Higher secant varieties of $\mathbb{P}^n \times \mathbb{P}^m$ embedded in bi-degree $(1, d)$

Alessandra Bernardi*, Enrico Carlini†, Maria Virginia Catalisano‡

Abstract

Let $X^{(n,m)}_{(1,d)}$ denote the Segre-Veronese embedding of $\mathbb{P}^n \times \mathbb{P}^m$ via the sections of the sheaf $\mathcal{O}(1, d)$. We study the dimensions of higher secant varieties of $X^{(n,m)}_{(1,d)}$ and we prove that there is no defective s^{th} secant variety, except possibly for n values of s. Moreover when $\binom{m+d}{d}$ is multiple of $(m + n + 1)$, the s^{th} secant variety of $X^{(n,m)}_{(1,d)}$ has the expected dimension for every s.

Introduction

The s^{th} higher secant variety of a projective variety $X \subset \mathbb{P}^N$ is defined to be the Zariski closure of the union of the span of s points of X (see Definition 1.1), we will denote it with $\sigma_s(X)$.

Secant varieties have been intensively studied (see for example [**AH**], [**BCS**], [**Gr**], [**Li**], [**St**], [**Za**]). One of the first problems of interest is the computation of their dimensions. In fact, there is an expected dimension for $\sigma_s(X) \subset \mathbb{P}^N$, that is, the minimum between N and $s(\dim X) + s - 1$. There are well known examples where that dimension is not attained, for instance, the variety of secant lines to the Veronese surface in $\mathbb{P}^5$. A variety X is said to be $(s-1)$-defective if there exists an integer $s \in \mathbb{N}$ such that the dimension of $\sigma_s(X)$ is less than the expected value. We would like to notice that only for Veronese varieties a complete list of all defective cases is given. This description is obtained using a result by J. Alexander and A. Hirschowitz [**AH**] recently reproposed with a simpler proof in [**BO**].

The interest around these varieties has been recently revived from many different areas of mathematics and applications when X is a variety parameterizing certain kind of tensors (for example Electrical Engineering - Antenna Array Processing [**ACCF**], [**DM**] and Telecommunications [**Ch**], [**dLC**] - Statistics -cumulant tensors, see [**McC**] -, Data Analysis - Independent Component Analysis [**Co1**], [**JS**] -; for other applications see also [**Co2**], [**CR**], [**dLMV**], [**SBG**] [**GVL**]).

One of the main examples is the one of Segre varieties. Segre varieties parameterize completely decomposable tensors (i.e. projective classes of tensors in $\mathbb{P}(V_1 \otimes \cdots \otimes V_t)$ that can be written as $v_1 \otimes \cdots \otimes v_t$, with $v_i \in V_i$ and V_i vector spaces for $i = 1, \ldots, t$). The s^{th} higher secant varieties of Segre varieties is therefore the closure of the sets of tensors that can be written as a linear combination of s completely decomposable tensors.

Segre-Veronese varieties can be described both as the embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ with the sections of the sheaf $\mathcal{O}(d_1, \ldots, d_t)$ into $\mathbb{P}^N$, for certain $d_1, \ldots, d_t \in \mathbb{N}$, with $N = \Pi_{i=1}^t \binom{n_i+d_i}{d} - 1$, both as a section of Segre varieties. Consider the Segre variety that naturally lives in $\mathbb{P}(V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t})$ with V_i vector spaces of dimensions $n_i + 1$ for $i = 1, \ldots, t$, then the Segre-Veronese variety is obtained intersecting that Segre variety with the projective subspaces $\mathbb{P}(S^{d_1}V_1 \otimes \cdots \otimes S^{d_t}V_t)$ of projective classes of partially symmetric tensors (where $S^{d_i}V_i \subset V_i^{\otimes d_i}$ is the subspace of completely symmetric tensors of $V_i^{\otimes d_i}$).

These two different ways of describing Segre-Veronese varieties allow us to translate problems about partially symmetric tensors into problems on forms of multi-degree $(d_1, \ldots, d_t)$ and viceversa. We will follow the description of Segre-Veronese variety as the variety parameterizing forms of certain multi-degree.

*CIRM–FBK c/o Università degli Studi di Trento, via Sommarive 14, 38050 Povo (Trento), Italy. E-mail address: `bernardi@fbk.eu`

†Dipartimento di Matematica Politecnico di Torino, Corso Duca Degli Abruzzi 24, 10129 Torino, Italy. E-mail address: `carlini@calvino.polito.it`

‡DIPTEM - Dipartimento di Ingegneria della Produzione, Termoenergetica e Modelli Matematici, Piazzale Kennedy, pad. D, 16129 Genoa, Italy. E-mail address: `catalisano@diptem.unige.it`

In this paper we will describe the s^{th} higher secant varieties of the embedding of $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^N$ ($N = (n+1)\binom{m+d}{d} - 1$), by the sections of the sheaf $\mathcal{O}(1,d)$, for almost all $s \in \mathbb{N}$ (see Theorem 2.3).

The higher secant varieties of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ are well known as they parameterize matrices of bounded rank (e.g., see [**Hr**]).

One of the first instance of the study of the two factors Segre-Veronese varieties is the one of $\mathbb{P}^1 \times \mathbb{P}^2$ embedded in bi-degree $(1,3)$ and appears in a paper by London [**Lo**], for a more recent approach see [**DF**] and [**CaCh**]. A first generalization for $\mathbb{P}^1 \times \mathbb{P}^2$ embedded in bi-degree $(1,d)$ is treated in [**DF**]. The general case for $\mathbb{P}^1 \times \mathbb{P}^2$ embedded in any bi-degree (d_1,d_2) is done in [**BD**]. In [**ChCi**] the case $\mathbb{P}^1 \times \mathbb{P}^n$ embedded i bi-degree $(d,1)$ is treated.

In [**CaCh**] one can find the defective cases $\mathbb{P}^2 \times \mathbb{P}^3$ embedded in bi-degree $(1,2)$, $\mathbb{P}^3 \times \mathbb{P}^4$ embedded in bi-degree $(1,2)$ and $\mathbb{P}^2 \times \mathbb{P}^5$ embedded in bi-degree $(1,2)$.

The paper [**CGG**] studies also the cases $\mathbb{P}^n \times \mathbb{P}^m$ with bi-degree $(n+1,1)$; $\mathbb{P}^1 \times \mathbb{P}^1$ with bi-degree (d_1,d_2) and $\mathbb{P}^2 \times \mathbb{P}^2$ with bi-degree $(2,2)$. In [**Ab**] the cases $\mathbb{P}^1 \times \mathbb{P}^m$ in bi-degree $(2d+1,2)$, $\mathbb{P}^1 \times \mathbb{P}^m$ in bi-degree $(2d,2)$, and $\mathbb{P}^1 \times \mathbb{P}^m$ in bi-degree $(d,3)$ can be found. A recent result on $\mathbb{P}^n \times \mathbb{P}^m$ in bi-degree $(1,2)$ is in [**AB**], where the authors prove the existence of two functions $\underline{s}(n,m)$ and $\overline{s}(n,m)$ such that $\sigma_s(X^{(n,m)}_{(1,2)})$ has the expected dimension for $s \leq \underline{s}(n,m)$ and for $s \geq \overline{s}(n,m)$. In the same paper it is also shown that $X^{(1,m)}_{(1,2)}$ is never defective and all the defective cases for $X^{(2,m)}_{(1,2)}$ are described.

The varieties $\mathbb{P}^n \times \mathbb{P}^m$ embedded in bi-degree $(1,d)$ are related to the study of Grassmann defectivity ([**DF**]). More precisely, one can consider the Veronese variety X obtained by embedding $\mathbb{P}^m$ in $\mathbb{P}^N$ using the d-uple Veronese embedding ($N = \binom{m+d}{d}$). Then consider, in $\mathbb{G}(n,N)$, the $(n,s-1)$-*Grassmann secant variety* of X, that is, the closure of the set of n-dimensional linear spaces contained in the linear span of s linearly independent points of X. The variety X is said to be $(n,s-1)$-*Grassmann defective* if the $(n,s-1)$-Grassmann secant variety of X has not the expected dimension. It is shown in [**DF**], following Terracini's ideas in [**Te1**], that X is $(n,s-1)$-Grassmann defective if and only if the s^{th} higher secant varieties of the embedding of $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^N$ ($N = (n+1)\binom{m+d}{d} - 1$), by the sections of the sheaf $\mathcal{O}(1,d)$, is $(s-1)$-defective. Hence, the result proved in this paper gives information about the Grasmann defectivity of Veronese varieties (see Remark 2.5).

The main result of this paper is Theorem 2.1 where we prove the regularity of the Hilbert function of a subscheme of $\mathbb{P}^{n+m}$ made of a d-uple $\mathbb{P}^{n-1}$, t projective subspaces of dimension n containing it, a simple $\mathbb{P}^{m-1}$ and a number of double points that is an integer multiple of $n-1$. This theorem, together with Theorem 1.1 in [**CGG**] (see Theorem 1.4 in this paper), gives immediately the regularity of the higher secant varieties of the Segre-Veronese variety that we are looking for.

More precisely, we consider (see Section 2) the case of $\mathbb{P}^n \times \mathbb{P}^m$ embedded in bi-degree $(1,d)$ for $d \geq 3$. We prove (see Theorem 2.3) that the s^{th} higher secant variety of such Segre-Veronese varieties have the expected dimensions for $s \leq s_1$ and for $s \geq s_2$, where

$$s_1 = \max\left\{ s \in \mathbb{N} \mid s \textit{ is a multiple of } (n+1) \textit{ and } s \leq \left\lfloor \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rfloor \right\},$$

$$s_2 = \min\left\{ s \in \mathbb{N} \mid s \textit{ is a multiple of } (n+1) \textit{ and } s \geq \left\lceil \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rceil \right\}.$$

1 Preliminaries and Notation

We will always work with projective spaces defined over an algebraically closed field K of characteristic 0. Let us recall the notion of higher secant varieties and some classical results which we will often use. For definitions and proofs we refer the reader to [**CGG**] .

Definition 1.1. Let $X \subset \mathbb{P}^N$ be a projective variety. We define the s^{th} *higher secant variety* of X, denoted by $\sigma_s(X)$, as the Zariski closure of the union of all linear spaces spanned by s points of X, i.e.:

$$\sigma_s(X) := \overline{\bigcup_{P_1,\ldots,P_s \in X} \langle P_1, \ldots, P_s \rangle} \subset \mathbb{P}^N.$$

When $\sigma_s(X)$ does not have the expected dimension, that is $\min\{N, s(\dim X+1)-1\}$, X is said to be $(s-1)$-*defective*, and the positive integer

$$\delta_{s-1}(X) = \min\{N, s(\dim X+1)-1\} - \dim\sigma_s(X)$$

is called the $(s-1)$-*defect* of X.

The basic tool to compute the dimension of $\sigma_s(X)$ is Terracini's Lemma ([**Te**]):

Lemma 1.2 (**Terracini's Lemma).** *Let X be an irreducible variety in $\mathbb{P}^N$, and let $P_1, \dots, P_s$ be s generic points on X. Then, the tangent space to $\sigma_s(X)$ at a generic point in $\langle P_1, \dots, P_s \rangle$ is the linear span in $\mathbb{P}^N$ of the tangent spaces T_{X,P_i} to X at P_i, $i = 1, \dots, s$, hence*

$$\dim\sigma_s(X) = \dim\langle T_{X,P_1}, \dots, T_{X,P_s} \rangle.$$

A consequence of Terracini's Lemma is the following corollary (see [**CGG**, Section 1] or [**AB**, Section 2] for a proof of it).

Corollary 1.3. *Let $X^{(n,m)}_{(1,d)} \subset \mathbb{P}^N$ be the Segre-Veronese variety image of the embedding of $\mathbb{P}^n \times \mathbb{P}^m$ by the sections of the sheaf $\mathcal{O}(1,d)$ into $\mathbb{P}^N$, with $N = (n+1)\binom{m+d}{d} - 1$. Then*

$$\dim\sigma_s\left(X^{(n,m)}_{(1,d)}\right) = N - \dim(I_Z)_{(1,d)} = H(Z,(1,d)) - 1,$$

where $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ is a set of s generic double points, I_Z is the multihomogeneous ideal of Z in $R = K[x_0, \dots, x_n, y_0, \dots, y_m]$, the multigraded coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$, and $H(Z,(1,d))$ is the multigraded Hilbert function of Z.

Now we recall the fundamental tool which allows us to convert certain questions about ideals of varieties in multiprojective space to questions about ideals in standard polynomial rings (for a more general statement see [**CGG**, Theorem 1.1]) .

Theorem 1.4. *Let $X^{(n,m)}_{(1,d)} \subset \mathbb{P}^N$ and $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ as in Corollary 1.3. Let $H_1, H_2 \subset \mathbb{P}^{n+m}$ be generic projective linear spaces of dimensions $n-1$ and $m-1$, respectively, and let $P_1, \dots, P_s \in \mathbb{P}^{n+m}$ be generic points. Denote by*

$$dH_1 + H_2 + 2P_1 + \cdots + 2P_s \subset \mathbb{P}^{n+m}$$

the scheme defined by the ideal sheaf $\mathcal{I}^d_{H_1} \cap \mathcal{I}_{H_2} \cap \mathcal{I}^2_{P_1} \cap \cdots \cap \mathcal{I}^2_{P_s} \subset \mathcal{O}_{P^{n+m}}$. Then

$$\dim(I_Z)_{(1,d)} = \dim(I_{dH_1+H_2+2P_1+\cdots+2P_s})_{d+1}$$

hence

$$\dim\sigma_s\left(X^{(n,m)}_{(1,d)}\right) = N - \dim(I_{dH_1+H_2+2P_1+\cdots+2P_s})_{d+1}.$$

Since we will make use of Castelnuovo's inequality several times, we recall it here (for notation and proof we refer to [**AH2**], Section 2).

Lemma 1.5 (**Castelnuovo's inequality).** *Let $H \subset \mathbb{P}^N$ be a hyperplane, and let $X \subset \mathbb{P}^N$ be a scheme. We denote by $Res_H X$ the scheme defined by the ideal $(I_X : I_H)$ and we call it the residual scheme of X with respect to H, while the scheme $Tr_H X \subset H$ is the schematic intersection $X \cap H$, called the trace of X on H. Then*

$$\dim(I_{X,\mathbb{P}^N})_t \le \dim(I_{Res_H X,\mathbb{P}^N})_{t-1} + \dim(I_{Tr_H X,H})_t.$$

2 Segre-Veronese embeddings of $\mathbb{P}^n \times \mathbb{P}^m$

Now that we have introduced all the necessary tools that we need for the main theorem of this paper we can state and prove it.

Theorem 2.1. *Let $d \geq 3$, $n, m \geq 1$ and let $s = (n+1)q$ be an integer multiple of $n+1$. Let $P_1, \ldots, P_s \in \mathbb{P}^{n+m}$ be generic points and $H_1 \simeq \mathbb{P}^{n-1}, H_2 \simeq \mathbb{P}^{m-1}$ be generic linear spaces in $\mathbb{P}^{n+m}$. Let $W_1, \ldots, W_t \subset \mathbb{P}^{n+m}$ be t generic linear spaces of dimension n containing H_1. Now consider the scheme*

$$\mathbb{X} := dH_1 + H_2 + 2P_1 + \cdots + 2P_s + W_1 + \cdots + W_t \tag{1}$$

Then for any $q, t \in \mathbb{N}$ the dimension of the degree $d+1$ piece of the ideal $I_{\mathbb{X}}$ is the expected one, that is

$$\dim(I_{\mathbb{X}})_{d+1} = \max\left\{(n+1)\binom{m+d}{d} - s(n+m+1) - t(n+1)\ ; 0\right\}.$$

Proof. We will prove the theorem by induction on n.

A hypersurface defined by a form of $(I_{dH_1})_{d+1}$ cuts on $W_i \simeq \mathbb{P}^n$ a hypersurface which has H_1 as a fixed component of multiplicity d. It follows that

$$\dim(I_{dH_1, W_i})_{d+1} = \dim(I_{\emptyset, W_i})_1 = n+1.$$

Hence the expected number of conditions that a linear space W_i imposes to the forms of $(I_{\mathbb{X}})_{d+1}$ is at most $n+1$. Moreover a double point imposes at most $n+m+1$ conditions. So, since by Theorem 1.4 with $Z = \emptyset$ we get

$$\dim(I_{dH_1+H_2})_{d+1} = \dim R_{(1,d)} = (n+1)\binom{m+d}{d},$$

(where $R = K[x_0, \ldots, x_n, y_0, \ldots, y_m]$), then we have

$$\dim(I_{\mathbb{X}})_{d+1} \geq (n+1)\binom{m+d}{d} - s(n+m+1) - t(n+1). \tag{2}$$

Now let $H \subset \mathbb{P}^{n+m}$ be a generic hyperplane containing H_2 and let $\widetilde{\mathbb{X}}$ be the scheme obtained from $\mathbb{X}$ by specializing the nq points $P_1, \ldots, P_{nq}$ on H, ($P_{nq+1}, \ldots, P_s$ remain generic points, not lying on H).

Since by the semicontinuity of the Hilbert Function $\dim(I_{\widetilde{\mathbb{X}}})_{d+1} \geq \dim(I_{\mathbb{X}})_{d+1}$, by (2) we have

$$\dim(I_{\widetilde{\mathbb{X}}})_{d+1} \geq (n+1)\binom{m+d}{d} - s(n+m+1) - t(n+1). \tag{3}$$

Let $V_i = \langle H_1, P_i \rangle \simeq \mathbb{P}^n$. Since the linear spaces V_i are in the base locus of the hypersurfaces defined by the forms of $(I_{\widetilde{\mathbb{X}}})_{d+1}$, we have

$$(I_{\widetilde{\mathbb{X}}})_{d+1} = (I_{\widetilde{\mathbb{X}}+V_1+\cdots+V_s})_{d+1}. \tag{4}$$

Consider the residual scheme of $(\widetilde{\mathbb{X}} + V_1 + \cdots + V_s)$ with respect to H:

$$Res_H(\widetilde{\mathbb{X}} + V_1 + \cdots + V_s) = dH_1 + W_1 + \cdots + W_t + P_1 + \cdots + P_{nq} + 2P_{nq+1} + \cdots + 2P_s + V_1 + \cdots + V_s$$

$$= dH_1 + W_1 + \cdots + W_t + 2P_{nq+1} + \cdots + 2P_s + V_1 + \cdots + V_s \subset \mathbb{P}^{n+m}.$$

Any form of degree d in $I_{Res_H(\widetilde{\mathbb{X}}+V_1+\cdots+V_s)}$ represents a cone whose vertex contains H_1. Hence if $\mathbb{Y} \subset \mathbb{P}^m$ is the scheme obtained by projecting $Res_H(\widetilde{\mathbb{X}} + V_1 + \cdots + V_s)$ from H_1 in a $\mathbb{P}^m$, we have:

$$\dim(I_{Res_H(\widetilde{\mathbb{X}}+V_1+\cdots+V_s)})_d = \dim(I_{\mathbb{Y}})_d. \tag{5}$$

Since the image by this projection of each W_i is a point, and for $1 \leq i \leq nq$ the image of $P_i + V_i$ is a simple point, and for $nq+1 \leq i \leq s$ the image of $2P_i + V_i$ is a double point, we have that $\mathbb{Y}$ is a scheme consisting of $t + nq$ generic points and q generic double points.

Now by the Alexander-Hirschowitz Theorem (see [**AH**]), since $d > 2$ and $t + nq > 1$ we have that the dimension of the degree d part of the ideal of q double points plus $t + nq$ simple points is always as expected. So we get

$$\dim(I_{\mathbb{Y}})_d = \max\left\{ \binom{m+d}{d} - q(m+1) - t - nq \ ; \ 0 \right\}. \tag{6}$$

Now let $n = 1$. In this case we have: $s = 2q$,

$$\dim(I_{Res_H(\widetilde{\mathbb{X}}+V_1+\cdots+V_s)})_d = \dim(I_{\mathbb{Y}})_d = \max\left\{ \binom{m+d}{d} - q(m+1) - t - q \ ; \ 0 \right\}, \tag{7}$$

moreover H_1 is a point, $H_1 \cap H$ is the empty set, the W_i and the V_i are lines, and V_i is the line H_1P_i.

Set $W_i' = W_i \cap H$, $V_i' = V_i \cap H$. Note that for $1 \leq i \leq q$ we have $V_i' = P_i$. The trace on H of $\widetilde{\mathbb{X}} + V_1 + \cdots + V_s$ is:

$$\begin{aligned} Tr_H(\widetilde{\mathbb{X}} + V_1 + \cdots + V_s) &= H_2 + 2P_1 + \cdots + 2P_q + W_1' + \cdots + W_t' + V_1' + \cdots + V_{2q}' = \\ &= H_2 + 2P_1 + \cdots + 2P_q + W_1' + \cdots + W_t' + V_{q+1}' + \cdots + V_{2q}' \subset H \simeq \mathbb{P}^m. \end{aligned}$$

So $Tr_H(\widetilde{\mathbb{X}} + V_1 + \cdots + V_s) \subset H$ is a scheme in $\mathbb{P}^m$ union of $H_2 \simeq \mathbb{P}^{m-1}$, plus q generic double points and $t+q$ generic simple points. As above, by [**AH**], since $d > 2$ and $t + q \geq 1$ we get

$$\begin{aligned} \dim(I_{Tr_H(\widetilde{\mathbb{X}}+V_1+\cdots+V_s)})_{d+1} &= \dim(I_{2P_1+\cdots+2P_q+W_1'+\cdots+W_t'+V_{q+1}'+\cdots+V_{2q}'})_d \\ &= \max\left\{ \binom{m+d}{d} - q(m+1) - t - q \ ; \ 0 \right\}. \end{aligned} \tag{8}$$

By Castelnuovo's inequality (see Lemma 1.5), by (7) and (8) we get

$$\dim(I_{\widetilde{\mathbb{X}}+V_1+\cdots+V_s})_{d+1} \leq \max\left\{ 2\binom{m+d}{d} - 2q(m+1) - 2t - 2q \ ; \ 0 \right\}, \tag{9}$$

so by (3), (4) and (9) we have

$$\dim(I_{\widetilde{\mathbb{X}}})_{d+1} = \max\left\{ 2\binom{m+d}{d} - 2q(m+2) - 2t \ ; \ 0 \right\}.$$

From here, by (2) and by the semicontinuity of the Hilbert Function we get

$$\dim(I_{\mathbb{X}})_{d+1} = \max\left\{ 2\binom{m+d}{d} - 2q(m+2) - 2t \ ; \ 0 \right\}$$

and the result is proved for $n = 1$.

Let $n > 1$.

Set: $H_1' = H_1 \cap H$; $W_i' = W_i \cap H$; $V_i' = V_i \cap H$. With this notation the trace of $\widetilde{\mathbb{X}} + V_1 + \cdots + V_s$ on H is:

$$Tr_H(\widetilde{\mathbb{X}} + V_1 + \cdots + V_s) = dH_1' + H_2 + 2P_1 + \cdots + 2P_{nq} + W_1' + \cdots + W_t' + V_1' + \cdots + V_s' \subset H \simeq \mathbb{P}^{n+m-1}.$$

Anaugously as above, observe that the linear spaces $V_i' = \langle H_1', P_i \rangle \simeq \mathbb{P}^n$ are in the base locus for the hypersurfaces defined by the forms of $(I_{dH_1'+2P_i})_{d+1}$, hence the parts of degree $d+1$ of the ideals of $Tr_H(\widetilde{\mathbb{X}} + V_1 + \cdots + V_s)$ and of $Tr_H(\widetilde{\mathbb{X}} + V_{nq+1} + \cdots + V_s)$ are equal. So we have

$$(I_{Tr_H(\widetilde{\mathbb{X}}+V_1+\cdots+V_s)})_{d+1} = (I_{Tr_H(\widetilde{\mathbb{X}}+V_{nq+1}+\cdots+V_s)})_{d+1} = (I_{\mathbb{T}})_{d+1},$$

where

$$\mathbb{T} = dH_1' + H_2 + 2P_1 + \cdots + 2P_{nq} + W_1' + \cdots + W_t' + V_{nq+1}' + \cdots + V_s' \subset \mathbb{P}^{n+m-1},$$

that is, $\mathbb{T}$ is union of the d-uple linear space $H_1' \simeq \mathbb{P}^{n-2}$, the linear space $H_2 \simeq \mathbb{P}^{m-1}$, $t+q$ generic linear spaces through H_1', and nq double points. Hence by the inductive hypothesis we have

$$\dim(I_{\mathbb{T}})_{d+1} = \max\left\{ n\binom{m+d}{d} - nq(n+m) - (t+q)n \;; 0 \right\}. \tag{10}$$

By (4), by Lemma 1.5, by (5), (6) and (10) we get

$$\begin{aligned}
\dim(I_{\tilde{\mathbb{X}}})_{d+1} &\leq \max\left\{ \binom{m+d}{d} - q(m+1) - t - nq \;;\; 0 \right\} + \max\left\{ n\binom{m+d}{d} - nq(n+m) - (t+q)n \;; 0 \right\} \\
&= \max\left\{ \binom{m+d}{d} - q(m+1) - t - nq \;;\; 0 \right\} + \max\left\{ n\left(\binom{m+d}{d} - q(m+1) - t - nq\right) \;; 0 \right\} \\
&= \max\left\{ (n+1)\left(\binom{m+d}{d} - q(m+1) - t - nq\right) \;; 0 \right\} \\
&= \max\left\{ (n+1)\binom{m+d}{d} - s(n+m+1) - t(n+1) \;; 0 \right\}.
\end{aligned}$$

Now the conclusion follows from (2) and the semicontinuity of the Hilbert Function and this ends the proof. □

Corollary 2.2. *Let $d \geq 3$, $n, m \geq 1$ and let*

$$s_1 := \max\left\{ s \in \mathbb{N} \mid s \text{ is a multiple of } (n+1) \text{ and } s \leq \left\lfloor \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rfloor \right\}$$

$$s_2 := \min\left\{ s \in \mathbb{N} \mid s \text{ is a multiple of } (n+1) \text{ and } s \geq \left\lceil \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rceil \right\}.$$

Let $P_1, \dots, P_s \in \mathbb{P}^{n+m}$ be generic points and $H_1 \simeq \mathbb{P}^{n-1}, H_2 \simeq \mathbb{P}^{m-1}$ be generic linear spaces in $\mathbb{P}^{n+m}$. Consider the scheme

$$\mathbb{X} := dH_1 + H_2 + 2P_1 + \cdots + 2P_s.$$

Then for any $s \leq s_1$ and any $s \geq s_2$ the dimension of $(I_{\mathbb{X}})_{d+1}$ is the expected one, that is

$$\dim(I_{\mathbb{X}})_{d+1} = \begin{cases} (n+1)\binom{m+d}{d} - s(n+m+1) & \text{for } s \leq s_1 \\ 0 & \text{for } s \geq s_2 \end{cases}$$

Proof. By applying Theorem 2.1, with $t = 0$, to the scheme $\mathbb{X} = dH_1 + H_2 + 2P_1 + \cdots + 2P_s$, we get that the dimension of $I(\mathbb{X})_{d+1}$ is the expected one for $s = (n+1)q$ and for any $q \in \mathbb{N}$. Hence if s_1 is the biggest integer multiple of $n+1$ such that $\dim(I_{\mathbb{X}})_{d+1} \neq 0$ we get that for that value of s the Hilbert function $H(I_{\mathbb{X}}, d+1)$ has the expected value. Now if for such s_1 we have that $(I_{\mathbb{X}})_{d+1}$ has the expected dimension than it has the expected dimension also for every $s \leq s_1$.

Now, if s_2 is the smallest integer multiple of $n+1$ such that $\dim(I_{\mathbb{X}})_{d+1} = 0$ then obviously such a dimension will be zero for all $s \geq s_2$. □

Theorem 2.3. *Let $d \geq 3$, $n, m \geq 1$, $N = (n+1)\binom{m+d}{d} - 1$ and let s_1, s_2 be as in Corollary 2.2.*

Then the variety $\sigma_s\left(X_{(1,d)}^{(n,m)}\right) \subset \mathbb{P}^N$ has the expected dimension for any $s \leq s_1$ and any $s \geq s_2$, that is

$$\dim \sigma_s\left(X_{(1,d)}^{(n,m)}\right) = \begin{cases} s(n+m+1) - 1 & \text{for } s \leq s_1 \\ N & \text{for } s \geq s_2 \end{cases}.$$

Proof. Let $H_1, H_2 \subset \mathbb{P}^{m+n}$ be projective subspaces of dimensions $n-1$ and $m-1$ respectively and let $P_1, \dots, P_s \in \mathbb{P}^{n+m}$ be s generic points of $\mathbb{P}^{n+m}$. Define $\mathbb{X} \subset \mathbb{P}^{m+n}$ to be the scheme $\mathbb{X} := dH_1 + H_2 + 2P_1 + \cdots + 2P_s$. Theorem 1.1 in [**CGG**] shows that $\dim \sigma_s \left(X^{(n,m)}_{(1,d)} \right)$ is the expected one if and only if $\dim(I_{\mathbb{X}})_{d+1}$ is the expected one. Therefore the conclusion immediately follows from Theorem 1.4 and Corollary 2.2. $\square$

Remark 2.4. If $\binom{m+d}{d}$ is multiple of $(m+n+1)$, say $\binom{m+d}{d} = h(m+n+1)$, we get

$$\left\lfloor \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rfloor = \left\lceil \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rceil = h(n+1)$$

so $s_1 = s_2$. Hence in this case the variety $\sigma_s \left(X^{(n,m)}_{(1,d)} \right)$ has the expected dimension for any s.

If $\binom{m+d}{d}$ is not multiple of $(m+n+1)$, it is easy to show that $s_2 - s_1 = n$. Thus there are at most n values of s for which the s^{th} higher secant varieties of $X^{(n,m)}_{(1,d)}$ can be defective.

Remark 2.5. Theorem 2.3 has a straightforward interpretation in terms of Grassmann defectivity. More precisely, we see that the d-uple Veronese embedding of $\mathbb{P}^m$ is not $(n, s-1)$-Grassmann defective when $s \le s_1$ or $s \ge s_2$.

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