A NEW APPROXIMATION ALGORITHM FOR THE MATCHING DISTANCE IN MULTIDIMENSIONAL PERSISTENCE

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ABSTRACT. Topological Persistence has proven to be a promising framework for dealing with problems concerning shape analysis and comparison. In this contexts, it was originally introduced by taking into account 1-dimensional properties of shapes, modeled by real-valued functions. More recently, Topological Persistence has been generalized to consider multidimensional properties of shapes, coded by vector-valued functions. This extension has led to introduce suitable shape descriptors, named the *multidimensional persistence Betti numbers functions*, and a distance to compare them, the so-called *multidimensional matching distance*.

In this paper we propose a new computational framework to deal with the multidimensional matching distance. We start by proving some new theoretical results, and then we use them to formulate an algorithm for computing such a distance up to an arbitrary threshold error.

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INTRODUCTION

In this paper, we present a computational framework for applying some tools coming from *multidimensional persistence* to shape analysis and comparison. Indeed, interpreting and comparing shapes are probably two of the most challenging issues in the fields of Computer Vision, Computer Graphics and Pattern Recognition. Nowadays, shape models convey a great amount of visual, semantic and digital information, and therefore finding suitable methods allowing for capturing, processing and representing such an information in a convenient way is definitely a desirable target [37, 38].

Persistence for shape analysis and comparison. In this context, methods deriving from Topological Persistence have recently gained a growing appeal. They focus on a topological exploration of a shape under study, with respect to some geometrical properties considered relevant for capturing the salient features of the shape itself [4, 8, 24, 29]. The assumption here is that the most important piece of information enclosed in geometrical data is usually the one that is "persistent" with respect to the defining parameters. More formally, the key idea is to model a shape as a space X, together with a real-valued function $\varphi : X \to \mathbb{R}$, called *filtering function*. The function φ plays the role of a descriptor for a shape property we consider relevant for the comparison or the analysis problem at hand. By studying the sublevel sets induced on X by φ, we can perform a topological exploration of the shape under study, focusing on the occurrence of meaningful topological events (e.g. the birth, or the merging, of connected components, holes, tunnels, voids and so on). Such an information can then be encoded in a parameterized version of the Betti numbers, known in the literature as *persistent Betti numbers* [25], a *rank invariant* [11], and, for the 0th homology, a *size function* [27, 30, 40]. The main point is that these shape descriptors can be represented in a very simple and compact way, by means of the so-called *persistent diagrams*. Moreover, they are stable with respect to a suitable distance, i.e. the *bottleneck distance* or *matching distance*. Thus, the tools offered by Topological Persistence nicely fit for dealing with shape analysis and comparison problems. Actually, in the last twenty years methods based on the previous guidelines have been successfully used in quite a lot of applications concerning shape analysis and comparison, see e.g. [5, 12, 14, 17, 22, 36, 39].

Motivations and prior works. A common scenario in applications is when two or more properties concur to define the shape of an object. Moreover, sometime it is desirable to study properties of a shape that are intrinsically multidimensional, such as the coordinate of a point in the 3-dimensional space, or the representation of color in the RGB model. Such considerations drove the attention to the so-called *multidimensional Topological Persistence* [8, 23, 29]. Here the term multidimensional, or equivalently n-dimensional, refers to the fact that the considered filtering functions take values in $\mathbb{R}^n$. This leads to consider the multidimensional extension of persistent Betti numbers, namely the *n-dimensional persistent Betti numbers*, hereafter n-dimensional PBNs.

Multidimensional persistence was firstly investigated in [28] as regards homotopy groups, and in [10] as regards homology modules. Another approach to the multidimensional setting is the one proposed in [2], based on the so-called *foliation method*. Focusing on the concept of n-dimensional 0th PBNs, the authors proved that, when $n > 1$, a foliation in half-planes can be given, such that the restriction of n-dimensional 0th PBNs to these half planes turns out to be 1-dimensional. This allowed the definition of a proven stable matching distance between n-dimensional PBNs, namely the *n-dimensional matching distance*. Such a result has been partially extended in [6], i.e. for any homology degree but restricted to the case of *max-tame* filtering functions, and then further refined in [13] for continuous filtering functions.

From the point of view of applications, the main problem in multidimensional persistence is that a *complete, discrete and stable* descriptor seems not to be available in the multidimensional setting, differently from what happens in the 1-dimensional situation [10]. Until now, the arising computational difficulties have been faced according to different strategies [3, 9, 19], but the work is still in progress.

In particular, in [3] the authors take a finite number of half-planes from the foliation proposed in [2] to obtain a computable approximation of the n-dimensional matching distance between 0th PBNs. They perform some experiments on the comparison of surfaces and volumetric objects in the 2- and 3-dimensional settings. Unfortunately, that work does not make clear how many and which half-planes one has to choose to get a reasonable approximation of the matching distance, which could require a huge number of calculations. A solution for this problem in the 2-dimensional setting is proposed in [1], in which a systematical procedure for half-planes selection is presented, giving rise to an algorithm for approximating the 2-dimensional matching distance between 0th PBNs up to an arbitrary threshold error.

Contribution of the paper. Following [1], this paper aims to solve the problem of obtaining good approximations for the n-dimensional matching distance between kth PBNs, for any dimension n and any homology degree k. More specifically, the main contributions of the present work are:

• New theoretical results (Theorem 2.3 and Theorem 2.4) concerning the matching distance corresponding to the leaves of the half-planes foliation. We show that, moving from one leave to the other, the change of the matching distance associated to these half-planes is bounded by a linear function of the distance between the considered leaves. This result extends to any dimension n the one obtained in [1] for the 2-dimensional setting of 0th PBNs. This is possible via the introduction of a suitable distance (Definition 2.1) on the space of parameters defining the half-planes foliation.

• As a by-product, we provide an algorithm to obtain an approximation of the n-dimensional matching distance up to an arbitrary threshold error, representing the maximum error we are disposed to accept in the computation. Our algorithm guarantees a systematical selection of the half-planes in the foliation. Moreover, the threshold allows us to module the computational costs, in order to find a good compromise between quality of results and running time.

The reminder of the paper is organized as follows. In Section 1 we review the standard facts about (multidimensional) persistence, with particular reference to PBNs and matching distance. Section 2 is devoted to present our approximation results. In Section 3 we introduce the algorithm for computing approximations of the multidimensional matching distance. Some discussions in Section 4 precede the final remarks and comments of Section 5.

1. PBNs: Definitions and first properties

In this section, we review the background on Persistent Homology and Topology. Recent surveys on this topic are [4, 23, 24, 29, 41]. However, we warn the reader that, differently for what happens in other papers about persistence, we shall assume that the considering filtering functions are *continuous* instead of tame, and we shall work with Čech homology instead of singular or simplicial homology. The reasons of considering continuous filtering functions is essentially that 1-dimensional reduction of multidimensional persistence is not possible in the setting of tame functions, as it was already observed in [6], but it luckily does in the wider setting of continuous functions. The choice of working with Čech homology is motivated by the fact that, having the continuity axiom, it allows us to prove the Representation Theorem 1.6, stating that the PBNs of a scalar-valued filtering function can be completely described by a persistence diagram. Even assuming tameness, this result would not hold for singular and simplicial theories, which guarantee a complete description of one-dimensional PBNs only outside a set of vanishing measure. In the framework of persistence, Čech homology has already been considered in [34, 35]. Moreover, the Čech approach to homology theory is currently being investigated for computational purposes [32].

Throughout the paper, the following relations $\preceq$ and $\prec$ are defined in $\mathbb{R}^n$: for $\vec{u} = (u_1, \dots, u_n)$ and $\vec{v} = (v_1, \dots, v_n)$, we say $\vec{u} \preceq \vec{v}$ (resp. $\vec{u} \prec \vec{v}$) if and only if $u_i \leq v_i$ (resp. $u_i < v_i$) for every index $i = 1, \dots, n$. Moreover, $\mathbb{R}^n$ is endowed with the usual $\max$-norm: $\|(u_1, u_2, \dots, u_n)\|_\infty = \max_{1 \leq i \leq n} |u_i|$.

We shall use the following notations: Δ^+ will be the open set $\{(\vec{u}, \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n : \vec{u} \prec \vec{v}\}$. For every n-tuple $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$, we shall set $u_* = \min_i u_i$ and, for every function $\vec{\varphi} : X \to \mathbb{R}^n$, we shall denote by $X\langle \vec{\varphi} \preceq \vec{u} \rangle$ the set $\{x \in X : \varphi_i(x) \leq u_i,\ i = 1, \dots, n\}$.

The next definition extends the concept of persistent homology group to a multidimensional setting.

Definition 1.1 (Persistent homology group)**.** Let $k \in \mathbb{Z}$. Let X be a topological space, and $\vec{\varphi} : X \to \mathbb{R}^n$ a continuous function. Let $\pi_k^{(\vec{u},\vec{v})} : \check{H}_k(X\langle \vec{\varphi} \preceq \vec{u} \rangle) \to \check{H}_k(X\langle \vec{\varphi} \preceq \vec{v} \rangle)$ be the homomorphism induced by the inclusion map $\pi^{(\vec{u},\vec{v})} : X\langle \vec{\varphi} \preceq \vec{u} \rangle \hookrightarrow X\langle \vec{\varphi} \preceq \vec{v} \rangle$ with $\vec{u} \preceq \vec{v}$, where $\check{H}_k$ denotes the kth Čech homology group. If $\vec{u} \prec \vec{v}$, the image of $\pi_k^{(\vec{u},\vec{v})}$ is called the *multidimensional kth persistent homology group of* $(X, \vec{\varphi})$ *at* $(\vec{u}, \vec{v})$, and is denoted by $\check{H}_k^{(\vec{u},\vec{v})}(X, \vec{\varphi})$.

In other words, the group $\check{H}_k^{(\vec{u},\vec{v})}(X, \vec{\varphi})$ contains all and only the homology classes of cycles born before or at $\vec{u}$ and still alive at $\vec{v}$. For details about Čech homology, the reader can refer to [26, Ch. IX].

In what follows, we shall work with coefficients in a field $\mathbb{K}$, so that homology groups are vector spaces. Therefore, they can be completely described by their dimension, leading to the following definition (cf. [11, 25]).

Definition 1.2 (Persistent Betti Numbers Function)**.** The function $\beta_{\vec{\varphi}} : \Delta^+ \to \mathbb{N} \cup \{\infty\}$ defined by

$$\beta_{\vec{\varphi}}(\vec{u}, \vec{v}) = \dim \operatorname{im} \pi_k^{(\vec{u},\vec{v})} = \dim \check{H}_k^{(\vec{u},\vec{v})}(X, \vec{\varphi})$$

will be called the *persistent Betti numbers function* of $\vec{\varphi}$, briefly PBNs.

Obviously, for each $k \in \mathbb{Z}$, we have different PBNs $\beta_{\vec{\varphi}}$ of $\vec{\varphi}$ (which should be denoted $\beta_{\vec{\varphi},k}$, say) but, for the sake of notational simplicity, we omit adding any reference to k. This will also apply to the notations used for other concepts in this paper, such as multiplicities and persistence diagrams.

It is possible to prove that, if X is a compact and locally contractible subspace of $\mathbb{R}^m$, the function $\beta_{\vec{\varphi}}$ never attains the value ∞ [7]. However, in order to stick as much as possible to the existing literature about persistence, in the present paper we shall confine ourselves to the weaker assumption that X is triangulable.

1.1. **1-dimensional PBNs.** Now we confine ourselves to the case $n = 1$. Indeed, our approach to the multidimensional setting of PBNs is based on a reduction to the 1-dimensional situation.

For the sake of simplicity, the symbols $\vec{\varphi}, \vec{u}, \vec{v}$ will be replaced by φ, u, v, respectively. We remark that Δ^+ reduces to be the set $\{(u, v) \in \mathbb{R}^2 : u < v\}$. Moreover, we use the following notations: $\Delta = \partial\Delta^+$, $\Delta^* = \Delta^+ \cup \{(u, \infty) : u \in \mathbb{R}\}$, and $\bar{\Delta}^* = \Delta^* \cup \Delta$.

Persistent diagrams and Representation Theorem. One of the main properties of 1-dimensional PBNs is that they admit a very simple and compact representation. More precisely, under the present assumption on X and φ, and making use of Čech homology, it is possible to prove that each 1-dimensional PBNs can be compactly described by a multiset of points, proper and at infinity, of the real plane. Due to the lack of a well-established terminology, we call them *proper cornerpoints* and *cornerpoints at infinity (or cornerlines)*, respectively.

Definition 1.3 (Proper cornerpoint)**.** For every point $p = (u, v) \in \Delta^+$, we define the number $\mu(p)$ as the minimum over all the positive real numbers ε, with $u + \varepsilon < v - \varepsilon$, of

$$\beta_\varphi(u + \varepsilon, v - \varepsilon) - \beta_\varphi(u - \varepsilon, v - \varepsilon) - \beta_\varphi(u + \varepsilon, v + \varepsilon) + \beta_\varphi(u - \varepsilon, v + \varepsilon).$$

The number μ will be called the *multiplicity* of p for β_φ. Moreover, we shall call a *proper cornerpoint for* β_φ any point $p \in \Delta^+$ such that the number $\mu(p)$ is strictly positive.

Definition 1.4 (Cornerpoint at infinity)**.** For every vertical line r, with equation $u = \bar{u}$, $\bar{u} \in \mathbb{R}$, we identify r with $(\bar{u}, \infty) \in \Delta^*$, and define the number $\mu(r)$ as the minimum over all the positive real numbers ε, with $\bar{u} + \varepsilon < 1/\varepsilon$, of

$$\beta_\varphi\left(\bar{u} + \varepsilon, 1/\varepsilon\right) - \beta_\varphi\left(\bar{u} - \varepsilon, 1/\varepsilon\right).$$

The number $\mu(r)$ will be called the *multiplicity* of r for β_φ. When this finite number is strictly positive, we call r a *cornerpoint at infinity for* β_φ.

The concept of cornerpoint allows us to introduce a representation of the PBNs, based on the following definition [13, 18].

Definition 1.5 (Persistence diagram)**.** The *persistence diagram* $D_\varphi \subset \bar{\Delta}^*$ is the multiset of all cornerpoints (both proper and at infinity) for β_φ, counted with their multiplicity, union the points of Δ, counted with infinite multiplicity.

The fundamental role of persistent diagrams is explicitly shown in the following Representation Theorem 1.6 [13, 18], claiming that they uniquely determine 1-dimensional PBNs (the converse also holds by definition of persistence diagram).

Theorem 1.6 (Representation Theorem)**.** *For every* $(\bar{u}, \bar{v}) \in \Delta^+$*, we have*

$$\beta_\varphi(\bar{u}, \bar{v}) = \sum_{\substack{(u,v)\in\Delta^* \\ u\le\bar{u},\, v>\bar{v}}} \mu((u, v)).$$

Roughly speaking, the Representation Theorem 1.6 claims that the value assumed by β_φ at a point $(\bar{u}, \bar{v}) \in \Delta^+$ equals the number of cornerpoints lying above and on the left of $(\bar{u}, \bar{v})$. By means of this theorem we are able to compactly represent 1-dimensional PBNs as multisets of cornerpoints and cornerpoints at infinity, i.e. as persistent diagrams.

Stability of 1-dimensional PBNs. As a consequence of the Representation Theorem 1.6 any distance between persistence diagrams induces a distance between 1-dimensional PBNs. This justifies the following definition [13, 18, 21].

Definition 1.7 (Matching distance)**.** Let X be a triangulable space endowed with continuous functions $\varphi, \varsigma : X \to \mathbb{R}$. The *matching distance* d_{match} between β_φ and β_ς is defined as

$$d_{match}\left(\beta_\varphi, \beta_\varsigma\right) = \min_{\gamma} \max_{p \in D_\varphi} \|p - \gamma(p)\|_{\overline{\infty}}, \tag{1.1}$$

where γ ranges over all multi-bijections (i.e. bijections between multisets) between D_φ and D_ς, and for every $p = (u, v), q = (u', v')$ in Δ^*,

$$\|p - q\|_{\overline{\infty}} = \min\left\{\max\left\{|u - u'|, |v - v'|\right\}, \max\left\{(v - u)/2, (v' - u')/2\right\}\right\},$$

with the convention about points at infinity that $\infty - y = y - \infty = \infty$ when $y \neq \infty$, $\infty - \infty = 0$, $\frac{\infty}{2} = \infty$, $|\infty| = \infty$, $\min\{c, \infty\} = c$ and $\max\{c, \infty\} = \infty$.

In plain words, $\|\cdot\|_{\overline{\infty}}$ measures the pseudo-distance between two points p and q as the minimum between the cost of moving one point onto the other and the cost of moving both points onto the diagonal, with respect to the max-norm and under the assumption that any two points of the diagonal have vanishing pseudo-distance (we recall that a pseudo-distance d is just a distance missing the condition $d(X, Y) = 0 \Rightarrow X = Y$, i.e. two distinct elements may have vanishing distance with respect to d). When the number of cornerpoints is finite, the matching of persistence diagrams is related to the bottleneck transportation problem, and the matching distance reduces to the bottleneck distance [18]. However, this is not always the case when working with continuous filtering functions. Indeed, such an assumption implies that the number of cornerpoints may be countably infinite. We remark that the matching distance is stable with respect to perturbations of the filtering functions, as the following Matching Stability Theorem states:

Theorem 1.8 (One-Dimensional Stability Theorem)**.** *Assume that X is a triangulable space, and $\varphi, \varsigma : X \to \mathbb{R}$ are two continuous functions. Then it holds that $d_{match}(\beta_\varphi, \beta_\varsigma) \leq \|\varphi - \varsigma\|_\infty$.*

For a proof of the previous theorem and more details about the matching distance the reader is referred to [13, 21] (see also [16, 18] for the bottleneck distance).

1.2. **The Foliation Method.** We now review the so called *foliation method*, leading to the definition of a stable distance for multidimensional PBNs [2, 6]. The key idea is that a foliation in half-planes of Δ^+ can be given, such that the restriction of the multidimensional PBNs function to these half-planes turns out to be a one-dimensional PBNs function in two scalar variables. This approach implies that the comparison of two multidimensional PBNs functions can be performed leaf by leaf by measuring the distance of appropriate one-dimensional PBNs functions. Therefore, the stability of multidimensional PBNs is a consequence of the one-dimensional PBNs' stability.

We start by recalling that the following parameterized family of half-planes in $\mathbb{R}^n \times \mathbb{R}^n$ is a foliation of Δ^+ (cf. [2, Prop. 1] and [15]).

Definition 1.9 (linearly admissible pairs)**.** For every vector $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ of $\mathbb{R}^n$ such that $\mu_i > 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^n \mu_i = 1$, and for every vector $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ of $\mathbb{R}^n$ such that $\sum_{i=1}^n \nu_i = 0$, we shall say that the pair $(\vec{\mu}, \vec{\nu})$ is *linearly admissible*. We shall denote the set of all linearly admissible pairs in $\mathbb{R}^n \times \mathbb{R}^n$ by $Ladm_n$. Given a linearly admissible pair $(\vec{\mu}, \vec{\nu})$, we define the half-plane $\pi_{(\vec{\mu},\vec{\nu})}$ of $\mathbb{R}^n \times \mathbb{R}^n$ by the following parametric equations:

$$\begin{cases} \vec{u} = s\vec{\mu} + \vec{\nu} \\ \vec{v} = t\vec{\mu} + \vec{\nu} \end{cases}$$

for $s, t \in \mathbb{R}$, with $s < t$.

Since these half-planes $\pi_{(\vec{l},\vec{b})}$ constitute a foliation of Δ^+, for each $(\vec{u}, \vec{v}) \in \Delta^+$ there exists one and only one $(\vec{\mu}, \vec{\nu}) \in Ladm_n$ such that $(\vec{u}, \vec{v}) \in \pi_{(\vec{\mu},\vec{\nu})}$. Observe that $\vec{\mu}$ and $\vec{\nu}$ only depend on $(\vec{u}, \vec{v})$.

A first property of this foliation is that the restriction of $\beta_{\vec{\varphi}}$ to each leaf can be seen as a particular one-dimensional PBNs function. Intuitively, on each half plane $\pi_{(\vec{\mu},\vec{\nu})}$ one can find the PBNs corresponding to the filtration of X obtained by sweeping the line through $\vec{u}$ and $\vec{v}$ parameterized by $\gamma_{(\vec{\mu},\vec{\nu})} : \mathbb{R} \to \mathbb{R}^n$, with $\gamma_{(\vec{\mu},\vec{\nu})}(\tau) = \tau\vec{\mu} + \vec{\nu}$.

A second property is that this filtration corresponds to the one given by the lower level sets of a certain scalar-valued continuous function. Both these properties are stated in the next theorem, analogous to [6, Thm. 2], and are intuitively shown in Figure 1.

Theorem 1.10 (Reduction Theorem)**.** *For every $(\vec{u},\vec{v}) \in \Delta^+$, let $(\vec{\mu},\vec{\nu})$ be the only linear admissible pair such that $(\vec{u},\vec{v}) = (s\vec{\mu}+\vec{\nu}, t\vec{\mu}+\vec{\nu}) \in \pi_{(\vec{\mu},\vec{\nu})}$. Setting $\mu_* = \min_i \mu_i$, let moreover $\varphi_{(\vec{\mu},\vec{\nu})} : X \to \mathbb{R}$ be the continuous filtering function defined by setting*

$$\varphi_{(\vec{\mu},\vec{\nu})}(x) = \mu_* \cdot \max_i \frac{\varphi_i(x) - \nu_i}{\mu_i}.$$

Then it holds that

$$\beta_{\vec{\varphi}}(\vec{u},\vec{v}) = \beta_{\frac{\varphi_{(\vec{\mu},\vec{\nu})}}{\mu_*}}(s,t)\,.$$

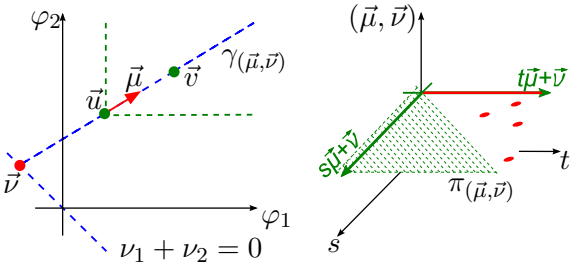

FIGURE 1. One-dimensional reduction of two-dimensional PBNs. Left: a one-dimensional filtration is constructed sweeping the line through $\vec{u}$ and $\vec{v}$. A unit vector $\vec{\mu}$ and a point $\vec{\nu}$ are used to parameterize this line as $\gamma_{(\vec{\mu},\vec{\nu})}(\tau) = \tau\vec{\mu}+\vec{\nu}$. Right: the persistence diagram of this filtration can be found on the leaf $\pi_{(\vec{\mu},\vec{\nu})}$ of the foliation.

As a consequence of the Reduction Theorem 1.10, it is possible to obtain an analogue of the distance d_{match} for the multidimensional case, denoted by D_{match}, having a particularly simple form, yet yielding the desired stability properties [2, 13].

Definition 1.11 (Multidimensional matching distance)**.** Let X be a triangulable space, and let $\vec{\varphi},\vec{\zeta}\colon X \to \mathbb{R}^n$ be continuous functions. For every $(\vec{\mu},\vec{\nu}) \in Ladm_n$, set $d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}},\beta_{\vec{\zeta}}\right) = d_{match}\left(\beta_{\varphi_{(\vec{\mu},\vec{\nu})}},\beta_{\zeta_{(\vec{\mu}\,,\,\vec{\nu})}}\right)$. The *multidimensional matching distance* D_{match} between $\beta_{\vec{\varphi}}$ and $\beta_{\vec{\zeta}}$ is then defined as

$$D_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\zeta}}\right) = \sup_{(\vec{\mu},\vec{\nu})\in Ladm_n} d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}},\beta_{\vec{\zeta}}\right).$$

2. NEW APPROXIMATION RESULTS

In this section we introduce some new theoretical results leading to the formulation of our algorithm for approximating D_{match}. All such results are formally proved in Appendix A.

In what follows, we shall assume that $n \geq 2$, and fix $c = \max\left\{\max_{x\in X}\left\|\vec{\varphi}(x)\right\|_\infty, \max_{x\in X}\left\|\vec{\zeta}(x)\right\|_\infty\right\}$. For every $\vec{\mu} = (\mu_1,\dots,\mu_n) \in \mathbb{R}^n$, the symbol μ_* is used to denote $\min_{i=1,\dots,n}\mu_i$.

We start by defining the following map d on the set $Ladm_n \times Ladm_n$.

Definition 2.1. We define the application $d : Ladm_n \times Ladm_n \to \mathbb{R}^+$ such that

$$d\left((\vec{\mu},\vec{\nu}),(\vec{\mu}',\vec{\nu}')\right) = \max\left\{\max_{i=1,\dots,n}\left|\frac{\mu_*}{\mu_i} - \frac{\mu'_*}{\mu'_i}\right|, \left\|\vec{\nu}-\vec{\nu}'\right\|_\infty\right\}.$$

We can prove the following proposition.

Proposition 2.2. *d is a distance on $Ladm_n$.*

Proof. See Appendix A. □

Before going on, let us analyze how open balls induced on $Ladm_n$ by d look like. For $r > 0$, the usual notation $B_r(p)$ denotes the open ball centered at the point p with radius r.

First of all, observe that we can identify the set $Ladm_n$ with the space product $M_n \times N_n$, being $M_n = \{\vec{\mu} \in \mathbb{R}^n : \sum_{i=1}^n \mu_i = 1, \text{ with } \mu_i > 0,\ i = 1,\dots n\}$ and $N_n = \{\vec{\nu} \in \mathbb{R}^n : \sum_{i=1}^n \nu_i = 0\}$. From the

definition of d, we can induce two different distances on M_n and N_n, say d_M and d_N respectively. More precisely, d_M takes each pair $(\vec{\mu}, \vec{\mu}') \in M_n \times M_n$ to $\max_{i=1,\dots,n} \left| \frac{\mu_*}{\mu_i} - \frac{\mu'_*}{\mu_i} \right|$. The distance d_N is simply the L_∞ distance, i.e. the one taking each pair $(\vec{\nu}, \vec{\nu}') \in N_n \times N_n$ to $\|\nu - \nu'\|_\infty$. As a consequence, an open ball of $Ladm_n$ induced by d, say $B_r((\vec{\mu}, \vec{\nu}))$, can be identified as the product $B_r(\vec{\mu}) \times B_r(\vec{\nu})$, with $B_r(\vec{\mu})$ an open ball of M_n induced by d_M, and $B_r(\vec{\nu})$ an open ball of N_n induced by d_N.

The next result arises from the observation that, at least in a wide subset of $Ladm_n$, the functions $\varphi_{(\vec{\mu},\vec{\nu})}$ and $\varsigma_{(\vec{\mu},\vec{\nu})}$ do not depend on all the components of $\vec{\varphi}$ and $\vec{\varsigma}$, respectively. Indeed, given two indexes $\bar{\imath}, \bar{\jmath} \in \{1, \dots, n\}$, with $\bar{\imath} \neq \bar{\jmath}$, it is quite easy to choose a linear admissible pair $(\vec{\mu}, \vec{\nu}) \in Ladm_n$ such that $\varphi_{\bar{\imath}}(x) - \mu_{\bar{\imath}} \leq 0$ and $\varphi_{\bar{\jmath}}(x) - \mu_{\bar{\jmath}} \geq 0$ for every $x \in X$, thus implying that $\varphi_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i \neq \bar{\imath}} \frac{\varphi_i - \nu_i}{\mu_i}$. The simplest example is when $n = 2$: In such a case, the elements of $Ladm_2$ are given by $(\vec{\mu}, \vec{\nu}) = ((a, 1-a), (b, -b))$, with $0 < a < 1$ and $b \in \mathbb{R}$. It is easy to check that, whenever $b \geq c$ (respectively $b \leq -c$) it holds that $\varphi_{(\vec{\mu},\vec{\nu})}(x) = \mu_* \cdot \frac{\varphi_2(x)+b}{1-a}$ (resp. $\varphi_{(\vec{\mu},\vec{\nu})}(x) = \mu_* \cdot \frac{\varphi_1(x)-b}{a}$) for every $x \in X$. Similar arguments hold for $\varsigma_{(\vec{\mu},\vec{\nu})}$. As a consequence, we can write

$$d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}) = \begin{cases} \frac{\mu_*}{a} \cdot d_{match}(\beta_{\varphi_1}, \beta_{\varsigma_1}), \text{ if } b \leq -c; \\ \frac{\mu_*}{1-a} \cdot d_{match}(\beta_{\varphi_2}, \beta_{\varsigma_2}), \text{ if } b \geq c, \end{cases} \tag{2.1}$$

the equality in (2.1) coming from the properties of the matching distance d_{match} (see also [15, Prop 2.3]). Based on the previous reasonings, the next result states how and when we can reduce the computation of $d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right)$ to a $(n-1)$-dimensional situation. Set $Ladm_n^+ = \{(\vec{\mu}, \vec{\nu}) \in Ladm_n : \|\vec{\nu}\|_\infty \geq (n-1)c\}$. Moreover, for every index $i \in \{1, \dots, n\}$, we denote by $\vec{\varphi}^i$ (respectively $\vec{\varsigma}^i$) the $\mathbb{R}^{n-1}$-valued function obtained from $\vec{\varphi}$ (resp. $\vec{\varsigma}$) by removing its i-th component. Similarly, the symbol $\vec{\mu}^i$ (resp. $\vec{\nu}^i$) will be used for the vector of $\mathbb{R}^{n-1}$ obtained from $\vec{\mu}$ (resp. $\vec{\nu}$) by removing its i-th component.

Theorem 2.3. *Assume that $(\vec{\mu}, \vec{\nu}) \in Ladm_n^+$. Then an index $\bar{\imath} \in \{1, \dots, n\}$ exists such that*

$$d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right) = \frac{\mu_*}{\min_{i \neq \bar{\imath}} \mu_i} \cdot d_{(\vec{\eta},\vec{\omega})}\left(\beta_{\vec{\varphi}^{\bar{\imath}}}, \beta_{\vec{\varsigma}^{\bar{\imath}}}\right), \tag{2.2}$$

with $(\vec{\eta}, \vec{\omega}) \in Ladm_{n-1}$ given by $\vec{\eta} = \vec{\mu}^{\bar{\imath}}/(1 - \mu_{\bar{\imath}})$ and $\vec{\omega} = \vec{\nu}^{\bar{\imath}} + \vec{\eta} \cdot \nu_{\bar{\imath}}$.

Proof. See Appendix A. □

We will show later how Theorem 2.3 can be used to sensibly decrease the computational costs in approximating D_{match}.

We proceed introducing a result which gives insights on how to bound the variation of $d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right)$ when moving from one leaf to another in $Ladm_n \setminus Ladm_n^+$.

Theorem 2.4. *Let $(\vec{\mu}, \vec{\nu}) \in Ladm_n \setminus Ladm_n^+$ and $(\vec{\mu}', \vec{\nu}') \in Ladm_n$, with $d\left((\vec{\mu}, \vec{\nu}), (\vec{\mu}', \vec{\nu}')\right) \leq \delta$. Then*

$$\left| d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right) - d_{(\vec{\mu}',\vec{\nu}')}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right) \right| \leq 2\delta(nc + 1).$$

Proof. See Appendix A. □

Remark 2.5. We observe that $d_{(\vec{\mu},\vec{\nu})}\left(\beta_\varphi, \beta_\varsigma\right) \leq 2c$ for every $(\vec{\mu}, \vec{\nu}) \in Ladm_n$ (this is a trivial consequence of Theorem 1.8); thus we have $\left| d_{(\vec{\mu},\vec{\nu})}\left(\beta_\varphi, \beta_\varsigma\right) - d_{(\vec{\mu}',\vec{\nu}')}\left(\beta_\varphi, \beta_\varsigma\right) \right| \leq 2c$. Now, if $\delta \geq \frac{1}{n}$ then $2c \leq 2\delta\,(nc + 1)$. Consequently, the inequality claimed by Theorem 2.4 is trivial when $\delta \geq \frac{1}{n}$.

3. Algorithm

In this section we show how the results proved in Theorem 2.3 and Theorem 2.4 can be exploited to develop an algorithm for approximating the multidimensional matching distance $D_{match}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right)$. We start by laying the general philosophy. Next we follow up by describing the details of the algorithm, first for the case $n = 2$ and then for any dimension greater than 2.

General approach. Definition 1.11 implies that, in general, a direct computation of $D_{match}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right)$ is not possible, since we should calculate the value $d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right)$ for an infinite number of pairs $(\vec{\mu}, \vec{\nu})$. On the other hand, if we choose a non-empty and finite subset $A_n \subseteq Ladm_n$, and substitute $\sup_{(\vec{\mu},\vec{\nu}) \in Ladm_n}$ with $\max_{(\vec{\mu},\vec{\nu}) \in A_n}$ in Definition 1.11, we get a computable pseudo-distance, say $\mathcal{D}_{match}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right)$, that can be effectively used in concrete applications.

Thinking of $\mathcal{D}_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right)$ as an approximation of $D_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right)$, we can argue that the larger the set $A_n \subseteq Ladm_n$, the smaller the difference between the two values. On the other hand, the smaller the set A_n, the faster the computation of $\mathcal{D}_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right)$. In this perspective, our goal is to find a set A_n that is a compromise between these two situations. Additionally, given an arbitrary real value $\varepsilon > 0$ as an error threshold, we want A_n depending on ϵ in a way that the output $\mathcal{D}_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right)$ satisfies the inequality $\left|D_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right) - \mathcal{D}_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right)\right| \leq \varepsilon$. This is actually what our algorithm is developed for, taking as input the error threshold ε and giving as output $\mathcal{D}_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right)$.

The 2-dimensional case. We start by providing a detailed treatment of the case $n = 2$, since our approach for higher dimensions is based on a reduction to the 2-dimensional situation. For a previous version of the algorithm in the case $n = 2$, the reader is referred to [1].

Let us fix a threshold error ε. By rescaling opportunely both $\vec{\varphi}$ and $\vec{\varsigma}$(and consequently ε), we can assume without loss of generality that $c = 1$. Before going on, for every $\delta > 0$ we introduce the concept of *δ-grid* on $L \subseteq Ladm_2$, i.e. a collection of points $\{p = (\vec{\mu},\vec{\nu}) \in Ladm_2\}$ such that (i) $B_\delta(p) \cap B_\delta(p') = \emptyset$ for every $p, p' \in \mathcal{G}$, $p \neq p'$ and (ii) $L \subseteq \cup_{p\in\mathcal{G}} \bar{B}_\delta(p)$, with $\bar{B}_\delta(p)$ the closure of $B_\delta(p)$. We say that a δ-grid is *finite* if it consists in a finite collection of points.

We need to fix δ. Remark 2.5 allows us to take δ smaller than $\frac{1}{2}$. Let us set $\delta = \frac{1}{4}$. We shall motivate our choice in a while. We also define a finite $\frac{1}{4}$-grid $\mathcal{G}$ on $Ladm_2 \setminus Ladm_2^+$, see Figure 2. To display the grid, we use the fact that $Ladm_2$ can be identified with the product space $M_2 \times N_2$, with $M_2 = \{\vec{\mu} = (a, 1-a),\ 0 < a < 1\}$ and $N_2 = \{\vec{\nu} = (b, -b),\ \nu \in \mathbb{R}\}$. Therefore we can represent $Ladm_2$ as the subset of the real plane given by $I \times \mathbb{R}$, I the open interval $\{a \in \mathbb{R} : 0 < a < 1\}$. In this perspective, $Ladm_2 \setminus Ladm_2^+ = \{(\vec{\mu},\vec{\nu}) : \|\vec{\nu}\|_\infty < 1\}$ is displayed as $I \times \{b \in \mathbb{R} : |b| < 1\}$. We shall describe later how to obtain $\mathcal{G}$ in quite a simple way.

We recall that our goal is to compute the largest value for $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}})$ on a suitable finite subset of $Ladm_2$. Equality (2.1) allows us to sensibly reduce the computation of $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}})$ on $Ladm_2^+$. Indeed, it implies that $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}) \leq d_{match}(\beta_{\varphi_1},\beta_{\varsigma_1})$ if $\vec{\nu} = (b,-b)$ is such that $b \leq -c$, while $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}) \leq d_{match}(\beta_{\varphi_2},\beta_{\varsigma_2})$ if $b \geq c$. Moreover, in the first case the value $d_{match}(\beta_{\varphi_1},\beta_{\varsigma_1})$ is achieved when $\vec{\mu} = (a, 1-a)$ is such that $a \leq \frac{1}{2}$, while in the second case the value $d_{match}(\beta_{\varphi_2},\beta_{\varsigma_2})$ is achieved when $a \geq \frac{1}{2}$. Thus, it is sufficient to consider the maximum between $d_{match}(\beta_{\varphi_1},\beta_{\varsigma_1})$ and $d_{match}(\beta_{\varphi_2},\beta_{\varsigma_2})$ in order to know the value $\max_{Ladm_2^+} d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}})$. We denote such a maximum by $\mathcal{D}_{ext}$.

COMPUTEDEXT($X, \vec{\varphi} = (\varphi_1,\varphi_2), \vec{\varsigma} = (\varsigma_1,\varsigma_2)$)
`for` i = 1 to 2
  compute β_{φ_i}; compute β_{ς_i}; compute $d_{match}\left(\beta_{\varphi_i},\beta_{\varsigma_i}\right)$;
`endfor`
$D_{ext} = \max\{d_{match}\left(\beta_{\varphi_1},\beta_{\varsigma_1}\right), d_{match}\left(\beta_{\varphi_2},\beta_{\varsigma_2}\right)\}$;
`return` D_{ext}.

Theorem 2.4 allows us to control the variation of $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}})$ in each set $(Ladm_2 \setminus Ladm_2^+) \cap \bar{B}_\delta(p_j)$, and hence in $Ladm_2 \setminus Ladm_2^+$. For every $p = (\vec{\mu},\vec{\nu}) \in \mathcal{G}$, we compute the value $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}})$, setting $\mathcal{D}_{int} = \max_{p\in\mathcal{G}} d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}})$.

COMPUTEDINT($X, \vec{\varphi} = (\varphi_1,\varphi_2), \vec{\varsigma} = (\varsigma_1,\varsigma_2), \mathcal{G}$)
$D_{int} = 0$;
`foreach` $p = (\vec{\mu},\vec{\nu})$ `in` $\mathcal{G}$
  compute $\beta_{\varphi_{(\vec{\mu},\vec{\nu})}}$; compute $\beta_{\varsigma_{(\vec{\mu},\vec{\nu})}}$; compute $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}})$;
`endfor`
$D_{int} = \max_{p\in\mathcal{G}} d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}})$;
`return` D_{int}.

The number $\mathcal{D}_{tot} = \max\{\mathcal{D}_{ext},\mathcal{D}_{int}\}$ is then a first approximation of $D_{match}(\beta_{\varphi},\beta_{\vec{\varsigma}})$. The next function COMPUTEDTOT shows how to refine the value $\mathcal{D}_{tot}$ to obtain an approximation of $D_{match}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}})$ up to the error threshold ε. First we briefly describe it.

If the inequality $2\delta\cdot(2c+1) \leq \varepsilon$ holds, by Definition 1.11 and by applying Theorem 2.4 it follows that $|D_{match}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}) - \mathcal{D}_{tot}| \leq \varepsilon$. Therefore COMPUTEDTOT ends, giving as output $\mathcal{D}_{match}(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}) = \mathcal{D}_{tot}$.

Otherwise, COMPUTEDTOT deletes each point $p \in \mathcal{G}$ such that $\mathcal{D}_{tot} - d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}) > 2\delta \cdot (2c+1)$. Indeed, Theorem 2.4 ensures that $\mathcal{D}_{tot}$ will not be achieved (or exceeded) by computing the values $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}})$ over the sets $\bar{B}_\delta(p)$. Moreover, the grid $\mathcal{G}$ is refined as follows: Each p still in $\mathcal{G}$ is replaced by four suitable points $p_1, \dots, p_4$, such that $\{p_j, j = 1, \dots, 4\}$ is a $\frac{\delta}{2}$-grid on $B_\delta(p)$. This refinement procedure, which will be described later, is performed by the function REFINE recalled in COMPUTEDTOT. Finally, $\mathcal{D}_{int}$ and $\mathcal{D}_{tot}$ are updated according to the new grid $\mathcal{G}$, δ is replaced by $\frac{\delta}{2}$, and the algorithm restarts by checking if the inequality $2\delta \cdot (2c+1) \leq \varepsilon$ holds.

COMPUTEDTOT$(\delta, \mathcal{G}, D_{ext}, D_{int})$
$tresh = 2\delta(2c+1)$; $\mathcal{D}_{tot} = \max\{\mathcal{D}_{ext}, \mathcal{D}_{int}\}$;
`while` $tresh \geq \varepsilon$
  `foreach` $p = (\vec{\mu}, \vec{\nu})$ `in` $\mathcal{G}$
    $\mathcal{G} \leftarrow \mathcal{G} \setminus p$;
    `if` $\mathcal{D}_{tot} - d_{(\vec{\mu}_j,\vec{\nu}_j)}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}) \leq tresh$
      $\mathcal{G} \leftarrow \mathcal{G} \cup$ REFINE(p, δ);
    `endif`
  `endfor`
  $\mathcal{D}_{int} =$ COMPUTEDINT$(X, \vec{\varphi} = (\varphi_1, \varphi_2), \vec{\varsigma} = (\varsigma_1, \varsigma_2), \mathcal{G})$;
  $\mathcal{D}_{tot} = \max\{\mathcal{D}_{ext}, \mathcal{D}_{int}\}$; $\delta \leftarrow \frac{\delta}{2}$; $tresh \leftarrow 2\delta(2c+1)$;
`endwhile`
`return` $\mathcal{D}_{tot}$.

To conclude, we describe how to construct a $\frac{1}{2^h}$-grid on $Ladm_2 \setminus Ladm_2^+$, with $h \geq 0$ integer number. Besides using it to construct the $\frac{1}{4}$-grid $\mathcal{G}$ required at the beginning of our algorithm (and thus justifying our initial choice for δ), such a procedure is preparatory to explain how function REFINE works.

When $h = 0$, we can simply take the point $p_0 = (\vec{\mu}_0, \vec{\nu}_0)$, with $\vec{\mu}_0 = (\frac{1}{2}, \frac{1}{2})$ and $\vec{\nu}_0 = (0, 0)$. The set $\{p_0\}$ is actually a 1-grid on $Ladm_2 \setminus Ladm_2^+$. Indeed, setting $B_1(p_0) = B_1(\vec{\mu}_0) \times B_1(\vec{\nu}_0)$, we have that $Ladm_2 \setminus Ladm_2^+ = B_1(p_0)$.

When $h > 0$, our $\frac{1}{2^h}$-grid consists in a collection of 4^h points, say $\{p_{ij} = (\vec{\mu}_i, \vec{\nu}_j) : i, j = 1, \dots, 2^h\}$, with $\vec{\mu}_i = (a_i, 1 - a_i)$ and $\vec{\nu}_j = (b_j, -b_j)$. From the definition of d we can deduce the following relations to determine the points p_{ij}:

$$(I) \quad \begin{cases} \frac{a_i}{1-a_i} = \frac{2i-1}{2^h}, \ i = 1, \dots, 2^{h-1}, \\ \frac{1-a_i}{a_i} = \frac{2^{h+1}-2i+1}{2^h}, \ i = 2^{h-1}+1, \dots, 2^h, \end{cases} \qquad (II) \quad b_j = \frac{2j-2^h-1}{2^h}, \ j = 1, \dots, 2^h.$$

We observe that $a_i < \frac{1}{2}$ for every $i = 1, \dots, 2^{h-1}$, while $a_i > \frac{1}{2}$ when $i = 2^{h-1}+1, \dots, 2^h$. Some examples of $\frac{1}{2^h}$-grid obtained by using equations (I) and (II) are displayed in Figure 2.

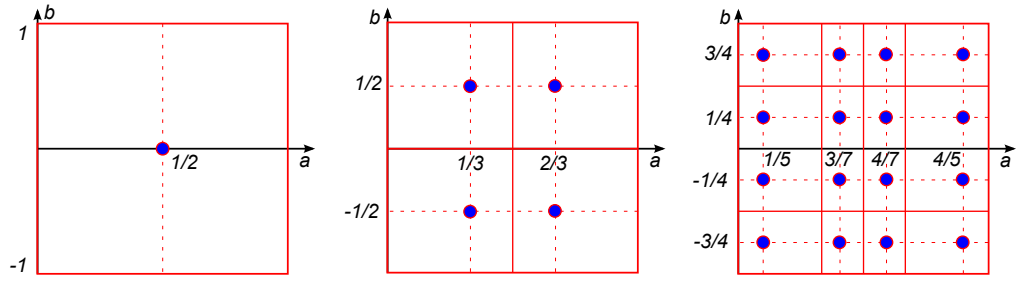

FIGURE 2. $\frac{1}{2^h}$-grids on $Ladm_2 \setminus Ladm_2^+$ for $h = 0$ (left), $h = 1$ (center), and $h = 2$ (right).

Equations (I) and (II) also lay the basis to build the function REFINE recalled in COMPUTEDTOT. Suppose that $p = (\vec{\mu}, \vec{\nu}) \in Ladm_2$ is a point of a $\frac{1}{2^h}$-grid, with $\vec{\mu} = (a, 1-a)$ and $\vec{\nu} = (b, -b)$. If recalled, function REFINE replaces p with a $\frac{1}{2^{h+1}}$-grid of $B_{\frac{1}{2^h}}(p)$, i.e. four points $p'_{ij} = (\vec{\mu}'_i, \vec{\nu}'_j)$ for $i, j = 1, 2$, with

$\vec{\mu}'_i = (a'_i, 1-a'_i)$, $\vec{\nu}'_j = (b'_j, -b'_j)$ and such that

$$\begin{cases} \frac{a'_i}{1-a'_i} = \frac{a}{1-a} + \frac{(-1)^i}{2^{h+1}} & \text{if } a < \frac{1}{2}, \\ \frac{1-a'_i}{a'_i} = \frac{1-a}{a} + \frac{(-1)^i}{2^{h+1}} & \text{if } a > \frac{1}{2}, \end{cases} \qquad b'_j = b + \frac{(-1)^j}{2^{h+1}}. \tag{3.1}$$

Observe that, by applying function REFINE to each point p of a $\frac{1}{2^h}$-grid of $Ladm_2 \setminus Ladm_2^+$, we have a $\frac{1}{2^{h+1}}$-grid of $Ladm_2 \setminus Ladm_2^+$. Indeed, by replacing h with $h+1$, we could obtain the solutions of (3.1) directly from (I) and (II). On the other hand, equations (3.1) allow us to replace just p, without computing other useless points.

The n-dimensional case. We now generalize our algorithm to the n-dimensional setting, with $n > 2$. Such an extension is partially based on a reduction to the 2-dimensional situation.

Similarly to the case $n = 2$, we aim at computing the largest value for $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}})$ over a suitable finite subset of $Ladm_n$. We fix a threshold error ε. By opportunely rescaling both $\vec{\varphi}$ and $\vec{\varsigma}$ (and consequently ε), we can assume without loss of generality that $c = 1$, so that $Ladm_n^+ = \{(\vec{\mu}, \vec{\nu}) \in Ladm_n : \|\vec{\nu}\|_\infty \geq n-1\}$. In $Ladm_n^+$, Theorem 2.3 allows us to reduce the computation of $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}})$ to a $(n-1)$-dimensional situation. Indeed, it implies that, for every $(\vec{\mu}, \vec{\nu}) \in Ladm_n^+$, there exists $(\vec{\eta}, \vec{\omega}) \in Ladm_{n-1}$ such that $d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right) \leq d_{(\vec{\eta},\vec{\omega})}\left(\beta_{\vec{\varphi}^{\bar{\imath}}}, \beta_{\vec{\varsigma}^{\bar{\imath}}}\right)$ for a suitable index $\bar{\imath} \in \{1, \dots, n\}$. On the other hand, it is possible to prove that, for every $\bar{\imath} \in \{1, \dots, n\}$ and every $(\vec{\eta}, \vec{\omega}) \in Ladm_{n-1}$, there always exists $(\vec{\mu}, \vec{\nu}) \in Ladm_n^+$ such that $d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right) = d_{(\vec{\eta},\vec{\omega})}\left(\beta_{\vec{\varphi}^{\bar{\imath}}}, \beta_{\vec{\varsigma}^{\bar{\imath}}}\right)$. As a consequence, the computation of $d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right)$ over the set $Ladm_n^+$ can be reduced to the one of the $(n-1)$-dimensional matching distances $D_{match}\left(\beta_{\vec{\varphi}^i}, \beta_{\vec{\varsigma}^i}\right)$, for $i = 1, \dots, n$.

Obviously, we can recursively repeat the same reasonings to progressively decrease the dimensionality of the problem. It turns out that computing the largest value for $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}})$ on $Ladm_n^+$ can be reduced to the 2-dimensional case, by considering the $\binom{n}{2}$ 2-dimensional matching distances $D_{match}\left(\beta_{\vec{\varphi}_{ij}}, \beta_{\vec{\varsigma}_{ij}}\right)$, with $\vec{\varphi}_{ij} = (\varphi_i, \varphi_j)$ and $\vec{\varsigma}_{ij} = (\varsigma_i, \varsigma_j)$ for every $i \neq j$.

Similarly to what happens in the 2-dimensional case, Theorem 2.4 allows us to control the variation of $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}})$ on the set $Ladm_n \setminus Ladm_n^+$. After fixing $\delta = \frac{1}{2^{\overline{m}}}$, with $\overline{m} = \min\{m \in \mathbb{N} : 2^m > n\}$ (see Remark 2.5), we can define a δ-grid $\mathcal{G}$ on $Ladm_n \setminus Ladm_n^+$ by extending equations (I) and (II) to the n-dimensional situation (actually, relations (II) should be additionally adapted to deal with the open interval $(-n+1, n-1)$ instead of $(-1, 1)$).

It turns out that for the n-dimensional case we only need to use functions COMPUTEDEXT, COMPUTEDINT and COMPUTEDTOT without any modification. Concerning function REFINE, it can be generalized to the n-dimensional situation starting from (3.1) to replace, when necessary, an element of $\mathcal{G}$ with 2^n points.

3.1. **Computational costs.** Let us fix a threshold error $\varepsilon > 0$ and observe that, for every $n \geq 2$, we can write $\varepsilon = n\varepsilon_n$, for a suitable real value ε_n. When $n = 2$, approximating $D_{match}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}})$ needs the computation of the 1-dimensional PBNs β_{φ_i}, β_{ς_i} for $i = 1, 2$, $\beta_{\varphi_{(\vec{\mu},\vec{\nu})}}$ and the associated 1-dimensional matching distance $d_{match}(\beta_{\varphi_i}, \beta_{\varsigma_i})$ and $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}})$, for each $(\vec{\mu}, \vec{\nu}) \in \mathcal{G}$ along with all its refinements. The running time for computing a 1-dimensional PBNs is cubic in the size of input data (e.g., number of simplices, nodes of a graph or voxels of a 3D image) in the worst case [24]. Computing the 1-dimensional matching distance between two 1-dimensional PBNs takes $O\left(q^{2.5}\right)$, being q the total number of cornerpoints of the two descriptors [4]. It turns out that the overall computational cost depends on the previous complexities, multiplied by the number of points in $\mathcal{G}$ along with all its refinements. The worst case cardinality for such a set is when there is no points cancellation. If so, at the hth iteration of the algorithm the cardinality of $\mathcal{G}$ corresponds to cover $I \times (-c, c)$ with open balls (w.r.t. the distance d) of radius $\delta = 1/2^h$, i.e. $O(4^h)$. In particular, since the algorithm ends as soon as $2\delta\,(2c+1) < \varepsilon$, at the last iteration we obtain $O(1/\varepsilon^2)$. This implies that the worst case complexity is at most proportional to $1/\varepsilon^2$. Additionally, we can estimate that the number of iterations of the algorithm is $\log_2 \frac{2c+1}{2\varepsilon}$.

When $n > 2$, we apply $\binom{n}{2}$ times the 2-dimensional framework to obtain the maximum value for $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}})$ on $Ladm_n^+$, thus giving a computational cost proportional to $(n/\varepsilon)^2 = 1/\varepsilon_n^2$. Moreover, we also need to compute $d_{(\vec{\mu},\vec{\nu})}(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}})$ for each $(\vec{\mu}, \vec{\nu}) \in \mathcal{G}$ along with all its refinements. Following similar considerations to the 2-dimensional situation, we get a running time that is proportional to $(n/\varepsilon)^n = (1/\varepsilon_n)^n$ in the worst case.

4. Discussions

As can be seen, the total running time of our algorithm is exponential in n in the worst case. On the other hand, it is worth noting that a number of strategies can be used to keep down the computational costs.

• From the Matching Stability Theorem 1.8 and from the definition of $\varphi_{(\vec{\mu},\vec{\nu})}$ and $\varsigma_{(\vec{\mu},\vec{\nu})}$ (cf. Theorem 1.10), we have that $d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right) \leq \max_{x\in X}\|\vec{\varphi}(x)-\vec{\varsigma}(x)\|_\infty$ and hence $\left|d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right) - d_{(\vec{\mu}',\vec{\nu}')}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right)\right| \leq \max_{x\in X}\|\vec{\varphi}(x)-\vec{\varsigma}(x)\|_\infty$, for every $(\vec{\mu},\vec{\nu}),(\vec{\mu}',\vec{\nu}') \in Ladm_n$. It follows that, if $\max_{x\in X}\|\vec{\varphi}(x)-\vec{\varsigma}(x)\|_\infty \leq \varepsilon$, we incur in the worst case. Whenever this happens, we simply prevent to run our algorithm, setting $\mathcal{D}_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right) = \max_{x\in X}\|\vec{\varphi}(x)-\vec{\varsigma}(x)\|_\infty$.

• Instead of fixing a threshold error ε, we could decide to fix the number of $(\vec{\mu},\vec{\nu}) \in Ladm_n \setminus Ladn_n^+$ we are disposed to consider during the computation of $\mathcal{D}_{match}\left(\beta_{\vec{\varphi}},\beta_{\vec{\varsigma}}\right)$. We recall that, even considering just one of these points, we would have a pseudo-distance between β_φ and $\beta_{\vec{\varsigma}}$ that can be effectively evaluated. Further, even working with a small number of points in $Ladm_n \setminus Ladn_n^+$ gave encouraging results in shape comparison applications [2, 3]. Nevertheless, the theoretical results underlying our algorithm provide a systematic procedure to select the points $(\vec{\mu},\vec{\nu})$, differently from what happened in previous works.

• Our computational approach perfectly fits in the general structure of *branch and bound algorithms*. Branch and bound is a general algorithm for finding optimal solutions of various optimization problems, especially in discrete and combinatorial optimization, see [31] for details. A number of techniques are available in literature, such as the *depth-first search* [20] or the *best-first search* [33], improving the efficiency of branch and bound algorithms in exploring the set of candidate solutions. In our case, applying one of these strategies, or a combination of them, would contribute in containing the computational costs.

Let us conclude by observing that a previous version of the algorithm has been tested in [1], to compare 2-dimensional 0th PBNs associated to 3D objects represented by surface models. Experiments show that, in practical cases, the algorithm is able to decimate the number of half-planes required to reach the desired approximation for the 2-dimensional matching distance. In particular, in the considered cases the algorithm cuts away from 55% to 96% of the total number of half-planes that should be considerd without the cancellation strategy. These results make us confident for future applications of the n-dimensional framework in shape comparison.

5. Conclusions

In this paper we present a novel theoretical and computational framework to get approximations of the matching distance between multidimensional PBNs. More precisely, starting from the so-called foliation method, we obtain new results to bound the matching distance associated to the leaves of a foliation defined on the domain of n-dimensional PBNs. These results follow the ones obtained in [1] for the 2-dimensional setting of 0th PBNs. Such an extension has been possible via the introduction of a suitable distance on the space of parameters identifying the leaves of the foliation. Moreover, we provide an algorithm to obtain approximations of the n-dimensional matching distance up to an arbitrary error threshold, representing the maximum error we are disposed to accept in the computation. We also investigate some possible directions to keep down the computational costs. Previous examples developed in [1] for the 2-dimensional matching distance show how such a framework could be used in Computer Vision, Computer Graphics and Pattern Recognition, to compare properties of shapes that can be modeled by vector-valued continuous functions. This is actually what we are planning to do, in order to test the feasibility of the proposed framework even in higher dimensions.

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A. Appendix

This appendix is devoted to formally prove the results presented in Section 2.

Proof of Proposition 2.2. We will show that d has actually all the properties of a distance.

(i): $d \geq 0$ by definition;

(ii): Let us prove that $d\left((\vec{\mu},\vec{\nu}),(\vec{\mu}',\vec{\nu}')\right) = 0 \Leftrightarrow (\vec{\mu},\vec{\nu}) = (\vec{\mu}',\vec{\nu}')$. Obviously, if $(\vec{\mu},\vec{\nu}) = (\vec{\mu}',\vec{\nu}')$ then $d\left((\vec{\mu},\vec{\nu}),(\vec{\mu}',\vec{\nu}')\right) = 0$. Conversely, suppose that $d\left((\vec{\mu},\vec{\nu}),(\vec{\mu}',\vec{\nu}')\right) = 0$. Then $\vec{\nu} = \vec{\nu}'$. Moreover, $\frac{\mu_*}{\mu_i} = \frac{\mu'_*}{\mu_i}$ for every $i = 1,\dots,n$, and hence $\frac{\mu_*}{\mu'_*} = \frac{\mu_i}{\mu'_i} = k$ for every $i = 1,\dots,n$. By contradiction, suppose that $k \neq 1$. Since $\mu_i = k\mu'_i$ for every $i = 1,\dots,n$, and given that $\sum_{i=1}^n \mu'_i = 1$ we would have $\sum_{i=1}^n \mu_i = \sum_{i=1}^n k\mu'_i = k \neq 1$, against the definition of $Ladm_n$. Therefore $k = 1$, implying that $\vec{\mu} = \vec{\mu}'$;

(iii): Symmetry is again by definition;

(iv): Let us prove the triangle inequality. For every $(\vec{\mu},\vec{\nu}),(\vec{\mu}',\vec{\nu}'),(\vec{\mu}'',\vec{\nu}'') \in Ladm_n$, it holds that

$$
\begin{aligned}
d\left((\vec{\mu},\vec{\nu}),(\vec{\mu}',\vec{\nu}')\right) &= \max\left\{\max_{i=1,\dots,n}\left|\frac{\mu_*}{\mu_i} - \frac{\mu'_*}{\mu'_i}\right|, \left\|\vec{\nu}-\vec{\nu}'\right\|_\infty\right\} \\
&= \max\left\{\max_{i=1,\dots,n}\left|\frac{\mu_*}{\mu_i} - \frac{\mu''_*}{\mu''_i} + \frac{\mu''_*}{\mu''_i} - \frac{\mu'_*}{\mu'_i}\right|, \left\|\vec{\nu}-\vec{\nu}''+\vec{\nu}''-\vec{\nu}'\right\|_\infty\right\} \\
&\leq \max\left\{\max_{i=1,\dots,n}\left|\frac{\mu_*}{\mu_i} - \frac{\mu''_*}{\mu''_i}\right| + \max_{i=1,\dots,n}\left|\frac{\mu''_*}{\mu''_i} - \frac{\mu'_*}{\mu'_i}\right|, \left\|\vec{\nu}-\vec{\nu}''\right\|_\infty + \left\|\vec{\nu}''-\vec{\nu}'\right\|_\infty\right\} \\
&\leq \max\left\{\max_{i=1,\dots,n}\left|\frac{\mu_*}{\mu_i} - \frac{\mu''_*}{\mu''_i}\right|, \left\|\vec{\nu}-\vec{\nu}''\right\|_\infty\right\} + \max\left\{\max_{i=1,\dots,n}\left|\frac{\mu''_*}{\mu''_i} - \frac{\mu'_*}{\mu'_i}\right|, \left\|\vec{\nu}''-\vec{\nu}'\right\|_\infty\right\} \\
&= d\left((\vec{\mu},\vec{\nu}),(\vec{\mu}'',\vec{\nu}'')\right) + d\left((\vec{\mu}'',\vec{\nu}''),(\vec{\mu}',\vec{\nu}')\right).
\end{aligned}
$$

□

Proof of Theorem 2.3. If $\|\vec{\nu}\|_\infty \geq (n-1)c$ it follows that an index $\bar{\imath} \in \{1,\dots,n\}$ exists, such that $\varphi_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq\bar{\imath}} \frac{\varphi_i-\nu_i}{\mu_i}$ and $\varsigma_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq\bar{\imath}} \frac{\varsigma_i-\nu_i}{\mu_i}$. Indeed, let us suppose that $\|\vec{\nu}\|_\infty = \nu_{\bar{\imath}} \geq (n-1)c$. Then $\sum_{i\neq\bar{\imath}} \nu_i = -\nu_{\bar{\imath}} \leq -(n-1)c$, implying that an index $\bar{\jmath} \neq \bar{\imath}$ exists, such that $\nu_{\bar{\jmath}} \leq -c$. Therefore $\varphi_{\bar{\jmath}}(x) - \nu_{\bar{\jmath}} \geq 0$ for every $x \in X$ (respectively $\varsigma_{\bar{\jmath}}(x) - \nu_{\bar{\jmath}} \geq 0$ for every $x \in X$) while $\varphi_{\bar{\imath}}(x) - \nu_{\bar{\imath}} \leq 0$ for every $x \in X$ (resp. $\varsigma_{\bar{\imath}}(x) - \nu_{\bar{\imath}} \leq 0$ for every $x \in X$), so that $\varphi_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq\bar{\imath}} \frac{\varphi_i-\nu_i}{\mu_i}$ (resp. $\varsigma_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq\bar{\imath}} \frac{\varsigma_i-\nu_i}{\mu_i}$). In a similar way it is possible to show that, if $\|\vec{\nu}\|_\infty = -\nu_{\bar{\jmath}} \geq (n-1)c$, i.e. $\nu_{\bar{\jmath}} \leq -(n-1)c$, then an index $\bar{\imath} \neq \bar{\jmath}$ exists, such that $\varphi_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq\bar{\imath}} \frac{\varphi_i-\nu_i}{\mu_i}$ (resp. $\varsigma_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq\bar{\imath}} \frac{\varsigma_i-\nu_i}{\mu_i}$).

By virtue of the previous reasonings, let us fix an index $\bar{\imath} \in \{1,\dots,n\}$ such that $\varphi_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq\bar{\imath}} \frac{\varphi_i-\nu_i}{\mu_i}$ and $\varsigma_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq\bar{\imath}} \frac{\varsigma_i-\nu_i}{\mu_i}$. Without loss of generality, we can assume $\bar{\imath} = n$, so that $\varphi_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq n} \frac{\varphi_i-\nu_i}{\mu_i}$ and $\varsigma_{(\vec{\mu},\vec{\nu})} = \mu_* \cdot \max_{i\neq n} \frac{\varsigma_i-\nu_i}{\mu_i}$. Now set $\eta_i = \frac{\mu_i}{1-\mu_n}$, $\omega_i = \nu_i + \eta_i\nu_n$ for every $i = 1,\dots,n-1$. Since $\sum_{i=1}^{n-1} \eta_i = 1$ and $\sum_{i=1}^{n-1} \omega_i = 0$, it follows that $((\eta_1,\dots,\eta_{n-1}),(\omega_1,\dots,\omega_{n-1})) = (\vec{\eta},\vec{\omega}) \in Ladm_{n-1}$. Moreover,

$$
\begin{aligned}
\varphi_{(\vec{\mu},\vec{\nu})} &= \mu_* \cdot \max_{i\neq n} \frac{\varphi_i - \nu_i}{\mu_i} \\
&= \mu_* \cdot \max_{i\neq n} \frac{\varphi_i - \omega_i + \eta_i\nu_n}{(1-\mu_n)\eta_i} \\
&= \frac{\mu_*}{1-\mu_n} \cdot \max_{i\neq n} \frac{\varphi_i - \omega_i}{\eta_i} + \frac{\mu_*}{1-\mu_n} \cdot \nu_n \\
&= \frac{\mu_*}{1-\mu_n} \cdot \frac{\eta_*}{\eta_*} \cdot \max_{i\neq n} \frac{\varphi_i - \omega_i}{\eta_i} + \frac{\mu_*}{1-\mu_n} \cdot \nu_n \\
&= \frac{\mu_*}{\min_{i\neq n}\mu_i} \cdot \eta_* \cdot \max_{i\neq n} \frac{\varphi_i - \omega_i}{\eta_i} + \frac{\mu_*}{1-\mu_n} \cdot \nu_n = \frac{\mu_*}{\min_{i\neq n}\mu_i} \cdot \varphi^n_{(\vec{\eta},\vec{\omega})} + \frac{\mu_*}{1-\mu_n} \cdot \nu_n,
\end{aligned}
$$

with $\varphi^n_{(\vec{\eta},\vec{\omega})} = \eta_* \cdot \max_{i\neq n} \frac{\varphi_i-\omega_i}{\eta_i}$. Analogously, setting $\varsigma^n_{(\vec{\eta},\vec{\omega})} = \max_{i\neq n} \frac{\varsigma_i-\omega_i}{\eta_i}$, it is possible to show that

$$\varsigma_{(\vec{\mu},\vec{\nu})} = \frac{\mu_*}{\min_{i\neq n}\mu_i} \cdot \varsigma^n_{(\vec{\eta},\vec{\omega})} + \frac{\mu_*}{1-\mu_n} \cdot \nu_n.$$

Therefore we can write

$$d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right) = \frac{\mu_*}{\min_{i\neq n}\mu_i} \cdot d_{(\vec{\eta},\vec{\omega})}\left(\beta_{\vec{\varphi}^n}, \beta_{\vec{\varsigma}^n}\right), \tag{A.1}$$

with the equality in (A.1) coming from the properties of the matching distance d_{match} (see also [15, Prop 2.3]). Analogous considerations hold even assuming $\bar{\imath} \neq n$. □

Proof of Theorem 2.4.

$$\begin{aligned}
\left|d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right) - d_{(\vec{\mu}',\vec{\nu}')}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right)\right| &= \left|d_{match}\left(\beta_{\varphi_{(\vec{\mu},\vec{\nu})}}, \beta_{\varsigma_{(\vec{\mu},\vec{\nu})}}\right) - d_{match}\left(\beta_{\varphi_{(\vec{\mu}',\vec{\nu}')}}, \beta_{\varsigma_{(\vec{\mu}',\vec{\nu}')}}\right)\right| \\
&\leq d_{match}\left(\beta_{\varphi_{(\vec{\mu},\vec{\nu})}}, \beta_{\varphi_{(\vec{\mu}',\vec{\nu}')}}\right) + d_{match}\left(\beta_{\varsigma_{(\vec{\mu},\vec{\nu})}}, \beta_{\varsigma_{(\vec{\mu}',\vec{\nu}')}}\right) \\
&\leq \max_{x\in X}\left|\varphi_{(\vec{\mu},\vec{\nu})}(x) - \varphi_{(\vec{\mu}',\vec{\nu}')}(x)\right| + \max_{x\in X}\left|\varsigma_{(\vec{\mu},\vec{\nu})}(x) - \varsigma_{(\vec{\mu}',\vec{\nu}')}(x)\right| \\
&\leq \max_{x\in X}\max_{i=1,\dots,n}\left|\frac{\mu_*}{\mu_i}\cdot(\varphi_i(x)-\nu_i) - \frac{\mu'_*}{\mu'_i}\cdot(\varphi_i(x)-\nu'_i)\right| + \qquad (A.2) \\
&+ \max_{x\in X}\max_{i=1,\dots,n}\left|\frac{\mu_*}{\mu_i}\cdot(\varsigma_i(x)-\nu_i) - \frac{\mu'_*}{\mu'_i}\cdot(\varsigma_i(x)-\nu'_i)\right|, \qquad (A.3)
\end{aligned}$$

with the first inequality coming from a trivial extension of the triangular inequality to the case of four elements, the second one from the Matching Stability Theorem 1.8 and the third one from the inequality $|\max_{i=1,\dots,n} u_i - \max_{i=1,\dots,n} v_i| \leq \max_{i=1,\dots,n}|u_i - v_i|$, for every $u_1,\dots,u_n,v_1,\dots,v_n \in \mathbb{R}$. Moreover, for every $i = 1,\dots,n$ we have

$$\begin{aligned}
\left|\frac{\mu_*}{\mu_i}\cdot(\varphi_i(x)-\nu_i) - \frac{\mu'_*}{\mu'_i}\cdot(\varphi_i(x)-\nu'_i)\right| &= \left|\frac{\mu_*}{\mu_i}\cdot(\varphi_i(x)-\nu_i) - \frac{\mu'_*}{\mu'_i}\cdot(\varphi_i(x)-\nu_i) + \right. \\
&+ \left.\frac{\mu'_*}{\mu'_i}\cdot(\varphi_i(x)-\nu_i) - \frac{\mu'_*}{\mu'_i}\cdot(\varphi_i(x)-\nu'_i)\right| \\
&\leq \left|\frac{\mu_*}{\mu_i}\cdot(\varphi_i(x)-\nu_i) - \frac{\mu'_*}{\mu'_i}\cdot(\varphi_i(x)-\nu_i)\right| + \\
&+ \left|\frac{\mu'_*}{\mu'_i}\cdot(\varphi_i(x)-\nu_i) - \frac{\mu'_*}{\mu'_i}\cdot(\varphi_i(x)-\nu'_i)\right| \\
&\leq |\varphi_i(x)-\nu_i|\cdot\left|\frac{\mu_*}{\mu_i} - \frac{\mu'_*}{\mu'_i}\right| + \frac{\mu'_*}{\mu'_i}\cdot|\nu_i-\nu'_i| \leq \delta(nc+1),
\end{aligned} \tag{A.4}$$

with the last inequality following from the following ones:

$$\max_{x\in X}\|\vec{\varphi}(x)\|_\infty \leq c, \quad \|\vec{\nu}\|_\infty \leq (n-1)c, \quad \left|\frac{\mu_*}{\mu_i} - \frac{\mu'_*}{\mu'_i}\right| \leq \delta, \quad \frac{\mu'_*}{\mu'_i} \leq 1, \quad \|\vec{\nu}-\vec{\nu}'\|_\infty \leq \delta.$$

Similarly, we can prove that, for every $i = 1,\dots,n$,

$$\left|\frac{\mu_*}{\mu_i}\cdot(\varsigma_i(x)-\nu_i) - \frac{\mu'_*}{\mu'_i}\cdot(\varsigma_i(x)-\nu'_i)\right| \leq \delta(nc+1), \tag{A.5}$$

and hence, substituting (A.2) and (A.3) respectively by (A.4) and (A.5) we have that

$$\left|d_{(\vec{\mu},\vec{\nu})}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right) - d_{(\vec{\mu}',\vec{\nu}')}\left(\beta_{\vec{\varphi}}, \beta_{\vec{\varsigma}}\right)\right| \leq 2\delta(nc+1).$$

□

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