FILTRATIONS INDUCED BY CONTINUOUS FUNCTIONS

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ABSTRACT. In Persistent Homology and Topology, filtrations are usually given by introducing an ordered collection of sets or a continuous function from a topological space to $\mathbb{R}^n$. A natural question arises, whether these approaches are equivalent or not. In this paper we study this problem and prove that, while the answer to the previous question is negative in the general case, the approach by continuous functions is not restrictive with respect to the other, provided that some natural stability and completeness assumptions are made. In particular, we show that every compact and stable 1-dimensional filtration of a compact metric space is induced by a continuous function. Moreover, we extend the previous result to the case of multi-dimensional filtrations, requiring that our filtration is also complete. Three examples show that we cannot drop the assumptions about stability and completeness.

INTRODUCTION

The concept of filtration is the start point for Persistent Topology and Homology. Actually, the main goal of these theories is to examine the topological and homological changes that happen when we go through a family of spaces that is totally ordered with respect to inclusion [4]. In literature, filtrations are usually given in two ways. The former consists of explicitly introducing a nested collection of sets (usually carriers of simplicial complexes), the latter of giving a continuous function from a topological space to $\mathbb{R}$ or $\mathbb{R}^n$, whose sub-level sets represent the elements of the considered filtration (cf., e.g., [3, 5]). An example of these two types of filtrations is shown in Figure 1. The two considered methods have produced two different approaches to study the concept of persistence. A natural question arises, whether these approaches are equivalent or not.

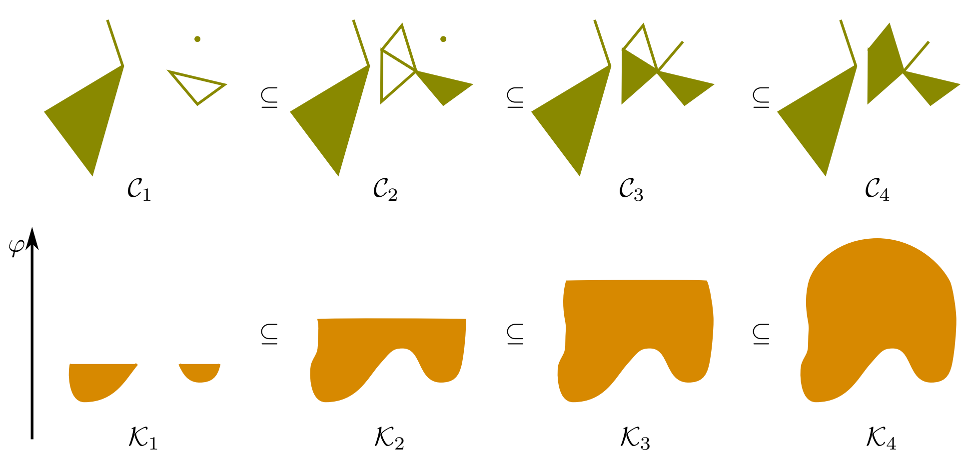

FIGURE 1. Examples of filtrations. First row: an ordered collection of simplicial complexes $\{\mathcal{C}_i\}$. Second row: the sub-level sets $\{\mathcal{K}_i\}$ of a real-valued continuous function φ.

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In this paper we study this problem and prove that, while the answer to the previous question is negative in the general case, the approach by continuous functions is not restrictive with respect to the other, provided that some natural stability and completeness assumptions are made.

The concept of stability of a filtration is motivated by the fact that in real applications we need to work with methodologies that are robust in the presence of noise. As a consequence, we have to require that the inclusions considered in our filtration persist under the action of small perturbations. For the same reason, we also need that a small change of the parameter in our filtration (whenever applicable) does not produce a large change of the associated set with respect to the Hausdorff distance. These assumptions are formalized by our definition of stable filtration (Definition 2.1).

In order to make our treatment as general as possible, we just require that the sets $\mathcal{K}_i$ ($i \in I$) in our filtration are compact subsets of a compact metric space K, and that the indexing set I is compact.

The paper starts by considering filtrations indexed by a 1-dimensional parameter. In this setting, after proving some lemmas, we show that every compact and stable 1-dimensional filtration of a compact metric space is induced by a continuous function (Theorem 2.8). In the last part of the paper, this result is extended to the case of multi-dimensional filtrations (Theorem 3.4), i.e. the case of filtrations indexed by an n-dimensional parameter (cf. [1, 2]). In order to do that, we need to assume also that our filtration is complete, i.e. compatible with respect to intersection (Definition 3.2). Three examples show that we cannot drop either the assumption about stability or the one concerning completeness (Examples 1, 2 and 3).

1. Preliminaries

In this section we give the preliminary concepts and the notation that will be used throughout the paper.

Let (K, d) be a non-empty compact metric space. Let us denote by $Comp(K)$ the set $\{\mathcal{K} : \mathcal{K} \text{ compact in } K\}$. Let us consider the Hausdorff distance d_H on $Comp(K) \setminus \{\emptyset\}$. Moreover, let I be a non-empty subset of $\mathbb{R}^n$ such that $I = I_1 \times I_2 \times \ldots \times I_n$. The following relation $\preceq$ is defined in I: for $i = (i_1, \ldots, i_n), i' = (i'_1, \ldots, i'_n) \in I$, we say $i \preceq i'$ if and only if $i_r \leq i'_r$ for every $r = 1, \ldots, n$.

Definition 1.1. An n-dimensional *filtration* of K is a family $\{\mathcal{K}_i \in Comp(K)\}_{i \in I}$ such that, $\emptyset, K \in \{\mathcal{K}_i\}_{i \in I}$, and $\mathcal{K}_i \subseteq \mathcal{K}_{i'}$ for every $i, i' \in I$, with $i \preceq i'$.

Definition 1.2. An n-dimensional filtration $\{\mathcal{K}_i\}_{i \in I}$ of K is *induced by a function* $\vec{\varphi} : K \to \mathbb{R}^n$ if $\mathcal{K}_i = \{P \in K : \vec{\varphi}(P) \preceq i\}$ for every $i \in I$.

Definition 1.3. We shall call *compact*, or *finite* any filtration $\{\mathcal{K}_i\}_{i \in I}$ with $I = I_1 \times I_2 \times \ldots \times I_n$ a compact, or finite subset of $\mathbb{R}^n$, respectively.

Remark 1.4. When I is bounded, the assumption that $\emptyset, K \in \{\mathcal{K}_i\}_{i \in I}$ is not so restrictive, since each family of compact sets verifying the last property in Definition 1.1 can be extended to a family containing $\emptyset$ and K, without losing that property. This assumption allows us a more concise exposition.

2. Mono-dimensional filtrations

This section is devoted to prove our main result in the case of filtrations indexed by a 1-dimensional parameter (Theorem 2.8). Therefore, in what follows, the symbol I will denote a non-empty subset of $\mathbb{R}$.

For every subset $X \subseteq K$, let us denote by $\overline{X}$, $int(X)$, ∂X, and X^c the closure, the interior, the boundary, and the complement of X in K, respectively. We recall that $int(X)^c = \overline{X^c}$.

Definition 2.1. We shall say that a compact 1-dimensional filtration $\{\mathcal{K}_i\}_{i\in I}$ of K is *stable with respect to the metric* d if the following statements hold:

(a) The functions $i \mapsto \mathcal{K}_i$ and $i \mapsto \overline{\mathcal{K}_i^c}$ are continuous, i.e. if $(i_m \in I)_{m\in\mathbb{N}}$ is a sequence converging to $\bar{\imath} \in I$, the sequence $(\mathcal{K}_{i_m})$ converges to $\mathcal{K}_{\bar{\imath}}$, and the sequence $(\overline{\mathcal{K}_{i_m}^c})$ converges to $(\overline{\mathcal{K}_{\bar{\imath}}^c})$ with respect to the Hausdorff distance d_H.

(b) For every set $\mathcal{K}_i$ and every $j \in I$ with $i < j$, $\mathcal{K}_i \subseteq int(\mathcal{K}_j)$.

Remark 2.2. Let us note that in the case $\{\mathcal{K}_i\}_{i\in I}$ is a finite 1-dimensional filtration of K, Definition 2.1 reduces to Definition 2.1 (b).

Remark 2.3. We observe that, in Definition 2.1 (a), the convergence of the sequence $(\mathcal{K}_{i_m})$ does not imply the convergence of the sequence $\left(\overline{\mathcal{K}_{i_m}^c}\right)$. Indeed, for example, let us consider the following compact filtration of the set $K = [0,1] \cup \{2\}$ (K is endowed with the Euclidean metric). We take $I = \{-1\} \cup [0,1] \cup \{2\}$ and set $\mathcal{K}_i = [0,i]$ for $i \in [0,1]$, while $\mathcal{K}_{-1} = \emptyset$ and $\mathcal{K}_2 = K$. It is immediate to check that the sequence $\left(\mathcal{K}_{1-1/m}\right)$ converges to $\mathcal{K}_1$, but the sequence $\left(\overline{\mathcal{K}_{1-1/m}^c}\right)$ does not converge to $\overline{\mathcal{K}_1^c}$.

The following Lemmas 2.4–2.7 provide meaningful characterizations of two functions $\alpha, \beta : K \to I$ which turn out to be useful in the proof of our main result.

Lemma 2.4. *Let $\{\mathcal{K}_i\}_{i\in I}$ be a compact and stable 1-dimensional filtration of K. For every $P \in K$, let $A(P) = \{i \in I, P \in \overline{\mathcal{K}_i^c}\} = \{i \in I, P \notin int(\mathcal{K}_i)\}$ and $B(P) = \{i \in I, P \in \mathcal{K}_i\}$. Then $A(P)$ and $B(P)$ are non-empty subsets of I. Moreover,* $\sup A(P) \in A(P)$ *and* $\inf B(P) \in B(P)$.

Proof. First of all let us observe that both $A(P)$ and $B(P)$ are non-empty because $i_{\min} = \min\{i \in I\} \in A(P)$ since $\emptyset = \mathcal{K}_{i_{\min}} \in \{\mathcal{K}_i\}_{i\in I}$, and $i_{\max} = \max\{i \in I\} \in B(P)$ since $K = \mathcal{K}_{i_{\max}} \in \{\mathcal{K}_i\}_{i\in I}$.

Let $\alpha(P) = \sup A(P)$. Because of the compactness of I, $\alpha(P) \in I$ and is finite. Let us show that $\alpha(P) \in A(P)$. Let (i_r) be a non-decreasing sequence of indices of $A(P)$ converging to $\alpha(P)$. From Definition 2.1 (a), it follows that $(\overline{\mathcal{K}_{i_r}^c})$ converges to $\overline{\mathcal{K}_{\alpha(P)}^c}$. We have to prove that $\alpha(P) \in A(P)$, i.e. $P \in \overline{\mathcal{K}_{\alpha(P)}^c}$. By contradiction, let us assume that $P \notin \overline{\mathcal{K}_{\alpha(P)}^c}$. Since $\overline{\mathcal{K}_{\alpha(P)}^c}$ is compact, $d_H(\{P\}, \overline{\mathcal{K}_{\alpha(P)}^c}) > 0$. Therefore, for any large enough index r, the inequality $d_H(\overline{\mathcal{K}_{\alpha(P)}^c}, \overline{\mathcal{K}_{i_r}^c}) < d_H(\{P\}, \overline{\mathcal{K}_{\alpha(P)}^c})$ holds. Hence $d_H(\{P\}, \overline{\mathcal{K}_{i_r}^c}) > 0$ for any large enough index r, contrarily to our assumption that $i_r \in A(P)$, i.e. $P \in \overline{\mathcal{K}_{i_r}^c}$.

Let $\beta(P) = \inf B(P)$. Because of the compactness of I, $\beta(P) \in I$ and is finite. Let us show that $\beta(P) \in B(P)$. Let (i_r) be a non-increasing sequence of indices of $B(P)$ converging to $\beta(P)$. From Definition 2.1 (a), it follows that $(\mathcal{K}_{i_r})$ converges to $\mathcal{K}_{\beta(P)}$. We have to prove that $\beta(P) \in B(P)$, i.e. $P \in \mathcal{K}_{\beta(P)}$. By contradiction, let us assume that $P \notin \mathcal{K}_{\beta(P)}$. Since $\mathcal{K}_{\beta(P)}$ is compact, $d_H(\{P\}, \mathcal{K}_{\beta(P)}) > 0$. Therefore, for any large enough index r, the inequality $d_H\left(\mathcal{K}_{\beta(P)}, \mathcal{K}_{i_r}\right) < d_H\left(\{P\}, \mathcal{K}_{\beta(P)}\right)$ holds. Hence $d_H(\{P\}, \mathcal{K}_{i_r}) > 0$ for any large enough index r, contrarily to our assumption that $i_r \in B(P)$, i.e. $P \in \mathcal{K}_{i_r}$. □

By virtue of the above Lemma 2.4, for every $P \in K$, we can define $\alpha(P) = \max A(P) \in I$ and $\beta(P) = \min B(P) \in I$. In plain words, for every $P \in K$, $\mathcal{K}_{\alpha(P)}$ is the largest compact $\mathcal{K}_i$ in the filtration such that $P \in \overline{\mathcal{K}_i^c} = int\left(\mathcal{K}_i\right)^c$, while

$\mathcal{K}_{\beta(P)}$ is the smallest compact $\mathcal{K}_j$ in the filtration such that $P \in \mathcal{K}_j$. In particular, $P \in \overline{\mathcal{K}^c_{\alpha(P)}} \cap \mathcal{K}_{\beta(P)}$.

Lemma 2.5. *Let $\{\mathcal{K}_i\}_{i\in I}$ be a compact and stable 1-dimensional filtration of K. Then the following statements hold:*

(1) *$\alpha(P) \le \beta(P)$ for every $P \in K$.*
(2) *If $P, Q \in K$ and $\alpha(P) < \alpha(Q)$, then $\beta(P) \le \alpha(Q)$.*
(3) *If $P, Q \in K$ and $\beta(P) < \beta(Q)$, then $\beta(P) \le \alpha(Q)$.*

Proof.

(1) To show that $\alpha(P) \le \beta(P)$, let us verify that, if $i_1 \in A(P)$ (i.e. $P \in \overline{\mathcal{K}^c_{i_1}}$) and $i_2 \in B(P)$ (i.e. $P \in \mathcal{K}_{i_2}$), then $i_1 \le i_2$. By contradiction, let us assume that $i_2 < i_1$. Then Definition 2.1 (b) implies that $\mathcal{K}_{i_2} \subseteq int(\mathcal{K}_{i_1})$. Since $P \in \mathcal{K}_{i_2}$, it follows that $P \in int(\mathcal{K}_{i_1})$, i.e. $P \notin \overline{\mathcal{K}^c_{i_1}}$, against the assumption $i_1 \in A(P)$.
(2) Since $\alpha(P) < \alpha(Q)$, it follows that $P \in int(\mathcal{K}_{\alpha(Q)})$, while $P \notin int\left(\mathcal{K}_{\alpha(P)}\right)$. In particular, $P \in \mathcal{K}_{\alpha(Q)}$. Therefore $\alpha(Q) \in B(P)$ and hence $\beta(P) \le \alpha(Q)$.
(3) Since $\beta(P) < \beta(Q)$, it follows that $Q \notin \mathcal{K}_{\beta(P)}$, while $Q \in \mathcal{K}_{\beta(Q)}$. In particular, $Q \notin int\left(\mathcal{K}_{\beta(P)}\right)$. Therefore $\beta(P) \in A(Q)$ and hence $\beta(P) \le \alpha(Q)$.

□

Remark 2.6. Let us observe that under the assumptions of compactness and stability of $\{\mathcal{K}_i\}_{i\in I}$, it follows that, for every $P \in K$ with $P \in \partial\mathcal{K}_i$ for a certain $i \in I$, $\alpha(P) = \beta(P) = i$. Indeed, from Lemma 2.5 (1), we have $\alpha(P) \le \beta(P)$ for every $P \in K$. On the other side, since $P \in \partial\mathcal{K}_i$ implies both that $P \in \mathcal{K}_i$, whence $\beta(P) \le i$, and that $P \notin int(\mathcal{K}_i)$, whence $\alpha(P) \ge i$, the equality is proved.

Lemma 2.7. *Let $\{\mathcal{K}_i\}_{i\in I}$ be a compact and stable 1-dimensional filtration of K. Then the following statements hold:*

(1) *The function α is everywhere upper semi-continuous.*
(2) *The function β is everywhere lower semi-continuous.*

Proof. Let us consider a sequence (P_r) of points in K converging to a point $P \in K$.

(1) Let $(\alpha(P_{r_k}))$ be a converging subsequence of $(\alpha(P_r))$. Let us set $L \stackrel{def}{=} \lim_k \alpha(P_{r_k})$. From the compactness of I, $L \in I$, and from Definition 2.1 (a), the sequence $(\overline{\mathcal{K}^c_{\alpha(P_{r_k})}})$ converges to the compact set $\overline{\mathcal{K}^c_L}$ with respect to d_H. Since $P = \lim_k P_{r_k}$, and $P_{r_k} \in \overline{\mathcal{K}^c_{\alpha(P_{r_k})}}$, we have that $P \in \overline{\mathcal{K}^c_L}$, and hence $\alpha(P) \ge L$. Therefore, the function α is everywhere upper semi-continuous.
(2) Let $(\beta(P_{r_k}))$ be a converging subsequence of $(\beta(P_r))$. Let us set $L \stackrel{def}{=} \lim_k \beta(P_{r_k})$. From the compactness of I, $L \in I$, and from Definition 2.1 (a), the sequence $(\mathcal{K}_{\beta(P_{r_k})})$ converges to the compact set $\mathcal{K}_L$ with respect to d_H. Since $P = \lim_k P_{r_k}$, and $P_{r_k} \in \mathcal{K}_{\beta(P_{r_k})}$, we have that $P \in \mathcal{K}_L$, and hence $\beta(P) \le L$. Therefore, the function β is everywhere lower semi-continuous.

□

Theorem 2.8. *Every compact and stable* 1*-dimensional filtration $\{\mathcal{K}_i\}_{i\in I}$ of a compact metric space K is induced by a continuous function $\varphi : K \to \mathbb{R}$.*

Proof. If $\{\mathcal{K}_i\}_{i\in I} = \{\mathcal{K}_{i_{\min}} = \emptyset, \mathcal{K}_{i_{\max}} = K\}$, then we can just take $\varphi : K \to \mathbb{R}$ such that $\varphi(P) = i_{\max}$ for every $P \in K$. This function is continuous and induces $\{\mathcal{K}_i\}_{i\in I}$.

Let us consider a proper filtration, i.e. a filtration $\{\mathcal{K}_i\}_{i\in I}$ such that at least one index $i' \in I$ exists with $i_{\min} < i' < i_{\max}$. We want to prove that there exists a continuous function inducing it.

Let us observe that $\mathcal{K}_{i_{\min}} = \emptyset$ and, because of the compactness of I, the value $i_1 = \inf\{I \setminus \{i_{\min}\}\} \le i'$ must belong to I. The empty set cannot be the limit of a sequence of compact non-empty sets with respect to the Hausdorff distance. Hence it must be $i_1 > i_{\min}$. Furthermore, $\overline{\mathcal{K}^c_{i_{\max}}} = \overline{K^c} = \emptyset$ and, because of the compactness of I, the value $i_2 = \sup\{I \setminus \{i_{\max}\}\} \ge i'$ must belong to I. The empty set cannot be the limit of a sequence of compact non-empty sets with respect to the Hausdorff distance. Hence it must be $\mathcal{K}_{i_2} \ne \mathcal{K}_{i_{\max}}$, so that $i_2 < i_{\max}$.

Now, let us fix a point $R \in int(\mathcal{K}^c_{i_2})$. Moreover, since our filtration is proper, we have that no point $P \in K$ exists such that $\alpha(P) = i_{\min}$ and $\beta(P) = i_{\max}$. Hence, for every $P \in K$, let us define the function $\varphi : K \to \mathbb{R}$ as follows, by recalling the inequality in Lemma 2.5 (1):

$$\varphi(P) = \begin{cases} \beta(P) & \text{if } i_{\min} = \alpha(P) \\ \dfrac{\alpha(P)\cdot d_H\left(\{P\},\overline{\mathcal{K}^c_{\beta(P)}}\right) + \beta(P)\cdot d_H\left(\{P\},\mathcal{K}_{\alpha(P)}\right)}{d_H\left(\{P\},\overline{\mathcal{K}^c_{\beta(P)}}\right) + d_H\left(\{P\},\mathcal{K}_{\alpha(P)}\right)} & \text{if } i_{\min} < \alpha(P) \le \beta(P) < i_{\max} \\ \dfrac{\alpha(P)\cdot d_H\left(\{P\},\{R\}\right) + \beta(P)\cdot d_H\left(\{P\},\mathcal{K}_{\alpha(P)}\right)}{d_H\left(\{P\},\{R\}\right) + d_H\left(\{P\},\mathcal{K}_{\alpha(P)}\right)} & \text{if } \beta(P) = i_{\max} \end{cases}$$

Before proceeding, we observe that $\alpha(P) \le \varphi(P) \le \beta(P)$ in all of the three cases in the definition of φ.

Let us prove that $\mathcal{K}_i = \{P \in K, \varphi(P) \le i\}$ for every $i \in I$.

Let us fix an index $i \in I$. If $P \in \mathcal{K}_i$ then $\beta(P) \le i$. Hence, according to the observation above, $\varphi(P) \le \beta(P) \le i$. Varying $i \in I$, this proves that $\mathcal{K}_i \subseteq \{P \in K, \varphi(P) \le i\}$ for every $i \in I$.

Let us show that $\mathcal{K}_i \supseteq \{P \in K, \varphi(P) \le i\}$ for every $i \in I$. If $P \notin \mathcal{K}_i$ then $P \in \mathcal{K}^c_i$, and hence $P \in \overline{\mathcal{K}^c_i}$, so that $i \le \alpha(P)$. Since $P \notin \mathcal{K}_i$, it follows that $\beta(P) > i$. Then, in all of the three cases in the definition of φ it is easy to show that $\varphi(P) > i$. Therefore, in any case it results that $\varphi(P) > i$.

Now, let us show that φ is continuous at any point $P \in K$.

First of all, let us examine the case $\alpha(P) = i_{\min}$ and the case $\beta(P) = i_{\max}$.

If $\alpha(P) = i_{\min}$ then (since $i_1 > i_{\min}$) $\beta(P) = i_1$, and $P \in int(\mathcal{K}_{i_1})$ because of Remark 2.6. So, there exists a neighborhood U of P such that $U \subseteq int(\mathcal{K}_{i_1})$. It follows that for any point $Q \in U$ the equalities $\alpha(Q) = i_{\min}$ and $\beta(Q) = i_1$ hold.

If $\beta(P) = i_{\max}$ then (since $i_2 < i_{\max}$) $\alpha(P) = i_2$, and $P \in int(\mathcal{K}^c_{i_2})$ because of Remark 2.6. So, there exists a neighborhood U of P such that $U \subseteq int(\mathcal{K}^c_{i_2})$. It follows that for any point $Q \in U$ the equalities $\alpha(Q) = i_2$ and $\beta(Q) = i_{\max}$ hold.

In both cases, φ is continuous at P.

In the rest of the proof, we shall assume that $i_{\min} < \alpha(P)$ and $\beta(P) < i_{\max}$.

In order to prove that φ is continuous at P, it will be sufficient to show that, if a sequence (P_r) converges to P and the sequence $(\varphi(P_r))$ is converging, then $\lim_r \varphi(P_r) = \varphi(P)$. This is due to the boundness of $\varphi(K)$.

Therefore, in what follows we shall assume that the sequences (P_r) and $(\varphi(P_r))$ are converging.

We recall that every real sequence admits either a strictly monotone or a constant subsequence. Hence, by possibly extracting a subsequence from (P_r) we can assume that each of the sequences $(\alpha(P_r)), (\beta(P_r))$ is either strictly monotone or constant. Obviously, this choice does not change the limits of the sequences (P_r) and $(\varphi(P_r))$. Let us consider the following two cases:

Case that $(\beta(P_r))$ is strictly monotone: If $(\beta(P_r))$ is strictly decreasing, then Lemma 2.5 (3) assures that $\beta(P_{r+1}) \leq \alpha(P_r)$. As a consequence,

$$\varphi(P_{r+1}) \leq \beta(P_{r+1}) \leq \alpha(P_r) \leq \varphi(P_r).$$

If $(\beta(P_r))$ is strictly increasing, then Lemma 2.5 (3) assures that $\beta(P_r) \leq \alpha(P_{r+1})$. As a consequence,

$$\varphi(P_r) \leq \beta(P_r) \leq \alpha(P_{r+1}) \leq \varphi(P_{r+1}).$$

In both cases, since the sequence $(\varphi(P_r))$ is converging, also the sequences $(\alpha(P_r))$, $(\beta(P_r))$ are converging and $\lim_r \alpha(P_r) = \lim_r \beta(P_r) = \lim_r \varphi(P_r)$. Let us call ℓ this limit.

The upper semi-continuity of the function α and the lower semi-continuity of the function β (Lemma 2.7) imply that $\alpha(P) \geq \ell \geq \beta(P)$. We already know that $\alpha(P) \leq \varphi(P) \leq \beta(P)$, and hence $\alpha(P) = \varphi(P) = \beta(P) = \ell$. Therefore, $\varphi(P) = \lim_r \varphi(P_r)$.

Case that $(\beta(P_r)) = L$ for every index r: If each element in the sequence $(\beta(P_r))$ is equal to a constant L then we know that, from the lower semi-continuity of β (Lemma 2.7 (2)), $\beta(P) \leq L \leq i_{max}$.

- If $\beta(P) < L$, then there is no $h \in I$ such that $\beta(P) < h < L$. Indeed, if such an index h existed, Definition 2.1 (b) would imply that $P \in \mathcal{K}_{\beta(P)} \subseteq int(\mathcal{K}_h)$. Since $P = \lim_r P_r$, we would have that $P_r \in \mathcal{K}_h$ for every large enough index r. As a consequence, the inequality $\beta(P_r) \leq h < L$ would hold, against the assumption $\beta(P_r) = L$ for every index r.

  Lemma 2.5 (1) assures that $\alpha(P_r) \leq \beta(P_r) = L$ for every index r. Then, since $(\alpha(P_r))$ is strictly monotone or constant, either $\alpha(P_r) = L$ for every index r or $\alpha(P_r) \leq \beta(P)$ for every index r. We observe that the case $\alpha(P_r) < \beta(P)$ cannot happen. Indeed, if the inequality $\alpha(P_r) < \beta(P)$ held, then the definition of α would imply that $P_r \in int\left(\mathcal{K}_{\beta(P)}\right) \subseteq \mathcal{K}_{\beta(P)}$. As a consequence, the inequality $L = \beta(P_r) \leq \beta(P)$ would hold, against the assumption $\beta(P) < L$.

  In summary, if $\beta(P) < L$, then either $\alpha(P_r) = L$ for every index r or $\alpha(P_r) = \beta(P)$ for every index r, i.e. $(\alpha(P_r))$ is a constant sequence.

  Let us consider the following two subcases:
  - Subcase $\alpha(P_r) = \beta(P_r) = L > \beta(P)$ for every r: In this case, the upper semi-continuity of α implies that $\alpha(P) \geq \lim_r \alpha(P_r) = L$, and hence that $\alpha(P) > \beta(P)$, contradicting Lemma 2.5 (1). So this case is impossible.
  - Subcase $\alpha(P_r) = \beta(P) < \beta(P_r) = L$ for every r: In this case, the upper semi-continuity of α implies that $\alpha(P) \geq \lim_r \alpha(P_r) = \beta(P)$. Since Lemma 2.5 (1) states that $\alpha(P) \leq \beta(P)$, we have $\alpha(P) = \beta(P)$. In summary, in this case, $\alpha(P_r) = \alpha(P) = \beta(P) < \beta(P_r) = L$ for every index r. From the definition of the function φ, it follows that $\varphi(P) = \alpha(P) = \beta(P)$. Moreover, since $\beta(P_r) = L \leq i_{\max}$ for every index r, the two cases below must be considered:
    If $L < i_{\max}$, then

$$\begin{aligned}\varphi(P_r) &= \frac{\alpha(P_r) \cdot d_H\left(\{P_r\}, \overline{\mathcal{K}^c_{\beta(P_r)}}\right) + \beta(P_r) \cdot d_H\left(\{P_r\}, \mathcal{K}_{\alpha(P_r)}\right)}{d_H\left(\{P_r\}, \overline{\mathcal{K}^c_{\beta(P_r)}}\right) + d_H\left(\{P_r\}, \mathcal{K}_{\alpha(P_r)}\right)} \\ &= \frac{\beta(P) \cdot d_H\left(\{P_r\}, \overline{\mathcal{K}^c_L}\right) + L \cdot d_H\left(\{P_r\}, \mathcal{K}_{\beta(P)}\right)}{d_H\left(\{P_r\}, \overline{\mathcal{K}^c_L}\right) + d_H\left(\{P_r\}, \mathcal{K}_{\beta(P)}\right)}.\end{aligned}$$

If $L = i_{\max}$, then

$$\begin{aligned}\varphi(P_r) &= \frac{\alpha(P_r)\cdot d_H\left(\{P_r\},\{R\}\right)+\beta(P_r)\cdot d_H\left(\{P_r\},\mathcal{K}_{\alpha(P_r)}\right)}{d_H\left(\{P_r\},\{R\}\right)+d_H\left(\{P_r\},\mathcal{K}_{\alpha(P_r)}\right)}\\ &= \frac{\beta(P)\cdot d_H\left(\{P_r\},\{R\}\right)+i_{\max}\cdot d_H\left(\{P_r\},\mathcal{K}_{\beta(P)}\right)}{d_H\left(\{P_r\},\{R\}\right)+d_H\left(\{P_r\},\mathcal{K}_{\beta(P)}\right)},\end{aligned}$$

with R a point in $int(\mathcal{K}^c_{i_2})$.

Since $P \in \mathcal{K}_{\beta(P)}$ and $\lim_r P_r = P$, we have $\lim_r d_H(\{P_r\},\mathcal{K}_{\beta(P)}) = 0$. Furthermore, if $L < i_{\max}$, then $\lim_r d_H(\{P_r\},\overline{\mathcal{K}^c_L}) = d_H(\{P\},\overline{\mathcal{K}^c_L})$; if $L = i_{\max}$, then $\lim_r d_H(\{P_r\},\{R\}) = d_H(\{P\},\{R\})$. Therefore, in both cases, $\lim_r \varphi(P_r) = \beta(P) = \varphi(P)$, i.e. φ is continuous at P.

- If $\beta(P) = L$, then $L < i_{max}$ (since we are assuming $\beta(P) < i_{\max}$). Recalling that $(\alpha(P_r))$ is either a strictly monotone or a constant bounded sequence, let $L' = \lim_r \alpha(P_r)$.

If the sequence $(\alpha(P_r))$ were strictly monotone, we could find two indexes r_1, r_2 such that $\alpha(P_{r_1}), \alpha(P_{r_2}) \neq L$ and $\alpha(P_{r_1}) < \alpha(P_{r_2})$. Lemma 2.5 assures that $\beta(P_{r_1}) \leq \alpha(P_{r_2}) \leq \beta(P_{r_2})$. Since $\beta(P_{r_1}) = \beta(P_{r_2}) = L$, it follows that $\alpha(P_{r_2}) = L$, against our assumption that $\alpha(P_{r_1}), \alpha(P_{r_2}) \neq L$. Therefore, the sequence $(\alpha(P_r))$ must be constant.

In summary, if $\beta(P_r) = \beta(P) = L$ for every index r, then $\alpha(P_r) = L'$ for every index r.

Since the function α is upper semi-continuous (Lemma 2.7 (1)), we have that $\alpha(P) \geq L'$. If the inequality $\alpha(P) > L'$ holds, then $\alpha(P_r) < \alpha(P)$ for every index r. Lemma 2.5 (2) assures that $\beta(P_r) \leq \alpha(P)$, and hence $\alpha(P) \geq L$. Lemma 2.5 (1) assures that $\alpha(P) \leq \beta(P)$, and hence $\alpha(P) \leq L$. Therefore, $\alpha(P) = L$.

In summary, if $\beta(P_r) = \beta(P) = L$ for every index r, then either $\alpha(P) = L'$ or $\alpha(P) = L$.

Therefore, we have to examine these last three cases:

$(i):$ $\beta(P_r) = \beta(P) = L > \alpha(P_r) = \alpha(P) = L'$ for every index r;
$(ii):$ $\beta(P_r) = \beta(P) = \alpha(P) = L > \alpha(P_r) = L'$ for every index r;
$(iii):$ $\beta(P_r) = \beta(P) = \alpha(P) = L = \alpha(P_r) = L'$ for every index r.

$(i):$ If $\beta(P_r) = \beta(P) = L > \alpha(P_r) = \alpha(P) = L'$ for every r, the definition of the function φ implies that

$$\begin{aligned}\varphi(P_r) &= \frac{\alpha(P_r)\cdot d_H(\{P_r\},\overline{\mathcal{K}^c_{\beta(P_r)}})+\beta(P_r)\cdot d_H(\{P_r\},\mathcal{K}_{\alpha(P_r)})}{d_H(\{P_r\},\overline{\mathcal{K}^c_{\beta(P_r)}})+d_H(\{P_r\},\mathcal{K}_{\alpha(P_r)})}\\ &= \frac{L'\cdot d_H(\{P_r\},\overline{\mathcal{K}^c_L})+L\cdot d_H(\{P_r\},\mathcal{K}_{L'})}{d_H(\{P_r\},\overline{\mathcal{K}^c_L})+d_H(\{P_r\},\mathcal{K}_{L'})}\end{aligned}$$

while

$$\varphi(P) = \frac{L'\cdot d_H(\{P\},\overline{\mathcal{K}^c_L})+L\cdot d_H(\{P\},\mathcal{K}_{L'})}{d_H(\{P\},\overline{\mathcal{K}^c_L})+d_H(\{P\},\mathcal{K}_{L'})}.$$

Therefore $\lim_r \varphi(P_r) = \varphi(P)$, and hence the function φ is continuous at P.

(ii) : If $\beta(P_r) = \beta(P) = \alpha(P) = L > \alpha(P_r) = L'$ for every index r, the definition of the function φ implies that

$$\varphi(P_r) = \frac{\alpha(P_r) \cdot d_H(\{P_r\}, \overline{\mathcal{K}^c_{\beta(P_r)}}) + \beta(P_r) \cdot d_H(\{P_r\}, \mathcal{K}_{\alpha(P_r)})}{d_H(\{P_r\}, \overline{\mathcal{K}^c_{\beta(P_r)}}) + d_H(\{P_r\}, \mathcal{K}_{\alpha(P_r)})}$$
$$= \frac{L' \cdot d_H(\{P_r\}, \overline{\mathcal{K}^c_L}) + L \cdot d_H(\{P_r\}, \mathcal{K}_{L'})}{d_H(\{P_r\}, \overline{\mathcal{K}^c_L}) + d_H(\{P_r\}, \mathcal{K}_{L'})}.$$

Recalling that $P \in \overline{\mathcal{K}^c_{\alpha(P)}} = \overline{\mathcal{K}^c_L}$ and $\lim_r P_r = P$, it follows that $\lim_r \varphi(P_r) = L$.

On the other hand

$$\varphi(P) = \frac{L \cdot d_H(\{P\}, \overline{\mathcal{K}^c_L}) + L \cdot d_H(\{P\}, \mathcal{K}_L)}{d_H(\{P\}, \overline{\mathcal{K}^c_L}) + d_H(\{P\}, \mathcal{K}_L)} = L.$$

Therefore $\lim_r \varphi(P_r) = \varphi(P)$, and hence the function φ is continuous at P.

(iii) : If $\beta(P_r) = \beta(P) = \alpha(P) = L = \alpha(P_r) = L'$ for every index r, the definition of the function φ implies that $\varphi(P_r) = \varphi(P) = L$ for every index r. Therefore $\lim_r \varphi(P_r) = \varphi(P)$, and hence the function φ is continuous at P also in this case.

□

Let us observe that, dropping the assumption of stability (Definition 2.1), Theorem 2.8 does not hold, as the following examples show. The first one does not verify property (a) in Definition 2.1, the second one does not verify property (b) in Definition 2.1.

Example 1. Let K be the closed interval $[0, 2]$, and $I = \{-1\} \cup [0, 1]$. Let us consider the compact sets

$$\mathcal{K}_i = \begin{cases} \emptyset & \text{if } i = -1 \\ \{0\} & \text{if } i = 0 \\ [0, i+1] & \text{if } i \in]0, 1]. \end{cases}$$

This filtration of K is not stable because, contrarily to Definition 2.1 (a), when the index i tends to 0, the compact sets $\mathcal{K}_i$ do not tend to $\mathcal{K}_0$.

Let us show that this filtration of the interval K cannot be induced by a continuous function $\varphi : K \to \mathbb{R}$. Indeed, if such a continuous function φ existed, we would have $\varphi(P) \leq \varepsilon$ for every $\varepsilon > 0$ and every $P \in [0, 1]$ since $[0, 1] \subseteq \mathcal{K}_\varepsilon$. Therefore, φ would take a non-positive value at each $P \in [0, 1]$, against the equality $\mathcal{K}_0 = \{0\}$.

Example 2. Let K be the disk filtered by the family $\{\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3\}$ in Figure 2, with $\mathcal{K}_0 = \emptyset$ and $\mathcal{K}_3 = K$. This filtration of K is not stable because, contrarily to Definition 2.1 (b), $\mathcal{K}_1 \nsubseteq int(\mathcal{K}_2)$, or also because, contrarily to Remark 2.6, $\alpha(\bar{P}) = 2 > \beta(\bar{P}) = 1$ for $\bar{P} = \partial\mathcal{K}_1 \cap \partial\mathcal{K}_2$. Let us show that this filtration of the disk K cannot be induced by a continuous function $\varphi : K \to \mathbb{R}$. Indeed, if such a continuous function φ existed, it should be that $\varphi(\bar{P}) \leq 1$, since $\bar{P} \in \mathcal{K}_1$. On the other hand, if we consider a sequence (P_r) of points of $\mathcal{K}_3 \setminus \mathcal{K}_2$ that converges to $\bar{P}$, we should have $\varphi(\bar{P}) = \lim_r \varphi(P_r) \geq 2$ (since $\varphi(P_r) > 2$ for every index r, given that $P_r \notin K_2$). This contradiction proves our statement.

3. Multi-dimensional filtrations

In this section, we extend the main result of Section 2 to n-dimensional filtrations, $n \geq 1$, i.e. to the case of filtrations indexed by an n-dimensional parameter. Therefore, in what follows, the symbol I will denote a compact subset $I_1 \times I_2 \times \ldots \times I_n$ of $\mathbb{R}^n$ and $p_j : I \to I_j$, $1 \leq j \leq n$, the projection of I onto the j-th component.

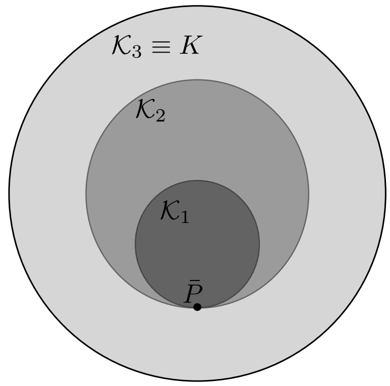

FIGURE 2. An example of non-stable 1-dimensional filtration of the disk K.

For every j with $1 \le j \le n$ and every fixed $h \in I_j$, let us set

$$\mathcal{K}^j_h = \mathcal{K}_{(\max I_1,\ldots,\max I_{j-1},h,\max I_{j+1},\ldots,\max I_n)} = \bigcup_{\substack{i\in I\\ p_j(i)=h}} \mathcal{K}_i.$$

We observe that $\{\mathcal{K}^j_h\}_{h\in I_j}$ is a compact 1-dimensional filtration of K.

Definition 3.1. We shall say that a compact n-dimensional filtration $\{\mathcal{K}_i\}_{i\in I}$ of K is *stable with respect to* d if the compact 1-dimensional filtrations $\{\mathcal{K}^1_{i_1}\}_{i_1\in I_1}$, $\{\mathcal{K}^2_{i_2}\}_{i_2\in I_2}$, …, $\{\mathcal{K}^n_{i_n}\}_{i_n\in I_n}$ are stable with respect to d.

Definition 3.2. A compact n-dimensional filtration $\{\mathcal{K}_i\}_{i\in I}$ of K will be said to be *complete* if, for every $i = (i_1,\ldots,i_n) \in I$, $\mathcal{K}_i = \mathcal{K}^1_{i_1} \cap \mathcal{K}^2_{i_2} \cap \ldots \cap \mathcal{K}^n_{i_n}$.

Remark 3.3. Let us observe that, setting $i_{min} = (\min I_1, \min I_2,\ldots,\min I_n)$ and $i_{max} = (\max I_1, \max I_2,\ldots,\max I_n)$, Definition 3.2 implies that $\emptyset = \mathcal{K}_{i_{min}} = \mathcal{K}^1_{\min I_1} \cap \mathcal{K}^2_{\min I_2} \cap \ldots \cap \mathcal{K}^n_{\min I_n}$, with $\mathcal{K}^j_{\min I_j} = \emptyset$, and $K = \mathcal{K}_{i_{max}} = \mathcal{K}^j_{\max I_j}$ for every $j = 1,\ldots,n$.

Theorem 3.4. *Every compact, stable and complete n-dimensional filtration $\{\mathcal{K}_i\}_{i\in I}$ of a compact metric space K is induced by a continuous function $\vec{\varphi} : K \to \mathbb{R}^n$.*

Proof. By Definition 3.2, the completeness of $\{\mathcal{K}_i\}_{i\in I}$ implies that, for every $i = (i_1, i_2,\ldots,i_n) \in I$, $\mathcal{K}_i$ is equal to $\mathcal{K}^1_{i_1} \cap \mathcal{K}^2_{i_2} \cap \ldots \cap \mathcal{K}^n_{i_n}$. Moreover, by Definition 3.1, the stability of $\{\mathcal{K}_i\}_{i\in I}$ implies the stability of the 1-dimensional filtrations $\{\mathcal{K}^1_{i_1}\}_{i_1\in I_1}$, $\{\mathcal{K}^2_{i_2}\}_{i_2\in I_2}$, …, $\{\mathcal{K}^n_{i_n}\}_{i_n\in I_n}$. Then, by Theorem 2.8, for every $\{\mathcal{K}^j_{i_j}\}_{i_j\in I_j}$, $j = 1,\ldots,n$, there exists a continuous function $\varphi_j : K \to \mathbb{R}$ such that $\mathcal{K}^j_{i_j} = \{P \in K : \varphi_j(P) \le i_j\}$ for every $i_j \in I_j$. Hence,

$$\begin{aligned}
&K_{(i_1,i_2,\ldots,i_n)} = \mathcal{K}^1_{i_1} \cap \mathcal{K}^2_{i_2} \cap \ldots \cap \mathcal{K}^n_{i_n}\\
&= \{P \in K : \varphi_1(P) \le i_1\} \cap \{P \in K : \varphi_2(P) \le i_2\} \cap \ldots \cap \{P \in K : \varphi_n(P) \le i_n\}\\
&= \{P \in K : \vec{\varphi}(P) = (\varphi_1, \varphi_2,\ldots,\varphi_n)(P) \preceq (i_1, i_2,\ldots,i_n)\}.
\end{aligned}$$

Therefore, the function $\vec{\varphi} : K \to \mathbb{R}^n$ induces $\{\mathcal{K}_i\}_{i\in I}$. Moreover, $\vec{\varphi}$ is continuous since its components $\varphi_1, \varphi_2,\ldots,\varphi_n : K \to \mathbb{R}$ are continuous. □

Let us observe that, without the assumption of completeness (Definition 3.2), Theorem 3.4 does not hold, as the following example shows.

Example 3. Let K be the rectangle in Figure 3, filtered by the family $\{\mathcal{K}_{(i_1,i_2)}\}$, with (i_1, i_2) varying in the set $I = \{0, 1, 2\} \times \{0, 1, 2\}$. From Remark 3.3, we have

$\mathcal{K}_{(0,i)} = \mathcal{K}_{(i,0)} = \emptyset$ for $i = 0, 1, 2$, and $\mathcal{K}_{(2,2)} = K$. We observe that $\{\mathcal{K}_{(i_1,i_2)}\}_{(i_1,i_2)\in I}$ is stable since the 1-dimensional filtrations $\{\mathcal{K}^1_{i_1}\}_{i_1\in\{0,1,2\}} = \{\mathcal{K}_{(0,2)}, \mathcal{K}_{(1,2)}, \mathcal{K}_{(2,2)}\}$, and $\{\mathcal{K}^2_{i_2}\}_{i_2\in\{0,1,2\}} = \{\mathcal{K}_{(2,0)}, \mathcal{K}_{(2,1)}, \mathcal{K}_{(2,2)}\}$ are stable with respect to d. However, $\{\mathcal{K}_{(i_1,i_2)}\}_{(i_1,i_2)\in I}$ is not complete since $\mathcal{K}_{(1,1)} \subsetneqq \mathcal{K}_{(2,1)} \cap \mathcal{K}_{(1,2)}$.

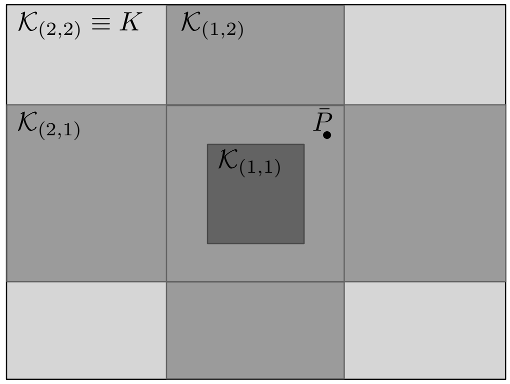

FIGURE 3. An example of non-complete 2-dimensional filtration of the rectangle K.

Let us show that this 2-dimensional filtration of the rectangle K cannot be induced by a continuous function $\vec{\varphi} : K \to \mathbb{R}^2$.

Let $\bar{P} \in \mathcal{K}_{(1,2)} \cap \mathcal{K}_{(2,1)} \setminus \mathcal{K}_{(1,1)}$ as in Figure 3. If there existed $\vec{\varphi} = (\varphi_1, \varphi_2) : K \to \mathbb{R}^2$ inducing this filtration, then $\varphi_1(\bar{P}) \le 1$ because $\bar{P} \in \mathcal{K}_{(1,2)}$, and $\varphi_2(\bar{P}) \le 1$ because $P \in \mathcal{K}_{(2,1)}$. Therefore, $\vec{\varphi}(\bar{P}) \preceq (1, 1)$. This means that $\bar{P}$ should belong to $\mathcal{K}_{(1,1)}$, giving a contradiction.

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