Some defective secant varieties to osculating varieties of Veronese surfaces

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Abstract. We consider the k-osculating varieties $O_{k,d}$ to the Veronese $d-$uple embeddings of $\mathbb{P}^2$. By studying the Hilbert function of certain zero-dimensional schemes $Y \subset \mathbb{P}^2$, we find the dimension of $O^s_{k,d}$, the $(s-1)^{th}$secant varieties of $O_{k,d}$, for $3 \leq s \leq 6$ and $s = 9$, and we determine whether those secant varieties are defective or not.

0. Introduction.

The problem of determining the dimension of the *higher secant varieties* of a projective variety is a classical subject of study. In the present paper we are concerned with the $(s-1)^{th}$ higher secant varieties of $O_{k,V_{n,d}}$, where $O_{k,V_{n,d}}$ is the k-osculating variety to the Veronese embedding $V_{n,d}$ of $\mathbb{P}^n$ into $\mathbb{P}^N$ ($N = \binom{d+n}{n} - 1$) via the complete linear system R_d, where $R = K[x_0, \ldots, x_n]$, and K is an algebraically closed field of characteristic zero.

This matter has been dealt with by several authors in the last few years (see [2], [3], [4], [5], [7]). We wish to mention E.Ballico and C.Fontanari. In [2] and [3] they study the higher secant varieties of $O_{k,V_{n,d}}$ for $n = 2$ and $k = 1, 2$, and they prove the following results:

0.1. Proposition. *For $k = 1$, the $(s-1)^{th}$ higher secant variety of the tangential variety to $V_{2,d}$ has the expected dimension, unless $s = 2$ and $d = 3$.*

0.2. Proposition. *For $k = 2$, the $(s-1)^{th}$ higher secant variety of 2-osculating variety to $V_{2,d}$ has the expected dimension, unless $s = 2$ and $d = 4$.*

In this note, for $n = 2$, for $3 \leq s \leq 6$ and $s = 9$, and for all k, we will determine the dimension of $O^s_{k,V_{2,d}}$. The methods for proving our results are similar to the ones used by Ballico and Fontanari. The basic idea is to use Terracini's Lemma (see [13]), then, via apolarity, the calculation of $\dim O^s_{k,V_{2,d}}$ is related to the evaluation of the Hilbert function of certain $0-$dimensional schemes $Y \subset \mathbb{P}^2$ supported at s generic points (see [5]), and this is done using geometric constructions, Bezout's theorem, and the Horace Method ([11]).

In the first section we fix some notation, and describe the relationship between the higher secant varieties we want to study, and the 0-dimensional schemes Y.

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In the second section we relate the Hilbert function of Y to the Hilbert function $H(X,d)$ of a scheme X of s generic $(k+1)$-fat points (Lemma 2.2, Proposition 2.6 to 2.9, and Proposition 2.12), and in Theorem 2.13 we prove the main result of this paper, i.e., for $s \leq 6$ and $s = 9$:

$$\dim O^s_{k,V_{2,d}} = \min\{H(X,d) + 2s,\ N+1\} - 1,$$

except when $s = 2$, $d = k+2$. In this case

$$\dim O^2_{k,V_{2,k+2}} = H(T,d) - 1 = N - 1,$$

where $H(T,d)$ is the Hilbert function of a scheme T of s generic $(k+2)$-fat points.

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1. Preliminaries and notation.

1.1. Definition. If $V \subset \mathbb{P}^N$ is an irreducible projective variety, an *m-fat point* on V is the $(m-1)^{th}$ infinitesimal neighborhood of a smooth point P in V, and it will be denoted by mP (i.e., the scheme mP is defined by the ideal sheaf $\mathcal{I}^m_{P,V} \subset \mathcal{O}_V$). If $\dim V = n$, then, mP is a 0-dimensional scheme of length $\binom{m-1+n}{n}$. If X is the union of the $(m-1)^{th}$ infinitesimal neighborhoods in V of s generic smooth points of V, we will say for short that X is union of s generic m-fat points on V.

1.2. Remark. In general it is a hard problem to determine the postulation for a union of m-fat points. If $V = \mathbb{P}^2$, there is a conjecture for the postulation of a generic union $X \subset \mathbb{P}^2$ of s m-fat points (e.g. see [10]): for $s \geq 10$ the conjecture says that X is regular in any degree d. This has been proved for $m \leq 20$ in [8], and, when s is a square, by L.Evain in [9]. For $s \leq 9$ all the defective cases are known (e.g., see [8] or [10]), more precisely, for any m and $s \leq 9$ the cases in which $X \subset \mathbb{P}^2$ is not regular are:

i) $s = 2$, and $m \leq d \leq 2m-2$;
ii) $s = 3$, and $\frac{3m}{2} \leq d \leq 2m-2$;
iii) $s = 5$, and $2m \leq d \leq \frac{5m-2}{2}$;
iv) $s = 6$, and $\frac{12m}{5} \leq d \leq \frac{5m-2}{2}$;
v) $s = 7$, and $\frac{21m}{8} \leq d \leq \frac{8m-2}{3}$;
vi) $s = 8$, and $\frac{48m}{17} \leq d \leq \frac{17m-2}{6}$.

Now we recall the notions of higher secant variety and k^{th} osculating variety.

1.3. Definition. Let $V \subset \mathbb{P}^N$ be a closed irreducible projective variety; the $(s-1)^{th}$ *higher secant variety* of V is the closure of the union of all linear spaces spanned by s points of V, and it will be denoted by V^s. Let $\dim V = n$; the *expected dimension* for V^s is

$$(\dagger) \qquad \operatorname{expdim} V^s = \min\{sn + s - 1,\ N\}$$

where the number $sn+s-1$ corresponds to ∞^{sn} choices of s points on V, plus ∞^{s-1} choices of a point on the $\mathbb{P}^{s-1}$ spanned by the s points. When $\dim V^s < \min\{sn+s-1,\ N\}$, the variety V^s is said to be *defective*, with *defect* $\delta = \min\{sn+s-1,\ N\} - \dim V^s$.

1.4. Definition. Let $V \subset \mathbb{P}^N$ be a variety, and let $P \in V$ be a smooth point; we define *the k^{th} osculating space to V at P*, and we denote it by $O_{k,V,P}$, as the linear space defined by the vanishing of all linear forms L such that $L|_V$ vanishes to order $k+1$ on V at P. Let $V_0 \subset V$ be the dense set of the smooth points where $O_{k,V,P}$ has maximal dimension. The k^{th} *osculating variety to* V is defined as:

$$O_{k,V} = \overline{\bigcup_{P \in V_0} O_{k,V,P}}.$$

1.5. Notation. Set $R = K[x,y,z] = \bigoplus R_d$. Let $V_d \subset \mathbb{P}^N$, $N = \binom{d+2}{2} - 1$, denote the *d-ple Veronese embedding* of $\mathbb{P}^2$, defined by the linear system R_d of all forms of a given degree d. Set $O_{k,d} = O_{k,V_d}$, so that the $(s-1)^{th}$ higher secant variety to the k^{th} osculating variety to the Veronese surface V_d will be denoted by $O^s_{k,d}$.

1.6. Remark. We have (see [5], Lemma 2.3) that the dimension of $O_{k,d}$ is always the expected one, that is

$$\dim O_{k,d} = \min\left\{ \binom{k+2}{2} + 1,\ \binom{d+2}{2} - 1 \right\}.$$

For $d \le k$ we immediately get $O_{k,d} = \mathbb{P}^N$, hence for $d \le k$ and for all s, we have $O^s_{k,d} = \mathbb{P}^N$.

Now we briefly recall how to associate to $O^s_{k,d}$, a zero dimensional scheme $Y \subset \mathbb{P}^2$ (see [5], Remark 2.2).

1.7. Remark. Let $\mathbb{P}^N = \mathbb{P}(R_d)$, and let $d \ge k+1$. A form $M \in R_d$ will denote, depending on the situation, a vector in R_d or a point in $\mathbb{P}^N$. We can view V_d as the image's closure of the map $(\mathbb{P}^2)^* \to \mathbb{P}^N$, where $L \mapsto L^d$, $L \in R_1$. Hence

$$V_d = \{L^d, \quad L \in R_1\}.$$

At the point $Q = L^d$ we have $O_{k,V_d,Q} = \{L^{d-k}F, \quad F \in R_k\}$ and $O_{k,d} = \bigcup_{Q \in V_d} O_{k,V_d,Q}$. So we have:

$$O_{k,d} = \{L^{d-k}F, \quad L \in R_1, \quad F \in R_k\}$$

hence

$$O^s_{k,d} = \{L_1^{d-k}F_1 + \cdots + L_s^{d-k}F_s, \quad L_i \in R_1, \quad F_i \in R_k, \quad i = 1, \ldots, s\}.$$

Let $P_i = L_i^{d-k}F_i$ be a generic point in $O_{k,d}$, and let $T_{O_{k,d},P_i}$ be the tangent space of $O_{k,d}$ at P_i. The affine cone over $T_{O_{k,d},P_i}$ is $W_i = < L_i^{d-k}R_k, L_i^{d-k-1}F_iR_1 >$.

Terracini's Lemma (see [13]) says that the tangent space of $O^s_{k,d}$ at a generic point of $< P_1, \ldots, P_s >$, $(P_1, \ldots, P_s \in O_{k,d})$, is the span of the tangent spaces of $O_{k,d}$ at P_i $(1 \le i \le s)$; if $T_{O_{k,d},P_i} = \mathbb{P}(W_i)$, then

$$\dim O^s_{k,d} = \dim < T_{O_{k,d},P_1}, \ldots, T_{O_{k,d},P_s} >= \dim < W_1, \ldots, W_s > -1$$

Now consider the ortogonal space $W_i^\perp \subset R_d$, $(1 \le i \le s)$ via the apolarity action (for the definition of $W_i^\perp$ see [5], Remark 2.5). It generates an ideal in R defining a scheme $Z_i(k,d) \subset \mathbb{P}^2$. Let Y be a generic union of s schemes $Z_i(k,d)$ in $\mathbb{P}^2$, $(1 \le i \le s)$. Since

$$\dim < W_1, \dots, W_s > -1 = N - \dim[< W_1, \dots, W_s >]^\perp = N - \dim(W_1^\perp \cap \dots \cap W_s^\perp) = N - h^0(\mathbb{P}^2, \mathcal{I}_Y(d)),$$

we have (see also [5], Remark 2.5):

$$(\ddagger) \qquad \dim O^s_{k,d} = N - h^0(\mathbb{P}^2, \mathcal{I}_Y(d)) = H(Y,d) - 1$$

where $H(Y,d)$ is the Hilbert function of Y in degree d.

Since for $d = k+1$, $O^2_{k,d} = \mathbb{P}^N$ (see [5], Proposition 3.4 C)), then for $s \ge 2$ we immediately get:

1.8. Proposition. *For $d = k+1$ and $s \ge 2$, we have $O^s_{k,d} = \mathbb{P}^N$.*

For $d \ge k+2$, the schemes $Z_i(k,d)$ are zero-dimensional, and do not depend on d, in fact we have the following lemma (see [5], Lemmata 2.6, 2.7, 2.8):

1.9. Lemma. *Let $Z(k,d) = Z_i(k,d)$ be one such scheme with support at P. For $d \ge k+2$, we have:*

i) $(k+1)P \subset Z(k,d) \subset (k+2)P$;

ii) the length of $Z(k,d)$ is $l(Z) = \binom{k+2}{2} + 2$;

iii) $Z(k,d) = Z(k,k+2)$.

Henceforth for $d \ge k+2$ we will denote $Z(k,d)$ by $Z(k)$, or Z, if k is obvious by the context.

From ($\ddagger$) and the lemma above it follows that for $d \ge k+2$ in order to study the dimension of $O^s_{k,d}$, we only need to study the postulation of unions of generic schemes $Z(k)$.

1.10. Remark. Let $d \ge k+2$. Recall that $Z(k)$ is defined by the ideal generated by $W^\perp \subset R_d$, where $W =< L^{d-k}R_k, L^{d-k-1}FR_1 >$, with $L \in R_1$ and $F \in R_k$. Now we want to give a specialization of the scheme $Z(k)$: put $L = x$ and $F = y^k$; we get

$$W =< x^{d-k}R_k, x^{d-k-1}y^kR_1 >$$

hence

$$\begin{aligned} W^\perp =< & x^{d-k-1}y^{k-1}z^2, \dots, x^{d-k-1}yz^k, x^{d-k-1}z^{k+1}, x^{d-k-2}y^{k+2}, x^{d-k-2}y^{k+1}z, \dots, \\ & x^{d-k-2}yz^{k+1}, x^{d-k-2}z^{k+2}, x^{d-k-3}y^{k+1}, x^{d-k-3}y^kz, \dots, x^{d-k-3}yz^k, x^{d-k-3}z^{k+1}, \dots, \\ & xy^{d-1}, xy^{d-2}z, \dots, xyz^{d-2}, xz^{d-1}, y^d, y^{d-1}z, \dots, yz^{d-1}, z^d > . \end{aligned}$$

Let I be the ideal generated by $W^\perp$. By a direct computation, it is easy to show that the saturation of I is the ideal

$$(I)^{sat} = (y,z)^{k+1} \cap ((y,z)^{k+2} + (z^2))$$

that defines a scheme supported at a point of $\mathbb{P}^2$, whose structure is given by the union of its k^{th} infinitesimal neighbourhood, with the intersection of its $(k+1)^{th}$ infinitesimal neighbourhood with a double line.

1.11. Notation. We fix the following notation:

i) let $P_1, \dots, P_s$ be s generic points in $\mathbb{P}^2$;

ii) let X be the union of s generic $(k+1)$-fat points in $\mathbb{P}^2$, with support in $P_1, \dots, P_s$;

iii) let T be the union of s generic $(k+2)$-fat points in $\mathbb{P}^2$, with support in $P_1, \dots, P_s$;

iv) let Z_i be a 0-dimensional scheme in $\mathbb{P}^2$, as defined in Remark 1.7, with support in P_i;

v) let $Y = Z_1 + \cdots + Z_s$;

vi) denote by $(k+1, k+2)P$ a 0-dimensional scheme whose defining ideal is $\wp^{k+1} \cap (\wp^{k+2} + l^2)$ where $\wp$ is the homogeneous ideal in $R = K[x, y, z]$ of a point $P \in \mathbb{P}^2$, and l is the ideal of a generic line through P; we call $(k+1, k+2)P$ a $(k+1, k+2)$ *point*;

vii) let $\mathcal{Z}_i$ be a $(k+1, k+2)$ point with support in P_i. By Remark 1.10, the scheme $\mathcal{Z}_i$ is a specialization of Z_i;

viii) let $\mathcal{Y} = \mathcal{Z}_1 + \cdots + \mathcal{Z}_s$ (so $\mathcal{Y}$ is a specialization of the scheme Y). We have

$$\deg \mathcal{Y} = \deg Y = s\left(\binom{k+2}{2} + 2\right) = \deg X + 2s;$$

ix) if $\mathcal{C} \subset \mathbb{P}^2$ is a curve, and Z is a zero-dimensional scheme, the scheme Z' defined by the ideal $(I_Z : I_{\mathcal{C}})$ is called the residual of Z with respect to $\mathcal{C}$, and denoted by $Res_{\mathcal{C}} Z$.

In the following lemma we determine the subscheme of a $(k+1, k+2)$ point with support in P, residual to a curve $\mathcal{C}$.

1.12. Lemma. *Let $\mathcal{Z}$ be a $(k+1, k+2)$ point, with support in P with defining ideal $\wp^{k+1} \cap (\wp^{k+2} + l^2)$, where $\wp$ is the ideal of P, and $l = (L)$ is the ideal of a generic line through P. Let $\mathcal{C} \subset \mathbb{P}^2$ be a curve having at P a singularity of multiplicity m, and having L as tangent direction with multiplicity t. Then $Res_{\mathcal{C}}(\mathcal{Z})$ is defined by the ideal*

$$I_{Res_{\mathcal{C}}(\mathcal{Z})} = \wp^{\max\{k+1-m;0\}} \cap (\wp^{\max\{k+2-m;0\}} + l^{\max\{2-t;0\}}).$$

$Res_{\mathcal{C}}(\mathcal{Z})$ is a fat point, or a $(k+1-m, k+2-m)$ point, except for $m < k+1$ and $t = 1$, more precisely:

$$Res_{\mathcal{C}}(\mathcal{Z}) = \begin{cases} 0P & for \;\; m \geq k+2, \;\; or \;\; m = k+1 \;\; and \;\; t \geq 2 \\ 1P & for \;\; m = k+1 \;\; and \;\; t \leq 1 \\ (k+1-m)P & for \;\; m < k+1 \;\; and \;\; t \geq 2 \\ 2P & for \;\; m = k \;\; and \;\; t = 0 \\ (k+1-m, k+2-m)P & for \;\; m < k \;\; and \;\; t = 0 \end{cases}.$$

Proof. Without loss of generality, we assume that $\wp = (x, y)$, $L = x$, and, by abuse of notation, that x, y are affine coordinates.

Let $x^t f_1 + f_2 = 0$ be an equation defining the curve $\mathcal{C}$, where f_1 is a homogeneous polynomial of degree $m - t$, $f_1 \notin (x)$, and $f_2 \in (x, y)^{m+1}$. We have to prove that

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y)^{\max\{k+1-m;0\}} \cap ((x, y)^{\max\{k+2-m;0\}} + (x^{\max\{2-t;0\}})).$$

This is obvious for $m \geq k+2$, and for $m = k+1, t \geq 2$, since in these cases $Res_{\mathcal{C}}(\mathcal{Z})$ is the emptyset.

Let $m = k+1$, $t \leq 1$. The equality above becomes

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y).$$

"$\subseteq$" : To prove this inclusion, let $g = a + h$, $a \in K$, $h \in (x, y)$. If $g \; (x^t f_1 + f_2) = (a + h) \; (x^t f_1 + f_2) \in ((x, y)^{k+2} + (x^2))$, since $f_2 \in (x, y)^{m+1}$, $h x^t f_1 \in (x, y)^{m+1}$, and $m + 1 = k + 2$, it follows that

$ax^t f_1 \in ((x,y)^{k+2} + (x^2))$. But f_1 is a homogeneous polynomial of degree $m-t$, $f_1 \notin (x)$, $t \leq 1$, so it easily follows that $a = 0$, hence $g \in (x,y)$. The reverse inclusion is obvious.

Since $I_{Res_{\mathcal{C}}(\mathcal{Z})} = (x,y)$, we have $Res_{\mathcal{C}}(\mathcal{Z}) = 1P$.

Now, let $m < k+1$, $t \geq 2$. In this case we have to prove that

$$((x,y)^{k+1} \cap ((x,y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x,y)^{k+1-m}.$$

If $g\ (x^t f_1 + f_2) \in (x,y)^{k+1}$, it immediately follows that $g \in (x,y)^{k+1-m}$, and the reverse inclusion is obvious. Moreover, since $I_{Res_{\mathcal{C}}(\mathcal{Z})} = (x,y)^{k+1-m}$, we have that $Res_{\mathcal{C}}(\mathcal{Z}) = (k+1-m)P$.

Let $m \leq k$, $t \leq 1$. Now we have to prove that

$$((x,y)^{k+1} \cap ((x,y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x,y)^{k+1-m} \cap ((x,y)^{k+2-m} + (x^{2-t})).$$

"$\subseteq$" : As in the previous case, if $g\ (x^t f_1 + f_2) \in (x,y)^{k+1}$, it follows that $g \in (x,y)^{k+1-m}$, so we can write

$$g = xg_1 + ay^{k+1-m} + g_2,$$

where $g_1 \in (x,y)^{k-m}$ is homogeneous of degree $k-m$, $g_2 \in (x,y)^{k+2-m}$, $a \in K$. In order to prove that

$$g\ (x^t f_1 + f_2) = (xg_1 + ay^{k+1-m} + g_2)(x^t f_1 + f_2) \in ((x,y)^{k+2} + (x^2)))$$

since $g_2 x^t f_1$, and $f_2 \in (x,y)^{k+2}$, it suffices to prove that

$$x^{t+1} g_1 f_1 + ax^t y^{k+1-m} f_1 \in ((x,y)^{k+2} + (x^2)).$$

Since $x^{t+1} g_1 f_1 + ax^t y^{k+1-m} f_1$ is homogeneous of degree $k+1$, and $f_1 \notin (x)$, we get that $x^{t+1} g_1 + ax^t y^{k+1-m} \in (x^2)$. For $t = 1$, this implies $a = 0$, so $g \in ((x,y)^{k+2-m} + (x))$. For $t = 0$ this implies $a = 0$, and $g_1 \in (x)$, so $g \in ((x,y)^{k+2-m} + (x^2))$.

"$\supseteq$" : This inclusion is obvious.

So we have proved that, for $m \leq k$ and $t \leq 1$:

$$I_{Res_{\mathcal{C}}(\mathcal{Z})} = \begin{cases} (x,y)^{k+1-m} \cap ((x,y)^{k+2-m} + (x)) = (x,y)^{k+1-m} \cap (x, y^{k+2-m}) & for\ \ m \leq k\ \ and\ \ t = 1 \\ (x,y) \cap ((x,y)^2 + (x^2)) = (x,y)^2 & for\ \ m = k\ \ and\ \ t = 0 \\ (x,y)^{k+1-m} \cap ((x,y)^{k+2-m} + (x^2)) & for\ \ m < k\ \ and\ \ t = 0 \end{cases},$$

hence for $m = k$ and $t = 0$ we have $Res_{\mathcal{C}}(\mathcal{Z}) = 2P$, for $m < k$ and $t = 0$ we have $Res_{\mathcal{C}}(\mathcal{Z}) = (k+1-m, k+2-m)P$, while for $m \leq k$ and $t = 1$, $Res_{\mathcal{C}}(\mathcal{Z})$ is the union of the fat point $(k+1-m)P$ with the intersection of the line $\{x = 0\}$ with the fat point $(k+2-m)P$. $\square$

2. Osculating varieties to Veronese surface and some of their higher secant varieties.

In this section we will compute the dimension of $O^s_{k,d}$ for $3 \leq s \leq 6$ and $s = 9$.

2.1. Remark. We recall that for $d \leq k+1$ and $s \geq 2$ (see Remark 1.6 and Proposition 1.8):

$$\dim O^s_{k,d} = N.$$

So we have to study the dimension of $O^s_{k,d}$ only for $d \geq k+2$. Since, for $d \geq k+2$ (see(‡))

$$\dim O^s_{k,d} = H(Y,d) - 1,$$

then, if we know the postulation of Y, we are done.

We wish to notice that, by (†), the expected dimension for $O^s_{k,d}$ is

$$\text{expdim } O^s_{k,d} = \min\{sn + s - 1,\ N\},$$

where $n = \dim O_{k,d} = \min\left\{\binom{k+2}{2} + 1,\ \binom{d+2}{2} - 1\right\} = \min\left\{\binom{k+2}{2} + 1,\ N\right\} = \min\left\{\frac{\deg Y}{s} - 1,\ N\right\}$ (see Remark 1.6 and Lemma 1.9 *ii*)). Hence it easily follows that

$$\text{expdim } O^s_{k,d} = \min\{\deg Y,\ N+1\} - 1 = \ \exp\ H(Y,d) - 1$$

where $\exp\ H(Y,d)$ is the expected value for $H(Y,d)$.

In the next lemmata we show that the postulation of Y is strictly related with the postulation of the specialized scheme $\mathcal{Y}$, and of the scheme of fat points X.

2.2. Lemma. *If the Hilbert function of the specialized scheme $\mathcal{Y}$ in degree d is*

$$H(\mathcal{Y},d) = \min\{H(X,d) + 2s, N+1\},$$

then

$$H(Y,d) = \min\{H(X,d) + 2s, N+1\}.$$

Proof. It follows from the obvious inequalities: $H(\mathcal{Y},d) \leq H(Y,d) \leq \min\ \{H(X,d) + 2s, N+1\}$. □

2.3. Lemma. *Let $s > 2$. Then:*

i) for $k = 1$, $\mathcal{Y} = Y = (2,3)P_1 + \cdots + (2,3)P_s$, and $H(\mathcal{Y},d) = \min\{\deg Y, N+1\}$;

ii) for $k = 2$, $\mathcal{Y} = (3,4)P_1 + \cdots + (3,4)P_s$, and $H(\mathcal{Y},d) = \min\{\deg Y, N+1\}$.

Proof. *i*) If $d = 2$ see [7], Proposition 3.3; for $d = 3$ see [7], Proposition 4.5; for $d \geq 4$ see [2], Theorem 1.

ii) follows from [3] Theorems 1 and 2. □

2.4. Lemma. *i) If $H(\mathcal{Y},d_0) = H(X,d_0) + 2s$, then for every $d \geq d_0$ we have*

$$H(\mathcal{Y},d) = H(X,d) + 2s;$$

ii) if $(I_{\mathcal{Y}})_{d_0} = (0)$, then for every $d \leq d_0$ we have $(I_{\mathcal{Y}})_d = (0)$.

Proof. *i*) Since $X \subset \mathcal{Y}$ and $H(\mathcal{Y},d_0) = H(X,d_0) + 2s$, then it easily follows that $\dim\left(I_X/I_{\mathcal{Y}}\right)_{d_0} = 2s$. Therefore there are $2s$ forms $f_1,\dots,f_{2s} \in (I_X)_{d_0}$ linearly independent modulo $(I_{\mathcal{Y}})_{d_0}$. Let $\{l = 0\}$ be a line not through any of the points $P_1,\dots,P_s$. The forms $f_1 l^{d-d_0},\dots,f_{2s} l^{d-d_0} \in (I_X)_d$ are linearly independent modulo $(I_{\mathcal{Y}})_d$, hence $\dim\left(I_X/I_{\mathcal{Y}}\right)_d \geq 2s$, so we have $H(\mathcal{Y},d) \geq H(X,d) + 2s$. Since obviously $H(\mathcal{Y},d) \leq H(X,d) + 2s$, then the conclusion follows.

ii) Obvious. $\square$

Now we will study the postulation of $\mathcal{Y}$ for each s separately ($s = 3, 4, 5, 6, 9$), but first we wish to mention the case $s = 2$.

2.5. Proposition. *For $s = 2$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \textit{for } k = 1: & \begin{cases} N + 1 & \textit{if } d \leq 2 \\ H(T, d) = 9 < \exp H(\mathcal{Y}, d) & \textit{if } d = 3 \\ H(X, d) + 4 = \deg Y & \textit{if } d \geq 4 \end{cases} \\ \textit{for } k = 2: & \begin{cases} N + 1 & \textit{if } d \leq 3 \\ H(T, d) = 14 < \exp H(\mathcal{Y}, d) & \textit{if } d = 4 \\ H(X, d) + 4 = \deg Y & \textit{if } d \geq 5 \end{cases} \\ \textit{for } \ k \geq 3: & \begin{cases} N + 1 & \textit{if } \ \ d \leq k + 1 \\ H(T, d) = N < \exp H(\mathcal{Y}, d) & \textit{if } \ \ d = k + 2 \\ H(X, d) + 4 < \exp H(\mathcal{Y}, d) & \textit{if } \ \ k + 3 \leq d \leq 2k \\ H(X, d) + 4 = \deg Y & \textit{if } \ d \geq 2k + 1 \end{cases} \end{cases} .$$

Proof. The case $d \leq k + 1$ follows from Lemma 2.4 *ii*), and [5], Proposition 3.4, *C*).

For $d = k + 2$, observe that the line L through P_1 and P_2 is a component of multiplicity at least $2(k + 1) - d = k$ for the curves defined by the forms both of $(I_{\mathcal{Y}})_d$ and of $(I_T)_d$. Since $Res_{kL}\mathcal{Y} = Res_{kL}T = 2P_1 + 2P_2$ (see Lemma 1.12), we get

$$\dim(I_{\mathcal{Y}})_{k+2} = \dim(I_T)_{k+2} = \dim(I_{2P_1+2P_2})_2 = 1$$

(the only curve is the $(k + 2)$-uple line through the two points). Thus $H(\mathcal{Y}, d) = H(T, d)$. Moreover, since T is not regular in degree $k + 2$ (see Remark 1.2), we get $H(\mathcal{Y}, d) < \exp H(\mathcal{Y}, d)$ (see [5], Corollary 3.5).

For $k = 1, 2$ and $d \geq k + 3$, see [5], Corollary 3.8. For $k \geq 3$, and $d \geq 2k + 1$ see [5], Proposition 3.9.

Now let $k \geq 3$, and $k + 3 \leq d \leq 2k$. For $d = k + 3$ the line L through P_1 and P_2 is a component of multiplicity at least $\nu = 2(k + 1) - d = k - 1$ for the curves defined by the forms of both $(I_{\mathcal{Y}})_d$, and $(I_X)_d$, hence from the case $k = 1$, $d = 4$, we get

$$\dim(I_{\mathcal{Y}})_{k+3} = \dim(I_{\mathcal{Y}'})_{k+3-(k-1)} = \dim(I_{\mathcal{Y}'})_4 = 15 - 10 = 5, \qquad \dim(I_X)_{k+3} = \dim(I_{X'})_4 = 9,$$

where $\mathcal{Y}' = Res_{\nu L}\mathcal{Y} = (2, 3)P_1 + (2, 3)P_2$ (see Lemma 1.12), and $X' = Res_{\nu L}X = 2P_1 + 2P_2$.

It follows that $H(\mathcal{Y}, k + 3) = H(X, k + 3) + 4$. Hence by Lemma 2.4 *i*), for every $d \geq k + 3$ we have

$$H(\mathcal{Y}, d) = H(X, d) + 4.$$

Since two $(k + 1)$-fat points impose independent conditions to curves of degree d if and only if $d \geq 2k + 1$ (see Remark 1.2), then, for $k + 3 \leq d \leq 2k$, we have $H(X, d) < \deg X$, thus

$$H(\mathcal{Y}, d) = H(X, d) + 4 < \deg X + 4 = \deg Y.$$

Moreover, since for $d = k + 3$, $\dim(I_{\mathcal{Y}})_{k+3} = 5$, then for $d \geq k + 3$, $\dim(I_{\mathcal{Y}})_d$ is positive, that is $H(\mathcal{Y}, d) < \binom{d+2}{2}$. It follows that if $k + 3 \leq d \leq 2k$, then $H(\mathcal{Y}, d) < \min\left\{\deg \mathcal{Y}, \binom{d+2}{2}\right\} = \exp H(\mathcal{Y}, d)$.

(For $k \geq 3$, and $k + 3 \leq d \leq 2k$, see also [5], Proposition 3.10). $\square$

2.6. Proposition. *For $s = 3$ we have:*

i)
$$H(\mathcal{Y}, d) = \begin{cases} N+1 & \text{if } d \le \lceil \frac{3(k+1)}{2} \rceil \\ H(X,d)+6 < \deg Y & \text{if } \lceil \frac{3(k+1)}{2} \rceil + 1 \le d \le 2k \\ H(X,d)+6 = \deg Y & \text{if } d \ge \max\{\lceil \frac{3(k+1)}{2} \rceil + 1;\ 2k+1\} \end{cases} .$$

ii)
$$H(\mathcal{Y}, d) < \exp H(\mathcal{Y}, d) \quad \text{iff} \quad \begin{cases} \lceil \frac{3(k+1)}{2} \rceil + 2 \le d \le 2k & \text{if } k+1 \text{ is even} \\ \lceil \frac{3(k+1)}{2} \rceil + 1 \le d \le 2k & \text{if } k+1 \text{ is odd} \end{cases} .$$

Proof. *i)* In case $d \le \lceil \frac{3(k+1)}{2} \rceil$, it suffices to prove that $(I_{\mathcal{Y}})_d = (0)$ for $d = \lceil \frac{3(k+1)}{2} \rceil$.

Let $\mathcal{C}$ be the curve formed by the three lines P_1P_2, P_1P_3, P_2P_3. For $d = \lceil \frac{3(k+1)}{2} \rceil$, the curve $\mathcal{C}$ is a fixed component, of multiplicity at least

$$\nu = 2(k+1) - d = \begin{cases} \frac{k+1}{2} & \text{if } k+1 \text{ is even} \\ \frac{k}{2} & \text{if } k+1 \text{ is odd} \end{cases}$$

for the curves defined by the forms of $(I_{\mathcal{Y}})_d$, so we have (see Lemma 1.12)

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-3\nu}$$

where

$$\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y} = \begin{cases} P_1+P_2+P_3 & \text{if } k+1 \text{ is even} \\ 2P_1+2P_2+2P_3 & \text{if } k+1 \text{ is odd} \end{cases}, \qquad d - 3\nu = \begin{cases} 0 & \text{if } k+1 \text{ is even} \\ 2 & \text{if } k+1 \text{ is odd} \end{cases}.$$

It immediately follows that $(I_{\mathcal{Y}})_d = (0)$.

Now let $d \ge \lceil \frac{3(k+1)}{2} \rceil + 1$. In order to prove that $H(\mathcal{Y}, d) = H(X, d) + 6$, by Lemma 2.4 it suffices to prove that $H(\mathcal{Y}, d) = H(X, d) + 6$ for $d = \lceil \frac{3(k+1)}{2} \rceil + 1$.

Let $d = \lceil \frac{3(k+1)}{2} \rceil + 1$. The curve $\mathcal{C}$ is a fixed component, with multiplicity at least

$$\nu = 2(k+1) - d = \begin{cases} \frac{k-1}{2} & \text{if } k+1 \text{ is even} \\ \frac{k-2}{2} & \text{if } k+1 \text{ is odd} \end{cases}$$

for the curves defined by the forms of both $(I_{\mathcal{Y}})_d$ and $(I_X)_d$, then we have

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-3\nu}\ , \qquad \dim(I_X)_d = \dim(I_{X'})_{d-3\nu}$$

where (see Lemma 1.12)

$$d - 3\nu = \begin{cases} 4 & \text{if } k+1 \text{ is even} \\ 6 & \text{if } k+1 \text{ is odd} \end{cases},$$

$$\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y} = \begin{cases} (2,3)P_1+(2,3)P_2+(2,3)P_3 & \text{if } k+1 \text{ is even} \\ (3,4)P_1+(3,4)P_2+(3,4)P_3 & \text{if } k+1 \text{ is odd} \end{cases},$$

$$X' = \begin{cases} 2P_1+2P_2+2P_3 & \text{if } k+1 \text{ is even} \\ 3P_1+3P_2+3P_3 & \text{if } k+1 \text{ is odd} \end{cases}.$$

Since it is well known that $\dim(I_{2P_1+2P_2+2P_3})_4 = 6$ and $\dim(I_{3P_1+3P_2+3P_3})_6 = 10$, we have

$$\dim(I_{X'})_{d-3\nu} = \begin{cases} 6 & \text{if } k+1 \text{ is even} \\ 10 & \text{if } k+1 \text{ is odd} \end{cases},$$

moreover, by Lemma 2.3 we get that

$$\dim(I_{\mathcal{Y}'})_{d-3\nu} = \begin{cases} 0 & \text{if } k+1 \text{ is even} \\ 4 & \text{if } k+1 \text{ is odd} \end{cases}.$$

It follows that $\dim(I_X)_d - \dim(I_{\mathcal{Y}})_d = 6$, hence $H(\mathcal{Y}, d) - H(X, d) = 6$.

Since three $(k+1)$-fat points impose independent conditions to curves of degree d if and only if $d \geq 2k+1$ (see Remark 1.2), then for $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$ we have $H(X, d) < \deg X$, while if $d \geq \max\left\{\lceil \frac{3(k+1)}{2} \rceil + 1;\ 2k+1\right\}$, then $H(X, d) = \deg X$. Since $\deg Y = \deg X + 6$ we get:

$$H(\mathcal{Y}, d) = \begin{cases} H(X, d) + 6 < \deg Y & \text{if } \lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k \\ H(X, d) + 6 = \deg Y & \text{if } d \geq \max\left\{\lceil \frac{3(k+1)}{2} \rceil + 1;\ 2k+1\right\} \end{cases}.$$

ii) For $d \leq \lceil \frac{3(k+1)}{2} \rceil$, or $d \geq \max\left\{\lceil \frac{3(k+1)}{2} \rceil + 1;\ 2k+1\right\}$, from *i)* we have $H(\mathcal{Y}, d) = \exp H(\mathcal{Y}, d)$.

If $k+1$ is even, and $d = \lceil \frac{3(k+1)}{2} \rceil + 1$, then $\dim(I_{\mathcal{Y}})_d = 0$, hence $H(\mathcal{Y}, d) = \binom{d+2}{2}$, the expected one.

If $k+1$ is even, and $d = \lceil \frac{3(k+1)}{2} \rceil + 2$, from *i)*, since $\dim(I_X)_{d-1} = 6$ implies $\dim(I_X)_d > 6$, we have:

$$\dim(I_{\mathcal{Y}})_d = \binom{d+2}{2} - H(\mathcal{Y}, d) = \binom{d+2}{2} - H(X, d) - 6 = \dim(I_X)_d - 6 > 0.$$

Hence, if $k+1$ is even, for $d = \lceil \frac{3(k+1)}{2} \rceil + 2$, and so also for $d \geq \lceil \frac{3(k+1)}{2} \rceil + 2$, we have $\dim(I_{\mathcal{Y}})_d > 0$, that is $H(\mathcal{Y}, d) < \binom{d+2}{2}$. Since, by *i)*, if $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$, then $H(\mathcal{Y}, d) < \deg Y$, it follows that for $\lceil \frac{3(k+1)}{2} \rceil + 2 \leq d \leq 2k$ we have $H(\mathcal{Y}, d) < \min\left\{\deg Y, \binom{d+2}{2}\right\} = \exp H(\mathcal{Y}, d)$.

If $k+1$ is odd and $d \geq \lceil \frac{3(k+1)}{2} \rceil + 1$, from the proof of *i)* we get $\dim(I_{\mathcal{Y}})_d > 0$, that is $H(\mathcal{Y}, d) < \binom{d+2}{2}$. Moreover, by *i)*, if $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$, then $H(\mathcal{Y}, d) < \deg Y$, and the conclusion immediately follows. □

2.7. Proposition. *For $s = 4$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \textit{for } \ k \leq 6: & \begin{cases} N+1 & \textit{if } \ d \leq 2k+2 \\ H(X, d) + 8 = \deg Y & \textit{if } \ d \geq 2k+3 \end{cases} \\ \textit{for } \ k \geq 6: & \begin{cases} N+1 & \textit{if } \ d \leq 2k+1 \\ H(X, d) + 8 = \deg Y & \textit{if } \ d \geq 2k+2 \end{cases} \end{cases}.$$

Proof. If $d \leq 2k+1$, by Bezout Theorem, each element of $(I_{\mathcal{Y}})_d$ is divisible by every form defining an irreducible conic through $P_1, \ldots, P_4$, hence $(I_{\mathcal{Y}})_d = (0)$.

Let $d = 2k+2$. Recall that the ideal of the scheme $\mathcal{Z}_i$ is $\wp_i^{k+1} \cap (\wp_i^{k+2} + l_i^2)$, where l_i defines a generic line L_i through P_i $(1 \leq i \leq 4)$ such that $\deg(\mathcal{Y} \cap L_i) = k+2$. Let $\mathcal{C}_i$ be the conic through $P_1, \ldots, P_4$, tangent in P_i to L_i. For the genericity of the L_i's, the conics $\mathcal{C}_1, \ldots, \mathcal{C}_4$ are irreducible and distinct. Bezout's Theorem implies that each conic $\mathcal{C}_i$ is a component of any curve defined by the forms of $(I_{\mathcal{Y}})_d$. By Lemma 1.12 we can determine $I_{Res_{\mathcal{C}_1+\cdots+\mathcal{C}_4}\mathcal{Y}}$, and it is an easy computation (wich will be omitted) that the intersection multiplicities of the curves defined by the forms of $(I_{Res_{\mathcal{C}_1+\cdots+\mathcal{C}_4}\mathcal{Y}})_{d-8}$ with a conic $\mathcal{C}_i$, is bigger than $2(d-8)$. Hence by Bezout's Theorem we get that each conic $\mathcal{C}_i$ is a component with multiplicity at least 2 of any curve defined by the forms of $(I_{\mathcal{Y}})_d$. So these curves have a component of degree 16. It follows that, if $(I_{\mathcal{Y}})_d \neq (0)$, then $d \geq 16$, that is $k \geq 7$. Thus, for $k \leq 6$, we have $(I_{\mathcal{Y}})_d = (0)$, that is $H(\mathcal{Y}, d) = N + 1$. Observe that for $k = 6$, we have $N + 1 = H(X, d) + 8 = \deg Y$, in fact in this case $d = 2k + 2 = 14$, $N + 1 = \binom{16}{2} = 120$, and, since four 7-fat points impose independent conditions to curves of degree 14 (see Remark 1.2), then $H(X, d) = 112$. If $k \geq 7$ we have

$$\dim(I_{\mathcal{Y}})_{2k+2} = \dim(I_{\mathcal{Y}'})_{2k+2-16},$$

where $\mathcal{Y}' = Res_{2\mathcal{C}_1+\cdots+2\mathcal{C}_4}\mathcal{Y} = (k-7)P_1 + \cdots + (k-7)P_4$ is a scheme of four $(k-7)$-fat points (see Lemma 1.12). Since $\mathcal{Y}'$ imposes independent conditions to curves of degree $2k - 14$ (see Remark 1.2), then $H(\mathcal{Y}, 2k+2) = \binom{2k+4}{2} - \dim(I_{\mathcal{Y}})_{2k+2} = \binom{2k+4}{2} - \dim(I_{\mathcal{Y}'})_{2k-14} = \binom{2k+4}{2} - \binom{2k-12}{2} + 4\binom{k-6}{2} = 4\binom{k+2}{2} + 8 = H(X, 2k+2) + 8 = \deg Y$.

Now let $d \geq 2k+3$. It suffices to prove that $H(\mathcal{Y}, 2k+3) = H(X, 2k+3) + 8 = \deg Y$ (see Lemma 2.4 *i)*), so let $d = 2k+3$. By induction on k. For $k = 1$ see Lemma 2.3. Let $k \geq 2$. Let $\mathcal{C}$ be an irreducible conic through $P_1, \ldots, P_4$, and let Q_1, Q_2, Q_3 be three points on $\mathcal{C}$. Let $\widetilde{\mathcal{Y}} = \mathcal{Y} + Q_1 + Q_2 + Q_3$. By Bezout's Theorem, the conic $\mathcal{C}$ is a fixed component for the curves of degree $2k+3$ through $\widetilde{\mathcal{Y}}$, then

$$\dim(I_{\widetilde{\mathcal{Y}}})_{2k+3} = \dim(I_{\widetilde{\mathcal{Y}}'})_{2k+1} = \binom{2k+3}{2} - H(\widetilde{\mathcal{Y}}', 2k+1),$$

where $\widetilde{\mathcal{Y}}' = Res_{\mathcal{C}}\widetilde{\mathcal{Y}} = Res_{\mathcal{C}}\mathcal{Y} = \sum_{i=1}^{4}(k, k+1)P_i$ (see Lemma 1.12). By the inductive hypothesis we have that $H(\widetilde{\mathcal{Y}}', 2k+1) = \deg \widetilde{\mathcal{Y}}' = 4\binom{k+1}{2} + 8$, hence

$$H(\widetilde{\mathcal{Y}}, 2k+3) = \binom{2k+5}{2} - \binom{2k+3}{2} + 4\binom{k+1}{2} + 8 = \deg \mathcal{Y} + 3 = \deg \widetilde{\mathcal{Y}}.$$

Hence $\widetilde{\mathcal{Y}}$ imposes independent conditions to curves of degree $2k+3$. Since $\mathcal{Y} \subset \widetilde{\mathcal{Y}}$, then also $\mathcal{Y}$ imposes independent conditions to curves of degree $2k+3$, that is $H(\mathcal{Y}, 2k+3) = \deg \mathcal{Y} = \deg Y$.

□

2.8. Proposition. *For $s = 5$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} N+1 & \text{if } d \leq 2k+3 \\ H(X, d) + 10 < \exp H(\mathcal{Y}, d) & \text{if } 2k+4 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1 \\ H(X, d) + 10 = \deg Y & \text{if } d \geq \max\left\{2k+4;\ \lfloor \frac{5(k+1)}{2} \rfloor\right\} \end{cases}.$$

Proof. Let $d \leq 2k+3$. If we prove that $(I_{\mathcal{Y}})_d = (0)$ for $d = 2k+3$ we are done. So let $d = 2k+3$. For $k = 1$ see Lemma 2.3. Let $k \geq 2$. Any curve defined by a nonzero element of $(I_X)_d$ has the conic $\mathcal{C}$ through $P_1, \ldots, P_5$ as a component of multiplicity at least $5(k+1) - 2d = k - 1$, where X is the fat point subscheme of 5 points of multiplicity $k+1$, hence the same is true for $\mathcal{Y}$ in place of X, since $X \subset \mathcal{Y}$, so we have:

$$\dim(I_{\mathcal{Y}})_{2k+3} = \dim(I_{\mathcal{Y}'})_{2k+3-2(k-1)} = \dim(I_{\mathcal{Y}'})_5,$$

where, by Lemma 1.12, $\mathcal{Y}' = Res_{(k-1)\mathcal{C}}\mathcal{Y} = (2,3)P_1 + \cdots + (2,3)P_5$. Since, by Lemma 2.3 *i)*, $\dim(I_{\mathcal{Y}'})_5 = 0$, then the conclusion follows.

Now let $d \geq 2k+4$. We have to prove that

$$H(\mathcal{Y}, d) = H(X, d) + 10.$$

By Lemma 2.4, it suffices to prove that $H(\mathcal{Y},d)=H(X,d)+10$ for $d=2k+4$, so let $d=2k+4$. For $k=1,2$ see Lemma 2.3. If $k=3$ (hence $d=10$), let Q be a point on the conic $\mathcal{C}$ through $P_1,\dots,P_5$. The scheme $\mathcal{Y}+Q$ imposes independent conditions to curves of degree 10. In fact, since the conic $\mathcal{C}$ is a fixed locus for $(I_{\mathcal{Y}+Q})_{10}$, from the case $k=2$ we get:

$$\dim(I_{\mathcal{Y}+Q})_{10}=\dim(I_{\mathcal{Y}'})_8=\binom{8+2}{2}-5(8)=5=\binom{10+2}{2}-5(12)-1=\binom{10+2}{2}-\deg(\mathcal{Y}+Q),$$

where $\mathcal{Y}'=Res_{\mathcal{C}}(\mathcal{Y}+Q)=(3,4)P_1+\cdots+(3,4)P_5$ (see Lemma 1.12). Since $\mathcal{Y}+Q$ imposes independent conditions to curves of degree 10, then $\mathcal{Y}$ and X also do the same. It follows that

$$H(\mathcal{Y},10)=\deg Y=\deg X+10=H(X,10)+10.$$

For $k\geq 4$, since $\mathcal{C}$ is a fixed component with multiplicity at least $(k-3)$ for curves defined both by $(I_{\mathcal{Y}})_{2k+4}$ and by $(I_X)_{2k+4}$, it follows that

$$\dim(I_{\mathcal{Y}})_{2k+4}=\dim(I_{\mathcal{Y}'})_{2k+4-2(k-3)}=\dim(I_{\mathcal{Y}'})_{10},\qquad \dim(I_X)_{2k+4}=\dim(I_{X'})_{10},$$

where (see Lemma 1.12)

$$\mathcal{Y}'=Res_{(k-3)\mathcal{C}}\mathcal{Y}=(4,5)P_1+\cdots+(4,5)P_5,\qquad X'=Res_{(k-3)\mathcal{C}}4P_1+\cdots+4P_5.$$

From the case $k=3$ it follows that

$$\dim(I_{\mathcal{Y}})_{2k+4}=6,\qquad \dim(I_X)_{2k+4}=16,$$

hence $H(\mathcal{Y},d)=H(X,d)+10$.

So we have proved that for $d\geq 2k+4$

$$H(\mathcal{Y},d)=H(X,d)+10.$$

Now, since $\dim(I_{\mathcal{Y}})_{2k+4}$ is positive, then $H(\mathcal{Y},d)<\binom{d+2}{2}$ for any $d\geq 2k+4$. Moreover, since five generic $(k+1)$-fat points impose independent conditions to curves of degree d if and only if $d\geq\lfloor\frac{5(k+1)}{2}\rfloor$ (see Remark 1.2), then for $2k+4\leq d\leq\lfloor\frac{5(k+1)}{2}\rfloor-1$, we have $H(X,d)<\deg X$, hence

$$H(\mathcal{Y},d)=H(X,d)+10<\min\left\{\deg X+10,\binom{d+2}{2}\right\}=\min\left\{\deg Y,\binom{d+2}{2}\right\}=\exp H(\mathcal{Y},d).$$

If $d\geq\max\left\{2k+4;\ \lfloor\frac{5(k+1)}{2}\rfloor\right\}$, then $H(X,d)=\deg X$, so $H(\mathcal{Y},d)=\deg Y$. $\square$

2.9. Proposition. *For $s=6$ we have:*

$$H(\mathcal{Y},d)=\begin{cases}\textit{for } k=1: & \begin{cases}N+1 & \textit{if } d\leq 6\\ H(X,d)+12=\deg Y & \textit{if } d\geq 7\end{cases}\\ \textit{for } k=2: & \begin{cases}N+1 & \textit{if } d\leq 8\\ H(X,d)+12=\deg Y & \textit{if } d\geq 9\end{cases}\\ \begin{array}{l}\textit{for } k\geq 3\\ k\equiv 2\ (mod\,5):\end{array} & \begin{cases}N+1 & \textit{if } d\leq\lceil\frac{12(k+1)}{5}\rceil-1\\ H(X,d)+12<\exp H(\mathcal{Y},d) & \textit{if } \lceil\frac{12(k+1)}{5}\rceil\leq d\leq\lfloor\frac{5(k+1)}{2}\rfloor-1\\ H(X,d)+12=\deg Y & \textit{if } d\geq\max\left\{\lceil\frac{12(k+1)}{5}\rceil;\ \lfloor\frac{5(k+1)}{2}\rfloor\right\}\end{cases}\\ \begin{array}{l}\textit{for } k\geq 3\\ k\not\equiv 2\ (mod\,5):\end{array} & \begin{cases}N+1 & \textit{if } d\leq\lceil\frac{12(k+1)}{5}\rceil\\ H(X,d)+12<\exp H(\mathcal{Y},d) & \textit{if } \lceil\frac{12(k+1)}{5}\rceil+1\leq d\leq\lfloor\frac{5(k+1)}{2}\rfloor-1\\ H(X,d)+12=\deg Y & \textit{if } d\geq\max\left\{\lceil\frac{12(k+1)}{5}\rceil+1;\lfloor\frac{5(k+1)}{2}\rfloor\right\}\end{cases}\end{cases}.$$

Proof. We start by proving four particular cases, that we need later in the proof.

2.10. Lemma. *We have:*

i) $\dim(I_{(8,9)P_1+\cdots+(8,9)P_6})_{20} = 3$*;*

ii) $\dim(I_{(6,7)P_1+\cdots+(6,7)P_6})_{15} = 0$*;*

iii) $\dim(I_{(5,6)P_1+\cdots+(5,6)P_6})_{13} = 3$*;*

iv) $\dim(I_{(4,5)P_1+\cdots+(4,5)P_6})_{11} = 6$.

Proof. Though all of the above equalities can be checked using CoCoA (see [6]), we prove *i)* by specializing the scheme $\mathcal{Y}$. The proofs of *ii), iii)*, and *iv)* by using a specialization of $\mathcal{Y}$ are similar, and we left them to the reader.

i) Let $Q \in \mathbb{P}^2$ be a generic point, and let $\mathcal{F} = \{F = 0\}$ be a rational integral curve of degree 5 passing through Q, and having at each P_i, $1 \le i \le 6$, an ordinary singularity of multiplicity 2, (so $F \in (I_{2P_1+\cdots+2P_6})_5$), and let $\{\widetilde{l_i} = 0\}$ be one of the two distinct lines contained in the tangent space $T_{\mathcal{F},P_i}$ to $\mathcal{F}$ at the point P_i. Recall that the defining ideal of $\mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$ is

$$I_{\mathcal{Y}} = (\wp_1^8 \cap (\wp_1^9 + l_1^2)) \cap \cdots \cap (\wp_6^8 \cap (\wp_6^9 + l_6^2)).$$

Specialize the scheme $\mathcal{Y}$ putting $l_i = \widetilde{l_i}$ for $i = 1,2,3,4$, and let $\mathcal{Y}^*$ be such specialization of $\mathcal{Y}$. Since the expected dimension of $(I_{\mathcal{Y}})_{20}$ is $\binom{20+2}{2} - \deg \mathcal{Y} = 231 - 228 = 3$, then if we prove that $\dim(I_{\mathcal{Y}^*})_{20} = 3$, we are done. It is easy to see that the curves defined by the forms of $(I_{\mathcal{Y}^*+Q})_{20}$ have the quintic $\mathcal{F}$ as fixed component with multiplicity at least 2, hence

$$\dim(I_{\mathcal{Y}^*+Q})_{20} = \dim(I_{\mathcal{W}})_{10}$$

where $\mathcal{W} = Res_{2\mathcal{F}}(\mathcal{Y}^* + Q) = 4P_1 + 4P_2 + 4P_3 + 4P_4 + (4,5)P_5 + (4,5)P_6$. Now let $\mathcal{W}^*$ be a specialization of $\mathcal{W}$ obtained by putting $l_i = \widetilde{l_i}$ for $i = 5,6$. Since the quintic $\mathcal{F}$ is as fixed component with multiplicity at least 2 for $(I_{\mathcal{W}^*+Q})_{10}$, and since $Res_{2\mathcal{F}}(\mathcal{W}^* + Q) = \emptyset$ (see Lemma 1.12) we have

$$\dim(I_{\mathcal{W}^*+Q})_{10} = \dim(I_{Res_{2\mathcal{F}}(\mathcal{W}^*+Q)})_0 = 1.$$

Thus for the specialized scheme $\mathcal{W}^*$ we have $\dim(I_{\mathcal{W}^*})_{10} = 2 = \binom{10+2}{2} - \deg \mathcal{W}^*$. Then $\mathcal{W}^*$, and so $\mathcal{W}$ also, imposes independent conditions to curves of degree 10. It follows that $\dim(I_{\mathcal{W}})_{10} = 2$. So $\dim(I_{\mathcal{Y}^*+Q})_{20} = 2$, hence $\dim(I_{\mathcal{Y}^*})_{20} = 3$, and we are done. □

Now let $k + 1 = 5q + r$, $(0 \le r \le 4)$. Thus $k \equiv 2 \pmod 5$ iff $r = 3$.

For $k = 1, 2$ see Lemma 2.3.

Let $k \ge 3$. Let $\mathcal{C}_i$ be the conic through $P_1, \ldots, \widehat{P_i}, \ldots, P_6$, ($i = 1, \ldots, 6$), and let $\mathcal{C} = \sum_{i=1}^{6} \mathcal{C}_i$. Observe that if $2d < 5(k+1)$, then the curves defined by the forms of $(I_{\mathcal{Y}})_d$, and by the forms of $(I_X)_d$ have the six conics $\mathcal{C}_i$ as fixed components with multiplicity at least $\nu = 5(k+1) - 2d$.

Then

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-12\nu}, \qquad \dim(I_X)_d = \dim(I_{X'})_{d-12\nu},$$

where

$$\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y} = (k+1-5\nu, k+2-5\nu)P_1 + \cdots + (k+1-5\nu, k+2-5\nu)P_6,$$

$$X' = Res_{\nu\mathcal{C}}X = (k+1-5\nu)P_1 + \cdots + (k+1-5\nu)P_6.$$

We split the proof in four cases.

Case 1): $k \equiv 2 \ (mod\ 5)$, and $d \le \lceil \frac{12(k+1)}{5} \rceil - 1 = 12q+7$.

In this case it suffices to prove that $(I_{\mathcal{Y}})_d = (0)$ for $d = 12q+7$. Since $2d = 2(12q+7) < 5(k+1) = 5(5q+3)$, then the curves defined by the forms of $(I_{\mathcal{Y}})_d$ should have a fixed locus of degree $12\nu = 12q+12$, and this is impossible, since $d = 12q+7$. It follows that $(I_{\mathcal{Y}})_d = (0)$.

Case 2): $k \equiv 2 \ (mod\ 5)$, and $d \ge \lceil \frac{12(k+1)}{5} \rceil = 12q+8$.

First we will prove that

$$H(\mathcal{Y}, d) = H(X, d) + 12.$$

By Lemma 2.4, it suffices to prove that $H(\mathcal{Y}, d) = H(X, d) + 12$, for $d = 12q+8$. Since $k \ge 3$, and $k+1 = 5q+3$, then we have $q \ge 1$. Let $q = 1$, so $d = 20$, $k+1 = 8$, $\mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$, and $X = 8P_1 + \cdots + 8P_6$. Since $\dim(I_{(8,9)P_1+\cdots+(8,9)P_6})_{20} = 3$ (see Lemma 2.10 *i)*), and six 8-fat points impose independent conditions to curves of degree 20 (see Remark 1.2), we have $\dim(I_X)_{20} = 15$. It follows that $H(\mathcal{Y}, d) = H(X, d) + 12$. If $q > 1$, then $\nu\mathcal{C} = \sum_{i=1}^{6} \nu\mathcal{C}_i$ is a fixed locus for $(I_{\mathcal{Y}})_d$ and $(I_X)_d$. Since $\nu = 5(k+1) - 2d = 5(5q+3) - 2(12q+8) = q-1$, we have $d - 12\nu = 12q+8-12(q-1) = 20$, and $k+1-5\nu = 5q+3-5(q-1) = 8$. So

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{20} = 3, \qquad \dim(I_X)_d = \dim(I_{X'})_{20} = 15,$$

where $\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$, $X' = Res_{\nu\mathcal{C}}X = 8P_1 + \cdots + 8P_6$.

So we have proved that $H(\mathcal{Y}, d) = H(X, d) + 12$.

Now, since for $d = 12q+8$, $\dim(I_{\mathcal{Y}})_d$ is positive (and in fact it is equal to $\dim(I_{\mathcal{Y}'})_{20} = 3$), then $H(\mathcal{Y}, d) < \binom{d+2}{2}$ for any $d \ge 12q+8$.

Since six generic $(k+1)$-fat points impose independent conditions to curves of degree d if and only if $d \ge \lfloor \frac{5(k+1)}{2} \rfloor$ (see Remark 1.2), then for $12q+8 \le d \le \lfloor \frac{5(k+1)}{2} \rfloor - 1$, we have $H(X, d) < \deg X$, hence

$$H(\mathcal{Y}, d) = H(X, d) + 12 < \min\left\{\deg X + 12, \binom{d+2}{2}\right\} = \min\left\{\deg Y, \binom{d+2}{2}\right\} = \exp H(\mathcal{Y}, d).$$

While for $d \ge \max\left\{12q+8;\ \lfloor \frac{5(k+1)}{2} \rfloor\right\}$, we have $H(X, d) = \deg X$, so $H(\mathcal{Y}, d) = H(X, d) + 12 = \deg X + 12 = \deg Y$. That is enough to finish the proof of this case.

Case 3): $k \not\equiv 2 \ (mod\ 5)$, and $d \le \lceil \frac{12(k+1)}{5} \rceil$.

By Lemma 2.4 we have only to prove that $H(\mathcal{Y}, d) = N+1$ for $d = \lceil \frac{12(k+1)}{5} \rceil = 12q + \lceil \frac{12r}{5} \rceil$. Since $k \ge 3$, we have $k+1 = 5q+r \ge 4$, hence $q \ge \frac{4-r}{5}$. As above, let $\nu = 5(k+1) - 2d$, $\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y}$, and let $d' = d - 12\nu$. We have:

r	$k+1$	d	ν	$\mathcal{Y}'$	d'
0	$5q$	$12q$	$q > 0$	$P_1 + \cdots + P_6$	0
1	$5q+1$	$12q+3$	$q-1 \ge 0$	$(6,7)P_1 + \cdots + (6,7)P_6$	15
2	$5q+2$	$12q+5$	$q > 0$	$(2,3)P_1 + \cdots + (2,3)P_6$	5
4	$5q+4$	$12q+10$	$q \ge 0$	$(4,5)P_1 + \cdots + (4,5)P_6$	10

Since for $\nu = 0$, we have $\mathcal{Y}' = \mathcal{Y}$ and $d' = d$, then for every $\nu \geq 0$ we have:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'}.$$

Now we will prove that $\dim(I_{\mathcal{Y}'})_{d'} = 0$.

For $r = 0$ it is obvious. For $r = 2$ see Lemma 2.3. For $r = 1$ by Lemma 2.10 *ii)*, we have $\dim(I_{(6,7)P_1+\cdots+(6,7)P_6})_{15} = 0$. For $r = 4$, let $\mathcal{F} = \{F = 0\}$ be a rational integral curve of degree 5 having at each P_i $(1 \leq i \leq 6)$ an ordinary singularity of multiplicity 2, $(F \in (I_{2P_1+\cdots+2P_6})_5)$. If there exists a form $G \neq 0$, $G \in (I_{(4,5)P_1+\cdots+(4,5)P_6})_{10}$, then $FG \neq 0$ and $FG \in (I_{(6,7)P_1+\cdots+(6,7)P_6})_{15}$, but this is impossible by the previous case $r = 1$.

Case 4): $k \not\equiv 2$ $(mod\ 5)$, and $d \geq \lceil \frac{12(k+1)}{5} \rceil + 1$.

First we will prove that

$$H(\mathcal{Y}, d) = H(X, d) + 12.$$

By Lemma 2.4, it suffices to prove that $H(\mathcal{Y}, d) = H(X, d) + 12$ for $d = \lceil \frac{12(k+1)}{5} \rceil + 1 = 12q + \lceil \frac{12r}{5} \rceil + 1$.

As usual, let $\nu = 5(k+1) - 2d$, $\mathcal{Y}' = Res_{\nu\mathcal{C}}\mathcal{Y}$, $X' = Res_{\nu\mathcal{C}}X$, and $d' = d - 12\nu$. We have:

r	$k+1$	d	ν	$k+1-5\nu$	$\mathcal{Y}'$	X'	d'
0	$5q$	$12q+1$	$q-2$	10	$\sum_{i=1}^6 (10,11)P_i$	$\sum_{i=1}^6 10P_i$	25
1	$5q+1$	$12q+4$	$q-3$	16	$\sum_{i=1}^6 (16,17)P_i$	$\sum_{i=1}^6 16P_i$	40
2	$5q+2$	$12q+6$	$q-2$	12	$\sum_{i=1}^6 (12,13)P_i$	$\sum_{i=1}^6 12P_i$	30
4	$5q+4$	$12q+11$	$q-2$	14	$\sum_{i=1}^6 (14,15)P_i$	$\sum_{i=1}^6 14P_i$	35

Since for $\nu = 0$, we have $\mathcal{Y}' = \mathcal{Y}$, $X' = X$, and $d' = d$, then for every $\nu \geq 0$ we have:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'}, \qquad \dim(I_X)_d = \dim(I_{X'})_{d'}.$$

It follows that

$$H(\mathcal{Y}, d) - H(X, d) = H(\mathcal{Y}', d') - H(X', d').$$

Hence in case $\nu \geq 0$ we have only to prove that:

(a) $H(\sum_{i=1}^6 (10,11)P_i,\ 25) = H(\sum_{i=1}^6 10P_i,\ 25) + 12;$

(b) $H(\sum_{i=1}^6 (12,13)P_i,\ 30) = H(\sum_{i=1}^6 12P_i,\ 30) + 12;$

(c) $H(\sum_{i=1}^6 (14,15)P_i,\ 35) = H(\sum_{i=1}^6 14P_i,\ 35) + 12;$

(d) $H(\sum_{i=1}^6 (16,17)P_i,\ 40) = H(\sum_{i=1}^6 16P_i,\ 40) + 12;$

Now we need the following lemma:

2.11. Lemma. *Let:*

$$\mathcal{Y} = (m, m+1)P_1 + \cdots + (m, m+1)P_6,$$

$$\widetilde{\mathcal{Y}} = (m+2, m+3)P_1 + \cdots + (m+2, m+3)P_6, \qquad \widetilde{X} = (m+2)P_1 + \cdots + (m+2)P_6.$$

If the integer $\eta = 5(d+5) - 12(m+2) + 1 \geq 0$, and $H(\mathcal{Y}, d) = \deg \mathcal{Y}$, then

i) $H(\widetilde{\mathcal{Y}}, d+5) = \deg(\widetilde{\mathcal{Y}})$, $H(\widetilde{X}, d+5) = \deg(\widetilde{X})$;
ii) $H(\widetilde{\mathcal{Y}}, d+5) = H(\widetilde{X}, d+5) + 12$.

Proof. *i)* Let $\mathcal{F}$ be (as above) a rational curve of degree 5 having at each P_i ($1 \leq i \leq 6$), an ordinary singularity of multiplicity 2. Let $Q_1, \dots, Q_\eta \in \mathcal{F}$ be generic points. Since $5(d+5) < 6(2(m+2)) + \eta$, by Bezout Theorem $\mathcal{F}$ is a fixed component for the curves defined by the forms of $(I_{\widetilde{\mathcal{Y}}+Q_1+\cdots+Q_\eta})_{d+5}$. It follows that

$$\dim(I_{\widetilde{\mathcal{Y}}+Q_1+\cdots+Q_\eta})_{d+5} = \dim(I_{\mathcal{Y}})_d.$$

Since $\binom{d+5+2}{2} - \deg(\widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta) = \frac{1}{2}(d+7)(d+6) - (\deg \mathcal{Y} + 6(m+2) + 6(m+1) + \eta) = \binom{d+2}{2} - \deg \mathcal{Y} = \binom{d+2}{2} - H(\mathcal{Y}, d) = \dim(I_{\mathcal{Y}})_d$, we have

$$\dim(I_{\widetilde{\mathcal{Y}}+Q_1+\cdots+Q_\eta})_{d+5} = \binom{d+5+2}{2} - \deg(\widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta)$$

hence $H(\widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta, d+5) = \deg(\widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta)$.

Since obviously $\widetilde{X} \subset \widetilde{\mathcal{Y}} \subset \widetilde{\mathcal{Y}} + Q_1 + \cdots + Q_\eta$, it follows that $H(\widetilde{\mathcal{Y}}, d+5) = \deg(\widetilde{\mathcal{Y}})$, and $H(\widetilde{X}, d+5) = \deg(\widetilde{X})$.

ii) Obvious. □

By *Case 2)* we know that $H(\sum_{i=1}^6 (8,9)P_i,\ 20) = H(\sum_{i=1}^6 8P_i,\ 20) + 12 = \deg(\sum_{i=1}^6 (8,9)P_i)$, so by Lemma 2.11 *ii)* we have $(a): H(\sum_{i=1}^6 (10,11)P_i,\ 25) = H(\sum_{i=1}^6 10P_i,\ 25) + 12$. Moreover, by Lemma 2.11 *i)*, $H(\sum_{i=1}^6 (10,11)P_i,\ 25) = \deg(\sum_{i=1}^6 (10,11)P_i)$, hence by Lemma 2.11 *ii)* we get $(b): H(\sum_{i=1}^6 (12,13)P_i,\ 30) = H(\sum_{i=1}^6 12P_i,\ 30) + 12$. Analogously, by Lemma 2.11, we have that $(b) \Rightarrow (c) \Rightarrow (d)$, so, for $\nu \geq 0$, we have proved that $H(\mathcal{Y}, d) = H(X, d) + 12$.

Now let $\nu < 0$. In this case, since $k + 1 = 5q + r \geq 3$, we are left with the folloving cases:

r	q	$k+1$	Y	X	d
0	1	5	$\sum_{i=1}^6 (5,6)P_i$	$\sum_{i=1}^6 5P_i$	13
1	1	6	$\sum_{i=1}^6 (6,7)P_i$	$\sum_{i=1}^6 6P_i$	16
1	2	11	$\sum_{i=1}^6 (11,12)P_i$	$\sum_{i=1}^6 11P_i$	28
2	1	7	$\sum_{i=1}^6 (7,8)P_i$	$\sum_{i=1}^6 7P_i$	18
4	0	4	$\sum_{i=1}^6 (4,5)P_i$	$\sum_{i=1}^6 4P_i$	11
4	1	9	$\sum_{i=1}^6 (9,10)P_i$	$\sum_{i=1}^6 9P_i$	23

hence we have to prove that:
(e) $H(\sum_{i=1}^6 (5,6)P_i,\ 13) = H(\sum_{i=1}^6 5P_i,\ 13) + 12$;
(f) $H(\sum_{i=1}^6 (6,7)P_i,\ 16) = H(\sum_{i=1}^6 6P_i,\ 16) + 12$;
(g) $H(\sum_{i=1}^6 (11,12)P_i,\ 28) = H(\sum_{i=1}^6 11P_i,\ 28) + 12$;
(h) $H(\sum_{i=1}^6 (7,8)P_i,\ 18) = H(\sum_{i=1}^6 7P_i,\ 18) + 12$;
(i) $H(\sum_{i=1}^6 (4,5)P_i,\ 11) = H(\sum_{i=1}^6 4P_i,\ 11) + 12$;

(l) $H(\sum_{i=1}^{6}(9,10)P_i,\ 23) = H(\sum_{i=1}^{6} 9P_i,\ 23) + 12$.

By Remark 1.2 and by Lemma 2.10 *iii)*, and *iv)*, it easily follows that (e) and (i) hold, moreover by Lemma 2.11 we have that $(e) \Rightarrow (h) \Rightarrow (l) \Rightarrow (g)$, and $(i) \Rightarrow (f)$, so we have proved that $H(\mathcal{Y}, d) = H(X, d) + 12$ also for $\nu < 0$.

Now observe that for $d = \lceil \frac{12(k+1)}{5} \rceil + 1$, $\dim(I_{\mathcal{Y}})_d$ is positive. In fact, as shown above, we have if $\nu \geq 0$:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'} = \binom{d'+2}{2} - \deg X' - 12 = \begin{cases} \binom{25+2}{2} - 6\binom{10+1}{2} - 12 & = 9 \qquad \text{for } r = 0 \\ \binom{40+2}{2} - 6\binom{16+1}{2} - 12 & = 33 \qquad \text{for } r = 1 \\ \binom{30+2}{2} - 6\binom{12+1}{2} - 12 & = 16 \qquad \text{for } r = 2 \\ \binom{35+2}{2} - 6\binom{14+1}{2} - 12 & = 24 \qquad \text{for } r = 4 \end{cases}$$

if $\nu < 0$:

$$\dim(I_{\mathcal{Y}})_d = \begin{cases} \binom{13+2}{2} - 6\binom{5+1}{2} - 12 & = 3 \qquad \text{for } r = 0,\ q = 1 \\ \binom{16+2}{2} - 6\binom{6+1}{2} - 12 & = 15 \qquad \text{for } r = 1,\ q = 1 \\ \binom{28+2}{2} - 6\binom{11+1}{2} - 12 & = 27 \qquad \text{for } r = 1,\ q = 2 \\ \binom{18+2}{2} - 6\binom{7+1}{2} - 12 & = 10 \qquad \text{for } r = 2,\ q = 1 \\ \binom{11+2}{2} - 6\binom{4+1}{2} - 12 & = 6 \qquad \text{for } r = 4,\ q = 0 \\ \binom{23+2}{2} - 6\binom{9+1}{2} - 12 & = 18 \qquad \text{for } r = 4,\ q = 1 \end{cases}.$$

Since $\dim(I_{\mathcal{Y}})_d > 0$ for $d = \lceil \frac{12(k+1)}{5} \rceil + 1$, we have that $\dim(I_{\mathcal{Y}})_d$ is positive for any $d \geq \lceil \frac{12(k+1)}{5} \rceil + 1$, and this means that $H(\mathcal{Y}, d) < \binom{d+2}{2}$ for any $d \geq \lceil \frac{12(k+1)}{5} \rceil + 1$. Moreover, since six generic $(k+1)$-fat points impose independent conditions to curves of degree d if and only if $d \geq \lfloor \frac{5(k+1)}{2} \rfloor$ (see Remark 1.2), then for $\lceil \frac{12(k+1)}{5} \rceil + 1 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$, we have $H(X, d) < \deg X$, hence

$$H(\mathcal{Y}, d) = H(X, d) + 12 < \min\left\{\deg X + 12, \binom{d+2}{2}\right\} = \min\left\{\deg Y, \binom{d+2}{2}\right\} = \exp H(\mathcal{Y}, d).$$

While for $d \geq \max\left\{\lceil \frac{12(k+1)}{5} \rceil + 1;\ \lfloor \frac{5(k+1)}{2} \rfloor\right\}$, we have $H(X, d) = \deg X$, so $H(\mathcal{Y}, d) = H(X, d) + 12 = \deg X + 12 = \deg Y$, and this finish the proof.

□

2.12. Proposition. *For $s = 9$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} k = 1: & \begin{cases} N + 1 & \text{if } d \leq 8 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 8 \end{cases} \\ k = 2: & \begin{cases} N + 1 & \text{if } d \leq 10 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 11 \end{cases} \\ k = 3: & \begin{cases} N + 1 & \text{if } d \leq 13 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 14 \end{cases} \\ k \geq 4: & \begin{cases} N + 1 & \text{if } d \leq 3k + 3 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 3k + 4 \end{cases} \end{cases}.$$

Proof. For $k = 1, 2$ the statement is known by [2] and [3].

Let $k = 3$, so $\mathcal{Y} = (4,5)P_1 + \cdots + (4,5)P_9$.

For $d = 13$, by CoCoA (see [6]), or by specializing the scheme $\mathcal{Y}$ it is easy to check that $\dim(I_{\mathcal{Y}})_{13} = 0$, hence for $d \leq 13$ the conclusion follows from Lemma 2.4.

Now let C be the unique (smooth) cubic curve passing through the support of $\mathcal{Y}$, i.e., through $P_1, \ldots, P_9$. Consider the following exact sequence, where $\mathcal{Y}' = Res_C \mathcal{Y}$:

$$0 \to \mathcal{I}_{\mathcal{Y}'}(d-3) \to \mathcal{I}_{\mathcal{Y}}(d) \to \mathcal{I}_{\mathcal{Y}\cap C, C}(d) \to 0$$

We have that $\mathcal{I}_{\mathcal{Y}\cap C, C}(d) = \mathcal{O}_C(dH - \mathcal{Y}\cap C)$, where H is a line section of C, and $\deg(\mathcal{O}_C(dH - \mathcal{Y}\cap C)) = 3d - 9(k+1)$.

Let $d = 14$. Since $k = 3$, we have $\deg(\mathcal{O}_C(dH - \mathcal{Y}\cap C)) = 14 \cdot 3 - 4 \cdot 9 = 6$. It follows that $h^1(\mathcal{O}_C(dH - \mathcal{Y}\cap C)) = 0$. Since $\mathcal{Y}' = (3,4)P_1 + \cdots + (3,4)P_9$, from the case $k = 2$ we get $h^1(\mathcal{I}_{\mathcal{Y}'}(d-3)) = h^1(\mathcal{I}_{\mathcal{Y}'}(11)) = 0$. So by the exact sequence above it follows that $h^1(\mathcal{I}_{(4,5)P_1+\cdots+(4,5)P_9}(14)) = 0$, which implies $H(\mathcal{Y}, 14) = \deg Y$. For $d > 14$ the conclusion follows from Lemma 2.4.

Let $k \geq 4$.

Now we proceed by induction on k. For $k = 4$, we have $\mathcal{Y} = (5,6)P_1 + \cdots + (5,6)P_9$, and $3k + 4 = 16$. By CoCoA (see [6]), or by specializing the scheme $\mathcal{Y}$ it is easy to check that $\dim(I_{\mathcal{Y}})_{16} = 0$. So, since $N + 1 = \binom{16+2}{2} = 9 \cdot 17 = \deg \mathcal{Y}$, it follows that $H(\mathcal{Y}, 16) = N + 1 = \deg \mathcal{Y}$. Hence by Lemma 2.4 it follows that for $d \leq 16$ we have $H(\mathcal{Y}, d) = N + 1$, while for $d \geq 16$ we have $H(\mathcal{Y}, d) = \deg \mathcal{Y}$.

Now let $k > 4$. We have:

$$\mathcal{Y} = (k+1, k+2)P_1 + \cdots + (k+1, k+2)P_9 \qquad \mathcal{Y}' = (k, k+1)P_1 + \cdots + (k, k+1)P_9.$$

Since obviously if $d \leq 3k+3$, then $d - 3 \leq 3(k-1) + 3$, and if $d \geq 3k+4$, then $d - 3 \geq 3(k-1) + 4$, by the induction hypothesis we have $H(\mathcal{Y}', d-3) = N' + 1$ for $d - 3 \leq 3(k-1) + 3$, ($N' = \binom{d-3+2}{2}$), and $H(\mathcal{Y}', d-3) = \deg \mathcal{Y}'$ for $d - 3 \geq 3(k-1) + 4$. That is:

$$h^0(\mathcal{I}_{Y'}(d-3)) = 0 \quad \text{for } \ d - 3 \leq 3(k-1) + 3,$$

$$h^1(\mathcal{I}_{Y'}(d-3)) = 0 \quad \text{for } \ d - 3 \geq 3(k-1) + 4.$$

Moreover, since $\deg(\mathcal{O}_C(dH - \mathcal{Y}\cap C)) = 3d - 9(k+1) \leq 0$ for $d \leq 3k+3$, and $\deg(\mathcal{O}_C(dH - \mathcal{Y}\cap C)) = 3d - 9(k+1) \geq 3$ for $d \geq 3k+4$, we have:

$$h^0(\mathcal{I}_{\mathcal{Y}\cap C, C}(d)) = 0 \quad \text{for } \ d \leq 3k+3,$$

$$h^1(\mathcal{I}_{\mathcal{Y}\cap C, C}(d)) = 0 \quad \text{for } \ d \geq 3k+4.$$

So whenever $d \leq 3k+3$, we get $h^0(\mathcal{I}_{Y'}(d-3)) = h^0(\mathcal{I}_{\mathcal{Y}\cap C, C}(d)) = 0$, which by the exact sequence above implies $h^0(\mathcal{I}_Y(d)) = 0$.

When $d \geq 3k+4$, we get $h^1(\mathcal{I}_{Y'}(d-3)) = h^1(\mathcal{I}_{\mathcal{Y}\cap C, C}(d)) = 0$, so by the exact sequence above we have $h^1(\mathcal{I}_Y(d)) = 0$, and we are done. $\square$

With all these partial results we have actually proved the main theorem of this paper:

2.13. Theorem. *For $s \le 6$, or $s = 9$, then*

$$\dim O^s_{k,d} = \min\left\{H(X,d) + 2s,\ \binom{d+2}{2}\right\} - 1$$

except when $s = 2$, $d = k+2$ where $\dim O^2_{k,k+2} = H(T,d) - 1 = \binom{d+2}{2} - 2$.

Proof. For $s = 1$, since $H(X,d) = \min\left\{\binom{k+2}{2}, \binom{d+2}{2}\right\}$, then the result follows from Remark 1.6.

For $s = 2$ and $d = k+2$, since $H(\mathcal{Y}, d) = H(T,d)$ (see Propositions 2.5), by the obvious inequalities $H(\mathcal{Y}, d) \le H(Y,d) \le H(T,d)$ we get

$$H(Y,d) = H(\mathcal{Y}, d) = H(T,d)$$

and the conclusion follows from Remark 1.7 (‡).

In the other cases by Lemma 2.2, and from Propositions 2.5 to 2.9, and Proposition 2.12, we have

$$H(Y,d) = H(\mathcal{Y}, d) = \min\{H(X,d) + 2s, N+1\},$$

hence from Remark 1.7 (‡) we get the conclusion. □

2.14. Corollary. *Let $\delta = \min\{\deg Y - 1,\ N\} - \dim O^s_{k,d}$ be the defect of $O^s_{k,d}$. If $s \le 6$, or $s = 9$, then $O^s_{k,d}$ is defective only in the following cases:*

i) $s = 2$, $d = k+2$, *with defect:* $\delta = 1$.

ii) $s = 2$, $k \ge 3$, $k+3 \le d \le 2k$, *with defect:* $\delta = \min\left\{\binom{2(k+1)-d}{2}; (d-k)^2 - 4\right\}$.

iii) $s = 3$, $k \ge 7$, k *odd*, $\lceil \frac{3(k+1)}{2} \rceil + 2 \le d \le 2k$, *with defect:* $\delta = \min\left\{3\binom{2(k+1)-d}{2}; \binom{2d-3k-1}{2} - 6\right\}$.

iv) $s = 3$, $k \ge 6$, k *even*, $\lceil \frac{3(k+1)}{2} \rceil + 1 \le d \le 2k$, *with defect:* $\delta = \min\left\{3\binom{2(k+1)-d}{2}; \binom{2d-3k-1}{2} - 6\right\}$.

v) $s = 5$, $k \ge 5$, $2k+4 \le d \le \lfloor \frac{5(k+1)}{2} \rfloor - 1$, *with defect* $\delta = \min\left\{\binom{5(k+1)-2d}{2};\ 5\binom{d-2k-1}{2} - 9\right\}$.

vi) $s = 6$, $k \equiv 2\ (mod\, 5)$, $k \ge 17$, $\lceil \frac{12(k+1)}{5} \rceil \le d \le \lfloor \frac{5(k+1)}{2} \rfloor - 1$, *with defect:*
$\delta = \min\left\{6\binom{5(k+1)-2d}{2};\ \binom{5d-12k-10}{2} - 12\right\}$.

vii) $s = 6$, $k \not\equiv 2\ (mod\, 5)$, $k \ge \begin{cases} 19 & \text{if } k \text{ odd} \\ 24 & \text{if } k \text{ even} \end{cases}$, $\lceil \frac{12(k+1)}{5} \rceil + 1 \le d \le \lfloor \frac{5(k+1)}{2} \rfloor - 1$, *with defect:*
$\delta = \min\left\{6\binom{5(k+1)-2d}{2};\ \binom{5d-12k-10}{2} - 12\right\}$.

Proof. First observe that: $k+3 \le 2k$ implies $k \ge 3$; if k is odd and $\lceil \frac{3(k+1)}{2} \rceil + 2 \le 2k$, then $3(k+1) + 4 \le 4k$, that is $k \ge 7$, while if k is even and $\lceil \frac{3(k+1)}{2} \rceil + 1 \le 2k$, then $k \ge 6$; from $2k+4 \le \lfloor \frac{5(k+1)}{2} \rfloor - 1$ we get $k \ge 5$; finally, for $k \equiv 2\ (mod\, 5)$, it is easy to compute that $\lceil \frac{12(k+1)}{5} \rceil \le \lfloor \frac{5(k+1)}{2} \rfloor - 1$ implies $k \ge 17$, while for $k \not\equiv 2\ (mod\, 5)$, if $\lceil \frac{12(k+1)}{5} \rceil + 1 \le \lfloor \frac{5(k+1)}{2} \rfloor - 1$, then $k \ge \begin{cases} 19 & \text{if } k \text{ odd} \\ 24 & \text{if } k \text{ even} \end{cases}$.

From what we have seen above, and by Remark 1.6, Propositions 2.5 to 2.9 and 2.12, we immediately get that $O^s_{k,d}$ is defective only in the cases *i)* to *vii)*.

For $s=2$ and $d=k+2$, since $\dim O^2_{k,k+2}=N-1$, while the expected dimension is N, we have $\delta=1$. In the other cases we know that $H(Y,d)=H(X,d)+2s$, so we have

$$\delta=\min\{\deg Y-1,\ N\}-\dim O^s_{k,d}=\min\{\deg Y-1,\ N\}-H(Y,d)+1$$

$$=\min\{\deg Y-H(X,d)-2s,\ N+1-H(X,d)-2s\}=\min\{\deg X-H(X,d),\ \dim(I_X)_d-2s\}.$$

For $s=2$, $k\geq 3$ and $k+3\leq d\leq 2k$, computing the dimension of $(I_X)_d$ by removing the line P_1P_2 $(2(k+1)-d)$ times, we get:

$$\dim(I_X)_d=\dim(I_{X'})_{2(d-k-1)}=\binom{2(d-k-1)+2}{2}-2\binom{d-k}{2}=(d-k)^2,$$

where $X'=(d-k-1)P_1+(d-k-1)P_2$, hence

$$\deg X-H(X,d)=2\binom{k+2}{2}-\binom{d+2}{2}+(d-k)^2=\binom{2(k+1)-d}{2},$$

$$\delta=\min\left\{\binom{2(k+1)-d}{2};(d-k)^2-4\right\}.$$

In cases $iii)$ and $iv)$, computing the dimension of $(I_X)_d$ by cutting off the three lines P_1P_2, P_1P_3, P_2P_3, $2(k+1)-d$ times each, we have:

$$\dim(I_X)_d=\dim(I_{X'})_{d-3(2k+2-d)}=\dim(I_{X'})_{2(2d-3k-3)}$$

$$=\binom{2(2d-3k-3)+2}{2}-3\binom{2d-3k-2}{2}=\binom{2d-3k-1}{2},$$

where $X'=\sum_{i=1}^{3}(k+1-2(2k+2-d))P_i=\sum_{i=1}^{3}(2d-3k-3)P_i$, and from here we easily get:

$$\deg X-H(X,d)=3\binom{k+2}{2}-\binom{d+2}{2}+\binom{2d-3k-1}{2}=3\binom{2(k+1)-d}{2},$$

$$\delta=\min\left\{3\binom{2(k+1)-d}{2};\binom{2d-3k-1}{2}-6\right\}.$$

For s=5, computing the dimension of $(I_X)_d$ (by cutting off the fixed conics), we get:

$$\dim(I_X)_d=\dim(I_{X'})_{d-2(5k+5-2d)}=\dim(I_{X'})_{5(d-2k-2)}$$

$$=\binom{5(d-2k-2)+2}{2}-5\binom{2d-4k-3}{2}=5\binom{d-2k-1}{2}+1,$$

where $X'=\sum_{i=1}^{5}(k+1-(5k+5-2d))P_i=\sum_{i=1}^{5}(2d-4k-4)P_i$, and from here we get:

$$\deg X-H(X,d)=5\binom{k+2}{2}-\binom{d+2}{2}+5\binom{d-2k-1}{2}+1=\binom{5(k+1)-2d}{2},$$

$$\delta=\min\left\{\binom{5(k+1)-2d}{2};\ 5\binom{d-2k-1}{2}-9\right\}.$$

Finally, for s=6, calculating the dimension of $(I_X)_d$ by removing every conic C_i (see the proof of Proposition 2.9) $(5(k+1)-2d)$ times, we get

$$\dim(I_X)_d = \dim(I_{X'})_{d-12(5k+5-2d)} = \dim(I_{X'})_{25d-60k-60}$$

$$= \binom{25d-60k-60+2}{2} - 6\binom{10d-24k-24+1}{2} = \binom{5d-12k-10}{2},$$

where $X' = \sum_{i=1}^{6}(k+1-5(5k+5-2d))P_i = \sum_{i=1}^{6}(10d-24k-24)P_i$, and from here we get:

$$\deg X - H(X,d) = 6\binom{k+2}{2} - \binom{d+2}{2} + \binom{5d-12k-10}{2} = 6\binom{5(k+1)-2d}{2},$$

$$\delta = \min\left\{6\binom{5(k+1)-2d}{2};\ \binom{5d-12k-10}{2} - 12\right\}.$$

□

2.15. Remark. Some examples, some computations, and a lack of geometric reasons, lead us to conjecture that also for $s=7$, and $s=8$ we have

$$\dim O^s_{k,d} = \min\left\{H(X,d)+2s,\ \binom{d+2}{2}\right\} - 1.$$

Unfortunately, by methods similar to the ones utilized for $s \le 6$, the proof splits into many cases, and becomes too long and tedious to justify including.

E.Ballico and C.Fontanari in [4] give partial results about the regularity of $O^s_{k,d}$ for $2 \le s \le 8$. Our Corollary 2.14, for $s \le 6$ or $s = 9$, improves the results of [4] and gives a complete classification of all the defective cases.

2.16. Remark. We wish to notice that there are no defective cases for $s=4$ or $s=9$.

In case $s=2$, $d=k+2$ defectivity is forced by the defectivity of T, in fact, since $Y \subset T$ implies that $H(Y,k+2) \le H(T,k+2)$, and since $H(T,k+2) = N < \exp H(Y,k+2) = N+1$, it follows that $H(Y,k+2) < \exp H(Y,k+2)$. In the other cases defectivity of $O^s_{k,d}$ is forced by the defectivity of X.

2.17. Remark. In light of Remarks 2.15 and 2.16, and the results of L.Evain (see Remark 1.2), we like to conjecture that if s is a square, then $O^s_{k,d}$ is regular in any degree d.

Anyway by the results of L.Evain, and by [5], Lemma 3.1, we easily deduce a partial result about the regularity of $O^s_{k,d}$:

If s is a square, and $N+1 \le \deg X$ or $N+1 \ge \deg T$, then $\dim O^s_{k,d}$ is as expected.

In fact if s is a square, by [9] we know that X and T have maximal Hilbert function. Hence if $N+1 \le \deg X$, then $\dim(I_X)_d = 0$, and if $N+1 \ge \deg T$, then $H(T,d) = \deg T$. Since $X \subset Y \subset T$, it follows that if $\dim(I_X)_d = 0$, then $H(Y,d) = N+1$, and if $H(T,d) = \deg T$, then $H(Y,d) = \deg Y$, and now the conclusion follows from Remark 2.1.

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