PARTIAL STRATIFICATION OF SECANT VARIETIES OF VERONESE VARIETIES VIA CURVILINEAR SUBSCHEMES

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Abstract. We give a partial stratification of the secant varieties of the order d Veronese variety of $\mathbb{P}^m$. We will focus on points lying on the span of curvilinear subschemes of Veronese varieties and we compute their symmetric rank for small border rank. We will also describe the structure of the Hilbert schemes of curvilinear subschemes of Veronese varieties.

Introduction

Let $\nu_d : \mathbb{P}^m \hookrightarrow \mathbb{P}^{\binom{m+d}{m}-1}$ be the order d Veronese embedding with $d \geq 3$. We write $X_{m,d} := \nu_d(\mathbb{P}^m)$. An element of $X_{m,d}$ can be described both as the projective class of a d-th power of a homogeneous linear form in $m+1$ variables and as the projective class of a completely decomposable symmetric d-modes tensor. In many applications like Chemometrics (see eg. [24]), Signal Processing (see eg. [20]), Data Analysis (see eg. [5]), Neuroimaging (see eg. [16]), Biology (see eg. [23]) and many others, the knowledge of the minimal decomposition of a tensor in terms of completely decomposable tensors turns out to be extremely useful. This kind of decomposition is strictly related with the concept of secant varieties of varieties parameterizing tensors (if the tensor is symmetric one has to deal with secant varieties of Veronese varieties).

Let $Y \subseteq \mathbb{P}^N$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ of characteristic zero.
For any point $P \in \mathbb{P}^N$ the *Y-rank $r_Y(P)$ of P* is the minimal cardinality of a finite set of points $S \subset Y$ such that $P \in \langle S \rangle$, where $\langle \ \rangle$ denote the linear span:

$$r_Y(P) := \min\{s \in \mathbb{N} \mid \exists\, S \subset Y, \sharp(S) = s, \text{ with } P \in \langle S \rangle\}. \tag{1}$$

If Y is the Veronese variety $X_{m,d}$ the Y-rank is also called the " symmetric tensor rank ". The minimal set of points $S \subset X_{m,d}$ that realizes the symmetric tensor rank of a point $P \in X_{m,d}$ is also said the set that realizes either the " CANDECOMP/PARAFAC decomposition " or the " canonical decomposition " of P.

The natural geometric object that one has to study in order to compute the symmetric tensor rank either of a symmetric tensor or of a homogeneous polynomial is the set that parameterizes points in $\mathbb{P}^N$ having $X_{m,d}$-rank smaller or equal than a fixed value $t \in \mathbb{N}$. For each integer $t \geq 1$ let the t-th secant variety $\sigma_t(X) \subseteq \mathbb{P}^N$ of a

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variety $X \subset \mathbb{P}^N$ be the Zariski closure in $\mathbb{P}^N$ of the union of all $(t-1)$-dimensional linear subspaces spanned by t points of $X \subset \mathbb{P}^N$:

$$\sigma_t(X) := \overline{\bigcup_{P_1,\ldots,P_t \in X} \langle P_1, \ldots, P_t \rangle} \tag{2}$$

For each $P \in \mathbb{P}^N$ the *border rank* $b_X(P)$ *of* P is the minimal integer t such that $P \in \sigma_t(X)$:

$$b_X(P) := \min\{t \in \mathbb{N} \mid P \in \sigma_t(X)\}. \tag{3}$$

We indicate with $\sigma_t^0(X)$ the set of the elements belonging to $\sigma_t(X)$ of fixed X-rank t:

$$\sigma_t^0(X) := \{P \in \sigma_t(X) \mid r_X(P) = t\} \tag{4}$$

Observe that if $\sigma_{t-1}(X) \neq \mathbb{P}^N$, then $\sigma_t^0(X)$ contains a non-empty open subset of $\sigma_t(X)$.

Some of the recent papers on algorithms that are able to compute the symmetric tensor rank of a symmetric tensor (see [9], [7], [10]) use the idea of giving a stratification of the t-th secant variety of the Veronese variety via the symmetric tensor rank. In fact, since $\sigma_t(X) = \overline{\sigma_t^0(X)}$, the elements belonging to $\sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ have X-rank strictly bigger than t. What some of the known algorithms for computing the symmetric rank of a symmetric tensor T do is firstly to test the equations of the secant varieties of the Veronese varieties (when known) in order to find the $X_{m,d}$-border rank of T, and secondly to use (when available) a stratification via the symmetric tensor rank of $\sigma_t(X_{m,d})$. For the state of the art on the computation of the symmetric rank of a symmetric tensor see [15], [10], [21] Theorem 5.1, [9] for the case of rational normal curves, and also [9] for the case $t = 2, 3$, moreover [7] for $t = 4$.

If a stratification of the secant varieties of the Veronese varieties via the symmetric rank could be done for all t, then we would get a stratification of $\mathbb{P}^N$ with strata labelled by pairs of integers (rank,border rank) and it would be possible to produce effective algorithms for the computation of the symmetric rank of any symmetric tensor.

What we propose to study here is the computation of the symmetric tensor rank of a particular class of the symmetric tensors whose symmetric border rank is strictly less than its symmetric rank. We will focus on those symmetric tensors that belong to the span of a reduced 0-dimensional curvilinear sub-scheme of the Veronese variety. We will indicate in Notation 6 this set as $\sigma_t^\dagger(X)$.

From now on in the introduction we take $X := X_{m,d} \subset \mathbb{P}^N$ with $N := \binom{m+d}{m} - 1$.

In Section 1 we will define a finite partial stratification of $\sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ by constructible subsets and $\sigma_t^\dagger(X)$ will turn out to be the union of these strata (thus $\sigma_t^\dagger(X)$ is constructible). For very low t (i.e. $t \leq \lfloor (d-1)/2 \rfloor$), we will describe the structure of $\sigma_t(X)^\dagger$: we will give its dimension, its codimension in $\sigma_t(X)$ and the dimension of each stratum (see Theorem 1). Moreover in the same theorem we will show that for such values of t, the symmetric border rank of the projective class of a homogeneous polynomial $[F] \in \sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ is computed by a unique 0-dimensional subscheme $W_F \subset X$ and that the generic $[F] \in \sigma_t(X)^\dagger$ is of the form $F = L^d M + L_1^d + \cdots + L_{t-2}^d$ with $L, L_1, \ldots, L_{t-2}, M$ linear forms. To compute the dimension of the 3 largest strata of our stratification we will use Terracini's lemma (see Propositions 1, 2 and 3).

We will also prove several results on the symmetric ranks of points $P \in \mathbb{P}^N$ whose border rank is computed by a scheme related to our stratification (see Proposition 5 and Theorem 2). In all cases that we will be able to compute, we will have

$$b_X(P) + r_X(P) \leq 3d - 2,$$

but we will need also additional conditions on the scheme computing $b_X(P)$ when $b_X(P) + r_X(P) \geq 2d + 2$.

1. The stratification

We begin this section by explaining the stratification of the curvilinear 0-dimensional subschemes of any smooth connected projective variety $Y \subset \mathbb{P}^r$.

Notation 1. For any integral projective variety $Y \subset \mathbb{P}^r$ let $\beta(Y)$ be the maximal positive integer such that every 0-dimensional scheme $Z \subset Y$ with $\deg(Z) \leq \beta(Y)$ is linearly independent, i.e. $\dim(\langle Z \rangle) = \deg(Z) - 1$.

Notation 2. Fix an integer $t \geq 2$. Let $A(t)$ be the set of all non-decreasing sequences $t_1 \geq t_2 \geq \cdots \geq t_s \geq 0$ such that $\sum_{i=1}^{s} t_i = t$ and $t_1 \geq 2$.
For each such sequence $\underline{t} = (t_1, \ldots, t_s)$ let $l(\underline{t})$ be the number of the non zero t_i's, for $i = 1, \ldots, s$.
Set $A'(t) := A(t) \sqcup \{(1, \ldots, 1)\}$ in which the string $(1, \ldots, 1)$ has t entries.
We say that $l((1, \ldots, 1)) = t$.

Definition 1. Let $Y \subset \mathbb{P}^r$ be a smooth and connected projective variety of dimension m. For every positive integer t let $\mathrm{Hilb}^t(Y)$ denote the Hilbert scheme of all degree t 0-dimensional subschemes of Y.

Remark 1. Observe that if $m \leq 2$, then $\mathrm{Hilb}^t(Y)$ is smooth and irreducible.

We now introduce now some subset of $\mathrm{Hilb}^t(Y)$ that will give the claimed stratification.

Notation and Remark 1. Let $Y \subset \mathbb{P}^r$ be a smooth connected projective variety of dimension m.

- For every positive integer t let $\mathrm{Hilb}^t(Y)_0$ be the set of all disjoint unions of t distinct points of Y.
  Observe that $\mathrm{Hilb}^t(Y)_0$ is a smooth and irreducible quasi-projective variety of dimension mt and also a dense open subset of it if $m \geq 2$ (see [18], [19]). Moreover for arbitrary $m = \dim(Y)$ the irreducible scheme $\mathrm{Hilb}^t(Y)_0$ is always open in $\mathrm{Hilb}^t(Y)$.
- Let $\mathrm{Hilb}^t(Y)_+$ be the closure of $\mathrm{Hilb}^t(Y)_0$ in $\mathrm{Hilb}^t(Y)_{red}$. The elements of $\mathrm{Hilb}^t(Y)_+$ are called the *smoothable* degree t subschemes of Y.
  If $t \gg m \geq 3$, then there are non-smoothable degree t subschemes of Y.
- An element $Z \in \mathrm{Hilb}^t(Y)$ is called *curvilinear* if at each point $P \in Z_{red}$ the Zariski tangent space of Z has dimension ≤ 1 (equivalently, Z is contained in a smooth subcurve of Y). Then for arbitrary m we can define $\mathrm{Hilb}^t(Y)_c$ that is the set of all degree t *curvilinear* subschemes.
  $\mathrm{Hilb}^t(Y)_c$ is a smooth open subscheme of $\mathrm{Hilb}^t(Y)_+$ which strictly contains $\mathrm{Hilb}^t(Y)_0$.

Fix now $O \in Y$ with $Y \subset \mathbb{P}^r$ being a smooth connected projective variety of dimension m. Following the introduction of [19] we state the corresponding result for the punctual Hilbert scheme of $\mathcal{O}_{Y,O}$.

Remark 2. For each integer $t > 0$ the part of the punctual Hilbert scheme parametrizing the degree t curvilinear subschemes of Y with P as its reduction is smooth, connected and of dimension $(t-1)(m-1)$.

Notation 3. Now fix an integer $s > 0$ and a non-decreasing sequence of integers $t_1 \ge \cdots \ge t_s > 0$ such that $t_1 + \cdots + t_s = t$ and $\underline{t} = (t_1, \dots, t_s)$. Let $\mathrm{Hilb}^t(Y)_c[s; t_1, \dots, t_s]$ denote the subset of $\mathrm{Hilb}^t(Y)_c$ parametrizing all elements of $\mathrm{Hilb}^t(Y)_c$ with s connected components of degree $t_1, \dots, t_s$ respectively. We also write it as $\mathrm{Hilb}^t(Y)_c[\underline{t}]$.

Remark 3. Since the support of each component $\mathrm{Hilb}^t(Y)_c[\underline{t}]$ varies in the m-dimensional variety $Y \subset \mathbb{P}^r$, the theorem on the punctual Hilbert scheme quoted in Remark 2 says that $\mathrm{Hilb}^t(Y)_c[s; t_1, \dots, t_s]$ is an irreducible algebraic set of dimension $ms + \sum_{i=1}^{s}(t_i - 1)(m-1) = mt + s - t$, i.e. of codimension $t - s$ in $\mathrm{Hilb}^t(Y)_c$.

Thus if $t \ge 2$ we have:

$$\mathrm{Hilb}^t(Y)_c = \sqcup_{\underline{t} \in A'(t)} \mathrm{Hilb}^t(Y)_c[\underline{t}] = \mathrm{Hilb}^t(Y)_0 \bigsqcup \sqcup_{\underline{t} \in A(t)} \mathrm{Hilb}^t(Y)_c[\underline{t}].$$

Each stratum $\mathrm{Hilb}^t(Y)_c[\underline{t}]$ is non-empty, irreducible and different elements of $A'(t)$ give disjoint strata.

Different strata may have the same codimension, but there is a unique stratum of codimension 1: it is the stratum with label $(2, 1, \dots, 1)$. This stratum parametrizes the disjoint unions of a tangent vector to Y and $t-2$ disjoint points of Y.

Notation 4. Take now a partial ordering $\preceq$ on $A'(t)$ writing $(a_1, \dots, a_x) \preceq (b_1, \dots, b_y)$ if and only if $\sum_{j=1}^{i} a_j \le \sum_{j=1}^{i} b_j$ for all integers $i \ge 1$.

It turns out that $\underline{a} \preceq \underline{b}$ if and only if the stratum $\mathrm{Hilb}^t(Y)_c[\underline{b}]$ is in the closure of the stratum $\mathrm{Hilb}^t(Y)_c[\underline{a}]$.

We recall now a lemma that we borrow form [12] and [9].

Lemma 1. *Let $Y \subset \mathbb{P}^r$ be a smooth and connected subvariety. Fix an integer k such that $k \le \beta(Y)$, where $\beta(Y)$ is defined in Notation 1, and $P \in \mathbb{P}^r$. Then $P \in \sigma_k(Y)$ if and only if there exists a smoothable 0-dimensional scheme $Z \subset Y$ such that $\deg(Z) = k$ and $P \in \langle Z \rangle$.*

Proof. As observed in [12], proof of Lemma 2.1.5, the condition $k \le \beta(Y)$ is the one need to apply [9], Proposition 11, to get the " only if " part. The same references also gives the " if " part, but we will write down the proof. Assume that Z is smoothable and $P \in \langle Z \rangle$. Call S_λ a family of reduced subsets converging to Z, then the condition " $k \le \beta(Y)$ " says that $\langle Z \rangle$ and all $\langle S_\lambda \rangle$ are $(k-1)$-dimensional and the family $\{\langle S_\lambda \rangle\}$ converges to $\langle Z \rangle$ in the appropriate Grassmannian $G(k-1, r)$. Since the incidence correspondence of $\mathbb{P}^r \times G(k-1, r)$ is closed and the projection $\mathbb{P}^r \times G(k-1, r) \to \mathbb{P}^r$ is a closed map, we get that $\langle Z \rangle$ is in the closure of $\cup_\lambda \langle S_\lambda \rangle$. The latter set is contained in $\cup_{t=1}^{k} \sigma_t^0(Y)$. □

The following lemma shows a very special property of the curvilinear subschemes.

Lemma 2. *Let $Y \subset \mathbb{P}^r$ be a smooth and connected subvariety. Let $W \subset Y$ be a linearly independent curvilinear subscheme of Y. Fix a general $P \in \langle W \rangle$. Then $P \notin \langle W' \rangle$ for any $W' \subsetneq W$.*

Proof. A key property of curvilinear schemes is that they have finitely many proper closed subschemes. Since the base field is algebraically closed, it is infinite. Thus $\mathbb{P}^{k-1}$ is not a finite union of proper linear subspaces. Thus $\cup_{W' \subsetneqq W} \langle W' \rangle \subsetneqq \langle W \rangle$ $\square$

We introduce the following Notation.

Notation 5. For each integral variety $Y \subset \mathbb{P}^r$ and each $Q \in Y_{reg}$ let $[2Q, Y]$ denote the first infinitesimal neighborhood of Q in Y, i.e. the closed subscheme of Y with $(\mathcal{I}_{Q,Y})^2$ as its ideal sheaf. We call any $[2Q, Y]$, with $Q \in Y_{reg}$, a double point of Y.

Remark 4. Observe that $[Q, Y]_{red} = \{Q\}$ and $\deg([Q, Y]) = \dim(Y) + 1$.

The following observation shows that Lemma 2 fails for some non-curvilinear subscheme.

Remark 5. Assume that $Y \subset \mathbb{P}^r$ is smooth and of dimension ≥ 2. Fix a smooth subvariety $N \subseteq Y$ such that $\dim(N) = 2$ and any $Q \in N$. Since N is embedded in $\mathbb{P}^r$, the linear space $\langle [2Q, Y] \rangle$ is a 2-dimensional space. Fix any $P \in \langle [2Q, Y] \rangle$. If $P = Q$, then $P \in \langle \{Q\} \rangle$. If $P \neq Q$, then the plane $\langle [2Q, Y] \rangle$ intersects $[2Q, Y]$ in a degree 2 subscheme $[2Q, Y]_P$ and $P \in \langle [2Q, Y]_P \rangle$ that is a line.

Notation 6. For any integer $t > 0$ let $\sigma_t(X)^\dagger$ denote the set of all $P \in \sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ such that there is a curvilinear degree t subscheme $Z \subset X_{reg}$ such that $P \in \langle Z \rangle$.

Remark 6. Let $X \subset \mathbb{P}^N$ be the Veronese variety $X_{m,d}$ with $N = \binom{n+d}{d} - 1$. Take $P \in \sigma_t(X)^\dagger$ and a curvilinear degree t subscheme $Z \subset X_{reg}$ such that $P \in \langle Z \rangle$. The curvilinear scheme Z has a certain number, s, of connected components of degrees $t_1, \dots, t_s$ respectively with $t_1 \geq \cdots \geq t_s$, but we cannot associate the string $(t_1, \dots, t_s)$ to P, because Z may not be unique. In fact the scheme Z is uniquely determined by P for an arbitrary $P \in \sigma_t(X)^\dagger$ only under very restrictive conditions (see eg. Theorem 1 for a sufficient condition). However, we think that it useful to see $\sigma_t(X)^\dagger$ as a union on the various strings $t_1 \geq \cdots \geq t_s$, even when this is not a disjoint union.

We recall the following definition.

Definition 2. Fix now integral and non-degenerate subvarieties $X_1, \dots, X_t \subset \mathbb{P}^r$ (repetitions are allowed). The join $J(X_1, \dots, X_t)$ of $X_1, \dots, X_t$ is the closure in $\mathbb{P}^r$ of the union of all $(t-1)$-dimensional vector spaces spanned by t linearly independent points $P_1, \dots, P_t$ with $P_i \in X_i$ for all i.

Remark 7. From Definition 2 we obviously have that $\sigma_t(X_1) = J(\underbrace{X_1, \dots, X_1}_{t})$.

Definition 3. Let $S(X_1, \dots, X_t) \subset X_1 \times \cdots \times X_t \times \mathbb{P}^r$ be the closure of the set of all $(P_1, P_2, \dots, P_t, P)$ such that $P \in \langle \{P_1, \dots, P_t\} \rangle$ and $P_i \in X_i$ for all i. We call $S(X_1, \dots, X_t)$ the abstract join of the subvarieties $X_1, \dots, X_t$ of $\mathbb{P}^r$.

Remark 8. The abstract join $S(X_1, \dots, X_t)$ is an integral projective variety and $\dim(S(X_1, \dots, X_t)) = t - 1 + \sum_{i=1}^{t} \dim(X_i)$. The projection of $X_1 \times \cdots \times X_t \times \mathbb{P}^r \to \mathbb{P}^r$ induces a proper morphism $u_{X_1,\dots,X_t} : S(X_1, \dots, X_t) \to \mathbb{P}^r$ such that $u_{X_1,\dots,X_t}(S(X_1, \dots, X_t)) = J(X_1, \dots, X_t)$. The embedded join has the expected dimension $t - 1 + \sum_{i=1}^{t} \dim(X_i)$ if and only if $u_{X_1,\dots,X_t}$ is generically finite.

Remind here the so called Terracini's Lemma (that is for sure the most used tool to study the dimensions of secant varieties).

Terracini's Lemma Let $Y \subset \mathbb{P}^r$ be a smooth projective variety and let $P \in \sigma_t(Y)$ be a generic point, i.e. $P \in \langle P_1, \dots, P_t\rangle$ for $P_1, \dots, P_t \in Y$ generic points. Then we have the following equality between Zariski tangent spaces: $T_P(\sigma_t(Y)) = \langle T_{P_1}(Y), \dots, T_{P_t}(Y)\rangle$.

In [1], part (2) of Corollary 1.11, it is proved that Terracini's lemma works also for joins. In fact it can be generalized as follows.

Terracini's Lemma for Joins Let $P \in \langle \{P_1, \dots, P_t\}\rangle$ with $(P_1, \dots, P_t)$ general in $X_1 \times \cdots \times X_t$, then the Zariski tangent space $T_P(J(X_1, \dots, X_t))$ may be identified with the linear span of the union of the first infinitesimal neighborhoods $[2P_i, X_i]$ of P_i in each X_i, $1 \leq i \leq t$.

We are now ready to apply this tool of curvilinear subschemes in the particular case of secant varieties of Veronese varieties.

2. Curvilinear subschemes and tangential varieties to Veronese varieties

From now on in this paper we fix integers $m \geq 2$, $d \geq 3$ and take $N := \binom{m+d}{m} - 1$ and $X := X_{m,d}$ the Veronese embedding of $\mathbb{P}^m$ into $\mathbb{P}^N$.

Definition 4. Let $\tau(X) \subseteq \mathbb{P}^N$ be the tangent developable of X, i.e. the closure in $\mathbb{P}^N$ of the union of all embedded tangent spaces T_PX, $P \in X_{reg}$:

$$\tau(X) := \overline{\bigcup_{P \in X} T_PX}$$

Remark 9. Obviously $\tau(X) \subseteq \sigma_2(X)$ and $\tau(X)$ is integral. Moreover Since $d \geq 3$, the variety $\tau(X)$ is a hypersurface of $\sigma_2(X)$.

Definition 5. For each integers $t \geq 3$ let $\tau(X,t) \subseteq \mathbb{P}^N$ be the join of $\tau(X)$ and $\sigma_{t-2}(X)$:

$$\tau(X,t) := J(\tau(X), \sigma_{t-2}(X)).$$

We recall that $\min\{n, t(m+1) - 2\}$ is the expected dimension of $\tau(X,t)$.

Here we fix integers d, t with $t \geq 2$, d not too small and look at $\tau(X,t)$ from many point of views.

Remark 10. The set $\tau(X,t)$ is nothing else than the closure inside $\sigma_t(X)$ of the largest stratum of our stratification. As recalled in Remark 3 this is the stratum given by $\mathrm{Hilb}^t(X)_c[(2,1,\cdots,1)]$ (see Notation 3).

Remark 11. For any integral projective scheme W, any effective Cartier divisor D of W and any closed subscheme Z of W the residual scheme $\mathrm{Res}_D(Z)$ of Z with respect to D is the closed subscheme of W with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. For every $L \in \mathrm{Pic}(W)$ we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes (L|D) \to 0 \tag{5}$$

From (5) we immediately get the following well-known result, often called the Castelnuovo's lemma.

Lemma 3. *Fix $L \in Pic(Y)$ for $Y \subset \mathbb{P}^r$ any integral projective variety. Then*

$$h^i(Y, \mathcal{I}_Z \otimes L) \leq h^i(Y, \mathcal{I}_{Res_D(Z)} \otimes L(-D)) + h^i(D, \mathcal{I}_{Z \cap D, D} \otimes (L|D))$$

for every $i \in \mathbb{N}$.

Notation 7. For any $Q \in \mathbb{P}^m$ and any integer $k \geq 2$ let kQ denote the $(k-1)$-infinitesimal neighborhood of Q in $\mathbb{P}^m$, i.e. the closed subscheme of $\mathbb{P}^m$ with $(\mathcal{I}_Q)^k$ as its ideal sheaf. The scheme kQ will be called a k-point of $\mathbb{P}^m$.

We give here the definition of a $(2,3)$-point as it is in [13], p. 977.

Definition 6. Fix a line $L \subset \mathbb{P}^m$ and a point $Q \in L$. The $(2,3)$ point of $\mathbb{P}^m$ associated to (Q, L) is the closed subscheme $Z(Q, L) \subset \mathbb{P}^m$ with $(\mathcal{I}_Q)^3 + (\mathcal{I}_L)^2$ as its ideal sheaf.

Remark 12. Notice that $2Q \subset Z(Q, L) \subset 3Q$.

Remark 13. Let $Z = Z_1 \sqcup Z(Q, L)$ be a closed subscheme of $\mathbb{P}^m$ for $Z_1 \subset \mathbb{P}^m$ a 0-dimensional scheme. Since $Z(Q, L) \subset 3Q$, if $h^1(\mathbb{P}^m, \mathcal{I}_{3Q \cup Z_1}(d)) = 0$, then $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) = 0$.

Lemma 4. *Fix an integer t such that $(m+1)(t-2)+2m < N$ with $N = \binom{m+d}{d} - 1$ and general $P_0, \dots, P_{t-2} \in \mathbb{P}^m$ and a general line $L \subset \mathbb{P}^m$ such that $P_0 \in L$. Set*

$$Z := Z(P_0, L) \bigcup (\cup_{i=1}^{t-2} 2P_i)$$

and

$$Z' := 3P_0 \bigcup (\cup_{i=1}^{t-2} 2P_i).$$

Then

(i) If $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) = 0$, then $\dim(\tau(X, t)) = t(m+1) - 2$.

(ii) If $h^1(\mathbb{P}^m, \mathcal{I}_{Z'}(d)) = 0$, then $\dim(\tau(X, t)) = t(m+1) - 2$.

Proof. If $t = 2$ then $\tau(X, t) = \tau(X)$ and the part (i) for this case is proved in [13]. The case $t \geq 3$ of part (i) follows from the case $t = 2$ and Terracini's lemma, because $\tau(X, t)$ is the join of $\tau(X)$ and $t-2$ copies of X. Part (ii) follows from part (i) and Remark 13. $\square$

Remark 14. Let $A \subset \mathbb{P}^m$, $m \geq 2$, be a connected curvilinear subscheme of degree 3. Up to a projective transformation there are two classes of such schemes: the collinear ones (i.e. A is contained in a line, i.e. $\nu_d(A)$ is contained in a degree d rational normal curve) and the non-collinear ones, i.e. the ones that are contained in a smooth conic of $\mathbb{P}^m$. The latter ones form a non-empty open subset of the corresponding stratum $(3, 0, \dots, 0)$ and, in this case, we will say that A is *not collinear*. The family of all such schemes A covers an integral variety of dimension $3m-2$ (that corresponds, after its embedding via the Veronese map ν_d, with the second osculating variety of $X_{m,d}$). In [8], Lemma 3.5, by using the theory of inverse systems, it is proved that the tangent space to the second osculating variety to Veronese variety is dominated by $4Q$, with $Q \in X_{m,d}$, exactly as $3Q$ dominates the tangent developable of $X_{m,d}$.

Lemma 5. *Fix integers $m \ge 2$ and $d \ge 5$. If $m \le 4$, then assume $d \ge 6$. Set $\alpha := \lfloor \binom{m+d-1}{m}/(m+1) \rfloor$. Let $Z_i \subset \mathbb{P}^m$, $i = 1, 2$, be a general union of i triple points and $\alpha - i$ double points. Then $h^1(\mathcal{I}_{Z_i}(d)) = 0$.*

Proof. Fix a hyperplane H of $\mathbb{P}^m$ and call E_i the disjoint union if i triple points of $\mathbb{P}^m$ with support on H with $i \in \{1, 2\}$. Thus $E_i \cap H$ is a disjoint union of i triple points of H. Since $d \ge 5$, we have $h^1(H, \mathcal{I}_{H \cap E_i}(d)) = 0$. Let $W_i \subset \mathbb{P}^m$ be a general union of $\alpha - i$ double points for $i \in \{1, 2\}$. Since W_i is general $W_i \cap H = \emptyset$.
If we prove that $h^1(\mathcal{I}_{E_i \cup W_i}(d)) = 0$, then, by semicontinuity, we get also that $h^1(\mathcal{I}_Z(d)) = 0$ for $i \in \{1, 2\}$.
By Lemma 3 it is sufficient to prove $h^1(\mathcal{I}_{\mathrm{Res}_{H(W_i \cup E_i)}}(d-1)) = 0$.
Since $W_i \cap H = \emptyset$, we have $\mathrm{Res}_H(W_i \cup E_i) = W_i \sqcup \mathrm{Res}_H(E_i)$. Thus $\mathrm{Res}_H(W_i \cup E_i)$ is a general union of α double points, with the only restriction that the reductions of two of these double points are contained in the hyperplane H. Any two points of $\mathbb{P}^m$, $m \ge 2$, are contained in some hyperplane. The group $\mathrm{Aut}(\mathbb{P}^m)$ acts transitively on the set of all hyperplanes of $\mathbb{P}^m$. The cohomology groups of projectively equivalent subschemes of $\mathbb{P}^m$ have the same dimension. Thus we may consider $W_i \sqcup \mathrm{Res}_H(E_i)$ as a general union of α double points of $\mathbb{P}^m$. Since $(m+1)\alpha \le \lfloor \binom{m+d-1}{m}/(m+1) \rfloor$, $d - 1 \ge 4$ and $d - 1 \ge 5$ if $m \le 4$, the very well known theorem of Alexander and Hirschowitz on the dimensions of all secant varieties to Veronese varieties gives $h^1(\mathcal{I}_{\mathrm{Res}_{H(W_i \cup E_i)}}(d-1)) = 0$ (see [2], [3], [4],[14], [11]) □

Lemma 6. *Fix integers $m \ge 2$ and $d \ge 6$. If $m \le 4$, then assume $d \ge 7$. Set $\beta := \lfloor \binom{m+d-2}{m}/(m+1) \rfloor$. Let $Z \subset \mathbb{P}^m$ be a general union of one quadruple point and $\beta - 1$ double points. Then $h^1(\mathcal{I}_Z(d)) = 0$.*

Proof. Fix a hyperplane H and call E a quadruple point of $\mathbb{P}^m$ with support on H. Thus $E \cap H$ is a quadruple point of H. Since $d \ge 2$, we have $h^1(H, \mathcal{I}_{H \cap E}(d)) = 0$. Let $W \subset \mathbb{P}^m$ be a general union of $\beta - 1$ double points. Since W is general, we have $W \cap H = \emptyset$.
If we prove that $h^1(\mathcal{I}_{E \cup W}(d)) = 0$ then, by semicontinuity, we get also that $h^1(\mathcal{I}_{E \cup W}(d)) = 0$. By Lemma 3 it is sufficient to prove $h^1(\mathcal{I}_{\mathrm{Res}_{H(W \cup E)}}(d-1)) = 0$.

Since $W \cap H = \emptyset$, we have $\mathrm{Res}_H(W \cup E) = W \sqcup \mathrm{Res}_H$. Thus $\mathrm{Res}_H(W \cup E)$ is a general union of $\beta - 1$ double points and one triple point with support on H. Since $\mathrm{Aut}(\mathbb{P}^m)$ acts transitively, the scheme $\mathrm{Res}_H(W \cup E)$ may be seem as a general disjoint union of $\beta - 1$ double points and one triple point. Now it is sufficient to apply the case $i = 1$ of Lemma 5 for the integer $d' := d - 1$. □

Proposition 1. *Set $\alpha := \lfloor \binom{m+d-1}{m}/(m+1) \rfloor$. Fix an integer $t \ge 3$ such that $t \le \alpha - 1$. There is a non-empty and irreducible codimension 1 algebraic subset Γ_1 of $\sigma_t(X)$ with the following property. For every $P \in \Gamma_1$ there is a scheme $Z_P \subset X$ such that $P \in \langle Z_P \rangle$ and Z_P has one connected component of degree 2 and $t - 2$ connected components of degree 1.*

Proof. Lemma 5 and Terracini's lemma (we recalled it in Section 1) give that the join $\tau(X, t)$ (see Definition 5) has the expected dimension. This is equivalent to say that the set of all points $P \in \langle Z_1 \cup \{P_1, \dots, P_{t-2}\} \rangle$ with Z_1 a tangent vector of X has the expected dimension, i.e. codimension 1 in $\sigma_t(X)$. Obviously $\tau(X, t) \neq \emptyset$ and $\Gamma_1 \neq \emptyset$. The set Γ_1 is irreducible, because it is an open subset of a join of irreducible subvarieties. □

The proof of Proposition 1 can be analogously repeated for the following two propositions: Proposition 2 and Proposition 3.

Proposition 2. *Set* $\alpha := \lfloor \binom{m+d-1}{m}/(m+1)\rfloor$. *Fix an integer* $t \ge 3$ *such that* $t \le \alpha - 2$. *There is a non-empty and irreducible codimension* 2 *algebraic subset* Γ_2 *of* $\sigma_t(X)$ *with the following property. For every* $P \in \Gamma_2$ *there is a scheme* $Z_P \subset X$ *such that* $P \in \langle Z_P\rangle$ *and* Z_P *has two connected components of degree* 2 *and* $t-4$ *connected components of degree* 1.

Proof. This proposition can be proved in the same way of Proposition 1 just quoting the case $i = 2$ of Lemma 5 instead of the case $i = 1$ of the same lemma. □

Proposition 3. *Set* $\beta := \lfloor \binom{m+d-2}{m}/(m+1)\rfloor$. *Fix an integer* $t \ge 3$ *such that* $t \le \beta - 1$. *There is a non-empty and irreducible codimension* 2 *algebraic subset* Γ_3 *of* $\sigma_t(X)$ *with the following property. For every* $P \in \Gamma_3$ *there is a scheme* $Z_P \subset X$ *such that* $P \in \langle Z_P\rangle$ *and* Z_P *has* $t-3$ *connected components of degree* 1 *and one connected component which is curvilinear, of degree* 3 *and non-collinear.*

Proof. This proposition can be proved in the same way of Proposition 1 just quoting Lemma 6 instead of Lemma 5 and using Remark 14. □

Remark 15. Observe that if we interpret the Veronese variety $X_{m,d}$ as the variety that parameterizes the projective classes of homogeneous polynomials of degree d in $m+1$ variables that can be written as d-th powers of linear forms then:

- The elements $F \in \Gamma_1$ can all be written in the following two forms:

$$F = L^{d-1}M + L_1^d + \cdots + L_{t-2}^d,$$

$$F = M_1^d + \cdots + M_d^d + L_1^d + \cdots + L_{t-2}^d;$$

- The elements $F \in \Gamma_2$ can all be written in the following two forms:

$$F = L^{d-1}M + L'^{d-1}M' + L_1^d + \cdots + L_{t-4}^d;$$

$$F = M_1^d + \cdots + M_d^d + M_1'^d + \cdots + M_d'^d + L_1^d + \cdots + L_{t-4}^d :$$

- The elements $F \in \Gamma_3$ can all be written in the following two forms:

$$F = L^{d-2}Q + L'^{d-1}M' + L_1^d + \cdots + L_{t-3}^d;$$

$$F = N_1^d + \cdots + N_{2d-1}^d + L_1^d + \cdots + L_{t-3}^d :$$

where $L, L'M, M'L_1, \ldots, L_{t-2}, M_1, \ldots, M_d, M_1', \ldots, M_d', N_1, \ldots, N_{2d-1}$ are all linear forms and Q is a quadratic form. Actually $M_1, \ldots, M_d$ and $M_1', \ldots, M_d'$ are binary forms (see [9]).

3. The ranks and border ranks of points of Γ_i

Here we compute the rank $r_X(P)$ for certain points $P \in \tau(X,t)$ when t is not too small with respect to d. If $t = 2$ we refer [9], Theorems 3, 4; if $t = 3$ we refer [6], Theorem 2.

We first handle the border rank.

Theorem 1. *Fix an integer* t *such that* $2 \le t \le \lfloor (d-1)/2\rfloor$. *For each* $P \in \sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X))$ *there is a unique* $W_P \in Hilb^t(X)$ *such that* $P \in \langle W_P\rangle$. *Set* $\sigma_t(X)^\dagger := \{P \in \sigma_t(X) \setminus (\sigma_t^0(X) \cup \sigma_{t-1}(X)) \mid \exists\, W_P \in Hilb^t(M)_c$ *s.t.* $P \in \langle W_P\rangle\}$ *as in Notation 6.*

(a) The constructible set $\sigma_t(X)^\dagger$ *is non-empty, irreducible and of dimension* $(m+1)t-2$. *For a general* $P \in \sigma_t(X)^\dagger$ *the associated* $W \subset X$ *computing* $b_X(P)$ *has a connected component of degree* 2 *(i.e. a tangent vector) and* $t-2$ *reduced connected components.*

(b) We have a set-theoretic partition $\sigma_t(X)^\dagger = \sqcup_{\underline{t}\in A(t)}\sigma(\underline{t})$, *where* $A(t)$ *is defined in Notation 1, in which each set* $\sigma(\underline{t})$ *is an irreducible and non-empty constructible subset of dimension* $(m+1)t-1-t+l(\underline{t})$, *where* $l(\underline{t})$ *is defined in Notation 2. The stratum* $\sigma(\underline{t})$, *with* $\underline{t} = (t_1,\dots,t_s)$, *is in the closure of the stratum* $\sigma(\underline{a})$ *with* $\underline{a} = (a_1,\dots,a_x)$ *if and only if for each integer* $i \ge 1$ *we have* $\sum_{j=1}^{i} t_j \ge \sum_{j=1}^{i} a_j$, *i.e. if and only if* $\underline{a} \preceq \underline{t}$.

(c) The complement of $\sigma_t(X)^\dagger$ *inside* $\sigma_t(X)\setminus(\sigma_t^0(X)\cup\sigma_{t-1}(X))$ *has codimension at least* 3 *if* $t \ge 3$, *or it is empty if* $t = 2$.

Proof. Fix $P \in \sigma_t(X)\setminus\sigma_{t-1}(X)$. Since $\beta(X) = d+1 \ge t$ ($\beta(X)$ is defined in Notation 1), Lemma 1 gives the existence of some $W \subset X$ such that $\deg(W) = t$, $P \in \langle W\rangle$ and W is smoothable. Since $2t \le d+1$, we can use [6], Lemma 1 to say that W is unique. Moreover, if $A \subset X$ is a degree t smoothable subscheme, $Q \in \langle A\rangle$ and $Q \notin \langle A'\rangle$ for any $A' \subsetneqq A$, then Lemma 1 gives $Q \in \sigma_t(X)\setminus\sigma_{t-1}(X)$. If A is curvilinear, then it is smoothable and $\cup_{A'\subsetneqq A}\langle A'\rangle \subsetneqq \langle A\rangle$. Hence each degree t curvilinear subscheme W of X contributes a non-empty open subset U_W of the $(t-1)$-dimensional projective space $\langle W\rangle$ and $U_{W_1}\cap U_{W_2} = \emptyset$ for all curvilinear W_1, W_2 such that $W_1 \ne W_2$. Thus

$$\sigma_t(X)^\dagger = \sqcup_{\underline{t}\in A(t)}(\sqcup_{W\in \mathrm{Hilb}^t(X)[\underline{t}]}U_W).$$

Each algebraic set $B_{\underline{t}} := \sqcup_{W\in\mathrm{Hilb}^t(X)[\underline{t}]}U_W$ is irreducible and of dimension $t-1+tm+l(\underline{t})-t$. This partition of $\sigma_t(X)^\dagger$ into non-empty irreducible constructible subsets is the partition claimed in part (b).

(i) Fix any two strata, say $B_{\underline{t_i}}$, $i = 1,2$. Since each $B_{\underline{t_i}}$ is irreducible, we get $\overline{B_{\underline{t_2}}} \subseteqq \overline{B_{\underline{t_1}}}$ if and only if $\overline{\mathrm{Hilb}^t(X)[\underline{t_2}]} \subseteqq \overline{\mathrm{Hilb}^t(X)[\underline{t_2}]}$, i.e. if and only if $\underline{t_1} \preceq \underline{t_2}$.

(ii) Here we prove part (c). Every element of $\mathrm{Hilb}^2(X)$ is either a tangent vector or the disjoint union of two points. Thus $\mathrm{Hilb}^2(X) = \mathrm{Hilb}^2(X)_c$. Hence we may assume $t \ge 3$. Fix $P \in \sigma_t(X)\setminus(\sigma_t^0(X)\cup\sigma_{t-1}(X))$ such that $P \notin \sigma_t(X)^\dagger$. By Lemma 1 there is a smoothable $W \subset X$ such that $\deg(W) = t$ and $P \in \langle W\rangle$. Since $2t \le \beta(X)$, such a scheme is unique. Thus it is sufficient to prove that the set $\mathbb{B}_t$ of all 0-dimensional smoothable schemes with degree t and not curvilinear have dimension at most $mt-3$.

Call $\mathbb{B}_t(s)$ the set of all $W \in \mathbb{B}_t$ with exactly s connected components.

First we assume that W is connected. Set $\{Q\} := W_{red}$. Since in the local Hilbert scheme of $\mathcal{O}_{X,Q}$ the smoothable colength t ideals are parametrized by an integral variety of dimension $(m-1)(t-1)$ and a dense open subset of it is formed by the ideals associated to a curvilinear subschemes, we have $\dim(\mathbb{B}_t(1)) \le m+(m-1)(t-1)-1 = mt-t = \dim(\mathrm{Hilb}^t(X)_c)-t$.

Now we assume $s \ge 2$. Let $W_1,\dots,W_s$ be the connected components of W, with at least one of them, say W_s, not curvilinear. Set $t_i = \deg(W_i)$. We have $t_1+\cdots+t_s = t$. Since W_s is not curvilinear, we have $t_s \ge 3$ and hence $t-s \ge 2$. Each W_i is smoothable. Thus each W_i, $i < s$, depends on at most $m+(m-1)(t_i-1) = mt_i+1-t_i$ parameters. We saw that $\mathbb{B}_{t_s}(1)$ depends on at most mt_s-t_s parameters. Thus $\dim(\mathbb{B}_t(s)) \le mt+s-1-t$. □

Proposition 4. *Assume $m \geq 2$. Fix integers d, t such that $2 \leq t \leq d$. Fix a curvilinear scheme $A \subset \mathbb{P}^m$ such that $\deg(A) = t$ and $\deg(A \cap L) \leq 2$ for every line $L \subset \mathbb{P}^m$. Set $Z := \nu_d(A)$. Fix $P \in \langle Z \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Then $b_X(P) = t$ and Z is the only 0-dimensional scheme W such that $\deg(W) \leq t$ and $P \in \langle W \rangle$.*

Proof. Since $t \leq d+1$, Z is linearly independent. Since Z is curvilinear, Lemma 2 gives the existence of many points $P' \in \langle Z \rangle$ such that $P' \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Let $W \subset X$ be a minimal degree subscheme such that $P \in \langle W \rangle$. Set $w := \deg(W)$. The minimality of w gives $w \leq t$. If $w = t$, then we assume $W \neq Z$. Now it is sufficient to show that these conditions give a contradiction. Write $Z := \nu_d(A)$ and $W = \nu_d(B)$ with A and B subschemes of $\mathbb{P}^m$, $\deg(A) = t$ and $\deg(B) = w$. We have $P \in \langle W \rangle \cap \langle Z \rangle$, then, since $W \neq Z$, by [6], Lemma 1, the scheme $W \cup Z$ is linearly dependent. We have $\deg(B \cup A) \leq t + w \leq 2d$. Since $W \cup Z$ is linearly dependent, we have $h^1(\mathcal{I}_{B \cup A}(d)) > 0$. Thus, by [9], Lemma 34, there is a line $R \subset \mathbb{P}^m$ such that $\deg(R \cap (B \cup A)) \geq d+2$. By assumption we have $\deg(R \cap A) \leq 2$. Thus $\deg(B \cap R) \geq d$. In our set-up we get $w = d$ and $B \subset R$. Since $P \in \langle W \rangle$, we get $P \in \langle \nu_d(R) \rangle$. That means that P belongs to the span of a rational normal curve. Then the border rank of P is computed by a curvilinear scheme which has length $\leq \lfloor (d+1)/2 \rfloor$, contradiction. $\square$

Proposition 5. *Fix a line $L \subset \mathbb{P}^m$ and set $D := \nu_d(L)$. Fix positive integers t_1, s_1, a 0-dimensional scheme $Z_1 \subset D$ such that $\deg(Z_1) = t_1$ and $S_1 \subset X \setminus D$ such that $\sharp(S_1) = s_1$. Assume $2 \leq t_1 \leq d/2$, $0 \leq s_1 \leq d/2$, that Z_1 is not reduced and $\dim(\langle D \cup S_1 \rangle) = d + s_1$. Fix $P \in \langle Z_1 \cup S_1 \rangle$ such that $P \notin \langle W \rangle$ for any $W \subsetneq Z_1 \cup S_1$. We have $\sharp(\langle Z_1 \rangle \cap \langle \{P\} \cup S_1 \rangle) = 1$. Set $\{Q\} := \langle Z_1 \rangle \cap \langle \{P\} \cup S_1 \rangle$. Then $b_X(P) = t_1 + s_1$, $r_X(P) = d + 2 + s_1 - t_1$, $Z_1 \cup S_1$ is the only subscheme of X computing $b_X(P)$ and every subset of X computing $r_X(P)$ contains S_1. If $2s_1 < d$, then every subset of X computing $r_X(P)$ is of the form $A \cup S_1$ with $A \subset D$, $\sharp(A) = d + 2 - s_1$ and A computing $r_D(Q)$.*

Proof. Obviously $b_X(P) \leq t_1 + s_1$. Since $P \in \langle Z_1 \cup S_1 \rangle \subset \langle D \cup S_1 \rangle$, $P \notin \langle S_1 \rangle$ and $\langle D \rangle$ has codimension s_1 in $\langle D \cup S_1 \rangle$, the linear subspace $\langle Z_1 \rangle \cap \langle \{P\} \cup S_1 \rangle$ is non-empty and 0-dimensional, $\{Q\}$. Since $\deg(Z_1) \leq d + 1 = \beta(X) = \beta(D)$ (where $\beta(X)$ is defined in Notation 1), the scheme Z_1 is linearly independent. Since $P \notin \langle W \rangle$ for any $W \subsetneq Z_1 \cup S_1$, we have $\langle Z_1 \rangle \cap \langle \{P\} \cup S_1 \rangle \neq \emptyset$. Since $\langle Z_1 \rangle \subset \langle D \rangle$, we get $\{Q\} = \langle Z_1 \rangle \cap \langle \{P\} \cup S_1 \rangle$. Hence Z_1 compute $b_D(Q)$ (Lemma 1). By Lemma 1 we also have $b_X(Q) = b_D(Q)$. Since Z_1 is not reduced, we have $r_D(Q) = d + 2 - t_1$ ([15] or [21], theorem 4.1, or [9]). We have $r_X(Q) = r_D(Q)$ ([22], Proposition 3.1, or [21], subsection 3.2). Write $Z_1 = \nu_d(A_1)$ and $S_1 = \nu_d(B_1)$ with $A_1, B_1 \subset \mathbb{P}^m$. Lemma 1 gives $b_X(P) \leq t_1 + s_1$. Assume $b_X(P) \leq t_1 + s_1 - 1$ and take $W = \nu_d(E)$ computing $b_X(Q)$ for certain 0-dimensional scheme $E \subset \mathbb{P}^m$. Thus $\deg(W) \leq 2t_1 + 2s_1 - 1$. Since $P \in \langle W \rangle \cap \langle Z_1 \cup S_1 \rangle$, by the already quoted [6], Lemma 1, we get $h^1(\mathbb{P}^m, \mathcal{I}_{E \cup A_1 \cup B_1}(d)) > 0$. Thus there is a line $R \subset \mathbb{P}^m$ such that $\deg(R \cap (E \cup Z_1 \cup S_1)) \geq d + 2$.
First assume $R = L$. Thus $L \cap (A_1 \cup B_1) = A_1$. Hence $\deg(E \cap L) \geq d + 2 - t_1$. Set $E' := E \cap L$, $E'' := E \setminus E'$, $W' := \nu_d(E')$ and $W'' := \nu_d(E'')$. Since $P \in \langle W' \cup W'' \rangle$, there is $O \in \langle W' \rangle$ such that $P \in \langle \{O\} \cup W'' \rangle$. Thus $b_X(P) \leq b_X(O) + \deg(W'')$. Since $O \in \langle D \rangle$, we have $r_X(O) \leq r_D(O) \leq \lfloor (d+2)/2 \rfloor < d + 2 - t_1 \leq \deg(W')$, contradicting the assumption that W computes $b_X(P)$.

Now assume $R \neq L$. Since the scheme $L \cap R$ has degree 1, while the scheme $A_1 \cap L$ has degree t_1, we get $\deg(R \cap E) \geq d + 2 - s_1 > (d+2)/2$. As above we get a contradiction.
Now assume $b_X(P) = t_1 + s_1$, but that $W \neq Z_1 \cup S_1$ computes $b_X(P)$. As above we get a line R such that $\deg(W \cup Z_1 \cup S_1) \geq d + 2$ and this line R must be L. Since $P \in \langle Z_1 \cup S_1 \rangle$, there is $U \in \langle D \rangle$ such that Z_1 computes the border D-rank of U and $P \in \langle U \cup S_1 \rangle$. Take $A \subset D$ computing $r_D(U)$. By [15] or [21], Theorem 4.1, or [9] we have $\sharp(A) = d + 2 - t_1$. Since $P \in \langle A \cup S_1 \rangle$ and $A \cap S_1 = \emptyset$, we have $r_X(P) \leq d+2+s_1-t_1$. Assume the existence of some $S \subset X$ computing $r_X(P)$ and such that $\sharp(S) \leq d+1+s_1-t_1$. Thus $\deg(S \cup S_1 \cup Z_1) \leq d+1+2s_1 \leq 2d+1$. Write $S = \nu_d(B)$. We proved that $Z_1 \cup S_1$ computes $b_X(P)$. By [6], Theorem 1, we have $B = B_1 \sqcup S_1$ with $B_1 = L \cap B$. Hence $\sharp(B_1) \leq d+1-t_1$. Since $P \in \langle B_1 \cup S_1 \rangle$, there is $V \in \langle B_1 \rangle$ such that $P \in \langle V \cup S_1 \rangle$. Hence $r_X(P) \leq r_X(V) + s_1$. Since B computes $r_X(P)$ and $V \in \langle B_1 \rangle$, we get $r_X(V) = \sharp(B_1)$ and that B_1 computes $r_X(V)$. Since $\nu_d(B_1) \subset D$, we have $V = Q$. Recall that $b_X(Q) = b_D(Q)$ and that Z_1 is the only subscheme of X computing $r_X(Q)$. We have $r_X(Q) = r_D(Q) = d + 2 - t_1$. Hence $\sharp(B_1) \geq d + 2 - t_1$, contradiction.
If $2s_1 < d$, then the same proof works even if $\sharp(B) = d + 2 + s_1 - t_1$ and prove that any set computing $r_X(P)$ contains S_1. □

Lemma 7. *Fix a hyperplane $M \subset \mathbb{P}^m$ and 0-dimensional schemes A, B such that B is reduced, $A \neq B$, $h^1(\mathcal{I}_A(d)) = h^1(\mathcal{I}_B(d)) = 0$ and $h^1(\mathbb{P}^m, \mathcal{I}_{Res_M(A\cup B)}(d-1)) = 0$. Set $Z := \nu_d(A)$, $S := \nu_d(B)$. Then $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d)) = h^1(M, \mathcal{I}_{(A\cup B)\cap M}(d))$ and Z and S are linearly independent. Assume the existence $P \in \langle Z \rangle \cap \langle S \rangle$ such that $P \notin \langle Z' \rangle$ for any $Z' \subsetneqq Z$ and $P \notin \langle S' \rangle$ for any $S' \subsetneqq S$. Set $F := (B \setminus (B \cap M)) \cap A$. Then $B = (B \cap M) \sqcup F$ and $A = (A \cap M) \sqcup F$.*

Proof. Since $h^1(\mathcal{I}_A(d)) = h^1(\mathcal{I}_B(d)) = 0$, both Z and S are linearly independent. Since $h^2(\mathcal{I}_{A\cup B}(d-1)) = 0$, the residual sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_M(A\cup B)}(d-1) \to \mathcal{I}_{A\cup B}(d) \to \mathcal{I}_{(A\cup B)\cap M}(d) \to 0.$$

gives $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d)) = h^1(M, \mathcal{I}_{(A\cup B)\cap M}(d))$. Assume the existence of P as in the statement. Set $B_1 := (B \cap M) \cup F$.

(a) Here we prove that $B = (B \cap M) \cup F$, i.e. $B = B_1$. Since $P \notin \langle S' \rangle$ for any $S' \subsetneqq S$, it is sufficient to prove $P \in \langle \nu_d(B_1) \rangle$. Since Z and S are linearly independent, Grassmann's formula gives $\dim(\langle Z \rangle \cap \langle S \rangle) = \deg(Z \cap S) - 1 + h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d))$. Since $\mathrm{Res}_M(A \cup B_1) \subseteq \mathrm{Res}_M(A \cup B)$ and $h^1(\mathbb{P}^m, \mathcal{I}_{\mathrm{Res}_M(A\cup B)}(d-1)) = 0$, we have $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B_1}(d)) = h^1(M, \mathcal{I}_{(A\cup B_1)\cap M}(d))$. Since $M \cap (A \cup B_1) = M \cap (A \cup B)$, we get $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B_1}(d)) = h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d))$. Since both schemes Z and $\nu_d(B)$ are linearly independent, Grassmann's formula gives $\dim(\langle Z \rangle \cap \langle \nu_d(B) \rangle) = \deg(A \cap B) - 1 + h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d))$. Since both schemes Z and $\nu_d(B_1)$ are linearly independent, Grassmann's formula gives $\dim(\langle Z \rangle \cap \langle \nu_d(B_1) \rangle) = \deg(A \cap B_1) - 1 + h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d))$. Since $A \cap B_1 = A \cap B$, we get $\dim(\langle Z \rangle \cap \langle S \rangle) = \dim(\langle Z \rangle \cap \langle \nu_d(B_1) \rangle$. Since $\langle Z \rangle \cap \langle \nu_d(B_1) \rangle \subseteq \langle Z \rangle \cap \langle S \rangle$, we get $\langle Z \rangle \cap \langle \nu_d(B_1) \rangle = \langle Z \rangle \cap \langle S \rangle$. Thus $P \in \langle \nu_d(B_1) \rangle$.

(b) In a very similar way we get $A = (A \cap M) \sqcup F$ (see steps (b), (c) and (d) of the proof of Theorem 1 in [6]). □

Theorem 2. *Assume $m \geq 3$. Fix integers $d \geq 5$ and $3 \leq t \leq d$. Fix a degree 2 connected subscheme $A_1 \subset L$ and a reduced set $A_2 \subset \mathbb{P}^m \setminus L$, such that $\sharp(A_2) = t - 2$ and $h^1(\mathbb{P}^m, \mathcal{I}_A(d)) = 0$, for $A := A_1 \cup A_2$. Set $Z_i := \nu_d(A_i)$, $i = 1, 2$, and*

$Z := Z_1 \cup Z_2$. *Assume that A is in linearly general position in $\mathbb{P}^m$. Fix $P \in \langle Z\rangle$ such that $P \notin \langle Z'\rangle$ for any $Z' \subsetneq Z$. Then $b_X(P) = t$ and $r_X(P) = d + t - 2$.*

Proof. Since $h^1(\mathbb{P}^m, \mathcal{I}_A(d)) = 0$, then the scheme Z is linearly independent. Proposition 4 gives $b_X(P) = t$. Fix a set $B \subset \mathbb{P}^m$ such that $S := \nu_d(B)$ computes $r_X(P)$. Assume $r_X(P) < d + t - 2$, i.e. $\sharp(S) \le d + t - 3$. Since $t \le d$, we have $r_X(P) + t \le 3d - 3$.

(a) Until step (g) we assume $m = 3$. We have $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup B}(d)) > 0$ ([6], Lemma 1). Hence $A \cup B$ is not in linearly general position (see [17], Theorem 3.2). Thus there is a plane $M \subset \mathbb{P}^3$ such that $\deg(M \cap (A \cup B)) \ge 4$. Among all such planes we take one, say M_1, such that the integer $x_1 := \deg(M_1 \cap (A \cup B))$ is maximal. Set $E_1 := A \cup B$ and $E_2 := \mathrm{Res}_{M_1}(E_1)$. Notice that $\deg(E_2) = \deg(E_1) - x_1$. Define inductively the planes $M_i \subset \mathbb{P}^3$, $i \ge 2$, the schemes E_{i+1}, $i \ge 2$, and the integers x_i, $i \ge 2$, by the condition that M_i is one of the planes such that the integer $x_i := \deg(M_i \cap E_i)$ is maximal and then set $E_{i+1} := \mathrm{Res}_{M_i}(E_i)$. We have $E_{i+1} \subseteq E_i$ (with strict inclusion if $E_i \ne \emptyset$) for all $i \ge 1$ and $E_i = \emptyset$ for all $i \gg 0$. For all integers t and $i \ge 1$ there is the residual exact sequence

$$0 \to \mathcal{I}_{E_{i+1}}(t-1) \to \mathcal{I}_{E_i}(t) \to \mathcal{I}_{E_i\cap M_i, M_i}(t) \to 0. \tag{6}$$

Let u be the minimal positive integer i such that and $h^1(M_i, \mathcal{I}_{M_i\cap E_i}(d+1-i)) > 0$. Use at most $r_X(P) + t$ times the exact sequences (6) to prove the existence of such an integer u. Any degree 3 subscheme of $\mathbb{P}^3$ is contained in a plane. Thus for any $i \ge 1$ either $x_i \ge 3$ or $x_{i+1} = 0$. Hence $x_i \ge 3$ for all $i \le u-1$. Since $r_X(P)+t \le 3d$, we get $u \le d$.

(b) Here we assume $u = 1$. Since A is in linearly general position, we have $\deg(M_1 \cap A) \le 3$. First assume $x_1 \ge 2d + 2$. Thus $\sharp(B) \ge \sharp(B \cap M_1) \ge 2d - 1 > d + t - 3$, contradiction. Hence $x_1 \le 2d + 1$. Since $h^1(M_1, \mathcal{I}_{M_1\cap E_1}(d)) > 0$, there is a line $T \subset M_1$ such that $\deg(T \cap E_1) \ge d+2$ ([9], Lemma 34). Since A is in linearly general position, we have $\deg(A \cap T) \le 2$. Thus $\deg(T \cap B) \ge d$. Assume for the moment $h^1(\mathbb{P}^3, \mathcal{I}_{E_2}(d-1)) > 0$. Hence $x_2 \ge d + 1$. Since by hypothesis $d \ge 4$, $x_2 \le x_1$ and $x_1 + x_2 \le 3d+1$, we have $x_2 \le 2d-1$. Hence [9], Lemma 34, applied to the integer $d-1$ gives the existence of a line $R \subset \mathbb{P}^3$ such that $\deg(E_2 \cap R) \ge d+1$. Since A is in linearly general position, we also get $\deg(R \cap E_2) \le 2$ and hence $\deg(R \cap B \cap E_2) \ge d - 1$. Thus $\sharp(S) \ge 2d - 1$, contradiction. Now assume $h^1(\mathbb{P}^3, \mathcal{I}_{E_2}(d-1)) = 0$. Lemma 7 gives the existence of a set $F \subset \mathbb{P}^3 \setminus M_1$ such that $A = (A\cap M_1) \sqcup F$ and $B = (B \cap M_1) \sqcup F$. Thus $\sharp(F) = \deg(A) - \deg(A\cap M_1) \ge t-1$. Since $\sharp(B \cap M_1) \ge d$, we obtained a contradiction.

(c) Here and in steps (d), (e), and (f) we assume $m = 3$ and $u \ge 2$. We first look at the possibilities for the integer u. Since every degree 3 closed subscheme of $\mathbb{P}^3$ is contained in a plane, either $x_i \ge 3$ or $x_{i+1} = 0$. Since $r_X(P)+t \le 3d-3$, we get $x_i = 0$ for all $i > d$. Hence $u \le d$. We have $x_u \ge d + 3 - u$ (e.g. by [9], Lemma 34). Since the sequence x_i, $i \ge 1$, is non-decreasing, we get $r_X(P)+2+t-2 \le u(d+3-u)$. Since the function $s \mapsto s(d + 3 - s)$ is concave in the interval $[2, d + 1]$, we get $u \in \{2, 3, d\}$.

(d) Here we assume $u = 2$. Since $3d + 1 \ge x_1 + x_2 \ge 2x_2$, we get $x_2 \le 2(d - 1) + 1$. Hence there is a line $R \subset \mathbb{P}^3$ such that $\deg(E_2 \cap R) \ge d + 1$. We claim that $x_1 \ge d + 1$. Indeed, since $A \cup B \nsubseteq R$, there is a plane $M \subset R$ such that $\deg(M \cap (A \cup B)) > \deg((A \cup B) \cap R) \ge d+1$. The maximality property of x_1 gives $x_1 \ge d + 2$. Since A is in linearly general position, we have $\deg(A \cap R) \le 2$ and

$\deg(A \cap M_1) \le 3$. Hence $\deg(B \cap E_2 \cap R) \ge d - 1$ and $r_X(P) \ge (x_1 - 3) + d - 1 \ge 2d - 2 \ge d + t - 2$, contradiction.

(e) Here we assume $u = 3$. Since $h^1(M_3, \mathcal{I}_{M_3 \cap E_3}(d-2)) > 0$, there is a line $R \subset M_3$ such that $\deg(E_3 \cap T) \ge d$. This is absurd, because $x_1 \ge x_2 \ge x_3 \ge d$ and $x_1 + x_2 + x_3 \le r_X(P) + t \le d + 2t - 3 \le 3d - 3$.

(f) Here we assume $u = d$. The condition " $h^1(\mathcal{I}_{M_d \cap E_d}(1)) > 0$ " says that either $M_d \cap E_d$ contains a scheme of length ≥ 3 contained in a line R or $x_d \ge 4$. Since $x_d \ge 3$, we have $r_X(P) + t \ge x_1 + \cdots + x_d \ge 3d$. Since $t \le d$ and $r_X(P) \le d + t - 3$, this is absurd.

(g) Here we assume $m > 3$. We make a similar proof, taking as M_i, $i \ge 1$, hyperplanes of $\mathbb{P}^m$. Any 0-dimensional scheme of degree at most m of $\mathbb{P}^m$ is contained in hyperplane. Hence either $x_i \ge m$ or $x_{i+1} = 0$. With these modification we repeat the proof of the case $m = 3$. □

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