SYMMETRIC TENSOR RANK WITH A TANGENT VECTOR: A GENERIC UNIQUENESS THEOREM

EDOARDO BALLICO, AND ALESSANDRA BERNARDI

Abstract. Let $X_{m,d} \subset \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, be the order d Veronese embedding of $\mathbb{P}^m$. Let $\tau(X_{m,d}) \subset \mathbb{P}^N$, be the tangent developable of $X_{m,d}$. For each integer $t \ge 2$ let $\tau(X_{m,d}, t) \subseteq \mathbb{P}^N$, be the joint of $\tau(X_{m,d})$ and $t-2$ copies of $X_{m,d}$. Here we prove that if $m \ge 2$, $d \ge 7$ and $t \le 1 + \lfloor \binom{m+d-2}{m}/(m+1) \rfloor$, then for a general $P \in \tau(X_{m,d}, t)$ there are uniquely determined $P_1, \dots, P_{t-2} \in X_{m,d}$ and a unique tangent vector ν of $X_{m,d}$ such that P is in the linear span of $\nu \cup \{P_1, \dots, P_{t-2}\}$, i.e. a degree d linear form f associated to P may be written as

$$f = L_{t-1}^{d-1} L_t + \sum_{i=1}^{t-2} L_i^d$$

with L_i, $1 \le i \le t$, uniquely determined (up to a constant) linear forms on $\mathbb{P}^m$.

1. Introduction

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. Fix integers $m \ge 2$ and $d \ge 3$. Let $j_{m,d} : \mathbb{P}^m \to \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, be the order d Veronese embedding of $\mathbb{P}^m$. Set $X_{m,d} := j_{m,d}(\mathbb{P}^m)$. We often write X instead of $X_{m,d}$. For each integer t such that $1 \le t \le N$ let $\sigma_t(X)$ denote the closure in $\mathbb{P}^N$ of the union of all $(t-1)$-dimensional linear subspaces spanned by t points of X (the t-secant variety of X). Let $\tau(X) \subseteq \mathbb{P}^N$ be the tangent developable of X, i.e. the closure in $\mathbb{P}^N$ of the union of all embedded tangent spaces $T_P X$, $P \in X$. Obviously $\tau(X) \subseteq \sigma_2(X)$ and $\tau(X)$ is integral. Since $d \ge 3$, the variety $\tau(X)$ is a hypersurface of $\sigma_2(X)$ ([6], Proposition 3.2). Fix integral positive-dimensional subvarieties $A_1, \dots, A_s \subset \mathbb{P}^N$, $s \ge 2$. The join $[A_1, A_2]$ is the closure in $\mathbb{P}^N$ of the union of all lines spanned by a point of A_1 and a different point of A_2. If $s \ge 3$ define inductively the join $[A_1, \dots, A_s]$ by the formula $[A_1, \dots, A_s] := [[A_1, \dots, A_{s-1}], A_s]$. The join $[A_1, \dots, A_s]$ is an integral variety and $\dim([A_1, \dots, A_s]) \le \min\{N, s - 1 + \sum_{i=1}^s \dim(A_i)\}$. The integer $\min\{N, s-1+\sum_{i=1}^s \dim(A_i)\}$ is called the *expected dimension* of the join $[A_1, \dots, A_s]$. Obviously $[A_1, \dots, A_s] = [A_{\sigma(1)}, \dots, A_{\sigma(s)}]$ for any permutation $\sigma : \{1, \dots, s\} \to \{1, \dots, s\}$. The secant variety $\sigma_t(X)$, $t \ge 2$, is the join of t copies of X. For each integers $t \ge 3$ let $\tau(X, t) \subseteq \mathbb{P}^N$ be the join of $\tau(X)$ and $\sigma_{t-2}(X)$ copies of X. We recall that $\min\{N, t(m+1) - 2\}$ is the expected dimension of $\tau(X, t)$, while $\min\{N, t(m+1) - 1\}$ is the expected dimension of $\sigma_t(X)$.

1991 *Mathematics Subject Classification.* 14N05; 14M17.

Key words and phrases. Veronese variety; tangential variety; join; weak defectivity.

The first author was partially supported by MIUR and GNSAGA of INdAM (Italy); the second author was partially supported by CIRM-FBK (TN-Italy), Marie-Curie FP7-PEOPLE-2009-IEF, INRIA Sophia Antipolis Mediterranée, Project Galaad (France) and Mittagh-Leffler Institut (Sweden).

In the range of triples (m, d, t) we will met in this paper both $\tau(X, t)$ and $\sigma_t(X)$ have the expected dimensions and hence $\tau(X, t)$ is a hypersurface of $\sigma_t(X)$.

After [3] it is natural to ask the following question.

Question 1. Assume $d \geq 3$ and $\tau(X, t) \neq \mathbb{P}^N$. Is a general point of $\tau(X, t)$ in the linear span of a unique set $\{P_0, P_1, \dots, P_{t-2}\}$ with $(P_0, P_1, \dots, P_{t-2}) \in \tau(X) \times X^{t-2}$?

For non weakly $(t-1)$-degenerate subvarieties of $\mathbb{P}^N$ the corresponding question is true by [7], Proposition 1.5. Here we answer it for a large set of triples of integers (m, d, t) and prove the following result.

Theorem 1. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m}/(m+1) \rfloor$. Assume $t \leq \beta + 1$. Let P be a general point of $\tau(X, t)$. Then there are uniquely determined points $P_1, \dots, P_{t-2} \in X$ and $Q \in \tau(X)$ such that $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$, i.e. there are uniquely determined points $P_1, \dots, P_{t-2} \in X$ and a unique tangent vector ν of X such that $P \in \langle \{P_1, \dots, P_{t-2}\} \cup \nu \rangle$.*

The existence part is obvious by the definition of the join as a closure of unions of certain linear subspaces, but the uniqueness part is non-trivial and interesting. Each point of Veronese variety X is of the form L^d for a unique (up to a constant) $L \in \mathbb{K}[x_0, \dots, x_m]_1$. Thus Theorem 1 may rephrased in the following way.

Theorem 2. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m}/(m+1) \rfloor$. Assume $t \leq \beta + 1$. Let P be a general point of $\tau(X, t)$ and f the homogeneous degree d form in $\mathbb{K}[x_0, \dots, x_m]$ associated to P. Then f may be written in a unique way*

$$f = L_{t-1}^{d-1} L_t + \sum_{i=1}^{t-2} L_i^d$$

with $L_i \in \mathbb{K}[x_0, \dots, x_m]_1$, $1 \leq i \leq t$.

In the statement of Theorem 2 the form f is uniquely determined only up to a non-zero scalar, and (as usual in this topic) “ uniqueness ” may allow not only a permutation of the forms $L_1, \dots, L_{t-2}$, but also a scalar multiplication of each L_i. To prove Theorem 1 and hence Theorem 2 we adapt the notion and the results on weakly defective varieties described in [4]. It is easy to adapt [4] to joins of different varieties instead of secant varieties of a fixed variety if a general tangent hyperplane is tangent only at one point ([5]). However, a general tangent space of $\tau(X)$ is tangent to $\tau(X)$ along a line, not just at the point of tangency. Hence a general hyperplane tangent to $\tau(X, t)$, $t \geq 2$, is tangent to $\tau(X, t)$ at least along a line. We prove the following result.

Theorem 3. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m}/(m+1) \rfloor$. Assume $t \leq \beta+1$. Let P be a general point of $\tau(X, t)$. Let $P_1, \dots, P_{t-2} \in X$ and $Q \in \tau(X)$ be the points such that $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$. Let ν be the tangent vector of X such that Q is a point of $\langle \nu \rangle \setminus \nu_{red}$. Let $H \subset \mathbb{P}^N$ be a general hyperplane containing the tangent space $T_P\tau(X, t)$ of $\tau(X, t)$. Then H is tangent to X only at the points $P_1, \dots, P_{t-2}, \nu_{red}$, the scheme $H \cap X$ has an ordinary node at each P_i, and H is tangent to $\tau(X) \setminus X$ only along the line $\langle \nu \rangle$.*

2. Preliminaries

Notation 1. Let Y be an integral quasi-projective variety and $Q \in Y_{reg}$. Let $\{kQ, Y\}$ denote the $(k-1)$-th infinitesimal neighborhood of Q in Y, i.e. the closed subscheme of Y with $(\mathcal{I}_Q)^k$ as its ideal sheaf. If $Y = \mathbb{P}^m$, then we write kQ instead of $\{kQ, \mathbb{P}^m\}$. The scheme $\{kQ, Y\}$ will be called a k-point of Y. We also say that a 2-point is a double point, that a 3-point is a triple point and a 4-point is a quadruple point.

We give here the definition of a $(2,3)$-point as it is in [6], p. 977.

Definition 1. Let $\mathfrak{q} \subset \mathbb{K}[x_0, \dots, x_m]$ be the reduced ideal of a simple point $Q \in \mathbb{P}^m$, and let $l \subset \mathbb{K}[x_0, \dots, x_m]$ be the ideal of a reduced line $L \subset \mathbb{P}^m$ through Q. We say that $Z(Q, L)$ is a $(2,3)$-point if it is the zero-dimensional scheme whose representative ideal is $(\mathfrak{q}^3 \cup l^2)$.

Remark 1. Notice that $2Q \subset Z(Q, L) \subset 3Q$.

We recall the notion of weak non-defectivity for an integral and non-degenerate projective variety $Y \subset \mathbb{P}^r$ (see [4]). For any closed subscheme $Z \subset \mathbb{P}^r$ set:

$$\mathcal{H}(-Z) := |\mathcal{I}_{Z,\mathbb{P}^r}(1)|. \tag{1}$$

Notation 2. Let $Z \subset \mathbb{P}^r$ be a zero-dimensional scheme such that $\{2Q, Y\} \subseteq Z$ for all $Q \in Z_{red}$. Fix $H \in \mathcal{H}(-Z)$ where $\mathcal{H}(-Z)$ is defined in (1). Let H_c be the closure in Y of the set of all $Q \in Y_{reg}$ such that $T_Q Y \subseteq H$.
The contact locus H_Z of H is the union of all irreducible components of H_c containing at least one point of Z_{red}.
We use the notation H_Z only in the case $Z_{red} \subset Y_{reg}$.

Fix an integer $k \geq 0$ and assume that $\sigma_{k+1}(Y)$ doesn't fill up the ambient space $\mathbb{P}^r$. Fix a general $(k+1)$-uple of points in Y i.e. $(P_0, \dots, P_k) \in Y^{k+1}$ and set

$$Z := \cup_{i=0}^{k} \{2P_i, Y\}. \tag{2}$$

The following definition of weakly k-defective variety coincides with the one given in [4].

Definition 2. A variety $Y \subset \mathbb{P}^r$ is said to be *weakly k-defective* if $\dim(H_Z) > 0$ for Z as in (2).

In [4], Theorem 1.4, it is proved that if $Y \subset \mathbb{P}^r$ is not weakly k-defective, then $H_Z = Z_{red}$ and that $\mathrm{Sing}(Y \cap H) = (\mathrm{Sing}(Y) \cap H) \cup Z_{red}$ for a general $Z = \cup_{i=0}^{k} \{2P_i, Y\}$ and a general $H \in \mathcal{H}(-Z)$. Notice that Y is weakly 0-defective if and only if its dual variety $Y^* \subset \mathbb{P}^{r*}$ is not a hypersurface.

In [5] the same authors considered also the case in which Y is not irreducible and hence its joins have as irreducible components the joins of different varieties.

Lemma 1. *Fix an integer $y \geq 2$, an integral projective variety Y, $L \in Pic(Y)$ and $P \in Y_{reg}$. Set $x := \dim(Y)$. Assume $h^0(Y, \mathcal{I}_{(y+1)P} \otimes L) = h^0(Y, L) - \binom{x+y-1}{x}$. Fix a general $F \in |\mathcal{I}_{yP} \otimes L|$. Then P is an isolated singular point of F.*

Proof. Let $u : Y' \to Y$ denote the blowing-up of P and $E := u^{-1}(P)$ the exceptional divisor. Since $\dim(Y) = x$, we have $E \cong \mathbb{P}^{x-1}$. Set $R := u^*(L)$. For each integer $t \geq 0$ we have $u_*(R(-tE)) \cong \mathcal{I}_{tP} \otimes L$. Thus the push-forward u_* induces an isomorphism between the linear system $|R(-tE)|$ on Y' and the linear system

$|\mathcal{I}_{tP} \otimes L|$ on Y. Set $M := R(-yE)$. Since $\mathcal{O}_{Y'}(E)|E \cong \mathcal{O}_E(-1)$ (up to the identification of E with $\mathbb{P}^{x-1}$), we have $R(-tE)|E \cong \mathcal{O}_E(t)$ for all $t \in \mathbb{N}$. Look at the exact sequence on Y':

$$0 \to M(-E) \to M \to \mathcal{O}_E(y) \to 0 \qquad (3)$$

By hypothesis $h^0(Y, \mathcal{I}_{yP} \otimes L) = h^0(Y, L) - \binom{x+y-2}{x}$. Thus our assumption implies $h^0(Y', M(-E)) = h^0(Y', M) - \binom{x+y-1}{x} + \binom{x+y-2}{x-1} = h^0(Y', M) - h^0(E, \mathcal{O}_E(y))$. Thus (3) gives the surjectivity of the restriction map $\rho : H^0(Y', M) \to H^0(E, M|E)$. Since $y \geq 0$, the line bundle $M|E$ is spanned. Thus the surjectivity of ρ implies that M is spanned at each point of E. Hence M is spanned in a neighborhood of E. Bertini's theorem implies that a general $F' \in |M|$ is smooth in a neighborhood of E. Since F is general and $|M| \cong |\mathcal{I}_{yP} \otimes L|$, P is an isolated singular point of F. □

3. Weak non-defectivity of $\tau(X, t)$

In this section we fix integers $m \geq 2$, $d \geq 3$ and set $N = \binom{m+d}{m} - 1$ and $X := X_{m,d}$. The variety $\tau(X)$ is 0-weakly defective, because a general tangent space of $\tau(X)$ is tangent to $\tau(X)$ along a line. Terracini's lemma for joins implies that a general tangent space of $\tau(X, t)$ is tangent to $\tau(X, t)$ at least along a line (see Remark 3). Thus $\tau(X, t)$ is weakly 0-defective. To handle this problem and prove Theorem 1 we introduce another definition, which is tailor-made to this particular case. As in [6] we want to work with zero-dimensional schemes on X, not on $\tau(X)$ or $\tau(X, t)$. We consider the corresponding notion in which $Y := X$, but Z is not the general disjoint union of t double points of X, but now Z is the general disjoint union of $t-2$ double points of $X := j_{m,d}(\mathbb{P}^m)$ and one $(2,3)$-points of $\mathbb{P}^m$ in the sense of [6] (see Definition 1). Also the double points will be seen as subschemes of $\mathbb{P}^m$. Notice that $\mathcal{H}(-\emptyset)$ (seen on $\mathbb{P}^m$) is just $|\mathcal{O}_{\mathbb{P}^m}(d)|$.

Remark 2. Fix $P \in X$ and $Q \in T_PX \setminus \{P\}$. Any two such pairs (P, Q) are projectively equivalent for the natural action of $\mathrm{Aut}(\mathbb{P}^m)$. We have $Q \in \tau(X)_{reg}$ and $T_Q\tau(X) \supset T_PX$. Set $D := \langle \{P, Q\} \rangle$. It is well-known that $D \setminus \{P\}$ is the the set of all $O \in \tau(X)_{reg}$ such that $T_Q\tau(X) = T_O\tau(X)$ (e.g. use that the set of all $g \in \mathrm{Aut}(\mathbb{P}^m)$ fixing P and the line containing P associated to the tangent vector induced by Q acts transitively on $T_PX \setminus D$).

Definition 3. Fix a general $(O_1, \dots, O_{t-2}, O) \in (\mathbb{P}^m)^{t-1}$ and a general line $L \subset \mathbb{P}^m$ such that $O \in L$. Set $Z := Z(O, L) \cup \bigcup_{i=1}^{t-2} 2O_i$. We say that the variety $\tau(X, t)$ is not *drip defective* if $\dim(H_Z) = 0$ for a general $H \in |\mathcal{I}_Z(d)|$.

We are now ready for the following lemma.

Lemma 2. *Fix an integer $t \geq 3$ such that $(m+1)t < n$. Let $Z_1 \subset \mathbb{P}^m$ be a general union of a quadruple point and $t-2$ double points. Let Z_2 be a general union of 2 triple points and $t-2$ double points. Fix a general disjoint union $Z = Z(O, L) \cup (\cup_{i=1}^{t-2} 2P_i)$, where $Z(O, L)$ is a $(2,3)$-point as in Definition 1 and O, L and $\{P_1, \dots, P_{t-2}\} \subset \mathbb{P}^m$ are general. Assume $h^1(\mathbb{P}^m, \mathcal{I}_{Z_1}(d)) = h^1(\mathbb{P}^m, \mathcal{I}_{Z_2}(d)) = 0$. Then:*

(i) $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) = 0$;

(ii) $\tau(X, t)$ is not drip defective;

(iii) a general $H \in \mathcal{H}(-Z)$ has an ordinary quadratic singularity at each P_i.

Proof. Set $W := 3O \cup (\cup_{i=1}^{t-2} 2P_i)$. The definition of a $(2,3)$-point gives that $Z(O, L) \subset 3O$. Thus $Z \subset W \subset Z_2$. Hence $h^1(\mathbb{P}^m, \mathcal{I}_Z(d)) \leq h^1(\mathbb{P}^m, \mathcal{I}_{Z_2}(d)) = 0$. Hence part (i) is proven.

To prove part (ii) of the lemma we need to prove that $\dim(H_Z) = 0$ for a general $H \in \mathcal{H}(-Z)$. Since $W \subsetneqq Z_1$ and $h^1(\mathbb{P}^m, \mathcal{I}_{Z_1}(d)) = 0$, we have $\mathcal{H}(-W) \neq \emptyset$. Since $W_{red} = Z_{red}$ and $Z \subset W$, to prove parts (ii) and (iii) of the lemma it is sufficient to prove $\dim((H_W)_c) = 0$ for a general $H_W \in \mathcal{H}(-W)$, where W is as above and $(H_W)_c$ is as in Notation 2. Assume that this is not true, therefore:

1. either the contact locus $(H_W)_c$ contains a positive-dimensional component J_i containing some of the P_i's, for $1 \leq i \leq t-2$,
2. or the contact locus $(H_W)_c$ contains a positive-dimensional irreducible component T containing Q.

Set $Z_3 := \cup_{i=1}^{t-3} 2P_i$ and $Z' := 3O \cup Z_3$.

(a) Here we assume the existence of a positive dimensional component $J_i \subset (H_W)_c$ containing one of the P_i's, say for example $J_{t-2} \ni P_{t-2}$. Thus a general $M \in |\mathcal{I}_W(d)|$ is singular along a positive-dimensional irreducible algebraic set containing P_{t-2}. Let $w : M \to \mathbb{P}^m$ denote the blowing-up of $\mathbb{P}^m$ at the points $O, P_1, \ldots, P_{t-3}$. Set $E_0 := w^{-1}(O)$ and $E_i := w^{-1}(P_i)$, $1 \leq i \leq t-3$. Let A be the only point of M such that $w(A) = P_{t-2}$. For each integer $y \geq 0$ we have $w_*(\mathcal{I}_{yA} \otimes w^*(\mathcal{O}_{\mathbb{P}^m}(d))(-3E_0 - 2E_1 - \cdots - 2E_{t-3})) = \mathcal{I}_{Z' \cup yP_{t-2}}(d)$. Applying Lemma 1 to the variety M, the line bundle $w^*(\mathcal{O}_{\mathbb{P}^m}(d))(-3E_0 - 2E_1 - \cdots - 2E_{t-3})$, the point A and the integer $y = 2$ we get a contradiction.

(b) Here we prove the non-existence of a positive-dimensional $T \subset (H_W)_c$ containing O. Let $w_1 : M_1 \to \mathbb{P}^m$ denote the blowing-up of $\mathbb{P}^m$ at the points $P_1, \ldots, P_{t-2}$. Set $E_i := w_1^{-1}(P_i)$, $1 \leq i \leq t-2$. Let $B \in M_1$ be the only point of M_1 such that $w_1(B) = O$. For each integer $y \geq 0$ we have $w_{1*}(\mathcal{I}_{yB} \otimes w_1^*(\mathcal{O}_{\mathbb{P}^m}(d))(-2E_1 - \cdots - 2E_{t-2})) = \mathcal{I}_{Z' \cup yO}(d)$. Since $h^1(\mathbb{P}^m, \mathcal{I}_{Z_2}(d)) = 0$ and $|\mathcal{I}_{Z_2}(d)| \subset |\mathcal{I}_Z(d)|$, Lemma 1 applied to the integer $y = 3$ gives a contradiction. $\square$

In [3], Lemmas 5 and 6, we proved the following two lemmas:

Lemma 3. *Fix integers $m \geq 2$ and $d \geq 5$. If $m \leq 4$, then assume $d \geq 6$. Set $\alpha := \lfloor \binom{m+d-1}{m}/(m+1) \rfloor$. Let $Z_i \subset \mathbb{P}^m$, $i = 1, 2$, be a general union of i triple points and $\alpha - i$ double points. Then $h^1(\mathcal{I}_{Z_i}(d)) = 0$.*

Lemma 4. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m}/(m+1) \rfloor$. Let $Z \subset \mathbb{P}^m$ be a general union of one quadruple point and $\beta - 1$ double points. Then $h^i(\mathcal{I}_Z(d)) = 0$.*

We will use the following set-up.

Notation 3. Fix any $Q \in \tau(X) \setminus X$. The point Q uniquely determines a point $E \in X$ and (up to a non-zero scalar) a tangent vector ν of X with $\nu_{red} = \{E\}$. We have $Q \in \langle \nu \rangle \setminus \{E\}$ and $T_Q\tau(X)$ is tangent to $\tau(X) \setminus X$ exactly along the line $\langle \nu \rangle = \langle \{E, Q\} \rangle$. Let $O \in \mathbb{P}^m$ be the only point such that $j_{n,d}(O) = E$. Let $u_O : \widetilde{X} \to \mathbb{P}^m$ be the blowing-up of O. Let $E := u_O^{-1}(O)$ denote the exceptional divisor. For all integers x, e set $\mathcal{O}_{\widetilde{X}}(x, eE) := u^*(\mathcal{O}_{\mathbb{P}^m}(x))(eE)$. Let $\mathcal{H}$ denote the linear system $|\mathcal{O}_{\widetilde{X}}(d, -3E)|$ on $\widetilde{X}$.

Remark 3. Since $d \geq 4$, the line bundle $\mathcal{O}_{\widetilde{X}}(d, -3E)$ is very ample, $u_*(\mathcal{O}_{\widetilde{X}}(d, -3E)) = \mathcal{I}_{3O}(1)$, $h^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(d, -3E)) = \binom{m+d}{m} - \binom{m+2}{3}$ and $h^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(d, -3E)) = 0$ for all $i > 0$.

Lemma 5. *The linear system $\mathcal{H}$ on $\widetilde{X}$ is not $(t-3)$-weakly defective. For a general $O_1, \dots, O_{t-2} \in \widetilde{X}$ a general $H \in |\mathcal{H}(-2O_1 - \cdots - 2O_{t-2})|$ is singular only at the points $O_1, \dots, O_{t-2}$ which are ordinary double points of H.*

Proof. Fix general $O_1, \dots, O_{t-2} \in \widetilde{X}$. Fix $j \in \{1, \dots, t-2\}$ and set $Z' := 3O_j \cup \bigcup_{i \neq j} 2O_i$, $Z'' := \cup_{i=1}^{t-2} 2O_i$ and $W := 3O_j \cup \bigcup_{i \neq j} 2O_i$. We have $u_*(\mathcal{I}_{Z'}(d, -3E)) \cong \mathcal{I}_{W \cup 3O}(1)$. The case $i = 2$ of Lemma 3 gives $h^1(\mathcal{I}_Z(d, -3E)) = 0$. Lemma 1 applied to a blowing-up of $\mathbb{P}^m$ at $\{O, O_1, \dots, O_{t-2}\} \setminus \{O_j\}$ shows that a general $H \in \mathcal{H}(-Z)$ has as an isolated singular point at O_j. Since this is true for all $j \in \{1, \dots, t-2\}$, $\mathcal{H}$ is not $(t-3)$-weakly defective (just by the definition of weak defectivity). The second assertion follows from the first one and [4], Theorem 1.4. $\square$

Now we can apply Lemmas 2, 3, 4 and 5 and get the following result.

Theorem 4. *Fix integers $m \geq 2$ and $d \geq 6$. If $m \leq 4$, then assume $d \geq 7$. Set $\beta := \lfloor \binom{m+d-2}{m}/(m+1) \rfloor$. Fix an integer t such that $2 \leq t \leq \beta + 1$. Then $\tau(X, t)$ is not drip defective.*

Proof. Fix general $P_1, \dots, P_{t-2}, O \in \mathbb{P}^m$ and a general line $L \subset \mathbb{P}^m$ such that $O \in L$. Set $Z := Z(O, L) \cup \bigcup_{i=1}^{t-2} 2P_i$, $W := 3O \cup \bigcup_{i=1}^{t-2} 2P_{t-2}$, $W' := 3O \cup 3O_1 \cup \bigcup_{i=2}^{t-2} 2P_{t-2}$ and $W'' := 4O \cup \bigcup_{i=1}^{t-2} 2P_{t-2}$. Take $O_i \in \widetilde{X}$ such that $u_O(O_i) = P_i$, $1 \leq i \leq t-2$. Since $u_{O*}(\mathcal{I}_{2O_1 \cup \cdots \cup 2O_{t-2}}(d, -4E)) \cong \mathcal{I}_W(d)$, Lemma 2 gives $h^1(\mathcal{I}_{2O_1 \cup \cdots \cup 2O_{t-2}}(d, -4E)) = 0$. Since $Z(O, L) \subset 3O$, the case $y = 3$ of Lemma 1 applied to the blowing-up of $\mathbb{P}^m$ at $O_1, \dots, O_{t-2}$ shows that a general $H \in |\mathcal{I}_W(d)|$ has an isolated singularity at O with multiplicity at most 3. $\square$

Recall that $\mathrm{Sing}(\tau(X)) = X$ and that for each $Q \in \tau(X) \setminus X$ there is a unique $O \in X$ and a unique tangent vector ν to X at O such that $Q \in \langle \nu \rangle$ and that $\langle \nu \rangle \setminus \{O\}$ is the contact locus of the tangent space $T_Q\tau(X)$ with $\tau(X) \setminus X$.

Let P be a general point of $\tau(X, t)$, i.e. fix a general $(P_1, \dots, P_{t-2}, Q) \in X^{t-2} \times \tau(X)$ and a general $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$.

Proof of Theorem 1. If $t = 2$, then Theorem 1 is true, because in this case the join is the join of a unique factor and $\dim(\tau(X)) = 2n$ (since $d \geq 2$). Thus we may assume $t \geq 3$. Fix a general $P \in \tau(X, t)$, say $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle$ with $(P_1, \dots, P_{t-2}, Q)$ general in $X^{t-2} \times \tau(X)$. Terracini's lemma for joins ([1], Corollary 1.10) gives $T_P\tau(X, t) = \langle T_{O_1}X \cup \cdots T_{O_{t-2}}X \cup T_Q\tau(X) \rangle$. Let O be the point of $\mathbb{P}^m$ such that $Q \in T_{j_{m,d}(O)}X$. Let $\mathcal{H}'$ (resp. $\mathcal{H}''$) be the set of all hyperplane $H \subset \mathbb{P}^N$ containing $T_Q\tau(X)$ (resp. $T_P\tau(X, t)$). We may see $\mathcal{H}'$ and $\mathcal{H}''$ as linear systems on the blowing-up $\widetilde{X}$ of $\mathbb{P}^m$ at O. We have $\mathcal{H}'' = \mathcal{H}'(-2P_1 - \cdots - 2P_{t-2})$ and $\mathcal{H} \subseteq \mathcal{H}'$, where $\mathcal{H}$ is defined in Notation 3. Take $O_i \in \widetilde{X}$, $1 \leq i \leq t-2$, such that $P_i = u(O_i)$ for all i. Since $(P_1, \dots, P_{t-2})$ is general in X^{t-2} for a fixed Q and $\mathcal{H} \subseteq \mathcal{H}'$, Lemma 5 gives that a general $H \in \mathcal{H}''$ intersects X in a divisor which, outside O, is singular only at $P_1, \dots, P_{t-2}$ and with an ordinary node at each P_i. Now assume $P \in \langle \{P'_1, \dots P'_{t-2}, Q'\} \rangle$ for some other $(P'_1, \dots, P'_{t-2}, Q') \in X^{t-2} \times \tau(X)$. Since P is general in $\tau(X, t)$ and $\tau(X, t)$ has the expected dimension, the $(t-1)$-ple $(P'_1, \dots, P'_{t-2}, Q')$ is general in $X^{t-2} \times \tau(X)$. Hence H is singular at each P'_i,

$1 \le i \le t-2$, and with an ordinary node at each P_i'. Since O is not an ordinary node of $H \cap X$, we get $\{P_1, \dots, P_{t-2}\} = \{P_1', \dots, P_{t-2}'\}$. Thus $O = O'$. Hence H is tangent to $\tau(X)_{reg}$ exactly along the line $\langle \{Q, O\} \rangle \setminus \{O\}$. Hence $Q' \in \langle \{Q, O\} \rangle$. Assume $Q \ne Q'$. Since P is general in $\tau(X, t)$, then $P \notin \tau(X, t-1)$. Hence $Q' \notin \langle \{O_1, \dots, O_{t-2}\} \rangle$ and $Q \notin \langle \{O_1, \dots, O_{t-2}\} \rangle$. Thus $\langle \{P_1, \dots, P_{t-2}, Q\} \rangle \cap \langle \{P_1, \dots, P_{t-2}, Q'\} \rangle = \langle \{P_1, \dots, P_{t-2}\} \rangle$ if $Q \ne Q'$. Since $P \in \langle \{P_1, \dots, P_{t-2}, Q\} \rangle \cap \langle \{P_1, \dots, P_{t-2}, Q'\} \rangle$, we got a contradiction. $\square$

Proof of Theorem 3. The case $t = 2$ is well-known and follows from the following fact: for any $O \in X$ and any $Q \in T_O X \setminus \{O\}$ the group $G_O := \{g \in \mathrm{Aut}(\mathbb{P}^n) : g(O) = O\}$ acts on $T_O X$ and the stabilizer $G_{O,Q}$ of Q for this action is the line $\langle \{O, Q\} \rangle$, while $T_O X \setminus \langle \{O, Q\} \rangle$ is another orbit for $G_{O,Q}$. Thus we may assume $t \ge 3$. Fix a general $P \in \tau(X, t)$ and a general hyperplane $H \supset T_P \tau(X, t)$. If H is tangent to $\tau(X)$ at a point $Q' \in \tau(X) \setminus X$, then it is tangent along a line containing Q'. Let $E \in X$ be the only point such that $Q' \in T_E X$. We get $T_E X \subset \tau(X, t)$ and that $H \cap T_E X$ is larger than the double point $2E \subset X$. Theorem 1 gives that Q, Q' and E are collinear, i.e H is tangent only along the line ν. $\square$

References

[1] B. Ådlandsvik, Joins and higher secant varieties. Math. Scand. **61** (1987), no. 2, 213–222.

[2] E. Ballico, On the weak non-defectivity of Veronese embeddings of projective spaces, Cent. Eur. J. Math. **3** (2005), no. 2, 183–187.

[3] E. Ballico and A. Bernardi, Partial stratification of secant varieties of Veronese varieties via curvilinear subschemes, arXiv: 1010.3546v1 [math.AG].

[4] L. Chiantini and C. Ciliberto, Weakly defective varieties. Trans. Amer. Math. Soc. **454** (2002), no. 1, 151–178.

[5] L. Chiantini and C. Ciliberto, On the dimension of secant varieties, J. Eur. Math. Soc. **12** (2010), no. 5, 1267–1291.

[6] M. V. Catalisano, A. V. Geramita and A. Gimigliano, On the secant varieties to the tangential varieties of a Veronesean. Proc. Amer. Math. Soc. **130** (2002), no. 4, 975–985.

[7] C. Ciliberto and F. Russo, Varieties with minimal secant degree and linear systems of maximal dimension on surfaces. Adv. Math. **206** (2006), no. 1, 1–50.

[8] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[9] M. Mella, Singularities of linear systems and the Waring problem. Trans. Amer. Math. Soc. **358** (2006), no. 12, 5523–5538.

[10] M. Mella, Base loci of linear systems and the Waring problem. Proc. Amer. Math. Soc. **137** (2009), no. 1, 91–98.

Dept. of Mathematics, University of Trento, 38123 Povo (TN), Italy

E-mail address: `ballico@science.unitn.it`

INRIA (Institut National de Recherche en Informatique et en Automatique) Project Galaad 2004 route del Lucioles, B.P. 93 06902 Sophia Antipolis Cedex FRANCE

E-mail address: `alessandra.bernardi@inria.fr`