Closed form integration of a hyperelliptic, odd powers, undamped oscillator

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Abstract

A known one-dimensional, undamped, anharmonic oscillator whose restoring force is an odd polynomial function of displacement, is solved exactly via the Gauss and Appell hypergeometric functions, revealing a new fully integrable nonlinear system. The closed form relationship linking the period T to the initial motion amplitude a can then play as a benchmark to all the approximate values of literature.

KEYWORD: Nonlinear oscillator, Hypergeometric functions, Duffing-type equation

AMS SUB. CLASS. 34A34, 34C15, 34C25, 70K05

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1 Introduction

Nonlinear Partial Differential equations describe a lot of real, and then non linear, phenomena in natural and applied sciences as Mechanics, Solid State Physics, Biology and Mathematical Finance. Among the many proposed methods to construct exact solutions to them, we shall focus on the subsidiary ordinary differential equations approach: this means that through a proper transformation, some nonlinear partial differential equations with strong nonlinearities can be reduced to an ordinary equation like:

$$f''(\xi) + lf(\xi) + mf^3(\xi) + nf^5(\xi) = 0, \quad l\,m\,n \neq 0 \tag{1}$$

which is a generalized Duffing-type equation. A meaningful sample could be the Pochhammer-Chree nonlinear partial differential equation:

$$u_{tt} - u_{ttxx} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0, \tag{2}$$

which describes the longitudinal deformation waves of amplitude u in an elastic rod through the space x and time t, to which one looks for some explicit solitary wave solution of type:

$$\begin{aligned} u(x,t) &= u(\xi); \\ \xi &= x - vt, \end{aligned} \tag{3}$$

being v a real constant. Plugging (3) in (2), integrating twice with respect to ξ and setting the integration constant to zero, it is easy seen that one obtains (1).

In this framework, Citterio and Talamo recently [5], devised a method for the approximation of the periodic solutions to strongly nonlinear oscillators selecting firstly an auxiliary system "elliptic core" which serves that under consideration. Solved the "core" in closed form, such a solution becomes a basis for building a set of trial functions till to the sought approximation. So, for a nonlinear problem like:

$$\ddot{x}(t) = \sum_{i\geq 1} a_{2i-1} x^{2i-1}, \tag{4}$$

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they assume the elliptic core as:

$$\ddot{x}(t) = a_1 x + a_3 x^3 + a_5 x^5. \tag{5}$$

Such a equation models an oscillator whose restoring force keeps the harmonic term which some higher power perturbations are added to. It has been by the Authors in [4] integrated through the Weierstraß elliptic function $\wp$. But it is also object of several recent papers, for instance [8] that ignores any exact solution at all, and tries an approximation homotopically. Someone else [9] in devising a further method for these oscillators, faces with a subcase $a_3 = 0$ of (5) believing that such a equation *has no known closed form solution*. Going back to [4] the particular oscillator [2] analyzed there, is:

$$\begin{cases} \ddot{x}(t) + x^3 + x^7 = 0, \\ x(0) = a > 0, \\ \dot{x}(0) = 0. \end{cases} \tag{6}$$

While the Authors develop [4], an approximate treatment of (6), our paper is aimed to provide the exact solution to the generalized-Duffing equation (6), facing with a hyperelliptic integral not reducible to elliptic, and then solved through the hypergeometric functions. They are built-in and suitable for both symbolic and numerical manipulations, what we did using Mathematica® 7.0 1.0.

2 Analysis of the third/seventh power undamped oscillator

The oscillator ruled by equation (6), is somewhat called after Atkinson, but reading [2] we did not find there any special effort about it. Anyway, the oscillator is not harmonic for being $a_5 = 0$. The pure harmonic is characterized by only the first power of displacement in its restoring force, so that it succeeds in oscillating at only one (its natural or "own") frequency, irrespective of its initial position. Anharmonicity is the realistic deviation of a real oscillator from such a idealized behavior. Nevertheless all the known oscillators, from atoms to the beams, become anharmonic when their "pump" amplitude exceeds some threshold, driving to nonlinear differential equations as a description of their behavior.

2.1 The nonlinear equation and its integration

The treatment of (6) is elementary, but not simple, and we refer to [1] pages 287-293 or [10] page 114. It is easy to see that all the solutions are periodic, and figure 1 shows the phase portrait consisting of infinitely many closed orbits, one for any a value.

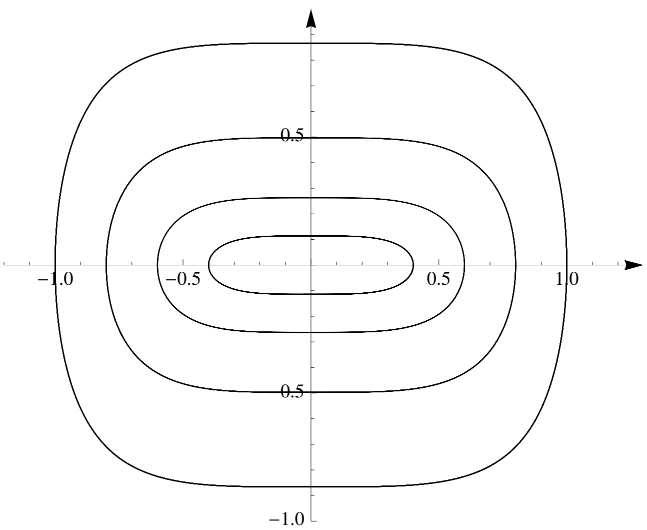

Figure 1: $0.4 \le a \le 1$: phase portrait of (6)

Equation (6) is so set to quadrature: in its first half-orbit for $-a \le x \le a$ we have

$$t = 2\int_x^a \frac{\mathrm{d}z}{\sqrt{(a^4-z^4)\,(a^4+2+z^4)}}$$

Thanks to the problem's symmetry, we are allowed to restrict our analysis to the first quarter, therefore: $0 \le x \le a$. The above integral is hypergeometric: making $v = z^4 \implies \mathrm{d}z = \frac{1}{4}\,v^{-3/4}\mathrm{d}v$, we have:

$$\begin{aligned} t &= \frac{1}{2}\int_{x^4}^{a^4} \frac{v^{-3/4}\mathrm{d}v}{\sqrt{(a^4-v)\,(a^4+2+v)}} \\ &= \frac{1}{2}\left\{\int_0^{a^4} \frac{v^{-3/4}\mathrm{d}v}{\sqrt{(a^4-v)\,(a^4+2+v)}} - \int_0^{x^4} \frac{v^{-3/4}\mathrm{d}v}{\sqrt{(a^4-v)\,(a^4+2+v)}}\right\} \end{aligned} \tag{7}$$

Let us normalize to the unit interval both integrals at right hand side of (7), the first through the substitution $v = a^4 u$ becomes:

$$\begin{aligned} &\int_0^{a^4} \frac{v^{-3/4}\mathrm{d}v}{\sqrt{(a^4-v)\,(a^4+2+v)}} = \\ &\frac{1}{a}\int_0^1 \frac{u^{-3/4}\,(1-u)^{-1/2}\,\mathrm{d}u}{(a^4+2+a^4u)^{1/2}} = \frac{1}{a\sqrt{a^4+2}}\int_0^1 \frac{u^{-3/4}\,(1-u)^{-1/2}\,\mathrm{d}u}{\left(1+\frac{a^4}{a^4+2}u\right)^{1/2}} \end{aligned}$$

so that:

$$\int_0^{a^4} \frac{v^{-3/4}\mathrm{d}v}{\sqrt{(a^4-v)\,(a^4+2+v)}} = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\frac{\sqrt{\pi}}{a\sqrt{a^4+2}}\ {}_2\mathrm{F}_1\left(\left.\begin{matrix}1/4;1/2\\3/4\end{matrix}\,\right|\, -\frac{a^4}{a^4+2}\right).$$

Where ${}_2\mathrm{F}_1$ if the Gauss hypergeometric function, see Appendix, equation (11) for further details. Normalizing the second integral $v = x^4 u$ at right hand side of (7) we get:

$$\int_0^{x^4} \frac{v^{-3/4}\mathrm{d}v}{\sqrt{(a^4-v)\,(a^4+2+v)}} = \frac{x}{a^2\sqrt{a^4+2}}\int_0^1 \frac{u^{-3/4}}{\left(1-\frac{x^4}{a^4}u\right)^{1/2}\left(1+\frac{x^4}{a^4+2}u\right)^{1/2}}\mathrm{d}u$$

so that:

$$\int_0^{x^4} \frac{v^{-3/4}\mathrm{d}v}{\sqrt{(a^4-v)\,(a^4+2+v)}} = \frac{4x}{a^2\sqrt{a^4+2}}\,\mathrm{F}_1\left(\left.\begin{matrix}1/4;1/2,1/2\\5/4\end{matrix}\,\right|\, \frac{x^4}{a^4}, -\frac{x^4}{a^4+2}\right)$$

Where F_1 if the Appell hypergeometric function, see Appendix, equation (12) for further details. In such a way, we computed time as a function of the bead position $x \ge 0$ along the first quarter of the phase plan orbit.

$$\begin{aligned} t = \frac{1}{2}&\left\{\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\frac{\sqrt{\pi}}{a\sqrt{a^4+2}}\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}1/4;1/2\\3/4\end{matrix}\,\right|\, -\frac{a^4}{a^4+2}\right) - \right.\\ &\left.\frac{4x}{a^2\sqrt{a^4+2}}\,\mathrm{F}_1\left(\left.\begin{matrix}1/4;1/2,1/2\\5/4\end{matrix}\,\right|\, \frac{x^4}{a^4}, -\frac{x^4}{a^4+2}\right)\right\} \end{aligned} \tag{8}$$

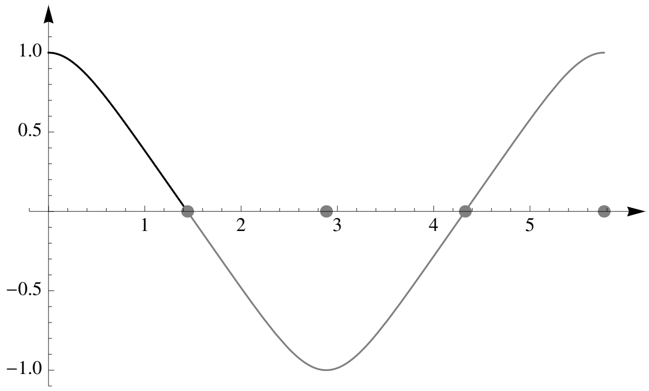

Figure 2: Overlapping of symbolic and numerical solutions: $a = 1$

The figure 2 shows the complete overlapping of our theoretical solution, specialized for $a = 1$, with the numerical one provided by the highly reliable routines of Mathematica®.

So the above formula (8) rules theoretically and solves completely the problem of our one-dimensional anharmonic, undamped, autonomous oscillator, and can then added to the (not rich) collection of highly non-linear systems completely integrable in closed form.

2.2 The oscillation period

By the previous formula for time, the oscillation period is immediately seen to be given by:

$$T(a) = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \frac{2\sqrt{\pi}}{a\sqrt{a^4+2}} \, {}_2F_1\left(\begin{array}{c} 1/4; 1/2 \\ 3/4 \end{array} \middle| -\frac{a^4}{a^4+2} \right) \tag{9}$$

For $a = 1$ it provides the value $T(1) = 5.76811851993611$ obtained simply asking Mathematica® to evaluate numerically the Gauss function.

Burton and Hamdan [3] believe our third/seventh power oscillator couldn't be expressed in terms of known functions, and use a time transformation method for getting an approximate solution. They also refer some other value of the period, always for $a = 1$, therefore: Burton-Hamdan, [3], 5.76678; Atkinson, [2], 5.76803; Sinha and Srinavasan, [11], 5.6199. So that Atkinson only provides three exact decimal digits. Since few nonlinear oscillators have known closed form expressions for their period, the results we obtained will go beyond the specific specific equation that is being investigated. Accordingly, let us provide a mechanical interpretation of the period behavior of our oscillator (6). We mean that increasing a, the energy stored in the oscillator will grow and the free motion will act with greater energy's availability, so that much more periods will occur in unity time: what is nothing but a frequency increase. Mathematically it is evident that the function $T(a)$ in (6) is decreasing, since it is the pointwise product of two postive decreasing functions: in fact for $0 < z < 1$ the function:

$$z \mapsto {}_2F_1\left(\begin{array}{c} 1/4; 1/2 \\ 3/4 \end{array} \middle| z \right)$$

is increasing and for $a > 0$

$$a \mapsto -\frac{a^4}{a^4+2}$$

is decreasing. Now let us provide the plot of $T(a)$, namely the behavior of period versus the initial amplitude a for our oscillator (6)

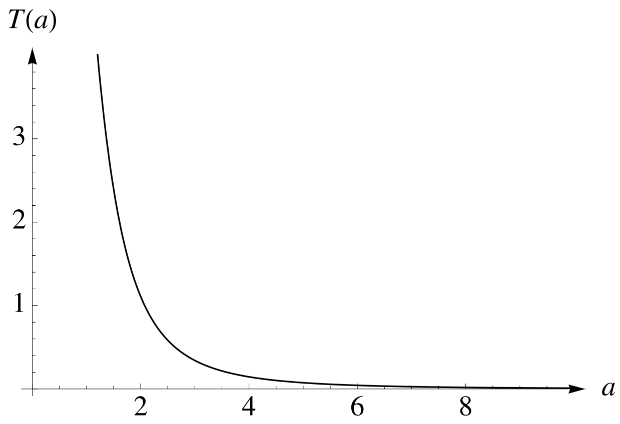

Figure 3: Plot of $T(a)$, $a \in [0,10]$ using (9)

Observe, finally, that formula (9) is itself a (convergent) power series expressing the period of (6): there is no need here to adapt the general methods exposed in [6], because they will deliver again the coefficients of the hypergeometric a power series, which are immediately obtained from (9):

$$T(a) = \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \sqrt{2\pi} \times \left(\frac{1}{a} - \frac{1}{3}a^3 + \frac{5}{28}a^7 - \frac{5}{44}a^{11} + \frac{85}{1056}a^{15} + \cdots \right)$$

Remark

About the relationship period/amplitude for nonlinear oscillators, it should be observed that often a completely different behavior holds. For instance, in the classic Duffing (powers 1 and 3) soft spring case ($\varepsilon < 0$) the restoring force increases with the extension less rapidly than the Hooke's law ($\varepsilon = 0$), and the period will increase with a. This is explained being the Duffing (soft!) oscillator a low term approximation of the simple pendulum:

$$\begin{cases} \ddot{\theta} + \dfrac{g}{L} \sin\theta = 0, \\ \theta(0) = \theta_0, \\ \dot{\theta}(0) = 0 \end{cases} \tag{10}$$

whose period grows with the initial amplitude θ_0 according to:

$$T = 4\sqrt{\frac{L}{g}} K(k) = 2\pi \sqrt{\frac{L}{g}} \left[1 + \frac{1}{4} \sin^2(\frac{\theta_0}{2}) + \frac{9}{64} \sin^4(\frac{\theta_0}{2}) + \cdots \right]$$

3 Conclusions

Based upon an article, [5], where a nonlinear unforced and undamped Duffing-type oscillator is proposed, this paper describes its closed form integration through the integral representation theorem of the Appell F_1 function, which is built in Mathematica® and then easy to be used. The period is then computed by means of the Gauss hypergeometric function ${}_2F_1$, achieving a benchmark to other results coming from cited approximate treatments. Finally, we provide a period a power series expansion and a interpretation of its decrease when the initial amplitude grows.

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Appendix

What follows is a subject of any textbook on Special Functions of Mathematical Physics (see for instance [12]), and then we will restrict to define only the hypergeometric functions and their integral representation theorem used in the paper. A slight idea of the link between the integral representation theorem and its use for evaluating the hyperelliptic integrals can be read on [7].

The best known among the hypergeometric functions is probably that with three parameter, one variable, named after K.F. Gauss, *Disquisitiones generales circa seriem infinitam*, 1812:

$$
{}_2\mathrm{F}_1\left(\begin{array}{c} a,\, b \\ c \end{array}\middle|\, z\right)=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{z^n}{n!},
$$

convergent within $|z|<1$, where $(a)_k=a(a+1)\cdots(a+k-1)=\Gamma(a+k)/\Gamma(a)$ is a Pochhammer product, with two numerator parameters a and b, and one denominator parameter c. The ${}_2\mathrm{F}_1$ integral representation theorem (Euler, 1748):

$$
{}_2\mathrm{F}_1\left(\begin{array}{c} a,\, b \\ c \end{array}\middle|\, z\right)=\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)}\int_0^1\frac{t^{a-1}(1-t)^{c-a-1}}{(1-zt)^b}\mathrm{d}t, \tag{11}
$$

where $\mathrm{Re}\, c>\mathrm{Re}\, a>0$, $|z|<1$, provides an extension to the region where the complex hypergeometric function is defined, namely for its analytical continuation, to the (almost) whole complex plane excluding the half-straight line $]1,\infty[$.

We finally recall the generalization to four parameter, two variable due to P. Appell *Sur les series hypergeometriques de deux variables* (1880), as hypergeometric function defined for $|x_1|<1$, $|x_2|<1$:

$$
\mathrm{F}_1\left(\begin{array}{c} a; b_1, b_2 \\ c \end{array}\middle|\, x_1, x_2\right)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{m_1+m_2}(b_1)_{m_1}(b_2)_{m_2}}{(c)_{m_1+m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!},
$$

whose integral representation theorem reads, if $\mathrm{Re}\, a>0$, $\mathrm{Re}(c-a)>0$, as:

$$
\mathrm{F}_1\left(\begin{array}{c} a; b_1, b_2 \\ c \end{array}\middle|\, x_1, x_2\right)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\frac{u^{a-1}\,(1-u)^{c-a-1}}{(1-x_1\,u)^{b_1}\,(1-x_2\,u)^{b_2}}\,\mathrm{d}u. \tag{12}
$$