, it is not restrictive to assume $P_n^{(0,0)} g \equiv 0$. By the triangle inequality and Remark 2.1, it suffices to prove

$$\sup_{Q_r^+(0,0)} |u| \le c r^\gamma, \qquad r \in]0, 1[,$$

where $\gamma = \alpha + n$ if $n = 0, 1$ and $\gamma = 2$ if $n = 2$. Recall the definition of $S_k^+(u)$ in (1.10). To prove Lemma 4.3 we show that there exists a positive $\tilde{c} = \tilde{c}\,(L, \alpha, M_1, M_2, M_3)$ such that (1.11) holds for all $k \in \mathbb{N}$. Indeed, if (1.11) holds then we see, by a simple iteration argument, that

$$S_k^+(u) \le \frac{\tilde{c}}{2^{k\gamma}}$$

and Lemma 4.3 follows. We first consider the case $n = 0$ and prove (1.11) with $\gamma = \alpha$. We assume that

$$(u, f, g, \psi) \in \mathscr{P}_0(L, Q^+, \alpha, M_1, M_2, M_3, M_4),$$

and, as in [10], divide the argument into three steps.

Step 1 (Setting up the argument by contradiction). We first note that by (4.6)

$$u(x,t) \ge -(C_\alpha + M_3)\, \|(x,t)\|_K^\alpha, \qquad (x,t) \in Q^+. \tag{4.8}$$

Assume that (1.11) is false. Then for every $j \in \mathbb{N}$, there exists a positive integer k_j and $(u_j, f_j, g_j, \psi_j) \in \mathscr{P}_0(L, Q^+, \alpha, M_1, M_2, M_3, M_4)$ such that $u_j(0,0) = 0 \geq \psi_j(0,0)$ and

$$S^+_{k_j+1}(u_j) > \max\left(\frac{j\,(C_\alpha + M_3)}{2^{(k_j+1)\alpha}}, \frac{S^+_{k_j}(u_j)}{2^\alpha}, \frac{S^+_{k_j-1}(u_j)}{2^{2\alpha}}, \ldots, \frac{S^+_0(u_j)}{2^{(k_j+1)\alpha}}\right). \tag{4.9}$$

Using the definition in (1.10) we see that there exists (x_j, t_j) in the closure of $Q^+_{2^{-k_j-1}}$ such that $|u_j(x_j, t_j)| = S^+_{k_j+1}(u_j)$ for every $j \geq 1$. Moreover from (4.8) it follows that $u_j(x_j, t_j) > 0$. Using (4.9) we can conclude, as $|u_j| \leq M_1$, that $j2^{-\alpha k_j}$ is bounded and hence that $k_j \to \infty$ as $j \to \infty$.

Step 2 (Constructing blow-ups). We define $(\tilde{x}_j, \tilde{t}_j) = \delta_{2^{k_j}}((x_j, t_j))$ and $\tilde{u}_j : Q^+_{2^{k_j}} \longrightarrow \mathbb{R}$ as

$$\tilde{u}_j(x,t) = \frac{u_j(\delta_{2^{-k_j}}(x,t))}{S^+_{k_j+1}(u_j)}. \tag{4.10}$$

Note that $(\tilde{x}_j, \tilde{t}_j)$ belongs to the closure of $Q^+_{1/2}$ and

$$\tilde{u}_j(\tilde{x}_j, \tilde{t}_j) = 1. \tag{4.11}$$

Moreover, we let $\tilde{L}_j = L_{2^{-k_j}}$, see (4.3) for the exact definition of this scaled operator, and

$$\tilde{f}_j(x,t) = 2^{-2k_j}\frac{f_j(\delta_{2^{-k_j}}(x,t))}{S^+_{k_j+1}(u_j)}, \quad \tilde{g}_j(x,t) = \frac{g_j(\delta_{2^{-k_j}}(x,t))}{S^+_{k_j+1}(u_j)}, \quad \tilde{\psi}_j(x,t) = \frac{\psi_j(\delta_{2^{-k_j}}(x,t))}{S^+_{k_j+1}(u_j)} \tag{4.12}$$

whenever $(x,t) \in Q^+_{2^{k_j}}$. Then, using (4.2) we see that

$$\begin{cases} \max\{\tilde{L}_j\tilde{u}_j - \tilde{f}_j, \tilde{\psi}_j - \tilde{u}_j\} = 0, & \text{in } Q^+_{2^{k_j}}, \\ \tilde{u}_j = \tilde{g}_j, & \text{on } \partial_P Q^+_{2^{k_j}}. \end{cases}$$

In the following we let $l \in \mathbb{N}$ be fixed and to be specified below. Then

$$(\tilde{u}_j, \tilde{u}_j, \tilde{f}_j, \tilde{\psi}_j) \in \mathscr{P}_0(\tilde{L}_j, Q^+_{2^l}, \alpha, \tilde{M}^j_1, \tilde{M}^j_2, \tilde{M}^j_3, \tilde{M}^j_4),$$

for some $\tilde{M}^j_1, \tilde{M}^j_2, \tilde{M}^j_3, \tilde{M}^j_4$. From (4.9) it follows that

$$\tilde{M}^j_1 = \sup_{Q^+_{2^l}} |\tilde{u}_j| = \frac{S^+_{k_j-l}(u_j)}{S^+_{k_j+1}(u_j)} \leq 2^{(l+1)\alpha} \quad \text{whenever} \quad k_j > l. \tag{4.13}$$

Furthermore by Remark 4.2 we have

$$\tilde{M}^j_2 \leq 2^{-2k_j}\frac{M_2}{S^+_{k_j+1}(u_j)}. \tag{4.14}$$

Moreover, we let

$$m_j = \max\left\{ \|\tilde{g}_j\|_{L^\infty(Q^+_{2^l})}, \sup_{Q^+_{2^l}} \tilde{\psi}_j \right\}. \tag{4.15}$$

Then, using (4.9) and the $C_K^{0,\alpha}$-regularity of g_j and ψ_j, we see that

$$\lim_{j\to\infty} \tilde{M}_2^j = \lim_{j\to\infty} m_j = 0. \tag{4.16}$$

Note that we here can not ensure the decay of $\tilde{M}_4^j$, as $j \to \infty$, as we only know that $\tilde{\psi}_j(0,0) \le 0$.

Step 3 (Completing the argument by contradiction). In the following we choose l suitably large to find a contradiction. We consider $j_0 \in \mathbb{N}$ such that $k_j > 2^l$ for $j \ge j_0$. We let

$$\partial_P^+ Q_{\tilde{R}}^+(0,0,1) = \partial_P Q_{\tilde{R}}^+(0,0,1) \cap \{t > 0\}, \qquad \partial_P^- Q_{\tilde{R}}^+(0,0,1) = \partial_P Q_{\tilde{R}}^+(0,0,1) \cap \{t = 0\}.$$

We consider the solution $\tilde{v}_j$ to

$$\begin{cases} \tilde{L}_j \tilde{v}_j = -\|\tilde{f}_j\|_{L^\infty(Q_{2^l}^+)} & \text{in } Q_{2^l}^+, \\ \tilde{v}_j = \tilde{M}_1^j & \text{on } \partial_P^+ Q_{2^l}^+, \\ \tilde{v}_j = m_j & \text{on } \partial_P^- Q_{2^l}^+, \end{cases} \tag{4.17}$$

and we prove that

$$\tilde{u}_j \le \tilde{v}_j \text{ in } Q_{2^l}^+. \tag{4.18}$$

By the maximum principle we have $\tilde{v}_j \ge m_j \ge \tilde{\psi}_j$ in $Q_{2^l}^+$. Furthermore

$$\tilde{L}_j(\tilde{v}_j - \tilde{u}_j) = -\|\tilde{f}_j\|_{L^\infty(Q_{2^l}^+)} + \tilde{f}_j \le 0 \quad \text{in} \quad \Omega := Q_{2^l}^+ \cap \{(x,t) :\ \tilde{u}_j(x,t) > \tilde{\psi}_j(x,t)\},$$

and $\tilde{v}_j \ge \tilde{u}_j$ on $\partial\Omega$. Hence (4.18) follows from the maximum principle. We next show that (4.18) contradicts (4.11). We write $\tilde{v}_j = w_j + \tilde{w}_j + \hat{w}_j$ on $Q_{2^l}^+(0,0,1)$ where

$$\begin{cases} \tilde{L}_j w_j = 0 & \text{in } Q_{2^l}^+(0,0,1), \\ w_j = 0 & \text{on } \partial_P^+ Q_{2^l}^+(0,0,1), \\ w_j = \tilde{v}_j & \text{on } \partial_P^- Q_{2^l}^+(0,0,1), \end{cases} \qquad \begin{cases} \tilde{L}_j \tilde{w}_j = 0 & \text{in } Q_{2^l}^+(0,0,1), \\ \tilde{w}_j = \tilde{v}_j & \text{on } \partial_P^+ Q_{2^l}^+(0,0,1), \\ \tilde{w}_j = 0 & \text{on } \partial_P^- Q_{2^l}^+(0,0,1), \end{cases}$$

$$\begin{cases} \tilde{L}_j \hat{w}_j = -\|\tilde{f}_j\|_{L^\infty(Q_{2^l}^-)} & \text{in } Q_{2^l}^+(0,0,1), \\ \hat{w}_j = 0 & \text{on } \partial_P Q_{2^l}^+(0,0,1). \end{cases}$$

By using the maximum principle, we see that

$$w_j \le m_j \quad \text{in } \ Q_{2^l}^+(0,0,1), \tag{4.19}$$

and

$$\|\hat{w}_j\|_{L^\infty\left(Q_{2^l}^+(0,0,1)\right)} \le \|\tilde{f}_j\|_{L^\infty(Q_{2^l}^+)} \le \tilde{M}_2^j. \tag{4.20}$$

We next use Lemma 2.2 in the cylinder $Q_{2^l}^+(0,0,1)$, with $R = 1$ and $\tilde{R} = 2^l$. By using (4.13), we get

$$\sup_{Q^+} \tilde{w}_j \le C_0 e^{-C_1 4^l} \sup_{\partial_P^+ Q_{2^l}^+(0,0,1)} \tilde{v}_j \le C_0 e^{-C_1 4^l} \tilde{M}_1^j \le C_0 e^{-C_1 4^l} 2^{(l+1)\alpha} \tag{4.21}$$

and we note that the right hand side in this inequality can be made arbitrarily small by choosing l large enough, independently of j. Combining (4.19), (4.20) and (4.21) we conclude that, for a suitably large l and j_0, we have

$$\sup_{Q^+} \tilde{v}_j \leq \frac{1}{2} \qquad \text{for any } j \geq j_0,$$

which contradicts (4.11) and (4.18). This proves the Lemma for $n = 0$.

The proof for $n = 1, 2$ is analogous. We follow Steps 1 and 2, and we realize that we need to show that (4.16) holds also for $n = 1, 2$. In both cases the same argument used above shows that $\tilde{M}_2^j \to 0$ as $j \to \infty$. We next prove that $m_j \to 0$ as $j \to \infty$. Consider first the case $n = 1$. Let (x, t), be any given point in $Q_{2^l}^+$ and let $\tilde{x} = E(-t)x$. Note that, by (2.1), we have that $(\tilde{x}, 0) = (x, t) \circ (0, t)^{-1} = (x, t) \circ (0, -t)$. Then, by (2.7), we find

$$\|(\tilde{x}, 0)\|_K \leq c\,(\|(x, t)\|_K + \|(0, t)\|_K) \leq 2c\|(x, t)\|_K.$$

As a consequence we see that

$$\tilde{\psi}_j(x, t) = \tilde{\psi}_j(x, t) - \tilde{\psi}_j(\tilde{x}, 0) + \tilde{\psi}_j(\tilde{x}, 0) \leq \left|\tilde{\psi}_j(x, t) - \tilde{\psi}_j(\tilde{x}, 0)\right| + \tilde{g}_j(\tilde{x}, 0) \tag{4.22}$$

where we have used the assumption that $\tilde{\psi}_j(\tilde{x}, 0) \leq \tilde{g}_j(\tilde{x}, 0)$ for all $(\tilde{x}, 0) \in Q_{2^l}^+$. However, by (4.12) we now note that

$$\left|\tilde{\psi}_j(x, t) - \tilde{\psi}_j(\tilde{x}, 0)\right| \leq 2^{(\alpha+1)(l-k_j)} \frac{M_4}{S_{k_j+1}^+(u_j)} \to 0 \text{ as } j \to \infty. \tag{4.23}$$

Furthermore, since $P_1^{(0,0)}\tilde{g}_j = 0$, we also have that

$$|\tilde{g}_j(\tilde{x}, 0)| \leq 2^{(\alpha+1)(l-k_j)} \frac{M_3}{S_{k_j+1}^+(u_j)} \to 0 \text{ as } j \to \infty. \tag{4.24}$$

Combined, (4.22)-(4.24) prove that $m_j \to 0$ as $j \to \infty$ also in the case $n = 1$. The case $n = 2$ is analogous. This completes the proof of the lemma. □

5 Proof of Theorems 1.2 and 1.3

The proof of Theorem 1.2 follows along the lines of the proof of Theorems 1.5, 1.6 and 1.7 in [10] by using Lemma 4.3 to estimate the solution near the initial state. Therefore we only give a detailed proof of part *i)* of Theorem 1.2. Concerning the proof of Theorem 1.3, we note that it could be achieved by simpler and more direct arguments: however here we use the same method of Theorem 1.2 relying on Lemma 3.2 instead of Lemma 4.3. Hence we omit any further detail concerning the proof of Theorem 1.3.

Proof of Theorem 1.2-i). By (2.8) there exists two positive constants C_1 and R such that $Q_{2R} \subseteq Q_{2RC_1}(x, t) \subset Q$ for every $(x, t) \in Q_{2R}^+$. By a standard covering argument and Remark 4.2, it suffices to consider the case $\Omega = Q^+$ and $\Omega' = Q_R^+$. We have to prove that

$$\sup_{\substack{(x,t),(\hat{x},\hat{t}) \in Q_R^+ \\ (x,t) \neq (\hat{x},\hat{t})}} \frac{|u(x, t) - u(\hat{x}, \hat{t})|}{d_K((x, t), (\hat{x}, \hat{t}))^\alpha} \leq c, \tag{5.1}$$

for some positive constant $c = c\left(\alpha, L, \|f\|_{C_K^{0,\alpha}(Q^+)}, \|g\|_{C_K^{0,\alpha}(Q^+)}, \|\psi\|_{C_K^{0,\alpha}(Q^+)}\right)$.

If $t = 0$ then we apply Lemma 4.3 on $Q^+_{2RC_1}(x, 0)$ with $n = 0$ and (5.1) follows. More precisely here we use the fact that

$$|u(\xi, \tau) - u(x, 0)| \le |u(\xi, \tau) - g(\xi, \tau)| + |g(\xi, \tau) - g(x, 0)| \le c\, d_K((\xi, \tau), (x, 0))^\alpha, \tag{5.2}$$

since $g \in C_K^{0,\alpha}(Q)$. Being the case $\hat{t} = 0$ analogous, we next consider both t and $\hat{t}$ strictly positive. We set $(\tilde{x}, 0) = (x, t) \circ (0, t)^{-1} = (E(-t)x, 0)$ so that $d_K((x, t), (\tilde{x}, 0)) = \sqrt{t}$, and we divide the proof in two cases.

Case 1. Assume $(\hat{x}, \hat{t}) \in Q^+_R \setminus Q_{\frac{\sqrt{t}}{2}}(x, t)$. Then we have

$$\begin{aligned} &d_K((\hat{x}, \hat{t}), (\tilde{x}, 0)) \le c\left(d_K((x, t), (\tilde{x}, 0)) + d_K((x, t), (\hat{x}, \hat{t}))\right) \le 2c\, d_K((x, t), (\hat{x}, \hat{t})), \\ &d_K((x, t), (\tilde{x}, 0)) \le 2\, d_K((x, t), (\hat{x}, \hat{t})). \end{aligned} \tag{5.3}$$

Thus

$$|u(x, t) - u(\hat{x}, \hat{t})| \le |u(x, t) - u(\tilde{x}, 0)| + |u(\hat{x}, \hat{t}) - u(\tilde{x}, 0)| \le$$

(by (5.2))

$$\le c_1\left(d_K((x, t), (\tilde{x}, 0))^\alpha + d_K((\hat{x}, \hat{t}), (\tilde{x}, 0))^\alpha\right)$$

(by (5.3))

$$\le c_2 d_K((x, t), (\hat{x}, \hat{t}))^\alpha.$$

Case 2. Assume $(\hat{x}, \hat{t}) \in Q^+_R \cap Q_{\frac{\sqrt{t}}{2}}(x, t)$ and note that $Q_{\sqrt{t}}(x, t) \subseteq Q^+_{2R}$. In this case we first note that, again by (5.2), we have

$$\|u - u(x, t)\|_{L^\infty\left(Q_{\sqrt{t}}(x,t)\right)} \le \|u - u(\tilde{x}, 0)\|_{L^\infty\left(Q_{\sqrt{t}}(x,t)\right)} + |u(x, t) - u(\tilde{x}, 0)| \le c t^{\frac{\alpha}{2}}.$$

Then we set

$$v(y, s) = \frac{u((x, t) \circ \delta_{\sqrt{t}}(y, s)) - u(x, t)}{t^{\frac{\alpha}{2}}}, \qquad (y, s) \in Q. \tag{5.4}$$

By the above estimate, $\|v\|_{L^\infty(Q)} \le c$ and therefore, by Theorem 1.1, we have

$$|v(y, s)| \le c_3 \|(y, s)\|_K^\alpha, \qquad \text{for any } (y, s) \in Q_{1/2}.$$

Scaling back we see that the above inequality can be equivalently written as

$$|u(x, t) - u(\hat{x}, \hat{t})| \le c_3 d_K((x, t), (\hat{x}, \hat{t}))^\alpha.$$

This completes the proof of Theorem 1.2-*i)*. □

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