A boundary estimate for non-negative solutions to Kolmogorov operators in non-divergence form

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Abstract: We consider non-negative solutions to a class of second order degnerate Kolmogorov equations in the form

$$\mathscr{L}u(x,t) = \sum_{i,j=1}^{m} a_{i,j}(x,t)\partial_{x_i x_j} u(x,t) + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} u(x,t) - \partial_t u(x,t) = 0,$$

where (x,t) belongs to an open set $\Omega \subset \mathbb{R}^N \times \mathbb{R}$, and $1 \le m \le N$. Let $\widetilde{z} \in \Omega$, let K be a compact subset of $\overline{\Omega}$, and let $\Sigma \subset \partial\Omega$ be such that $K \cap \partial\Omega \subset \Sigma$. We give some sufficient geometric conditions for the validity of the following Carleson type inequality. There exists a positive constant C_K, only depending on $\Omega, \Sigma, K, \widetilde{z}$ and on $\mathscr{L}$, such that

$$\sup_K u \le C_K\, u(\widetilde{z}),$$

for every non-negative solution u of $\mathscr{L}u = 0$ in Ω such that $u_{|\Sigma} = 0$.

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1 Introduction

In the study of local Fatou theorems, Carleson proves in [6] the following estimate for positive harmonic functions. *Let $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain with Lipschitz constant M, let $w \in \partial D$, $0 < r < r_0$, and suppose that u is a non-negative continuous harmonic function in $\bar{D} \cap B(w,2r)$. Suppose that $u = 0$ on $\partial D \cap B(w,2r)$. Then there exists a positive constant $c = c(n,M)$ and a point $a_{\widetilde{r}}(w)$ satisfying $|a_{\widetilde{r}}(w) - w| = \widetilde{r}$, $\operatorname{dist}(a_{\widetilde{r}}(w), \partial D) > \widetilde{r}/M$, such that if $\widetilde{r} = r/c$, then*

$$\max_{D \cap B(w,\widetilde{r})} u \le c\, u(a_{\widetilde{r}}(w)).$$

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The above estimate is now referred to as Carleson estimate. Important generalization of this results to more general second order elliptic and parabolic equations have been given by Caffarelli, Fabes, Mortola and Salsa in [5], and by Salsa in [17], respectively. The purpose of this paper is to establish a general version of the Carleson's results for non-negative solutions to operators of Kolmogorov type.

Our research is a part of a thorough study of the boundary behavior for non-negative solutions to operators of Kolmogorov type (see [9], [11], [16]), motivated by several applications to Physics and Finance.

Throughout the paper we consider a class of second order differential operators of Kolmogorov type of the form

$$\mathscr{L} = \sum_{i,j=1}^{m} a_{i,j}(z)\partial_{x_i x_j} + \sum_{i=1}^{m} a_i(z)\partial_{x_i} + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j} - \partial_t, \tag{1.1}$$

where $z = (x,t) \in \mathbb{R}^N \times \mathbb{R}$, $1 \le m \le N$ and the coefficients $a_{i,j}$ and a_i are bounded continuous functions. The matrix $B = (b_{i,j})_{i,j=1,\dots,N}$ has real constant entries. Concerning structural assumptions on the operator $\mathscr{L}$ we assume the following.

[H.1] The matrix $A_0(z) = (a_{i,j}(z))_{i,j=1,\dots,m}$ is symmetric and uniformly positive definite in $\mathbb{R}^m$: there exists a positive constant Λ such that

$$\Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{m} a_{i,j}(z)\xi_i\xi_j \le \Lambda|\xi|^2, \qquad \forall\, \xi \in \mathbb{R}^{m_0},\ z \in \mathbb{R}^{N+1}.$$

[H.2] The *constant coefficients* operator

$$\mathscr{K} = \sum_{i,j=1}^{m} a_{i,j}\partial_{x_i x_j} + \sum_{i,j=1}^{N} b_{i,j}x_i\partial_{x_j} - \partial_t \tag{1.2}$$

is hypoelliptic, i.e. every distributional solution of $\mathscr{K}u = f$ is a smooth classical solution, whenever f is smooth. Here $A_0 = (a_{i,j})_{i,j=1,\dots,m}$ is a constant, symmetric and positive matrix.

[H.3] The coefficients $a_{i,j}$ and a_i belong to the space $C_K^{0,\alpha}(\mathbb{R}^{N+1})$ of Hölder continuous functions (defined in (2.5) below), for some $\alpha \in]0,1]$.

Note that the operator $\mathscr{K}$ can be written as

$$\mathscr{K} = \sum_{i=1}^{m} X_i^2 + Y,$$

where

$$X_i = \sum_{j=1}^{m} \bar{a}_{i,j}\partial_{x_j}, \quad i = 1,\dots,m, \qquad Y = \langle x, B\nabla\rangle - \partial_t, \tag{1.3}$$

and the $\bar{a}_{i,j}$'s are the entries of the unique positive matrix $\bar{A}_0$ such that $A_0 = \bar{A}_0^2$.

We recall that hypothesis **[H.2]** is equivalent to Hörmander condition [12]:

$$\text{rank Lie}\,(X_1, \dots, X_m, Y)\,(z) = N + 1, \qquad \forall\, z \in \mathbb{R}^{N+1}. \tag{1.4}$$

It is known that the natural framework to study operators satisfying a Hörmander condition is the analysis on Lie group. In particular, the relevant Lie group related to the operator $\mathscr{K}$ in (1.2) is defined using the group law

$$(x, t) \circ (\xi, \tau) = (\xi + \exp(-\tau B^T)x, t + \tau), \qquad (x, t), (\xi, \tau) \in \mathbb{R}^{N+1}. \tag{1.5}$$

The vector fields $X_1, \dots, X_m$ and Y are left-invariant with respect to the group law (1.5), in the sense that

$$X_j\,(u(\zeta \circ \cdot\,)) = (X_j u)\,(\zeta \circ \cdot\,), \quad j = 1, \dots, m, \qquad Y\,(u(\zeta \circ \cdot\,)) = (Y u)\,(\zeta \circ \cdot\,) \tag{1.6}$$

for every $\zeta \in \mathbb{R}^{N+1}$ (hence $\mathscr{K}\,(u(\zeta \circ \cdot\,)) = (\mathscr{K} u)\,(\zeta \circ \cdot\,)$).

We next introduce the integral trajectories of Kolmogorov equations. We say that a path $\gamma : [0, T] \to \mathbb{R}^{N+1}$ is *$\mathscr{L}$-admissible* if it is absolutely continuous and satisfies

$$\gamma'(s) = \sum_{j=1}^{m} \omega_j(s) X_j(\gamma(s)) + \lambda(s) Y(\gamma(s)), \qquad \text{for a.e. } \ s \in [0, T], \tag{1.7}$$

where $\omega_j \in L^2([0, T])$ for $j = 1, \dots, m$, and λ is a non-negative measurable function. We say that γ *connects z_0 to z* if $\gamma(0) = z_0$ and $\gamma(T) = z$. Concerning the problem of the *existence* of admissible paths, we recall that it is a *controllability problem*, and that [**H**.**2**] is equivalent to the following *Kalman condition*:

$$\text{rank}\Big(\bar{A} \;\; B^T\bar{A} \;\; \cdots \;\; \big(B^T\big)^{N-1}\bar{A}\Big) = N. \tag{1.8}$$

Here $\bar{A}$ is the $N \times N$ matrix defined by

$$\begin{pmatrix} \bar{A}_0 & 0 \\ 0 & 0 \end{pmatrix}$$

and $\bar{A}_0$ is the $m \times m$ constant matrix introduced in (1.3). We recall that (1.8) is a sufficient condition for the existence of a solution of (1.7), in the case of $\Omega = \mathbb{R}^N \times]T_0, T_1[$ (see [14], Theorem 5, p. 81).

We denote by

$$\mathbf{A}_{z_0}(\Omega) = \big\{ z \in \Omega \mid \text{there exists an } \mathscr{L}\text{-admissible } \gamma : [0, T] \to \Omega \text{ connecting } z_0 \text{ to } z \big\}, \tag{1.9}$$

and we define $\mathscr{A}_{z_0} = \mathscr{A}_{z_0}(\Omega) = \overline{\mathbf{A}_{z_0}(\Omega)}$ as the closure (in $\mathbb{R}^{N+1}$) of $\mathbf{A}_{z_0}(\Omega)$. We will refer to $\mathscr{A}_{z_0}$ as the *attainable set*.

We recall that **[H.2]** is equivalent to the following structural assumption on B [13]: *there exists a basis for* $\mathbb{R}^N$ *such that the matrix* B *has the form*

$$\begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_\kappa \\ * & * & * & \cdots & * \end{pmatrix} \tag{1.10}$$

where B_j *is a* $m_{j-1} \times m_j$ *matrix of rank* m_j *for* $j \in \{1, \dots, \kappa\}$, $1 \le m_\kappa \le \ldots \le m_1 \le m_0 = m$ *and* $m + m_1 + \ldots + m_\kappa = N$, *while* $*$ *represents arbitrary matrices with constant entries.* Based on (1.10), we introduce the family of dilations $(\delta_r)_{r>0}$ on $\mathbb{R}^{N+1}$ defined by

$$\delta_r := (D_r, r^2) = \mathrm{diag}(rI_m, r^3 I_{m_1}, \dots, r^{2\kappa+1} I_{m_\kappa}, r^2), \tag{1.11}$$

where I_k, $k \in \mathbb{N}$, is the k-dimensional unit matrix. To simplify our presentation, we will also assume the following technical condition:

[H.4] the operator $\mathscr{K}$ in (1.2) is δ_r-homogeneous of degree two, i.e.

$$\mathscr{K} \circ \delta_r = r^2 (\delta_r \circ \mathscr{K}), \qquad \forall\, r > 0.$$

We explicitly remark that **[H.4]** is satisfied if (and only if) all the blocks denoted by $*$ in (1.10) are null (see [13]).

We next introduce some definitions based on the dilations (1.11) and on the translations (1.5). For any given $z_0 \in \mathbb{R}^{N+1}, \bar{x} \in \mathbb{R}^N, \bar{t} \in \mathbb{R}^+$ we consider an open neighborhood $U \subset \mathbb{R}^N$ of $\bar{x}$, and we denote by $Z^-_{\bar{x},\bar{t},U}(z_0)$ and $Z^+_{\bar{x},\bar{t},U}(z_0)$ the following *tusk-shaped* cones

$$\begin{aligned} Z^-_{\bar{x},\bar{t},U}(z_0) &= \left\{ z_0 \circ \delta_s(x, -\bar{t}) \mid x \in U,\, 0 < s \le 1 \right\}; \\ Z^+_{\bar{x},\bar{t},U}(z_0) &= \left\{ z_0 \circ \delta_s(x, \bar{t}) \mid x \in U,\, 0 < s \le 1 \right\}. \end{aligned} \tag{1.12}$$

In the sequel, aiming to simplify the notations, we shall write $Z^\pm(z_0)$ instead of $Z^\pm_{\bar{x},\bar{t},U}(z_0)$. Note that $Z^-(z_0)$ and $Z^+(z_0)$ are cones with the same vertex at z_0, while the basis of $Z^-(z_0)$ is at the time level $t_0 - \bar{t} < t_0$, and the basis of $Z^+(z_0)$ is at the time level $t_0 + \bar{t} > t_0$.

Definition 1.1 *Let* Ω *be an open subset of* $\mathbb{R}^{N+1}$ *and let* $\Sigma \subset \partial\Omega$.

i) We say that Σ *satisfies the* uniform exterior cone condition *if there exist* $\bar{x} \in \mathbb{R}^N, \bar{t} > 0$ *and an open neighborhood* $U \subseteq \mathbb{R}^N$ *of* $\bar{x}$ *such that*

$$Z^-(z_0) \cap \Omega = \emptyset \qquad \textit{for every} \quad z_0 \in \Sigma,$$

where $Z^-(z_0) = Z^-_{\bar{x},\bar{t},U}(z_0)$;

ii) we say that $Z^+_{\bar{x},\bar{t},U}(z_0)$ *satisfies the* Harnack connectivity condition *if* $z_0 \circ \delta_{s_0}(\bar{x}, \bar{t}) \in \mathrm{Int}\left(\mathscr{A}_{z_0 \circ (\bar{x},\bar{t})}(Z^+(z_0))\right)$ *for some* $s_0 \in]0, 1[$;

iii) we say that Σ satisfies the uniform interior cone condition *if there exist $\bar{x} \in \mathbb{R}^N, \bar{t} > 0$ and an open neighborhood $U \subseteq \mathbb{R}^N$ of $\bar{x}$ such that*

$$Z^+(z_0) \subset \Omega \qquad \textit{for every} \quad z_0 \in \Sigma,$$

where $Z^+(z_0) = Z^+_{\bar{x},\bar{t},U}(z_0)$ satisfies ii).

We point out that, by its very construction, $Z^+_{\bar{x},\bar{t},U}(z_0)$ satisfies the Harnack connectivity condition for every $z_0 \in \mathbb{R}^{N+1}$ if $Z^+_{\bar{x},\bar{t},U}(w_0)$ does satisfy it for some $w_0 \in \mathbb{R}^{N+1}$.

We are now ready to formulate our main result.

Theorem 1.2 *Let $\mathscr{L}$ be an operator in the form* (1.1)*, satisfying assumptions* **[H.1-4]**. *Let Ω be an open subset of $\mathbb{R}^{N+1}$, let Σ be an open subset of $\partial\Omega$, let K be a compact subset of $\overline{\Omega}$ and let $\widetilde{z} \in \Omega$. Assume that $\partial\Omega \cap K \subset \Sigma$, and that $K \subset \mathrm{Int}(\mathscr{A}_{\widetilde{z}})$ (with respect to the topology of $\overline{\Omega}$). Suppose that Σ satisfies both interior and exterior uniform cone condition and that there exist an open set $V \subset \mathbb{R}^{N+1}$ and a positive constant $\bar{c}$, such that*

- *i) $K \cap \Sigma \subseteq V$,*
- *ii) for every $z \in V \cap \Omega$ there exists a pair $(w, s) \in \Sigma \times \mathbb{R}^+$ with $z = w \circ \delta_s(\bar{x}, \bar{t})$, and $d_K(w \circ \delta_s(\bar{x}, \bar{t}), \Sigma) \geq \bar{c}\, s$.*

Then there exists a positive constant C_K, only depending on $\Omega, \Sigma, K, \widetilde{z}$ and on $\mathscr{L}$, such that

$$\sup_K u \leq C_K\, u(\widetilde{z}),$$

for every non-negative solution u of $\mathscr{L}u = 0$ in Ω such that $u_{|\Sigma} = 0$.

Remark 1.3 *The exterior cone condition yields the existence of barrier functions for the boundary value problem (see Manfredini [15]), then it gives an uniform continuity modulus of the solution near the boundary. We also note that, when $\mathscr{L}$ is an uniformly parabolic operator, then assumptions i) and ii) made in Theorem 1.2 are satisfied by $\mathrm{Lip}\big(1, \frac{1}{2}\big)$ surfaces.*

Next proposition provides us with a simple sufficient condition for these assumptions in the case of degenerate operators $\mathscr{L}$. We say that a bounded open set Ω is *regular* if $\Omega = \mathrm{Int}\big(\overline{\Omega}\big)$ and its boundary is covered by a finite set of manifolds. In the following ν denotes the outer normal on $\partial\Omega$.

Proposition 1.4 *Let $\Omega \subset \mathbb{R}^{N+1}$ be a bounded open regular set, let Σ be an open subset of $\partial\Omega$. Let $\widetilde{z} \in \Omega, \widetilde{w} \in \Sigma$ be such that $\widetilde{w} \in \mathrm{Int}(\mathscr{A}_{\widetilde{z}})$ (with respect to the topology of $\overline{\Omega}$). Assume that there exists an open neighborhood $W \subset \mathbb{R}^{N+1}$ of $\widetilde{w}$ such that $\Sigma \cap W$ is a N-dimensional C^1 manifold, and suppose that either*

- *a) $(\nu_1(\widetilde{w}), \dots, \nu_m(\widetilde{w})) \neq 0$,*

or

b) $(\nu_1(z),\dots,\nu_m(z)) = 0$ *at every* $z \in W \cap \Sigma$, *and* $\langle Y(\widetilde{w}), \nu(\widetilde{w})\rangle > 0$.

Then there exists an open neighborhood $\widetilde{W} \subset W$ *of* $\widetilde{w}$ *such that*

i) $\Sigma \cap \widetilde{W}$ *satisfies both interior and exterior uniform cone conditions,*

ii) $\widetilde{W} \subset \mathrm{Int}(\mathscr{A}_{\widetilde{z}})$,

iii) for any $z \in \widetilde{W} \cap \Omega$ *there exists a pair* $(w, s) \in \Sigma \times \mathbb{R}^+$ *with* $z = w \circ \delta_s(\bar{x}, \bar{t})$, *and* $d_K(w \circ \delta_s(\bar{x}, \bar{t}), \Sigma) \geq \bar{c}\, s$, *for some positive constant* $\bar{c}$.

As a consequence of Proposition 1.4, the assumptions made in Theorem 1.2 are satisfied by any compact set $K \subset \widetilde{W} \cup \Sigma$. Then there exists a positive constant C_K such that $\sup_K u \leq C_K\, u(\widetilde{z})$, for every non-negative solution u of $\mathscr{L}u = 0$ in Ω such that $u_{|\Sigma} = 0$.

Remark 1.5 *The above condition* $(\nu_1(w),\dots,\nu_m(w)) \neq 0$ *can be used also in the case of cylinders. More precisely, if* $\Omega = \widetilde{\Omega} \cap \{(x,t) \in \mathbb{R}^{N+1} \mid t > t_0\}$, *and* $\widetilde{\Omega}$ *satisfies condition a) of Proposition 1.4 at some point* $w = (x_0, t_0) \in \Sigma$, *then cones build at every point of* $\partial\widetilde{\Omega}$ *can be used for* $\partial\Omega$ *as well.*

This paper is organized as follows. In Section 2 we recall some notations and a Harnack type inequality for Kolmogorov equations. Then we prove in Theorem 2.4 a geometric version of the Harnack inequality, formulated in terms of $\mathscr{L}$-admissible paths. In Section 3 we prove some results about the behavior of the solution to $\mathscr{L}u = 0$ near the boundary of its domain. In Section 4 we show that the uniform Harnack connectivity condition required in Theorem 1.2 is not a technical assumption but it is needed by the strong degeneracy of Kolmogorov operators. Section 5 is devoted to the Proof of Theorem 1.2 and Proposition 1.4.

2 Preliminaries and Interior Harnack inequalities

In this Section we introduce some notations, then we state some Harnack type inequalities for Kolmogorov equations.

We split the coordinate $x \in \mathbb{R}^N$ as

$$x = \left(x^{(0)}, x^{(1)}, \dots, x^{(\kappa)}\right), \qquad x^{(0)} \in \mathbb{R}^m, \quad x^{(j)} \in \mathbb{R}^{m_j}, \quad j \in \{1,\dots,\kappa\}. \tag{2.1}$$

Based on this we define

$$|x|_K = \sum_{j=0}^{\kappa} \left|x^{(j)}\right|^{\frac{1}{2j+1}}, \qquad \|(x,t)\|_K = |x|_K + |t|^{\frac{1}{2}}.$$

We note that $\|\delta_r z\|_K = r\|z\|_K$ for every $r > 0$ and $z \in \mathbb{R}^{N+1}$. We recall the following pseudo-triangular inequality: there exists a positive constant $\mathbf{c}$ such that

$$\|z^{-1}\|_K \leq \mathbf{c}\|z\|_K, \qquad \|z \circ \zeta\|_K \leq \mathbf{c}(\|z\|_K + \|\zeta\|_K), \qquad z, \zeta \in \mathbb{R}^{N+1}. \tag{2.2}$$

We also define the *quasi-distance* d_K by setting

$$d_K(z,\zeta) := \|\zeta^{-1} \circ z\|_K, \qquad z,\zeta \in \mathbb{R}^{N+1}, \tag{2.3}$$

and the *ball*

$$\mathcal{B}_K(z_0,r) := \{z \in \mathbb{R}^{N+1} \mid d_K(z,z_0) < r\}. \tag{2.4}$$

Note that from (2.2) it directly follows

$$d_K(z,\zeta) \le \mathbf{c}(d_K(z,w) + d_K(w,\zeta)), \qquad z,\zeta,w \in \mathbb{R}^{N+1}.$$

We say that a function $f : \Omega \to \mathbb{R}$ is Hölder continuous of exponent $\alpha \in]0,1]$, in short $f \in C_K^{0,\alpha}(\Omega)$, if there exists a positive constant C such that

$$|f(z) - f(\zeta)| \le C\, d_K(z,\zeta)^\alpha, \qquad \text{for every } \ z,\zeta \in \Omega. \tag{2.5}$$

For any positive R and $(x_0,t_0) \in \mathbb{R}^{N+1}$, we put $Q^- = \big(B_1(\frac{1}{2}\, e_1) \cap B_1(-\frac{1}{2}\, e_1)\big) \times [-1,0]$, and $Q_R^-(x_0,t_0) = (x_0,t_0) \circ \delta_R\,(Q^-)$. For $\alpha,\beta,\gamma,\theta \in \mathbb{R}$, with $0 < \alpha < \beta < \gamma < \theta^2$, we set

$$\begin{aligned}
\widetilde{Q}_R^-(x_0,t_0) &= \big\{(x,t) \in Q_{\theta R}^-(x_0,t_0) \mid t_0 - \gamma R^2 \le t \le t_0 - \beta R^2\big\}, \\
\widetilde{Q}_R^+(x_0,t_0) &= \big\{(x,t) \in Q_{\theta R}^-(x_0,t_0) \mid t_0 - \alpha R^2 \le t \le t_0\big\}.
\end{aligned}$$

We recall the following invariant Harnack inequality for non-negative solutions u of $\mathscr{L} u = 0$.

Theorem 2.1 (Theorem 1.2 in [10]) *Under assumptions* **[H.1-3]**, *there exist constants* $R_0 > 0$, $M > 1$ *and* $\alpha,\beta,\gamma,\theta \in]0,1[$, *with* $0 < \alpha < \beta < \gamma < \theta^2$, *depending only on the operator* $\mathscr{L}$, *such that*

$$\sup_{\widetilde{Q}_R^-(x_0,t_0)} u \le M \inf_{\widetilde{Q}_R^+(x_0,t_0)} u,$$

for every non-negative solution u *of* $\mathscr{L} u = 0$ *in* $Q_R^-(x_0,t_0)$ *and for any* $R \in]0,R_0]$, $(x_0,t_0) \in \mathbb{R}^{N+1}$.

Remark 2.2 *As noticed in Section 1, unlike the uniform parabolic case, in Theorem 2.1 the constants* $\alpha,\beta,\gamma,\theta$ *cannot be arbitrarily chosen. Indeed, according to [7, Proposition 4.5], the cylinder* $\widetilde{Q}_R^-(x_0,t_0)$ *has to be contained in* $\mathrm{Int}\big(\mathscr{A}_{(x_0,t_0)}\big)$.

We next formulate and prove a non-local Harnack inequality which is stated in terms of $\mathscr{L}$-admissible paths. This result is the analogous of [7, Theorem 3.2] for operators satisfying **[H.1-3]**. Note that here, unlike in [7, Theorem 3.2], we don't require assumption **[H.4]**. We first introduce some notations based on (2.3). For any $z \in \mathbb{R}^{N+1}$ and $H \subset \mathbb{R}^{N+1}$, we define

$$d_K(z,H) := \inf\{d_K(z,\zeta) \mid \zeta \in H\}.$$

Finally, for any open set $\Omega \subset \mathbb{R}^{N+1}$ and for any $\varepsilon \in]0,1[$, we define

$$\Omega_\varepsilon = \{z \in \Omega \mid d_K(z,\partial\Omega) \ge \varepsilon\}. \tag{2.6}$$

Theorem 2.3 *Let $\mathscr{L}$ be an operator in the form* (1.1)*, satisfying assumptions* **[H.1-3]**. *Let Ω be an open subset of $\mathbb{R}^{N+1}$, and let $\varepsilon \in]0,1]$ be so small that $\Omega_\varepsilon \neq \emptyset$. Consider a $\mathscr{L}$-admissible path γ, contained in Ω_ε, with $\inf_{[0,T]} \lambda > 0$. Then there exists a positive constant $C(\gamma, \varepsilon)$, that also depends on the constants appearing in* **[H.1-3]***, such that*

$$u(\xi, \tau) \leq C(\gamma, \varepsilon)\, u(x,t), \qquad (x,t) = \gamma(0), \quad (\xi, \tau) = \gamma(T),$$

for every non-negative solution u of $\mathscr{L}u = 0$ in Ω. Moreover

$$C(\gamma, \varepsilon) = \exp\left(c_0 + c_1 \frac{t-\tau}{\varepsilon^2} + c_2 \int_0^T \frac{\omega_1^2(s) + \cdots + \omega_m^2(s)}{\lambda(s)}\, ds \right),$$

where c_0, c_1 and c_2 are positive constants only depending on the operator $\mathscr{L}$.

Proof. We follow the same argument used in [7, Theorem 3.2]. We summarize the proof for the reader's convenience.

We first assume $\lambda \equiv 1$, so that $T = t - \tau$. We claim that there exists a finite sequence $\sigma_0, \sigma_1, \ldots, \sigma_k \in [0, t-\tau]$ with $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_k = t - \tau$, such that

$$u(\gamma(\sigma_j)) \leq M\, u(\gamma(\sigma_{j-1})), \quad j = 1, \ldots, k, \tag{2.7}$$

where $M > 1$ is the constant in Theorem 2.1. Hence

$$u(\gamma(t-\tau)) \leq M^k u(\gamma(0)), \tag{2.8}$$

and the claim follows by establishing a suitable bound for k. In order to apply Theorem 2.1, we have to show that there exist $r_0, r_1, \ldots, r_{k-1} \in]0, R_0]$, with

$$Q^-_{r_j}(\gamma(\sigma_j)) \subset \Omega, \quad \gamma(\sigma_{j+1}) \in \widetilde{Q}^-_{r_j}(\gamma(\sigma_j)) \qquad j = 0, 1, \ldots, k-1. \tag{2.9}$$

Since $\gamma([0, t-\tau]) \subset \Omega_\varepsilon$, there exists $\mu \in \left]0, \min\left\{1, \frac{R_0}{\varepsilon}\right\}\right[$ such that

$$Q^-_{\mu\varepsilon}(\gamma(\sigma)) \subset \mathcal{B}_K(\gamma(\sigma), \varepsilon) \subset \Omega, \qquad \text{for every } \ \sigma \in [0, t-\tau]. \tag{2.10}$$

Moreover, in [3, Lemma 2.2] it is shown that there exists a positive constant h, only dependent on $\mathscr{L}$, such that, for any $0 \leq a < b \leq t - \tau$,

$$\int_a^b |\omega(s)|^2 ds \leq h \quad \Rightarrow \quad \gamma(b) \in \widetilde{Q}^-_r(\gamma(a)), \ \text{ with } r = \sqrt{\frac{b-a}{\beta}}. \tag{2.11}$$

We are now in position to choose the σ_j's. We set $\sigma_0 = 0$, and we recursively define

$$\sigma_{j+1} = \min\left\{ \sigma_j + \beta(\mu\varepsilon)^2, \inf\left\{ \sigma \in]\sigma_j, t-\tau] : \int_{\sigma_j}^{\sigma} \frac{|\omega(s)|^2}{h}\, ds > 1 \right\} \right\}. \tag{2.12}$$

Note that, as the L^2 norm of ω is assumed to be finite, there exists a integer $j =: k-1$ such that the integral in (2.12) does not exceed 1. In this case we agree to set $\sigma_k = t - \tau$. Then, we let

$$r_j = \sqrt{\frac{\sigma_{j+1} - \sigma_j}{\beta}}, \qquad j = 0, \ldots, k-1.$$

The sequences $\{\sigma_j\}_{j=0}^{k}$ and $\{r_j\}_{j=0}^{k-1}$ satisfy (2.9). Indeed, we have $r_j \le \mu\varepsilon$, so that the first part of (2.9) follows from (2.10). On the other hand, since $0 \le \sigma_j < \sigma_{j+1} \le t-\tau$ and $\int_{\sigma_j}^{\sigma_{j+1}} |\omega(s)|^2\, ds \le h$, also the second requirement of (2.9) is fulfilled by (2.11).

In order to estimate k, the definition in (2.12) yields that

$$k-1 < \sum_{j=0}^{k-1} \int_{\sigma_j}^{\sigma_{j+1}} \left(\frac{|\omega(s)|^2}{h} + \frac{1}{\beta(\mu\varepsilon)^2} \right) ds \le 2k,$$

and therefore

$$k \le 1 + \frac{t-\tau}{c\,\varepsilon^2} + \frac{1}{h} \int_0^{t-\tau} |\omega(s)|^2 ds. \tag{2.13}$$

Hence, in the case $\lambda \equiv 1$, the proof is a direct consequence of (2.8) and (2.13), by setting

$$c_0 = \log(M), \qquad c_1 = \frac{\log(M)}{c}, \qquad c_2 = \frac{\log(M)}{h}. \tag{2.14}$$

Next consider any measurable function $\lambda : [0,T] \to \mathbb{R}$ such that $\inf_{[0,T]} \lambda > 0$, and set

$$\varphi : [0,T] \to [0, t-\tau], \qquad \varphi(s) = \int_0^s \lambda(\rho) d\rho, \; s \in [0,T].$$

Then, the function $\widetilde{\gamma}(s) := \gamma(\varphi^{-1}(s))$ satisfies

$$\begin{aligned} &\widetilde{\gamma} : [0, t-\tau] \to \Omega, \quad \widetilde{\gamma}(0) = (x,t), \quad \widetilde{\gamma}(t-\tau) = (\xi, \tau) \\ &\widetilde{\gamma}'(s) = \sum_{j=1}^{m} \frac{\omega_j(\varphi^{-1}(s))}{\lambda(\varphi^{-1}(s))} X_j(\widetilde{\gamma}(s)) + Y(\widetilde{\gamma}(s)), \quad \text{for a.e. } s \in [0, t-\tau]. \end{aligned}$$

By applying the first part of the proof to $\widetilde{\gamma}$, we obtain

$$\int_0^{t-\tau} \left(\frac{\omega_1(\varphi^{-1}(s))}{\lambda(\varphi^{-1}(s))} \right)^2 + \cdots + \left(\frac{\omega_m(\varphi^{-1}(s))}{\lambda(\varphi^{-1}(s))} \right)^2 ds = \int_0^T \frac{\omega_1^2(\rho) + \cdots + \omega_m^2(\rho)}{\lambda(\rho)} d\rho.$$

This accomplishes the proof. □

Theorem 2.4 *Let $\mathscr{L}$ be an operator in the form* (1.1)*, satisfying assumptions* **[H.1-3]**. *Let Ω be an open subset of $\mathbb{R}^{N+1}$ and let $z_0 \in \Omega$. For every compact set $K \subseteq \mathrm{Int}(\mathscr{A}_{z_0})$, there exists a positive constant C_K, only dependent on Ω, z_0, K and on the operator $\mathscr{L}$, such that*

$$\sup_K u \le C_K \, u(z_0),$$

for every non-negative solution u of $\mathscr{L}u = 0$ in Ω.

Proof. Let K be a compact subset of $\mathrm{Int}(\mathscr{A}_{z_0})$. Then, if $(x,t) \in K$, we have

$$Q_r^-(\bar{x}, \bar{t}) \subset \mathscr{A}_{z_0}, \qquad (\bar{x}, \bar{t}) = (x,t) \circ \left(0, r^2 \frac{\beta + \gamma}{2}\right),$$

for a sufficiently small $r \in]0, R_0]$. Here β, γ are as in Theorem 2.1, which gives

$$\sup_{\widetilde{Q}_r^-(\bar{x},\bar{t})} u \le M \inf_{\widetilde{Q}_r^+(\bar{x},\bar{t})} u.$$

Note that $\widetilde{Q}_r^-(\bar{x},\bar{t})$ is a neighborhood of (x,t). We next show that there exists a positive constant $\widetilde{C}$ only depending on (x,t) such that

$$\inf_{\widetilde{Q}_r^+(\bar{x},\bar{t})} u \le \widetilde{C}\, u(z_0). \tag{2.15}$$

The proof of Theorem 2.4 will follow from a standard covering argument.

We prove (2.15). There exists a $\mathscr{L}$-admissible path $\gamma : [0,T] \to \Omega$ defined by $\omega_1, ..., \omega_m, \lambda$ and connecting z_0 to $(\bar{x},\bar{t}) \in \mathrm{Int}(\mathscr{A}_{z_0})$. For every positive ε, denote by γ_ε the solution to

$$\begin{aligned}
&\gamma_\varepsilon : [0,T] \to \mathbb{R}^{N+1}, \qquad \gamma_\varepsilon(0) = z_0,\\
&\gamma_\varepsilon'(s) = \sum_{j=1}^m \omega_j(s) X_j(\gamma_\varepsilon(s)) + (\lambda(s)+\varepsilon) Y(\gamma_\varepsilon(s)), \quad \text{for a.e. } s \in [0,T].
\end{aligned}$$

In particular, since γ_ε converges uniformly to γ as $\varepsilon \to 0$, and $\gamma([0,T])$ is a compact subset of Ω, it is possible to choose ε such that $\gamma_\varepsilon([0,T]) \subset \Omega$. Note that $\gamma_\varepsilon(T) = (x_\varepsilon, \bar{t} - \varepsilon T)$, then $\gamma_\varepsilon(T) \in \widetilde{Q}_r^+(\bar{x},\bar{t})$, provided that ε is suitably small. Since $\inf_{[0,T]}(\lambda(s)+\varepsilon) \ge \varepsilon$, Theorem 2.3 implies that there exists a constant $C(\gamma,\varepsilon) > 0$ such that

$$u(\gamma_\varepsilon(T)) \le C(\gamma,\varepsilon)\, u(z_0).$$

This gives (2.15) and ends the proof. $\square$

3 Basic Boundary estimates

In this section we prove some results on the behavior of the solution to $\mathscr{L}u = 0$ near the boundary of its domain. We fist recall the definition of the ball $\mathcal{B}_K(z_0, r)$ in (2.4).

Lemma 3.1 *Let $\Omega \subseteq \mathbb{R}^{N+1}$ be an open set, and let Σ be an open subset of $\partial\Omega$ satisfying exterior uniform cone condition i) in Definition 1.1. Then, for every $\theta \in]0,1[$ there exists $\rho_\theta \in]0,1]$ such that*

$$\sup_{\Omega \cap \mathcal{B}_K(z_0, r\rho_\theta)} u \le \theta \sup_{\Omega \cap \mathcal{B}_K(z_0, r)} u \tag{3.1}$$

for every non-negative solution u of $\mathscr{L}u = 0$ in Ω such that $u_{|\Sigma} = 0$, and for every $z_0 \in \Sigma$ and $r > 0$ such that $\mathcal{B}_K(z_0,r) \cap \partial\Omega \subset \Sigma$.

Proof. We rely on a standard local barrier argument. Let $\bar{x} \in \mathbb{R}^N, \bar{t} > 0$ and $U \subseteq \mathbb{R}^N$ be such that

$$Z^-(z_0) = Z^-_{\bar{x},\bar{t},U}(z_0) \subset \mathbb{R}^{N+1} \setminus \Omega \qquad \text{for every} \quad z_0 \in \Sigma.$$

Then, [15, Theorem 6.3] implies that every $z_0 \in \Sigma$ is a $\mathscr{L}$-regular point in the sense of the abstract potential theory (see, *e.g.*, [1],[8]). We next show that the uniform cone condition gives a uniform estimate of the continuity modulus of the solution u near the boundary.

By [1, Satz 4.3.3] (see also [8, Proposition 2.4.5]) and the Lie group invariance, there exists a neighborhood V_0 of 0 and a barrier function $w : V_0 \setminus Z^-(0) \to \mathbb{R}$ such that

$$w(0) = 0, \quad \mathscr{L}w \le 0 \quad \text{in } \mathrm{Int}(V_0 \setminus Z^-(0)), \quad w > 0 \ \text{ in } \overline{V_0 \setminus Z^-(0)} \setminus \{0\}.$$

It is not restrictive to assume that $V_0 = \mathcal{B}_K(0, R)$ for some positive R, and

$$\inf_{\partial \mathcal{B}_K(0,R) \setminus Z^-(0)} w = 1. \tag{3.2}$$

Being w continuous at 0, for every $\theta \in]0, 1[$ there exists $\rho_\theta \in]0, 1]$ such that

$$\sup_{\mathcal{B}_K(0,R\rho_\theta) \setminus Z^-(0)} w < \theta. \tag{3.3}$$

Let $z_0 \in \Sigma$, and let $r > 0$ be such that $\mathcal{B}_K(z_0, r) \cap \partial\Omega \subset \Sigma$. We consider the function

$$v(z) = w(\delta_{R/r}(z_0^{-1} \circ z)).$$

Since $Z^-(0)$ is invariant with respect to dilations, (3.2) and (3.3) read as

$$\inf_{\partial \mathcal{B}_K(z_0,r) \setminus Z^-(z_0)} v = 1, \qquad \sup_{\mathcal{B}_K(z_0,r\rho_\theta) \setminus Z^-(z_0)} v < \theta. \tag{3.4}$$

Let $u \ge 0$ be a solution to $\mathscr{L}u = 0$ in Ω, $u_{|\Sigma} = 0$. The classical maximum principle together with the first equation in (3.4) yield

$$u \le v \sup_{\Omega \cap \mathcal{B}_K(z_0,r)} u \qquad \text{in } \Omega \cap \mathcal{B}_K(z_0, r).$$

Then, the claim directly follows from the second assertion of (3.4). $\square$

Proposition 3.2 *Let $\mathscr{L}$ be an operator in the form* (1.1)*, satisfying assumptions* **[H.1-4]**. *Let Ω be an open subset of $\mathbb{R}^{N+1}$, and let $z_0 \in \partial\Omega$. Suppose that there exists a cone $Z^+_{\bar{x},\bar{t},U}(z_0) \subset \Omega$, satisfying the Harnack connectivity condition ii) in Definition 1.1. Then there exist two positive constants C and β, such that*

$$u(z_0 \circ \delta_s(\bar{x}, \bar{t})) \le \frac{C}{\|\delta_s(\bar{x}, \bar{t})\|_K^\beta} \sup_{r \in [s_0,1]} u(z_0 \circ \delta_r(\bar{x}, \bar{t})) \qquad 0 < s < s_0,$$

for every non-negative solution u of $\mathscr{L}u = 0$ in Ω.

Proof. For any positive ρ we set $\widetilde{Z}^+(z_0) = \mathrm{Int}\big(z_0 \circ \delta_\rho(Z^+_{\bar{x},\bar{t},U}(0))\big)$. Since $Z^+(z_0)$ is a bounded set and Ω is open, there exists $\rho > 1$ such that

$$z_0 \circ (\bar{x}, \bar{t}) \in \widetilde{Z}^+(z_0) \subset \Omega \quad \text{and} \quad \mathscr{A}_{z_0 \circ (\bar{x},\bar{t})}(Z^+(z_0)) \subset \mathscr{A}_{z_0 \circ (\bar{x},\bar{t})}\big(\widetilde{Z}^+(z_0)\big).$$

Then, by applying Theorem 2.4 to the compact set $K = \{z_0 \circ \delta_{s_0}(\bar{x}, \bar{t})\}$, there exists a positive constant $\widetilde{C} = \widetilde{C}(z_0, s_0, \bar{x}, \bar{t}, U)$ such that

$$u(z_0 \circ \delta_{s_0}(\bar{x}, \bar{t})) \leq \widetilde{C}\, u(z_0 \circ (\bar{x}, \bar{t})), \tag{3.5}$$

for every solution $u \geq 0$ of $\mathscr{L} u = 0$ in $\widetilde{Z}^+(z_0)$.

We are now in position to conclude the proof. For a given $s \in]0, s_0[$, the function

$$u_s : \widetilde{Z}^+(z_0) \to \mathbb{R}, \qquad u_s = u\big(z_0 \circ \delta_{s/s_0}(z_0^{-1} \circ \cdot)\big)$$

is a non-negative solution to $\mathscr{L}_s u_s = 0$, where

$$\mathscr{L}_s = \sum_{i,j=1}^{m} a_{i,j}\big(z_0 \circ \delta_{s/s_0}(z_0^{-1} \circ z)\big)\partial_{x_i x_j} + \sum_{i=1}^{m} \frac{s}{s_0}\, a_i\big(z_0 \circ \delta_{s/s_0}(z_0^{-1} \circ z)\big)\partial_{x_i} + \sum_{i,j=1}^{N} b_{i,j} x_i \partial_{x_j} - \partial_t.$$

Since $\mathscr{L}_s$ satisfies assumptions **[H.1-3]**, then (3.5) also applies to u_s. As a consequence,

$$u(z_0 \circ \delta_s(\bar{x}, \bar{t})) = u_s(z_0 \circ \delta_{s_0}(\bar{x}, \bar{t})) \leq \widetilde{C}\, u_s(z_0 \circ (\bar{x}, \bar{t})) = \widetilde{C}\, u(z_0 \circ \delta_{s/s_0}(\bar{x}, \bar{t})). \tag{3.6}$$

Now let n be the unique positive integer such that $s_0^{n+1} \leq s < s_0^n$. By applying n times (3.6) we find

$$u(z_0 \circ \delta_s(\bar{x}, \bar{t})) \leq \widetilde{C}^n\, u(z_0 \circ \delta_r(\bar{x}, \bar{t})), \qquad r = s/(s_0)^n.$$

On the other hand, the δ_r-homogeneity of the norm $\|\cdot\|_K$ yields

$$n = \frac{\ln \big\|\delta_{s_0^n}(\bar{x}, \bar{t})\big\|_K - \ln \|(\bar{x}, \bar{t})\|_K}{\ln s_0},$$

so that

$$\widetilde{C}^n = C\,\big\|\delta_{s_0^n}(\bar{x}, \bar{t})\big\|_K^{-\beta}, \tag{3.7}$$

with $C = \exp\big(-\frac{\ln \widetilde{C}}{\ln s_0} \ln \|(\bar{x}, \bar{t})\|_K\big)$, and $\beta = -\frac{\ln \widetilde{C}}{\ln s_0} > 0$.

Finally, since $s < s_0^n$ and $\beta > 0$, from (3.7) it follows that $\widetilde{C}^n < C\,\big\|\delta_s(\bar{x}, \bar{t})\big\|_K^{-\beta}$, so that

$$u\big(z_0 \circ \delta_s(\bar{x}, \bar{t})\big) \leq \frac{C}{\big\|\delta_s(\bar{x}, \bar{t})\big\|_K^{\beta}} \sup_{r \in [s_0, 1]} u(z_0 \circ \delta_r(\bar{x}, \bar{t})).$$

This accomplishes the proof. $\square$

4 About the Harnack connectivity condition

We next give some comments about the Harnack connectivity condition required in Proposition 3.2.

When $\mathscr{L}$ is an uniformly parabolic operator, it is easy to see that $\mathscr{A}_{(\bar{x},\bar{t})}(Z^+(0,0)) = Z^+(0,0)$, provided that U is connected. Hence $\delta_{s_0}(\bar{x}, \bar{t}) \in \operatorname{Int}\big(\mathscr{A}_{(\bar{x},\bar{t})}(Z^+(0,0))\big)$ is trivially satisfied for any $s_0 \in]0,1[$ and the statement of Proposition 3.2 restores the usual parabolic bound (see (3.5) in [17]):

$$u(x_0 + s\bar{x}, t_0 + s^2\bar{t}) \leq \frac{C}{s^\beta \|(\bar{x},\bar{t})\|^\beta} u(x_0 + \bar{x}, t_0 + \bar{t}) \qquad 0 < s < 1. \tag{4.1}$$

When considering degenerate Kolmogorov equations, the Harnack connectivity condition is not always satisfied, as the following Example 4.1 shows. Moreover, this assumption is relevant. Indeed, in Remark 4.2 we give an example of a domain such that the analogous of (4.1) fails.

Consider the simplest degenerate Kolmogorov equation in the form (1.2),

$$\partial_t u = \partial_{x_1}^2 u + x_1 \partial_{x_2} u, \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{R}, \tag{4.2}$$

and note that it can be written in terms of vector fields (1.3) as follows

$$X^2 u + Yu = 0, \qquad X = \partial_{x_1}, \quad \text{and} \quad Y = x_1 \partial_{x_2} - \partial_t.$$

Recall that the composition law and the dilations related to the operator in (4.2) are

$$(x,y,t) \circ (\xi,\eta,\tau) = (x+\xi, y+\eta - x\tau, t+\tau), \qquad \delta_r(x,y,t) = \big(rx, r^3 y, r^2 t\big),$$

respectively. Example 4.1 shows that we can easily find a cone $Z^+(0,0)$ and a point $(\bar{x},\bar{t})$ such that $\delta_s(\bar{x},\bar{t}) \notin \mathrm{Int}\big(\mathscr{A}_{(\bar{x},\bar{t})}(Z^+(0,0))\big)$ for every positive s.

Example 4.1 *In the setting of the Kolmogorov operator in* (4.2)*, we let* $(\bar{x},\bar{t}) = (1,0,1)$ *and* $Z^+(0,0,0) \subset \big\{(x,t) \in \mathbb{R}^3 \mid x_1 > 0\big\}$*. Then* $\mathscr{A}_{(\bar{x},\bar{t})}\left(Z^+(0,0,0)\right) \subset \big\{(x,t) \in \mathbb{R}^3 \mid x_2 \geq 0\big\}$ *and* $\delta_s(\bar{x},\bar{t}) = (s,0,s^2)$*.*

We consider the attainable set of $(0,0,0)$ in the following open set

$$\Omega =]-R,R[\times]-1,1[\times]-1,1[, \tag{4.3}$$

where R is a given positive constant. A direct computation shows that

$$\mathscr{A}_{(0,0,0)} = \big\{(x,t) \in \overline{\Omega} : |x_2| \leq M|t|\big\}, \tag{4.4}$$

In [7, Proposition 4.5] it is proved that there exists a non-negative solution of (4.2) such that $u \equiv 0$ in $\mathscr{A}_{(0,0,0)}$, and $u > 0$ in $\Omega \setminus \mathscr{A}_{(0,0,0)}$. As a consequence, a Harnack inequality as stated in Theorem 2.4 cannot hold in a set K that is not contained in $\mathrm{Int}\big(\mathscr{A}_{(0,0,0)}\big)$.

The following remark deals with the boundary behavior of a positive solution to $\mathscr{L}u = 0$.

Remark 4.2 *Let* Ω *be the set defined in* (4.3)*, with* $R \in \big]1,\frac{3}{2}\big[$*. Let* u *be the function built in [7, Proposition 4.5], which solves* (4.2) *and satisfies* $u \equiv 0$ *in* $\mathscr{A}_{(0,0,0)}(\Omega)$*,* $u > 0$ *in* $\Omega \setminus \mathscr{A}_{(0,0,0)}(\Omega)$*. Let* $z_0 = (0,1,-1), (\bar{x},\bar{t}) = (0,-1,1)$*, and let* $U =]-R,R[\times]-2,0[$*. Then the cone* $Z^+_{\bar{x},\bar{t},U}(z_0) \subset \Omega$*, but the following inequality*

$$\sup_{s\in[0,1]} u(z_0 \circ \delta_s(\bar{x},\bar{t})) \leq \frac{C}{\|(\bar{x},\bar{t})\|_K^\beta} u(z_0 \circ (\bar{x},\bar{t})),$$

which is the analogous of (4.1)*, does not hold.*

Indeed, if we set

$$\eta(s) = z_0 \circ \delta_s(\bar{x}, \bar{t}) = \big(0, 1 - s^3, s^2 - 1\big)\,,$$

there exists a $\widetilde{s} \in]0,1[$ *such that* $\eta(s) \in \mathscr{A}_{(0,0,0)}(\Omega)$ *for every* $s \in]0, \widetilde{s}]$*, and* $\eta(s) \notin \mathscr{A}_{(0,0,0)}(\Omega)$ *for every* $s \in]\widetilde{s}, 1[$*. On the other hand, we have* $z_0 \circ (\bar{x}, \bar{t}) = (0,0,0)$*, so that* $u(\eta(s)) > 0 = u(z_0 \circ (\bar{x}, \bar{t}))$ *for every* $s \in]\widetilde{s}, 1[$*.*

Hence, the assumption on $s_0 \in]0,1[$ *made in Proposition 3.2 cannot be avoided.*

5 Proof of our main results

Proof of Theorem 1.2. We denote by $\widetilde{K}$ and $\widetilde{\widetilde{K}}$ the following compact sets:

$$\widetilde{K} = \big\{ z \in \overline{\Omega} \mid d_K(z, K) \le \rho \big\}, \qquad \widetilde{\widetilde{K}} = \big\{ z \in \overline{\Omega} \mid d_K(z, \widetilde{K}) \le \rho \big\},$$

where ρ is a positive constant such that $\partial\Omega \cap \widetilde{\widetilde{K}} \subset \Sigma \cap V$, and that $\widetilde{\widetilde{K}} \subset \mathrm{Int}(\mathscr{A}_{\widetilde{z}})$ with respect to the topology of $\overline{\Omega}$. We also require that

$$\rho < \min\big\{\mathbf{c^{-1}}, \rho_\theta\big\}, \tag{5.1}$$

where $\mathbf{c}$ is the constant in (2.2), $\rho_\theta \in]0,1]$ is the constant in Lemma 3.1 related to any given $\theta \in]0, \mathbf{c}^{-\beta}[$, and β is as in Proposition 3.2.

We next claim that Ω satisfies an uniform interior cone condition with respect to a suitable $\widetilde{Z}^+(w) = Z^+_{\widetilde{x},\widetilde{t},\widetilde{U}}(w)$, such that

$$\widetilde{Z}^+(w) \subset \mathcal{B}_K(w, \mathbf{c}^{-1}\rho^2) \qquad \text{for every} \quad w \in \mathbb{R}^{N+1}. \tag{5.2}$$

To this aim, we set $(\widetilde{x}, \widetilde{t}) = \delta_\eta(\bar{x}, \bar{t})$ and $\widetilde{U} = D_\eta(U)$ for some $\eta \in]0,1[$. Note that $\widetilde{Z}^+(w) = w \circ \delta_\eta\big(Z^+_{\bar{x},\bar{t},U}(0)\big)$. Then, since $Z^+_{\bar{x},\bar{t},U}(0)$ is bounded, we can choose a small η such that $\delta_\eta\big(Z^+_{\bar{x},\bar{t},U}(0)\big) \subset \mathcal{B}_K(0, \mathbf{c}^{-1}\rho^2)$. This proves (5.2). As a plain consequence we have

$$\widetilde{Z}^+(w) \subseteq \widetilde{\widetilde{K}} \qquad \text{for every} \quad w \in \widetilde{K} \cap \Sigma. \tag{5.3}$$

Moreover, the Lie group invariance implies $w \circ \delta_{s_0}(\widetilde{x}, \widetilde{t}) \in \mathrm{Int}\big(\mathscr{A}_{(\widetilde{x},\widetilde{t})}(Z^+(w))\big)$ with the same $s_0 \in]0,1[$ as $Z^+_{\bar{x},\bar{t},U}(w)$. We also remark that condition *ii)* can be equivalently stated in terms of $(\widetilde{x}, \widetilde{t})$, since

$$d_K(w \circ \delta_s(\widetilde{x}, \widetilde{t}), \Sigma) = d_K(w \circ \delta_{s\eta}(\bar{x}, \bar{t}), \Sigma) \ge \widetilde{c}\, s \quad \text{with} \quad \widetilde{c} = \bar{c}\, \eta. \tag{5.4}$$

We finally remark that Proposition 3.2 applies to $\widetilde{Z}^+(w)$ with the same β as $Z^+(w)$.

Recalling notation (2.6), we set

$$K^\varepsilon = K \setminus \Omega_\varepsilon = \{ z \in K \mid d_K(z, \partial\Omega) < \varepsilon \}. \tag{5.5}$$

We next choose a sufficiently small $\varepsilon \in]0, \widetilde{c}\, s_0[$ such that if $z \in \widetilde{\widetilde{K}}$ satisfies $d_K(z, \Sigma) < \varepsilon$, then $z \in V$. In particular, if $z \in K^\varepsilon$ we have $z \in V$, so that there exists a unique $(w, s) \in \Sigma \times \mathbb{R}^+$ with $z = w \circ \delta_s(\widetilde{x}, \widetilde{t})$. We note that

$$d_K(z, \Sigma) \le d_K(z, w) = d_K(w \circ \delta_s(\widetilde{x}, \widetilde{t}), w) = s \|(\widetilde{x}, \widetilde{t})\|_K, \quad \text{and} \quad \widetilde{c}\, s \le d_K(z, \Sigma) < \varepsilon.$$

Since $\widetilde{c}\, s < \varepsilon < \widetilde{c}\, s_0$, we have $s < s_0 < 1$, then $z = w \circ \delta_s(\widetilde{x}, \widetilde{t}) \in \widetilde{Z}^+(w)$. By (5.2) and (2.2) we get $w \in \mathcal{B}_K(z, \rho)$, hence $w \in \widetilde{K} \cap \Sigma \subset V$.

In conclusion, if $z \in K^\varepsilon$, then there exists a unique pair $(w, s) \in \big(\widetilde{K} \cap \Sigma\big) \times]0, s_0[$ such that $z = w \circ \delta_s(\widetilde{x}, \widetilde{t})$. Moreover, if $w \in \widetilde{K} \cap \Sigma$ and $s \in [s_0, 1]$, by using (5.3) and (5.4) we find

$$w \circ \delta_s(\widetilde{x}, \widetilde{t}) \in \widetilde{\widetilde{K}} \cap \Omega_\varepsilon. \tag{5.6}$$

We are now in position to conclude the proof of Theorem 1.2. By Theorem 2.4, there exists a positive constant $\widetilde{C} = \widetilde{C}\big(\widetilde{\widetilde{K}} \cap \Omega_\varepsilon, \widetilde{z}, \Omega\big)$ such that

$$\sup_{\widetilde{\widetilde{K}} \cap \Omega_\varepsilon} u \le \widetilde{C}\, u(\widetilde{z}). \tag{5.7}$$

It is not restrictive to assume $u(\widetilde{z}) \neq 0$, otherwise the statement would be a plain consequence of Bony's maximum principle [2, Théorème 3.2]. Hence, up to a multiplication by a positive constant, we can suppose $\widetilde{C}\, u(\widetilde{z}) = 1$.

We fix a constant $\lambda > 1$, which will be suitably chosen later. By contradiction, we suppose that there exists $z_1 \in K$ satisfying $u(z_1) > \lambda$. Since $K \subset \widetilde{\widetilde{K}}$, we have $z_1 \in K^\varepsilon$. Then, there exists a unique $(w_1, s_1) \in \big(\widetilde{K} \cap \Sigma\big) \times]0, s_0[$ such that $z_1 = w_1 \circ \delta_{s_1}(\widetilde{x}, \widetilde{t})$. From Proposition 3.2 it follows that

$$\lambda < u(z_1) = u(w_1 \circ \delta_{s_1}(\widetilde{x}, \widetilde{t})) \le \frac{C}{\|\delta_{s_1}(\widetilde{x}, \widetilde{t})\|_K^\beta} \sup_{s \in [s_0, 1]} u(w_1 \circ \delta_s(\widetilde{x}, \widetilde{t})). \tag{5.8}$$

Hence, (5.6), (5.7) and (5.8) give

$$\rho_1 := s_1 \|(\widetilde{x}, \widetilde{t})\|_K < C^{\frac{1}{\beta}} \lambda^{-\frac{1}{\beta}}. \tag{5.9}$$

Since $s_1 < s_0 < 1$, we have $z_1 \in \widetilde{Z}^+(w_1) \subset \mathcal{B}_K(w_1, \mathbf{c}^{-1}\rho^2)$, by (5.2). Then $\rho_1 = d_K(z_1, w_1) < \mathbf{c}^{-1}\rho^2$. Since $\rho < \rho_\theta$ we have $\rho_\theta^{-1}\rho_1 < \rho$, then $\mathcal{B}_K(w_1, \rho_\theta^{-1}\rho_1) \cap \partial\Omega \subset \Sigma$. By Lemma 3.1, we then have

$$\lambda < u(z_1) \le \sup_{\Omega \cap \mathcal{B}_K(w_1, \rho_1)} u \le \theta \sup_{\Omega \cap \mathcal{B}_K(w_1, \rho_\theta^{-1}\rho_1)} u.$$

Hence, there exists $z_2 \in \overline{\Omega \cap \mathcal{B}_K(w_1, \rho_\theta^{-1}\rho_1)}$ such that

$$u(z_2) > \lambda\, \theta^{-1}. \tag{5.10}$$

We next show that $z_2 \in \widetilde{K}$, provided that λ is big enough. We have

$$d_K(z_2, z_1) \le \mathbf{c}\,(d_K(z_2, w_1) + d_K(w_1, z_1)) \le \mathbf{c}\left(\rho_\theta^{-1}\rho_1 + \mathbf{c}\rho_1\right) < \mathbf{c}\left(\rho_\theta^{-1} + \mathbf{c}\right) C^{\frac{1}{\beta}} \lambda^{-\frac{1}{\beta}}.$$

If we choose $\lambda > C\left(\mathbf{c}\left(\rho_\theta^{-1} + \mathbf{c}\right)\right)^\beta \rho^{-\beta}$ we obtain $z_2 \in \mathcal{B}_K(z_1, \rho) \subset \widetilde{K}$.

We note that $z_2 \in \widetilde{K}^\varepsilon$ by (5.10), then $z_2 \in V$. Hence there exists a unique $(w_2, s_2) \in \Sigma \times]0, s_0[$ such that $z_2 = w_2 \circ \delta_{s_2}(\widetilde{x}, \widetilde{t})$. We next show that, if λ is sufficiently large, then $w_2 \in \widetilde{K} \cap \Sigma$. We set $\rho_2 = s_2 \|(\widetilde{x}, \widetilde{t})\|_K$, and we see

$$d_K(w_2, z_1) \le \mathbf{c}\left(d_K(w_2, z_2) + d_K(z_2, z_1)\right) < \mathbf{c}^2\Big(\rho_2 + \left(\rho_\theta^{-1} + \mathbf{c}\right) C^{\frac{1}{\beta}} \lambda^{-\frac{1}{\beta}}\Big).$$

Since $s_2 < s_0 < 1$ and $\widetilde{Z}^+(w_2) \subset \mathcal{B}_K(w_2, \mathbf{c}^{-1}\rho^2)$, we have $\rho_2 < \mathbf{c}^{-1}\rho^2$. Then

$$d_K(w_2, z_1) < \mathbf{c}\,\Big(\rho^2 + \mathbf{c}\left(\rho_\theta^{-1} + \mathbf{c}\right) C^{\frac{1}{\beta}} \lambda^{-\frac{1}{\beta}}\Big) < \rho,$$

provided that $\lambda > C\Big(\frac{\mathbf{c}(\rho_\theta^{-1} + \mathbf{c})}{\rho(\mathbf{c}^{-1} - \rho)}\Big)^\beta$. Note that last expression is well defined, since $\rho < \mathbf{c}^{-1}$.

We next show that $s_2 \|(\widetilde{x}, \widetilde{t})\|_K < C^{\frac{1}{\beta}} \lambda^{-\frac{1}{\beta}} \theta^{\frac{1}{\beta}}$. Indeed, as in (5.8), it is easy to see that

$$\lambda\,\theta^{-1} < u(z_2) \le \frac{C}{\|\delta_{s_2}(\widetilde{x}, \widetilde{t})\|_K^\beta} \sup_{s \in [s_0, 1]} u(w_2 \circ \delta_s(\widetilde{x}, \widetilde{t})) \le \frac{C}{(s_2\|(\widetilde{x}, \widetilde{t})\|_K)^\beta}.$$

We next iterate the above argument. We set $\mathbf{c}_0 := \mathbf{c}\left(\rho_\theta^{-1} + \mathbf{c}\right)$, and we prove that, if

$$\lambda > \frac{C}{\theta}\left(\frac{\mathbf{c}_0 \sum_{k=1}^\infty \big(\theta^{\frac{1}{\beta}}\mathbf{c}\big)^k}{\rho(\mathbf{c}^{-1} - \rho)}\right)^\beta, \tag{5.11}$$

then there exists a sequence $\{z_j\}$ such that $z_j = w_j \circ \delta_{s_j}(\widetilde{x}, \widetilde{t})$, $0 < s_j < s_0$,

$$z_j \in \widetilde{K}^\varepsilon, \quad w_j \in \widetilde{K} \cap \Sigma, \quad u(z_j) > \lambda\,\theta^{1-j}, \quad s_j\|(\widetilde{x}, \widetilde{t})\|_K < C^{\frac{1}{\beta}} \lambda^{-\frac{1}{\beta}} \theta^{\frac{j-1}{\beta}}, \tag{5.12}$$

for every $j \in \mathbb{N}$. Note that the series in (5.11) is convergent since $\theta \in]0, \mathbf{c}^{-\beta}[$, and that $\rho < \mathbf{c}^{-1}$ by (5.1). As a consequence of last inequality in (5.12) we get $d_k(z_j, \Sigma) \le d_k(z_j, w_j) \to 0$ as $j \to \infty$, then $u(z_j)$ vanishes as $j \to \infty$, and we reach a contradiction. This shows that

$$\sup_K u \le \lambda\,\widetilde{C}\,u(\widetilde{z}),$$

and the proof is accomplished.

We next prove (5.12) by induction. The claim has previously been proved for $j = 1$ and $j = 2$. Assume that (5.12) is satisfied for $j = k$, and set $\rho_k = s_k\|(\widetilde{x}, \widetilde{t})\|_K$. Since $s_k < s_0 < 1$ we have $\rho_k < \mathbf{c}^{-1}\rho^2$, then $\rho_\theta^{-1}\rho_k < \rho$ and $\mathcal{B}_K(w_k, \rho_k) \cap \partial\Omega \subset \Sigma$. Hence, by Lemma 3.1 there exists $z_{k+1} \in \overline{\Omega \cap \mathcal{B}_K(w_k, \rho_k)}$ such that $u(z_{k+1}) > \lambda\,\theta^{-k}$.

We next show that $z_{k+1} \in \widetilde{K}^\varepsilon$. We have

$$d_K(z_{k+1}, z_1) \le \mathbf{c}\left(d_K(z_{k+1}, z_2) + d_K(z_2, z_1)\right) \le \sum_{j=1}^k \left(\mathbf{c}^j d_K(z_{j+1}, z_j)\right). \tag{5.13}$$

We also have

$$d_K(z_{j+1}, z_j) \le \mathbf{c}\left(d_K(z_{j+1}, w_j) + d_K(w_j, z_j)\right) \le \mathbf{c}\left(\rho_\theta^{-1}\rho_j + \mathbf{c}\rho_j\right) < \mathbf{c}_0 C^{\frac{1}{\beta}} \lambda^{-\frac{1}{\beta}} \theta^{\frac{j-1}{\beta}},$$

by (5.12). By using the above inequality in (5.13) we find

$$d_K(z_{k+1}, z_1) < \mathbf{c}_0 C^{\frac{1}{\beta}} (\lambda\, \theta)^{-\frac{1}{\beta}} \sum_{j=1}^{\infty} \left(\theta^{\frac{1}{\beta}} \mathbf{c}\right)^j < \rho,$$

because of our choice of λ in (5.11). This proves that $z_{k+1} \in \widetilde{K}$. Since $u(z_{k+1}) > \lambda\, \theta^{-k} > 1$, we actually have $z_{k+1} \in \widetilde{K}^{\varepsilon}$. In particular, $z_{k+1} \in V$, then there exists a unique $(w_{j+1}, s_{j+1}) \in \Sigma \times]0, s_0[$ such that $z_{j+1} = w_{j+1} \circ \delta_{s_{j+1}}(\widetilde{x}, \widetilde{t})$.

We next show that $w_{k+1} \in \widetilde{K} \cap \Sigma$. We see that

$$\begin{aligned} d_K(w_{k+1}, z_1) &\le \mathbf{c}\left(d_K(w_{k+1}, z_{k+1}) + d_K(z_{k+1}, z_1)\right) \\ &< \mathbf{c}\left(\mathbf{c}\rho_{k+1} + \mathbf{c}_0 C^{\frac{1}{\beta}} (\lambda\, \theta)^{-\frac{1}{\beta}} \sum_{j=1}^{\infty} \left(\theta^{\frac{1}{\beta}} \mathbf{c}\right)^j\right) \\ &< \mathbf{c}\left(\rho^2 + \mathbf{c}_0 C^{\frac{1}{\beta}} (\lambda\, \theta)^{-\frac{1}{\beta}} \sum_{j=1}^{\infty} \left(\theta^{\frac{1}{\beta}} \mathbf{c}\right)^j\right) < \rho, \end{aligned}$$

again by (5.11).

From Proposition 3.2 it follows that

$$\lambda\, \theta^{-k} < u(z_{k+1}) \le \frac{C}{\|\delta_{s_{k+1}}(\widetilde{x}, \widetilde{t})\|_K^{\beta}} \sup_{s \in [s_0, 1]} u(w_{k+1} \circ \delta_s(\widetilde{x}, \widetilde{t})) \le \frac{C}{(s_{k+1} \|(\widetilde{x}, \widetilde{t})\|_K)^{\beta}},$$

then $s_{k+1} \|(\widetilde{x}, \widetilde{t})\|_K < C^{\frac{1}{\beta}} \lambda^{-\frac{1}{\beta}} \theta^{\frac{k}{\beta}}$. This proves (5.12) for $j = k+1$. $\square$

Proof of Proposition 1.4. In both cases *a)* and *b)*, there exist an open neighborhood $W_1 \subseteq W$ of $\widetilde{w}$ and a C^1 function $F : W_1 \to \mathbb{R}$ such that $\Sigma \cap W_1 = \{z \in W_1 : F(z) = 0\}$, and $\nu = \nabla F \neq 0$ in $\Sigma \cap W_1$. Moreover, it is possible to choose F such that $\Omega \cap W_1 = \{z \in W_1 : F(z) < 0\}$. By Dini's theorem, there exist an open neighborhood $O \subset \mathbb{R}^N$ of 0, an open neighborhood $W_2 \subseteq W$ of $\widetilde{w}$, and a C^1 diffeomorphism $\varphi : O \to W_2 \cap \Sigma$ with $\varphi(0) = \widetilde{w}$, such that the Jacobian matrix $J_\varphi(0)$ has rank N. It is not restrictive to suppose that $W_1 = W_2 = W$, and that $W \subset \mathrm{Int}(\mathscr{A}_{\widetilde{z}})$. We next rely on the function φ to check the uniform cone conditions.

Assume that *a)* is satisfied. We first prove that the interior uniform cone condition holds in a neighborhood of $\widetilde{w}$. To this aim we carefully choose a point $\widetilde{x}$, a neighborhood $\widetilde{U}$ of $\widetilde{x}$, and we consider the cone $Z^+_{\widetilde{x},1,\widetilde{U}}(w)$, for every $w \in \Sigma \cap W$. As noticed in Example 4.1, the Harnack connectivity condition is not trivially satisfied by any cone Z^+. According to notation (2.1), we first choose the first component $\widetilde{x}^{(0)}$, then the other ones $\widetilde{x}^{(1)}, \dots, \widetilde{x}^{(\kappa)}$ in order to guarantee the existence of an $s_0 \in]0, 1[$ as required in Definition 1.1 *iii)*. We actually prove that, in this setting, the condition is fulfilled for *any* $s_0 \in]0, 1[$.

We choose $\widetilde{x}^{(0)} \in \mathbb{R}^m$ such that

$$\sum_{j=1}^{m} \widetilde{x}_j \nu_j(\widetilde{w}) < 0. \tag{5.14}$$

We consider the solution $\gamma : [0,1] \to \mathbb{R}^{N+1}$ of

$$\gamma'(s) = \sum_{j=1}^{m} \omega_j X_j(\gamma(s)) + Y(\gamma(s)), \qquad \gamma(0) = (\widetilde{x}, 1), \tag{5.15}$$

where the constant vector $(\omega_1, \ldots, \omega_m)$ equals $c\,\bar{A}_0^{-1}\widetilde{x}^{(0)}$, for a suitable constant c which will be chosen in the sequel. Note that γ is a $\mathscr{L}$-admissible path. Since (5.15) is equivalent to

$$\gamma'(s) = c\,\widetilde{x}^{(0)} + (B^T - \partial_t)\gamma(s), \qquad \gamma(0) = (\widetilde{x}, 1), \tag{5.16}$$

a direct computation shows that

$$\gamma(s) = \left(\exp\left(sB^T\right)\widetilde{x} + c\int_0^s \exp\left((s-\tau)B^T\right)(\widetilde{x}^{(0)}, 0, \ldots, 0)\, d\tau,\ 1 - s \right). \tag{5.17}$$

By using the same block decomposition as in (1.10), the matrix

$$\exp\left(sB^T\right) = E(-s) = (E_{ij}(-s))_{i,j=0,\ldots,\kappa}$$

is given by

$$\begin{aligned}
&E_{0,0}(-s) = I_m, \qquad\qquad E_{j,j}(-s) = I_{m_j}, \qquad j = 1, \ldots, \kappa, \\
&E_{j,k}(-s) = \frac{s^{j-k}}{(j-k)!}\, B_j^T \ldots B_{\kappa+1}^T, \qquad j = 1, \ldots \kappa, \quad k = 0, \ldots j-1.
\end{aligned}$$

We fix any $r \in]0,1[$, and we set $s_0 = \sqrt{1-r}$. We aim to choose c and $(\widetilde{x}^{(1)}, \ldots, \widetilde{x}^{(\kappa)})$ such that

$$\gamma(r) = \delta_{s_0}(\widetilde{x}, 1). \tag{5.18}$$

The equality between the time variables follows from $r = 1 - s_0^2$. Moreover, we find

$$\begin{aligned}
&\gamma^{(0)}(r) = \widetilde{x}^{(0)} + cr\widetilde{x}^{(0)} = s_0\widetilde{x}^{(0)}; \\
&\gamma^{(1)}(r) = rB_1^T\widetilde{x}^{(0)} + \widetilde{x}^{(1)} + c\frac{r^2}{2}\, B_1^T\widetilde{x}^{(0)} = s_0^3\widetilde{x}^{(1)}; \\
&\qquad \ldots \\
&\gamma^{(\kappa)}(r) = \sum_{i=1}^{\kappa} \frac{r^i}{i!}\, B_\kappa^T \ldots B_{\kappa-i+1}^T \widetilde{x}^{(\kappa-i)} + \widetilde{x}^{(\kappa)} \\
&\qquad\qquad + c\,\frac{r^{\kappa+1}}{(\kappa+1)!}\, B_\kappa^T \ldots B_1^T\widetilde{x}^{(0)} = s_0^{2\kappa+1}\widetilde{x}^{(\kappa)}.
\end{aligned}$$

As a consequence, in order to satisfy (5.18), it is sufficient to choose $c = \frac{s_0 - 1}{r}$, and

$$\widetilde{x}^{(j)} = \frac{1}{c \sum_{k=0}^{2j} s_0^k} \left(\sum_{i=1}^{j} \frac{r^{i-1}}{i!} B_j^T \dots B_{j-i+1}^T \widetilde{x}^{(j-i)} + c \frac{r^j}{(j+1)!} B_j^T \dots B_1^T \widetilde{x}^{(0)} \right), \tag{5.19}$$

for every $j = 1, \dots, \kappa$.

In order to prove that Σ satisfies an uniform interior cone condition for every w in a neighborhood of $\widetilde{w}$, we have to show that the values of the path γ in (5.16) and (5.18) belong to the cone $Z^+_{\widetilde{x},1,\widetilde{U}}(w)$. To this aim, we fix a suitably small r and we show that $\gamma(s)$ belongs to a prescribed neighborhood of $(\widetilde{x}, 1)$ for any $s \in [0, r]$. Since the definition of $\widetilde{x}$ depends on r, we make a careful construction in order to avoid recursive arguments. We define $\widetilde{y} \in \mathbb{R}^N$ by setting

$$\widetilde{y}^{(0)} = \widetilde{x}^{(0)}, \qquad \widetilde{y}^{(j)} = -\frac{2}{(2j+1)} B_j^T \widetilde{y}^{(j-1)}, \qquad j = 1, \dots, \kappa,$$

and, by (5.19), we find $\widetilde{x} = \widetilde{x}(r) \to \widetilde{y}$ as $r \to 0$. By (5.17) and the above fact, we have: for every neighborhood $\widetilde{V}$ of $(\widetilde{y}, 1)$ there exists $r \in]0, 1[$ such that

$$\gamma(s) \in \widetilde{V} \quad \text{for any} \quad s \in [0, r]. \tag{5.20}$$

We next use the point $(\widetilde{y}, 1)$ to build the needed interior cone. We first find a suitable neighborhood $\widetilde{U}$ of $\widetilde{y}$ to define the cone $Z^+_{\widetilde{y},1,\widetilde{U}}(\varphi(y))$. Then, we choose a small positive r such that, according to (5.19), $\widetilde{x} = \widetilde{x}(r) \in \widetilde{U}$, so that $Z^+_{\widetilde{x},1,\widetilde{U}}(\varphi(y)) = Z^+_{\widetilde{y},1,\widetilde{U}}(\varphi(y))$. With the aim to find a such $\widetilde{U}$, we first remark that the function $\Phi_{\widetilde{y},1} : O \times] -\sigma, \sigma[\to \mathbb{R}^{N+1}$,

$$\Phi_{\widetilde{y},1}(y, s) = \varphi(y) \circ \delta_s(\widetilde{y}, 1) = (D_s \widetilde{y} + \exp(-s^2 B^T) \varphi_x(y), \varphi_t(y) + s^2), \tag{5.21}$$

is a local diffeomorphism. Indeed, we have

$$\det J_{\Phi_{\widetilde{y},1}}(0, 0) = \det \Big(J_\varphi(0) \quad (\widetilde{x}^{(0)}, 0, \dots, 0)^T \Big) \neq 0, \tag{5.22}$$

since $\operatorname{rank} J_\varphi(0) = N$ and, by (5.14),

$$\langle (\widetilde{x}^{(0)}, 0, \dots, 0), \nu(\widetilde{w}) \rangle \neq 0.$$

By choosing $\sigma > 0$ so small as $\Phi_{\widetilde{y},1}(y, s) \in W$, we define

$$g :] -\sigma, \sigma[\to \mathbb{R}, \qquad g(s) = F\big(\Phi_{\widetilde{y},1}(y, s)\big).$$

Note that $g'(0) = \langle \nabla F(\varphi(y)), (\widetilde{x}^{(0)}, 0, \dots, 0) \rangle$ tends to $\sum_{j=1}^m \widetilde{x}_j \nu_j(\widetilde{w}) < 0$ as $y \to 0$, so that $g(s) < 0$ for $s > 0$. Then, there exist $\sigma_1 > 0$ and a neighborhood O_1 of $0 \in \mathbb{R}^N$ such that $\Phi_{\widetilde{y},1}(y, s) \in \Omega$ for $(y, s) \in O_1 \times]0, \sigma_1]$.

Next, let $\widehat{W}$ be an open neighborhood of $(\hat{y}, \hat{t}) := \delta_{\sigma_1}(\widetilde{y}, 1)$. For any $(\eta, \tau) \in \widehat{W}$ we consider the function $\Phi_{\eta,\tau}(y, s)$ defined as in (5.21). We have

$$\frac{d}{ds} \Phi_{\eta,\tau}(y, s)|_{s=0} = (\eta^{(0)}, 0, \dots, 0),$$

and
$$\langle(\eta^{(0)},0,\dots,0),\nu(\varphi(y))\rangle \to \sigma_1\langle(\widetilde{x}^{(0)},0,\dots,0),\nu(\widetilde{w})\rangle < 0,$$
as $(\eta,\tau)\to(\hat{y},\hat{t})$ and $y\to 0$. We choose $\widehat{W}$ suitably small, a $\sigma_2>0$ and a neighborhood O_2 of $0\in\mathbb{R}^N$ such that
$$\Phi_{\eta,\tau}(y,s)\in\Omega\cap W, \qquad \text{for every } \ (\eta,\tau)\in\widehat{W} \ \text{ and } \ (y,s)\in O_2\times\,]0,\sigma_2].$$
Since the function $(x,\rho)\mapsto\delta_\rho(x,1)$ is continuous, there exists a neighborhood $\widetilde{U}$ of $\widetilde{y}$ and $\varepsilon\in]0,1[$ such that
$$\widehat{V}:=\big\{\delta_\rho(x,1)\mid x\in\widetilde{U},\ \sigma_1(1-\varepsilon)<\rho<\sigma_1(1+\varepsilon)\big\}\subset\widehat{W}.$$
The set $\widetilde{V}:=\delta_{1/\sigma_1}\big(\widehat{V}\big)$ is an open neighborhood of $(\widetilde{y},1)$. Then, we choose $r\in]0,2\varepsilon-\varepsilon^2[$ such that (5.20) holds. In particular, $\gamma(0)=(\widetilde{x},1)\in\widetilde{V}$, where $\widetilde{x}=\widetilde{x}(r)\in\widetilde{U}$ is as in (5.19). Moreover, by (5.17),
$$\gamma(s)\in\widetilde{V}\cap\{(x,t)\mid t\in[s_0^2,1]\}=\big\{\delta_\rho(x,1)\mid x\in\widetilde{U},\ 1-\varepsilon<\rho\le 1\big\}, \tag{5.23}$$
for all $s\in[0,r]$.

We are now in position to verify that *i), ii), iii)* hold under condition *a)*. We set
$$\widetilde{W}=\big\{\varphi(y)\circ\delta_\rho(\widetilde{x},1)\mid y\in O_2,\ 0\le\rho\le\sigma_1\sigma_2\big\}.$$
Since $\sigma_1<1$, we have $\widetilde{W}\subset W\subset \mathrm{Int}(\mathscr{A}_{\widetilde{z}})$, then condition *ii)* of Proposition 1.4 is satisfied. We set $(\bar{x},\bar{t})=\delta_{\sigma_1\sigma_2}(\widetilde{x},1)$, and $U=D_{\sigma_1\sigma_2}\big(\widetilde{U}\big)$. In order to check the first statement of *iii)*, we consider the function
$$G(y,s)=\varphi(y)\circ\delta_s(\bar{x},\bar{t})=\Phi_{\widetilde{x},1}(y,\sigma_1\sigma_2 s).$$
As in (5.22), it is easy to check that it is a local diffeomorphism. Then, by shrinking a bit σ_2 if necessary, the function G is surjective onto $\widetilde{W}$, and this proves the first statement of *iii)*.

We next show that $Z^+_{\bar{x},\bar{t},U}(\varphi(y))$ satisfies the uniform interior cone condition, for every $y\in O_2$. We have
$$\begin{aligned} Z^+_{\bar{x},\bar{t},U}(\varphi(y)) &= \big\{\varphi(y)\circ\delta_{\rho/\sigma_2}(x,\bar{t})\mid x\in U,\ 0<\rho\le\sigma_2\big\}\\ &=\big\{\varphi(y)\circ\delta_\rho(\delta_{\sigma_1}(x,1))\mid x\in\widetilde{U},\ 0<\rho\le\sigma_2\big\}\\ &\subset\big\{\varphi(y)\circ\delta_\rho(\eta,\tau)\mid(\eta,\tau)\in\widehat{W},\ 0<\rho\le\sigma_2\big\}\subseteq\Omega.\end{aligned}$$
Note that, by construction, there exists $\bar{r}>0$ such that, for every $s\in]0,\frac{1}{2}\sigma_1\sigma_2[$, the ball $\mathcal{B}_K(\varphi(y)\circ\delta_s(\bar{x},\bar{t}),\bar{r}s)$ is contained in $Z^+_{\bar{x},\bar{t},U}(\varphi(y))$. Thus, we get $d_K(\varphi(y)\circ\delta_s(\bar{x},\bar{t}),\Sigma)\ge\bar{r}s$, for every $y\in O_2$. This concludes the proof of condition *iii)*.

We finally check the Harnack connectivity condition. We claim that
$$\varphi(y)\circ\delta_{s_0}(\bar{x},\bar{t})\in\mathrm{Int}\Big(\mathscr{A}_{\varphi(y)\circ(\bar{x},\bar{t})}\big(Z^+_{\bar{x},\bar{t},U}(\varphi(y))\big)\Big), \quad \text{for every} \quad y\in O_2. \tag{5.24}$$

For any $y \in O_2$, the path defined as

$$\gamma_y(s) = \varphi(y) \circ \delta_{\sigma_1\sigma_2}(\gamma(s)), \qquad s \in [0, r],$$

is an $\mathscr{L}$-admissible curve connecting $\varphi(y) \circ (\bar{x}, \bar{t})$ to $\varphi(y) \circ \delta_{s_0}(\bar{x}, \bar{t})$. Moreover, we have

$$\gamma_y(s) \in Z^+_{\bar{x},\bar{t},U}(\varphi(y)), \quad \text{for every} \quad s \in [0, r]. \tag{5.25}$$

Indeed, by (5.23),

$$\begin{aligned}\gamma_y([0, r]) &\subseteq \left\{\varphi(y) \circ \delta_{\sigma_1\sigma_2}\big(\delta_\rho(x, 1)\big) \mid x \in \widetilde{U},\ 1 - \varepsilon < \rho \leq 1\right\} \\ &= \left\{\varphi(y) \circ \delta_\rho(x, \bar{t}) \mid x \in U,\ 1 - \varepsilon < \rho \leq 1\right\} \subset Z^+_{\bar{x},\bar{t},U}(\varphi(y)).\end{aligned}$$

We have shown that

$$\varphi(y) \circ \delta_{s_0}(\bar{x}, \bar{t}) \in \mathbf{A}_{\varphi(y)\circ(\bar{x},\bar{t})}\big(Z^+_{\bar{x},\bar{t},U}(\varphi(y))\big).$$

In order to get (5.24), it is enough to prove that

$$\varphi(y) \circ \delta_{s_0}(\bar{x}, \bar{t}) \in \mathrm{Int}\Big(\mathbf{A}_{\varphi(y)\circ(\bar{x},\bar{t})}\big(Z^+_{\bar{x},\bar{t},U}(\varphi(y))\big)\Big), \quad \text{for every} \quad y \in O_2. \tag{5.26}$$

To this aim we first recall that the map $L^\infty([0, 1], \mathbb{R}^m) \ni \omega \mapsto \gamma(1)$, where γ is the solution to (5.15), is differentiable (see for instance [4, Theorem 3.2.6]). Moreover its differential is surjective by (1.8), which is a controllability condition for the system (5.15). Therefore, by the Implicit Function Theorem, as ω varies in a neighborhood of the constant vector $c\,\bar{A}_0^{-1}\widetilde{x}^{(0)}$, the image of the map $\omega \mapsto \gamma(1)$ covers a neighborhood of $\varphi(y) \circ \delta_{s_0}(\bar{x}, \bar{t})$. This proves (5.26) and, then, (5.24). This proves that the uniform interior cone condition on $\widetilde{W} \cap \Sigma$ is satisfied.

In order to show the uniform exterior cone condition, it is sufficient to consider the function $\Phi^-_{\widetilde{x},1} : O \times] - \sigma, \sigma[\to \mathbb{R}^{N+1}$,

$$\Phi^-_{\widetilde{x},1}(y, s) = \varphi(y) \circ (-\delta_s(\widetilde{x}, 1)) = (-D_s\,\widetilde{x} + \exp(s^2 B^T)\varphi_x(y), \varphi_t(y) - s^2).$$

By following the same argument as before, we plainly check that $Z^-_{-\bar{x},\bar{t},-U}(\varphi(y)) \subseteq \mathbb{R}^{N+1} \setminus \Omega$. This implies the uniform exterior cone condition in $\Sigma \cap \widetilde{W}$. This accomplishes the proof of Proposition 1.4 under the assumption *a)*.

We next suppose that condition *b)* is verified. In this case, we set $\bar{x} = 0$ to define the cone $Z^+_{\bar{x},\bar{t},U}(\varphi(y))$. We next choose a neighborhood U of 0 and a positive $\bar{t}$ such that $Z^+_{0,\bar{t},U}(\varphi(y)) \subseteq \Omega$. To this aim, we introduce the function $\Psi : O \times] - \widetilde{\sigma}, \widetilde{\sigma}[\to \mathbb{R}^{N+1}$,

$$\Psi(y, s) = \varphi(y) \circ (0, s) = (\exp(-sB^T)\varphi_x(y), \varphi_t(y) + s). \tag{5.27}$$

It is a local diffeomorphism. Indeed,

$$\det J_\Psi(0, 0) = \det\big(J_\varphi(0) \quad -Y(\varphi(0))\big) \neq 0$$

since $\mathrm{rank}\, J_\varphi(0) = N$ and $\langle Y(\widetilde{w}), \nu(\widetilde{w})\rangle > 0$. If $\widetilde{\sigma}$ is small enough, we have $\Psi(y, s) \in W$, and then we can define

$$h :] - \widetilde{\sigma}, \widetilde{\sigma}[\to \mathbb{R}, \qquad h(s) = F(\Psi(y, s)).$$

Note that $h'(0) = \langle \nabla F(\varphi(y)), -Y(\varphi(y))\rangle$ tends to $-\langle \nu(\widetilde{w}), Y(\widetilde{w})\rangle < 0$ as $y \to 0$, hence $h(s) < 0$ for $s > 0$. As a consequence, there exists $\widetilde{\sigma}_1 > 0$ and a neighborhood $\widetilde{O}_1$ of 0 such that $\Psi(y,s) \in \Omega$ for $(y,s) \in \widetilde{O}_1 \times]0, \widetilde{\sigma}_1]$. We recall notation (5.21) and we note that

$$\Psi(y,s) = \varphi(y) \circ \delta_{\sqrt{s}}(0,1) = \Phi_{0,1}\left(y, \sqrt{s}\right) \quad \text{for any } s \in]0, \widetilde{\sigma}[.$$

Let $\widehat{W}$ be an open neighborhood of $(0, \widetilde{\sigma}_1)$ such that $\varphi(y) \circ (\xi, \tau) \in \Omega$ for any $(\xi, \tau) \in \widehat{W}$.

In order to build an inner cone, we need to show that there exist $\sigma_3 > 0$ and a neighborhood O_3 of 0 such that $\Phi_{\xi,\tau}\left(y, \sqrt{s}\right) \in \Omega$ for every $(\xi, \tau) \in \widehat{W}$ and $(y, \sqrt{s}) \in O_3 \times]0, \sigma_3]$.

We next prove that there exists an open neighborhood $W_1 = I_1 \times I_2 \times \ldots \times I_N \times I_t$ of $\widetilde{w}$ such that

$$\Omega \cap W_1 = I_1 \times \ldots \times I_m \times \Omega', \quad \text{with} \quad \Omega' \in \mathbb{R}^{N-m+1}. \tag{5.28}$$

To this aim, we show that for every $v \in \mathbb{R}^m$ there exists $\varepsilon > 0$ satisfying

$$w + s(v^{(0)}, 0, \ldots, 0) \in \Sigma, \qquad \text{for any} \quad s \in] - \varepsilon, \varepsilon[\quad \text{and} \quad w \in W_1. \tag{5.29}$$

Since $\nu = (\nu_x, \nu_t) \neq 0$ in $\Sigma \cap W$, in the following argument it is not restrictive to assume $\nu_t(w) \neq 0$ for every $w \in \Sigma \cap W$. By Dini's theorem, if $w = (w_x, w_t)$, there exist two neighborhoods $V_x \subset \mathbb{R}^N, V_t \subset \mathbb{R}$ of w_x, w_t with $V_x \times V_t \subset W$, and $\psi \in C^1(V_x)$ such that $\Sigma \cap (V_x \times V_t) = \{(x, \psi(x)) : x \in V_x\}$. We set

$$f(s) = F\big(w_x + s(v^{(0)}, 0, \ldots, 0), \psi(w_x + s(v^{(0)}, 0, \ldots, 0))\big), \qquad s \in] - \varepsilon, \varepsilon[,$$

where $\varepsilon > 0$ is so small as $w_x + s(v_1, \ldots, v_m) \in V_x$. Then $f \equiv 0$, and

$$\begin{aligned} f'(s) = &\big\langle \nabla F\big(w_x + s(v^{(0)}, 0, \ldots, 0), \psi(w_x + s(v^{(0)}, 0, \ldots, 0))\big), (v^{(0)}, 0, \ldots, 0)\big\rangle \\ &+ \partial_t F(\ldots) \left\langle \nabla \psi(w_x + s(v^{(0)}, 0, \ldots, 0)), (v^{(0)}, 0, \ldots, 0)\right\rangle = 0, \qquad s \in] - \varepsilon, \varepsilon[. \end{aligned}$$

As a consequence,

$$\frac{d}{ds}\psi(w_x + s(v^{(0)}, 0, \ldots, 0)) = 0, \qquad s \in] - \varepsilon, \varepsilon[,$$

and this proves (5.29), being

$$\psi(w_x + s(v^{(0)}, 0, \ldots, 0)) = \psi(w_x) = w_t, \qquad s \in] - \varepsilon, \varepsilon[.$$

With (5.28) at hands, we consider the projections π_1 and π_2

$$\pi_1(x_1, \ldots, x_N, t) = (x_1, \ldots, x_m), \qquad \pi_2(x_1, \ldots, x_N, t) = (x_{m+1}, \ldots, x_N, t).$$

We claim that there exists a neighborhood O_3 of $0 \in \mathbb{R}^N$ such that

$$\pi_2\left(Z^+_{0,\bar{t},U}(\varphi(y))\right) \subseteq \Omega', \qquad \text{for every} \quad y \in O_3. \tag{5.30}$$

Note that

$$\pi_1\left(\Phi_{\xi,\tau}\left(y, \sqrt{s}\right)\right) \in I_1 \times \ldots \times I_m, \tag{5.31}$$

for every $(\xi,\tau)\in\widehat{W}$ and $(y,s)\in\widetilde{O}_1\times]0,\widetilde{\sigma}_1]$. Then, by (5.28) it is sufficient to see that $\pi_2\left(\Phi_{\xi,\tau}\left(y,\sqrt{s}\right)\right)\in\Omega'$ for every $(\xi,\tau)\in\widehat{W}$ and $(y,\sqrt{s})\in O_3\times]0,\sigma_3]$. We have

$$\frac{d}{ds}\pi_2\left(\Phi_{\xi,\tau}\left(y,\sqrt{s}\right)\right)|_{s\to 0}=-\tau\,\pi_2\big(Y(\varphi(y)\big),$$

and

$$-\tau\big\langle\pi_2\big(Y(\varphi(y)\big),\nu(\varphi(y))\big\rangle\to-\widetilde{\sigma}_1\langle Y(\widetilde{w}),\nu(\widetilde{w})\rangle<0,$$

as $(\xi,\tau)\to(0,\widetilde{\sigma}_1)$ and $y\to 0$. Then, if $\widehat{W}$ is suitably small, there exist $\sigma_3>0$ and a neighborhood O_3 of 0 such that

$$\pi_2\left(\Phi_{\xi,\tau}\left(y,\sqrt{s}\right)\right)\in\Omega',\qquad\text{for every}\quad(\xi,\tau)\in\widehat{W}\quad\text{and}\quad\left(y,\sqrt{s}\right)\in O_3\times]0,\sigma_3].\tag{5.32}$$

We set $\bar{t}=\sigma_3^2$, $r=\sigma_3(\widetilde{\sigma}_1)^{-\frac{1}{2}}$ and we choose a neighborhood U of the origin such that $U\times\{\bar{t}\}\subset\delta_r\big(\widehat{W}\big)$. We finally put

$$\widetilde{W}=\big\{\varphi(y)\circ\delta_\rho(\xi,\bar{t})\mid y\in O_3,\ 0\le\rho\le 1,\xi\in U\big\}.$$

It is now easy to check that *i), ii), iii)* hold under condition *b)*. Indeed, (5.30) plainly follows from (5.32). Then $Z^+_{0,\bar{t},U}(\varphi(y))\subseteq\Omega$ for every $y\in O_3$, by (5.30) and (5.31). Note that the function $s\mapsto\varphi(y)\circ\delta_{\sqrt{1-s}}(0,1)$ is an $\mathscr{L}$-admissible path, then the Harnack connectivity condition *ii)* in Definition 1.1 is trivially satisfied. In order to show the uniform exterior cone condition, it is sufficient to consider the function $\Phi^-_{0,1}(y,s)=\varphi(y)\circ(-\delta_s(0,1))$ instead of Φ, and to follow the same argument as in the case *a)*.

The first assertion in *iii)* follows from the fact that the function Ψ in (5.27) is a local diffeomorphism. Finally, *ii)* and second assertion in *iii)* can be easily verified, arguing as in the case *a)*. □

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