

REEB GRAPHS OF SURFACES ARE STABLE UNDER FUNCTION PERTURBATIONS

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ABSTRACT. Reeb graphs are combinatorial signatures that capture shape properties from the perspective of a chosen function. One of the most important questions is whether Reeb graphs are robust against function perturbations that may occur because of noise and approximation errors in the data acquisition process. In this work we tackle the problem of stability providing an editing distance between Reeb graphs of orientable surfaces in terms of the cost necessary to transform one graph into another by edit operations. Our main result is that changes in the functions, measured by the maximum norm, imply not greater changes in this distance, yielding the stability property under function perturbations.

INTRODUCTION

In shape comparison, a widely used scheme is to measure the dissimilarity between signatures associated with each shape rather than match shapes directly [16, 13, 22].

Reeb graphs are signatures describing shapes from topological and geometrical perspectives. In this framework, shapes are modeled as spaces X endowed with scalar functions f . The role of f is to explore geometrical properties of the space X . The Reeb graph of $f : X \rightarrow \mathbb{R}$ is obtained by shrinking each connected component of a level set of f to a single point [17].

Reeb graphs have been used as an effective tool for shape analysis and description tasks since [21, 20]. The Reeb graph has a number of characteristics that make it useful as a search key for 3D objects. First, a Reeb graph always consists of a one-dimensional graph structure and does not have any higher dimension components such as the degenerate surface that can occur in a medial axis. Second, by defining the function appropriately, it is possible to construct a Reeb graph that is invariant to translation and rotation, or even more complicate isometries of the shape.

One of the most important questions is whether Reeb graphs are robust against perturbations that may occur because of noise and approximation errors in the data acquisition process. Heuristics have been developed so that the Reeb graph turns out to be resistant to connectivity changes caused by simplification, subdivision and remesh, and robust against noise and certain changes due to deformation [8, 3].

In this paper we tackle the robustness problem for Reeb graphs from a theoretical point of view. The main idea is to generalize to the case of surfaces the techniques developed in [5] to prove the stability of Reeb graphs of curves against function perturbations. Indeed the case of surfaces appears as the most interesting area of applications of the Reeb graph as a shape descriptor.

To this end, we introduce a combinatorial dissimilarity measure, called an *editing distance*, between Reeb graphs of surfaces in terms of the cost necessary to transform one

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graph into another by edit operations. The editing distance turns out to have all the properties of a pseudometric. Our main result is that changes in the functions, measured by the supremum norm, imply not greater changes in this editing distance, yielding the stability property under function perturbations.

In the literature, some other comparison methodologies have been proposed to compare Reeb graphs and estimate the similarity of the shapes described by Reeb graph.

In [8] the authors propose a Multiresolutional Reeb Graph (MRG) based on geodesic distance. Similarity between 3D shapes is calculated using a coarse-to-fine strategy while preserving the topological consistency of the graph structures to provide a fast and efficient estimation of similarity and correspondence between shapes.

In [3] the authors discuss a method for measuring the similarity and recognizing sub-part correspondences of 3D shapes, based on the synergy of a structural descriptor, like the Extended Reeb Graph, with a geometric descriptor, like spherical harmonics.

Only recently the problem of Reeb graph stability has been investigated from the theoretical point of view.

In [5] an editing distance between Reeb graphs of curves endowed with Morse functions is introduced and shown to yield stability. Importantly, despite the combinatorial nature of this distance, it coincides with the natural pseudo-distance between shapes [6], thus showing the maximal discriminative power for this sort of distances.

The work in [2] about a stable distance for merge trees is also pertinent to the stability problem for Reeb graphs: merge trees are known to determine contour trees, that are Reeb graphs for simple domains.

Recently a functional distortion distance between Reeb graphs has been proposed in the preprint [1], with proven stable and discriminative. The functional distortion distance is intrinsically continuous, whereas the editing distance we propose is combinatorial.

Outline. The paper is organized as follows. In Section 1 we recall the basic properties of labeled Reeb graphs of orientable surfaces. In Section 2 we define the editing deformations between labeled Reeb graphs, and show that through a finite sequence of these deformations we can always transform a Reeb graph into another. In Section 3 we associate a cost with each type of editing deformation and define the editing distance as the infimum cost we have to bear to transform one graph into another. Eventually, Section 4 illustrates the robustness of Reeb graphs with respect to the editing distance.

1. LABELED REEB GRAPHS OF ORIENTABLE SURFACES

Hereafter, \mathcal{M} denotes a connected, closed (i.e. compact and without boundary), orientable, smooth surface of genus g , and \mathcal{F} the set of C^∞ real functions on \mathcal{M} .

For $f \in \mathcal{F}$, we denote by $K(f)$ the set of its critical points. If $p \in K(f)$, then the real number $f(p)$ is called a *critical value* of f , and the set $\{q \in \mathcal{M} : q \in f^{-1}(f(p))\}$ is called a *critical level* of f . Otherwise, if $p \in \mathcal{M} \setminus K(f)$, then $f(p)$ is called a *regular value*. Moreover, a critical point p is called *non-degenerate* if the Hessian matrix of f at p is non-singular. The *index* of a non-degenerate critical point p of f is the dimension of the largest subspace of the tangent space to \mathcal{M} at p on which the Hessian is negative definite. In particular, the index of a point $p \in K(f)$ is equal to 0, 1, or 2 depending on whether p is a minimum, a saddle, or a maximum point of f .

A function $f \in \mathcal{F}$ is called a *Morse function* if all its critical points are non-degenerate. Besides, a Morse function is said to be *simple* if each critical level contains exactly one

critical point. The set of simple Morse functions will be denoted by \mathcal{F}^0 , as a reminder that it is a sub-manifold of \mathcal{F} of co-dimension 0 (see also Section 4).

Definition 1.1. Let $f \in \mathcal{F}^0$, and define on \mathcal{M} the following equivalence relation: for every $p, q \in \mathcal{M}$, $p \sim q$ whenever p, q belong to the same connected component of $f^{-1}(f(p))$. The quotient space \mathcal{M} / \sim is a finite and connected simplicial complex of dimension 1 known as the *Reeb graph* associated with f .

Throughout the paper, Reeb graphs are regarded as combinatorial graphs and not as topological spaces. The Reeb graph associated with f will be denoted by Γ_f , its vertex set by $V(\Gamma_f)$, and its edge set by $E(\Gamma_f)$. Moreover, if $v_1, v_2 \in V(\Gamma_f)$ are adjacent vertices, i.e., connected by an edge, we will write $e(v_1, v_2) \in E(\Gamma_f)$.

The critical points of f correspond bijectively to the vertices of Γ_f . For this reason, in the following, we will often identify vertices with the corresponding critical points. In particular, the maxima and minima of f correspond to vertices of degree 1, while saddle points to vertices of degree 3 (the degree of a vertex is the number of edges which connect this vertex to the graph). Our assumption that \mathcal{M} is orientable ensures the absence of vertices of degree 2. Moreover, if \mathcal{M} has genus g , Γ_f has exactly g linearly independent cycles. We will denote a cycle of length m in the graph by an m -cycle.

Let us observe that, if p, q, r denote the number of minima, maxima, and saddle points of f , from the relationships between the Euler characteristic of \mathcal{M} , $\chi(\mathcal{M})$, and p, q, r , i.e. $\chi(\mathcal{M}) = p + q - r$, and between $\chi(\mathcal{M})$ and the genus g of \mathcal{M} , i.e. $\chi(\mathcal{M}) = 2 - 2g$, it follows that the cardinality of $V(\Gamma_f)$, which is $p + q + r$, is also equal to $2(p + q + g - 1)$, i.e. is even in number. The minimum number of vertices of a Reeb graph is achieved whenever $p = q = 1$, and consequently $r = 2g$. In this case the cardinality of $V(\Gamma_f)$ is equal to $2g + 2$.

Definition 1.2. We shall call *minimal* a Reeb graph Γ_f with $p = q = 1$. Moreover, we say that Γ_f is *canonical* if it is minimal and all its cycles, if any, are 2-cycles.

Examples of minimal and canonical Reeb graphs are displayed in Figure 1. In particular, in a minimal Reeb graph, the vertices of degree 1 represent the global minimum and maximum of f , respectively; the remaining $2g$ vertices are of degree 3 and are connected each other in some way to form g linearly independent cycles representing the g holes of \mathcal{M} .

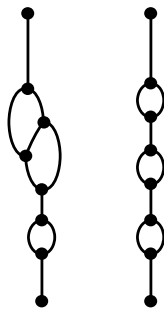


FIGURE 1. Examples of minimal Reeb graphs. The graph on the right is also canonical.

We want to underline that our definition of canonical Reeb graph is slightly different from the one in [10]. This choice has been done to simplify the proof of Proposition 2.7.

In what follows, we label the vertices of Γ_f by equipping each of them with the value of f at the corresponding critical point. We denote such a labeled graph by (Γ_f, ℓ_f) , where $\ell_f : V(\Gamma_f) \rightarrow \mathbb{R}$ is the restriction of $f : \mathcal{M} \rightarrow \mathbb{R}$ to $K(f)$. In a labeled Reeb graph, each vertex v of degree 3 has at least two of its adjacent vertices, say v_1, v_2 , such that $\ell_f(v_1) < \ell_f(v) < \ell_f(v_2)$. An example is displayed in Figure 2.

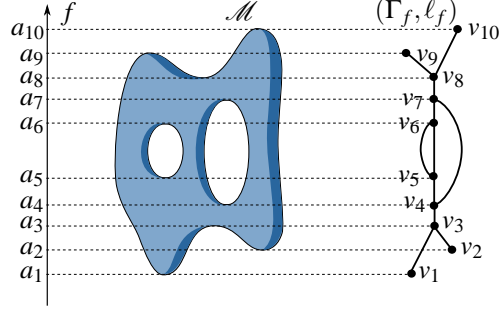


FIGURE 2. Left: the height function $f : \mathcal{M} \rightarrow \mathbb{R}$; center: the surface \mathcal{M} of genus $g = 2$; right: the associated labeled Reeb graph (Γ_f, ℓ_f) . Here labels are represented by the heights of the vertices.

To facilitate the reader, in all the figures of this paper we shall adopt the convention of representing f as the height function, so that $\ell_f(v_a) < \ell_f(v_b)$ if and only if v_a is lower than v_b in the picture.

Let us consider the realization problem, i.e. the problem of constructing a function $f \in \mathcal{F}^0$ from a graph on an even number of vertices, all of which are of degree 1 or 3, appropriately labeled. This result requires the following definition.

Definition 1.3. We shall say that two labeled Reeb graphs $(\Gamma_f, \ell_f), (\Gamma_g, \ell_g)$ are *isomorphic*, and we write $(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, if there exists a graph isomorphism $\Phi : V(\Gamma_f) \rightarrow V(\Gamma_g)$ such that, for every $v \in V(\Gamma_f)$, $f(v) = g(\Phi(v))$ (i.e. Φ preserves edges and vertices labels).

Proposition 1.4 (Realization theorem). *Let (G, ℓ) be a labeled graph, where G is a graph with m linearly independent cycles, on an even number of vertices, all of which are of degree 1 or 3, and $\ell : V(G) \rightarrow \mathbb{R}$ is an injective function such that, for any vertex v of degree 3, at least two among its adjacent vertices, say w, w' , are such that $\ell(w) < \ell(v) < \ell(w')$. Then an orientable closed surface \mathcal{M} of genus $g = m$, and a simple Morse function $f : \mathcal{M} \rightarrow \mathbb{R}$ exist such that $(\Gamma_f, \ell_f) \cong (G, \ell)$.*

Proof. Under our assumption on the degree of vertices of G , \mathcal{M} and f can be constructed as in the proof of Thm. 2.1 in [15]. \square

We now face with the uniqueness problem, up to isomorphism of labeled graphs. First of all we review the following two relations of equivalence on functions.

Definition 1.5. Let $\mathcal{D}(\mathcal{M})$ be the set of self-diffeomorphisms of \mathcal{M} . Two functions $f, g \in \mathcal{F}^0$ are called *right-equivalent* (briefly, *R-equivalent*) if there exists $\xi \in \mathcal{D}(\mathcal{M})$ such that $f = g \circ \xi$. Moreover, f, g are called *right-left equivalent* (briefly, *RL-equivalent*) if there

exist $\xi \in \mathcal{D}(\mathcal{M})$ and an orientation preserving self-diffeomorphism η of \mathbb{R} such that $f = \eta \circ g \circ \xi$.

These equivalence relations on functions are mirrored by Reeb graphs isomorphisms.

Proposition 1.6 (Uniqueness theorem). *If f, g are simple Morse functions on a closed surface, then*

- (1) f and g are RL-equivalent if and only if their Reeb graphs Γ_f and Γ_g are isomorphic by an isomorphism that preserves the vertex order;
- (2) f and g are R-equivalent if and only if their labeled Reeb graphs (Γ_f, ℓ_f) and (Γ_g, ℓ_g) are isomorphic.

Proof. Given two RL-equivalent functions $f, g \in \mathcal{F}^0$, it is immediate to see that there is a graph isomorphism Φ between Γ_f and Γ_g . Furthermore, Φ also preserves the vertex-order, i.e., for every $v, w \in V(\Gamma_f)$, $f(v) < f(w)$ if and only if $g(\Phi(v)) < g(\Phi(w))$. The converse is not so straightforward. Its proof follows from [12] (see also [19, Thm. 6.1]).

As for the second statement, two R-equivalent functions are, in particular, RL-equivalent. Therefore, their Reeb graphs are isomorphic by an isomorphism that preserves the vertex order. Since f and g necessarily have the same critical values, this isomorphism also preserves labels. Vice-versa, if (Γ_f, ℓ_f) and (Γ_g, ℓ_g) are isomorphic, by (1) it holds that there exist $\xi \in \mathcal{D}(\mathcal{M})$ and an orientation preserving self-diffeomorphism η of \mathbb{R} such that $f = \eta \circ g \circ \xi$. Let us set $h = g \circ \xi$. The function h belongs to \mathcal{F}^0 , and has the same critical points as f and also the same indexes. Moreover, h and f have the same values at each critical point because (Γ_f, ℓ_f) and (Γ_g, ℓ_g) are isomorphic and thus the labels are the same. Hence, applying [11, Lemma 1], it follows that there exists a self-diffeomorphism ξ' of \mathcal{M} such that $f = h \circ \xi'$. Thus $f = g \circ \xi \circ \xi'$, yielding that f and g are R-equivalent. \square

2. EDITING DEFORMATIONS BETWEEN LABELED REEB GRAPHS

In this section we list the editing deformations admissible to transform a labeled Reeb graph of an orientable surface into another. We introduce at first elementary deformations, then, by virtue of the Realization theorem (Proposition 1.4), the deformations obtained by their composition.

Elementary deformations allow to transform a Reeb graph into another with either a different number of vertices ((B) and (D)), or with the same number of vertices endowed with different labels ((R) and (K_{*i*}), $i = 1, 2, 3$), and can be described as follows.

Let (Γ_f, ℓ_f) be a labeled Reeb graph with n vertices. We call T an *elementary deformation* of (Γ_f, ℓ_f) if T transforms (Γ_f, ℓ_f) in one and only one of the ways described next, with the convention of denoting the open interval with endpoints a, b by $]a, b[$.

- (B) Fix $e(v_1, v_2) \in E(\Gamma_f)$, with $\ell_f(v_1) < \ell_f(v_2)$. Then T transforms (Γ_f, ℓ_f) into a labeled graph (G, ℓ) according to the following rule: G is the new graph on $n + 2$ vertices, obtained deleting the edge $e(v_1, v_2)$ and inserting two new vertices u_1, u_2 and the edges $e(v_1, u_1), e(u_1, u_2), e(u_1, v_2)$; moreover, $\ell : V(G) \rightarrow \mathbb{R}$ is defined by extending ℓ_f from $V(\Gamma_f)$ to $V(G) = V(\Gamma_f) \cup \{u_1, u_2\}$ in such a way that $\ell|_{V(\Gamma_f)} \equiv \ell_f$, and either $\ell_f(v_1) < \ell(u_1) < \ell(u_2) < \ell_f(v_2)$, with $\ell^{-1}(] \ell(u_1), \ell(u_2) [) = \emptyset$, or $\ell_f(v_1) < \ell(u_2) < \ell(u_1) < \ell_f(v_2)$, with $\ell^{-1}(] \ell(u_2), \ell(u_1) [) = \emptyset$ (see Table 1, row 1).
- (D) Assume $e(v_1, u_1), e(u_1, u_2), e(u_1, v_2) \in E(\Gamma_f)$, u_2 of degree 1, and either $\ell_f(v_1) < \ell_f(u_1) < \ell_f(u_2) < \ell_f(v_2)$, with $\ell_f^{-1}(] \ell_f(u_1), \ell_f(u_2) [) = \emptyset$, or $\ell_f(v_1) < \ell_f(u_2) < \ell_f(u_1) < \ell_f(v_2)$, with $\ell_f^{-1}(] \ell_f(u_2), \ell_f(u_1) [) = \emptyset$. Then T transforms (Γ_f, ℓ_f) into a labeled graph (G, ℓ) according to the following rule: G is the new graph on $n - 2$

vertices, obtained deleting u_1, u_2 and the edges $e(v_1, u_1), e(u_1, u_2), e(u_1, v_2)$, and inserting an edge $e(v_1, v_2)$; moreover, $\ell : V(G) \rightarrow \mathbb{R}$ is defined as the restriction of ℓ_f to $V(\Gamma_f) \setminus \{u_1, u_2\}$ (see Table 1, row 1).

- (R) T transforms (Γ_f, ℓ_f) into a labeled graph (G, ℓ) according to the following rule: $G = \Gamma_f$, and $\ell : V(G) \rightarrow \mathbb{R}$ induces the same vertex-order as ℓ_f except for at most two vertices, say u_1, u_2 , for which, if $\ell_f(u_1) < \ell_f(u_2)$ and $\ell_f^{-1}([\ell_f(u_1), \ell_f(u_2)]) = \emptyset$, then $\ell(u_1) > \ell(u_2)$, and $\ell^{-1}([\ell(u_2), \ell(u_1)]) = \emptyset$ (see Table 1, row 2).
- (K₁) Assume $e(v_1, u_1), e(u_1, u_2), e(u_1, v_4), e(u_2, v_2), e(u_2, v_3) \in E(\Gamma_f)$, with two among v_2, v_3, v_4 possibly coincident, and either $\ell_f(v_1) < \ell_f(u_1) < \ell_f(u_2) < \ell_f(v_2), \ell_f(v_3), \ell_f(v_4)$, with $\ell_f^{-1}([\ell_f(u_1), \ell_f(u_2)]) = \emptyset$, or $\ell_f(v_2), \ell_f(v_3), \ell_f(v_4) < \ell_f(u_2) < \ell_f(u_1) < \ell_f(v_1)$, with $\ell_f^{-1}([\ell_f(u_2), \ell_f(u_1)]) = \emptyset$. Then T transforms (Γ_f, ℓ_f) into a labeled graph (G, ℓ) according to the following rule: G is the new graph on n vertices, obtained deleting the edges $e(v_1, u_1), e(u_2, v_2)$, and inserting the edges $e(v_1, u_2), e(u_1, v_2)$; moreover, $\ell : V(G) \rightarrow \mathbb{R}$ is defined as ℓ_f on $V(\Gamma_f) \setminus \{u_1, u_2\}$, and either $\ell_f(v_1) < \ell(u_2) < \ell(u_1) < \ell_f(v_2), \ell_f(v_3), \ell_f(v_4)$, with $\ell^{-1}([\ell(u_2), \ell(u_1)]) = \emptyset$, or $\ell_f(v_2), \ell_f(v_3), \ell_f(v_4) < \ell(u_1) < \ell(u_2) < \ell_f(v_1)$, with $\ell^{-1}([\ell(u_1), \ell(u_2)]) = \emptyset$ (see Table 1, row 3).
- (K₂) Assume $e(v_1, u_1), e(u_1, u_2), e(v_2, u_1), e(u_2, v_3), e(u_2, v_4) \in E(\Gamma_f)$, with u_1, u_2 of degree 3, v_2, v_3 possibly coincident with v_1, v_4 , respectively, and $\ell_f(v_1), \ell_f(v_2) < \ell_f(u_1) < \ell_f(u_2) < \ell_f(v_3), \ell_f(v_4)$, with $\ell_f^{-1}([\ell_f(u_1), \ell_f(u_2)]) = \emptyset$. Then T transforms (Γ_f, ℓ_f) into a labeled graph (G, ℓ) according to the following rule: G is the new graph on n vertices, obtained deleting the edges $e(v_1, u_1), e(u_2, v_3)$, and inserting the edges $e(u_1, v_3), e(v_1, u_2)$; moreover, $\ell : V(G) \rightarrow \mathbb{R}$ is defined as ℓ_f on $V(\Gamma_f) \setminus \{u_1, u_2\}$, and $\ell_f(v_1), \ell_f(v_2) < \ell(u_2) < \ell(u_1) < \ell_f(v_3), \ell_f(v_4)$, with $\ell^{-1}([\ell(u_2), \ell(u_1)]) = \emptyset$ (see Table 1, row 4).
- (K₃) Assume $e(v_1, u_2), e(u_1, u_2), e(v_2, u_1), e(u_1, v_3), e(u_2, v_4) \in E(\Gamma_f)$, with u_1, u_2 of degree 3, v_2, v_3 possibly coincident with v_1, v_4 , respectively, and $\ell_f(v_1), \ell_f(v_2) < \ell_f(u_2) < \ell_f(u_1) < \ell_f(v_3), \ell_f(v_4)$, with $\ell_f^{-1}([\ell_f(u_2), \ell_f(u_1)]) = \emptyset$. Then T transforms (Γ_f, ℓ_f) into a labeled graph (G, ℓ) according to the following rule: G is the new graph on n vertices, obtained deleting the edges $e(v_1, u_2), e(u_1, v_3)$, and inserting the edges $e(v_1, u_1), e(u_2, v_3)$; moreover, $\ell : V(G) \rightarrow \mathbb{R}$ is defined as ℓ_f on $V(\Gamma_f) \setminus \{u_1, u_2\}$, and $\ell_f(v_1), \ell_f(v_2) < \ell(u_1) < \ell(u_2) < \ell_f(v_3), \ell_f(v_4)$, with $\ell^{-1}([\ell(u_1), \ell(u_2)]) = \emptyset$ (see Table 1, row 4).

We shall denote by $T(\Gamma_f, \ell_f)$ the result of the elementary deformation T applied to (Γ_f, ℓ_f) .

Let us observe that, by virtue of the above elementary deformations, the vertex-order induced by f can change only for the vertices u_i compared with the others and remains the same among the vertices v_i .

Proposition 2.1. *Let T be an elementary deformation of (Γ_f, ℓ_f) , and let $(G, \ell) = T(\Gamma_f, \ell_f)$. Then (G, ℓ) is isomorphic to a labeled Reeb graph (Γ_g, ℓ_g) , with $g \in \mathcal{F}^0$.*

Proof. The claim follows from Propositions 1.4. □

As a consequence of Proposition 2.1, we can apply elementary deformations iteratively. This fact is used in the next Definition 2.2.

Given an elementary deformation T of (Γ_f, ℓ_f) and an elementary deformation S of $T(\Gamma_f, \ell_f)$, the juxtaposition ST means applying first T and then S .

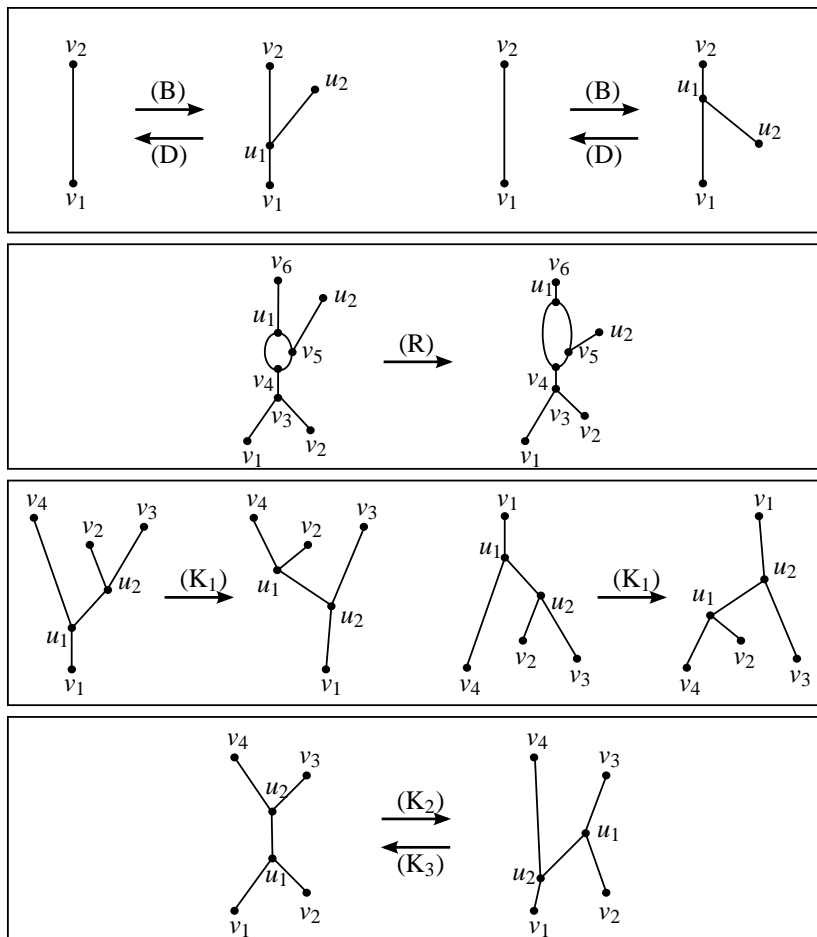


TABLE 1. Elementary deformations of a labeled Reeb graph.

Definition 2.2. We shall call *deformation* of (Γ_f, ℓ_f) any finite ordered sequence $T = (T_1, T_2, \dots, T_r)$ of elementary deformations such that T_1 is an elementary deformation of (Γ_f, ℓ_f) , T_2 is an elementary deformation of $T_1(\Gamma_f, \ell_f)$, ..., T_r is an elementary deformation of $T_{r-1}T_{r-2} \cdots T_1(\Gamma_f, \ell_f)$. We shall denote by $T(\Gamma_f, \ell_f)$ the result of the deformation T applied to (Γ_f, ℓ_f) . Moreover, we shall call *identical deformation* any deformation such that $T(\Gamma_f, \ell_f) \cong (\Gamma_f, \ell_f)$.

Let us observe that the identical deformation can be considered as a particular elementary deformation of type (R).

We now introduce the concept of inverse deformation.

Definition 2.3. Let T be a deformation such that $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$. Then we denote by T^{-1} , and call it the *inverse* of T , the deformation such that $T^{-1}(\Gamma_g, \ell_g) \cong (\Gamma_f, \ell_f)$ defined as follows:

- If T is elementary of type (D) deleting two vertices, then T^{-1} is of type (B) inserting the same vertices, with the same labels, and viceversa;

- If T is elementary of type (R) relabeling vertices of $V(\Gamma_f)$, then T^{-1} is again of type (R) relabeling these vertices in the inverse way;
- If T is elementary of type (K₁) relabeling two vertices, then T^{-1} is again of type (K₁) relabeling the same vertices in the inverse way;
- If T is elementary of type (K₂) relabeling two vertices, then T^{-1} is of type (K₃) relabeling the same vertices in the inverse way, and viceversa;
- If $T = (T_1, \dots, T_r)$, then $T^{-1} = (T_r^{-1}, \dots, T_1^{-1})$.

We prove that, for every two labeled Reeb graphs, a finite number of elementary deformations always allows us to transform any of them into the other one, up to isomorphism. We first need two lemmas which are widely inspired by [10, Lemma 1 and Theorem 1], respectively.

Lemma 2.4. *Let (Γ_f, ℓ_f) be a labeled Reeb graph with n vertices, $n \geq 4$.*

- Let $u, v \in V(\Gamma_f)$ correspond to two minima or two maxima of f . There exists a deformation T such that u and v are adjacent to the same vertex w in $T(\Gamma_f, \ell_f)$.*
- Let C be an m -cycle in Γ_f , $m \geq 2$. There exists a deformation T such that C is a 2-cycle in $T(\Gamma_f, \ell_f)$.*

Proof. Let us prove statement (i) assuming that u, v correspond to two minima of f . The other case is analogous.

Let us consider a path γ on Γ_f having u, v as endpoints, whose length is $m > 2$, and the finite sequence of vertices through which it passes is (w_0, w_1, \dots, w_m) , with $w_0 = u, w_m = v$, and $w_i \neq w_j$ for $i \neq j$. We want to show that there exists a deformation T such that in $T(\Gamma_f, \ell_f)$ the path γ is reduced to be of length 2, i.e. u, v are adjacent to the same vertex w .

Let $w_i \neq u, v$ with $\ell_f(w_i) = \max_{j=0, \dots, m} \{\ell_f(w_j)\}$. It exists because u, v are minima of f .

It is easy to observe that, in a neighborhood of w_i , possibly after a finite sequence of deformations of type (R), the graph gets one of the configurations shown in Figure 3 (a) – (e) (left).

As it can be seen, through a finite sequence of deformations of type (K₁) and/or (K₃), possibly together with deformations of type (R), the path γ , which has length m , can be transformed into a simple path of length $m - 1$. Iterating this procedure, we deduce the desired claim.

The proof of statement (ii) is analogous to that of statement (i), provided that γ is taken to be an m -cycle with $u \equiv v$ of degree 3, and $f(u) = \min_{j=0, \dots, m-1} \{f(w_j)\}$. \square

Remark 2.5. We observe that if the vertex w in Figure 3 (a) – (e) (left) is of degree 1, then, in the cases (a), (b), (d), (e), it can be deleted with its adjacent vertex through a deformation of type (D), possibly after a deformation of type (R), and, in the case (c), it can be deleted via the composition of (K₃) with (R). This is an alternative way to decrease the length of the path γ by one.

Lemma 2.6. *Every labeled Reeb graph (Γ_f, ℓ_f) can be transformed into a canonical one through a finite sequence of elementary deformations.*

Proof. Our proof is in two steps: first we show how to transform an arbitrary Reeb graph into a minimal one; then how to reduce a minimal Reeb graph to the canonical form.

The first step is by induction on $s = p + q$, with p and q denoting the number of minima and maxima of f . If $s = 2$, then Γ_f is already minimal (see Definition 1.2). Let us assume that any Reeb graph with $s \geq 2$ vertices of degree 1 can be transformed into a minimal one through a certain deformation. Let Γ_f have $s + 1$ vertices of degree 1. Thus, at least

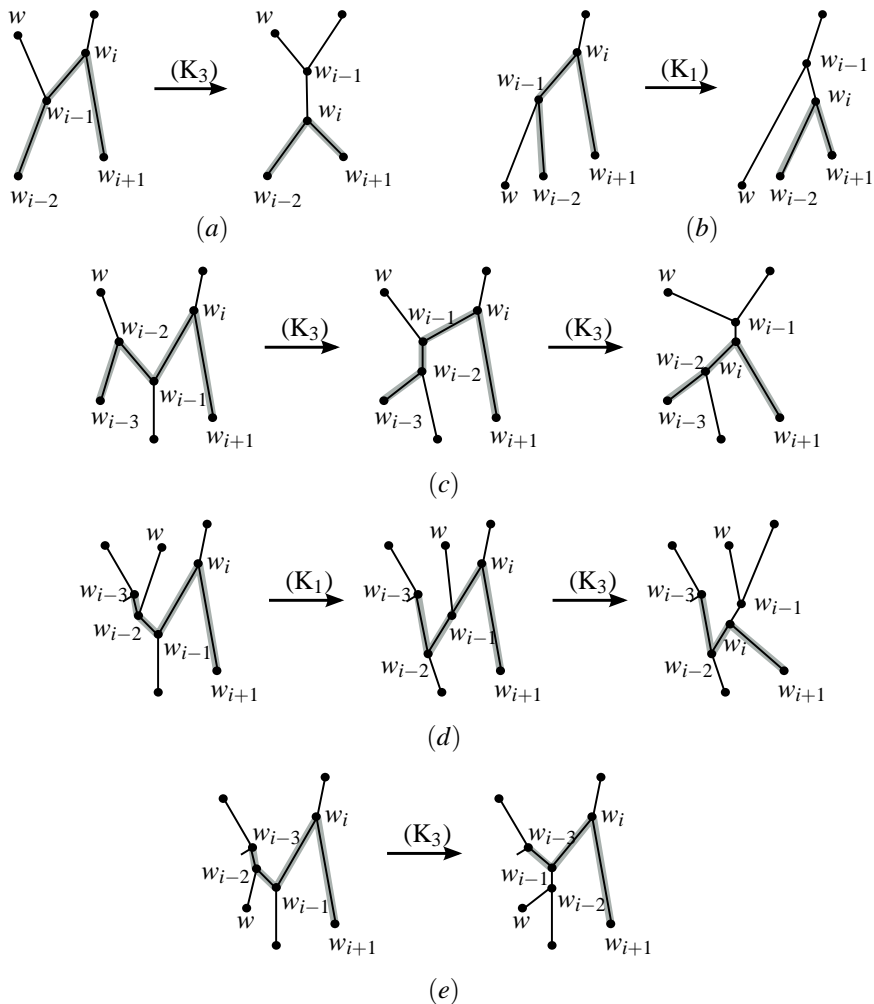


FIGURE 3. Possible configurations of a simple path on a labeled Reeb graph in a neighborhood of its maximum point, and elementary deformations which reduce its length.

one between p and q is greater than one. Let $p > 1$ (the case $q > 1$ is analogous). By Lemma 2.4 (i), if u, v correspond to two minima of f , we can construct a deformation T such that in $T(\Gamma_f, \ell_f)$ these vertices are both adjacent to a certain vertex w of degree 3. Let $T(\Gamma_f, \ell_f) = (\Gamma, \ell)$, with $\ell(u) < \ell(v) < \ell(w)$. If there exists a vertex $w' \in \ell^{-1}(] \ell(v), \ell(w) [)$, since v, w' cannot be adjacent, we can apply a deformation of type (R) relabeling only v , and get a new labeling ℓ' such that $\ell'(w')$ is not contained in $] \ell'(v), \ell'(w) [$. Possibly repeating this procedure finitely many times, we get a new labeling, that for simplicity we still denote by ℓ , such that $\ell^{-1}(] \ell(v), \ell(w) [) = \emptyset$. Hence, through a deformation of type (D) deleting v, w , the resulting labeled Reeb graph has s vertices of degree 1. Hence, by the inductive hypothesis, it can be transformed into a minimal Reeb graph.

Now we prove the second step. Let Γ_f be a minimal Reeb Graph, i.e. $p = q = 1$. The total number of splitting saddles (i.e. vertices of degree 3 for which there are two higher adjacent vertices) of Γ_f is \mathfrak{g} . If $\mathfrak{g} = 0$, then Γ_f is already canonical. Let us consider the case $\mathfrak{g} \geq 1$. Let $v \in V(\Gamma_f)$ be a splitting saddle such that, for every cycle C containing v , $\ell_f(v) = \min_{w \in C} \{\ell_f(w)\}$, and let C be one of these cycles. By Lemma 2.4 (ii), there exists a deformation T that transforms C into a 2-cycle, still having v as the lowest vertex. Let v' be the highest vertex in this 2-cycle. We observe that no other cycles of $T(\Gamma_f)$ contain v and v' , otherwise the initial assumption on $\ell_f(v)$ would be contradicted. Hence v, v' and the edges adjacent to them are not touched when applying again Lemma 2.4 (ii) to reduce the length of another cycle. Therefore, iterating the same argument on a different splitting saddle, after at most \mathfrak{g} iterations (actually at most $\mathfrak{g} - 1$ would suffice) Γ_f is transformed into a canonical Reeb graph. \square

Proposition 2.7. *Let (Γ_f, ℓ_f) and (Γ_g, ℓ_g) be two labeled Reeb graphs. Then the set of all the deformations T such that $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$ is non-empty. This set of deformations will be denoted by $\mathcal{F}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$.*

Proof. By Lemma 2.6 we can find two deformations T_f and T_g transforming (Γ_f, ℓ_f) and (Γ_g, ℓ_g) , respectively, into canonical labeled Reeb graphs. Moreover, $T_f(\Gamma_f, \ell_f)$ can be transformed into a graph isomorphic to $T_g(\Gamma_g, \ell_g)$ through an elementary deformation of type (R), say T_R . Thus $(\Gamma_g, \ell_g) \cong T_g^{-1} T_R T_f(\Gamma_f, \ell_f)$. \square

3. EDITING DISTANCE BETWEEN LABELED REEB GRAPHS

In this section we introduce an editing distance between labeled Reeb graphs, in terms of the cost necessary to transform one graph into another. We begin by defining the cost of a deformation.

Definition 3.1. Let T be an elementary deformation such that $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$.

- If T is of type (B) inserting the vertices $u_1, u_2 \in V(\Gamma_g)$, then we define the associated cost as

$$c(T) = \frac{|\ell_g(u_1) - \ell_g(u_2)|}{2}.$$

- If T is of type (D) deleting the vertices $u_1, u_2 \in V(\Gamma_f)$, then we define the associated cost as

$$c(T) = \frac{|\ell_f(u_1) - \ell_f(u_2)|}{2}.$$

- If T is of type (R) relabeling the vertices $v \in V(\Gamma_f) = V(\Gamma_g)$, then we define the associated cost as

$$c(T) = \max_{v \in V(\Gamma_f)} |\ell_f(v) - \ell_g(v)|.$$

- If T is of type (K_i) , with $i = 1, 2, 3$, relabeling the vertices $u_1, u_2 \in V(\Gamma_f)$, then we define the associated cost as

$$c(T) = \max\{|\ell_f(u_1) - \ell_g(u_1)|, |\ell_f(u_2) - \ell_g(u_2)|\}.$$

Moreover, if $T = (T_1, \dots, T_r)$ is a deformation such that $T_r \cdots T_1(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, we define the associated cost as $c(T) = \sum_{i=1}^r c(T_i)$.

Proposition 3.2. *For every deformation T such that $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, $c(T^{-1}) = c(T)$.*

Proof. It is sufficient to observe that, for every deformation $T = (T_1, \dots, T_r)$ such that $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, Definitions 3.1 and 2.3 imply the following equalities:

$$c(T) = \sum_{i=1}^r c(T_i) = \sum_{i=1}^r c(T_i^{-1}) = c(T^{-1}).$$

□

Theorem 3.3. *For every two labeled Reeb graphs (Γ_f, ℓ_f) and (Γ_g, ℓ_g) , we set*

$$d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = \inf_{T \in \mathcal{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))} c(T).$$

Then d is a pseudometric on isomorphism classes of labeled Reeb graphs.

Proof. The coincidence axiom can be verified by observing that the identical deformation, if obtained as a particular elementary deformation of type (R), has a cost equal to 0; the symmetry is a consequence of Proposition 3.2; the triangle inequality can be proved in the standard way. □

In order to deduce that d is actually a metric, we need to prove that $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = 0$ implies $(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$. Nevertheless, for simplicity, we will refer to d as to the *editing distance*.

In the next Section 4 we prove that the editing distance between two labeled Reeb graphs is upper-bounded by the C^0 -norm evaluated at the difference between the corresponding functions. We observe that such a result is strictly related to how the cost of an elementary deformation of type (R) has been defined. See, for instance, Example 1.

Example 1. Let $f, g : \mathcal{M} \rightarrow \mathbb{R}$ with $f, g \in \mathcal{F}^0$ illustrated in Figure 4. Let $f(q_i) - f(p_i) = a$, $i = 1, 2, 3$. Let us show that $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \frac{a}{2}$. For every $0 < \varepsilon < \frac{a}{2}$, we can apply

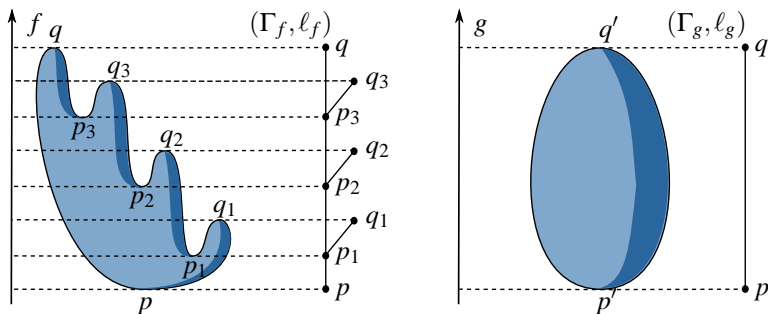


FIGURE 4. The functions $f, g \in \mathcal{F}^0$ considered in Example. Even in this case $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = D(f, g) = \frac{1}{2}(f(q_1) - f(p_1))$.

to (Γ_f, ℓ_f) a deformation of type (R), that relabels the vertices p_i, q_i , $i = 1, 2, 3$, in such a way that $\ell_f(p_i)$ is increased by $\frac{a}{2} - \varepsilon$, and $\ell_f(q_i)$ is decreased by $\frac{a}{2} - \varepsilon$, composed with three deformations of type (D) that delete p_i with q_i , $i = 1, 2, 3$. Thus, since the total cost is equal to $\frac{a}{2} - \varepsilon + 3\varepsilon$, by the arbitrariness of ε , it holds that $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \frac{a}{2}$.

4. STABILITY

This section is devoted to proving that Reeb graphs of orientable surfaces are stable under function perturbations. More precisely, it will be shown that arbitrary changes in simple Morse functions imply smaller changes in the editing distance between the associated labeled Reeb graphs. Formally:

Theorem 4.1. *For every $f, g \in \mathcal{F}^0$, $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \|f - g\|_{C^0}$, where $\|f - g\|_{C^0} = \max_{p \in \mathcal{M}} |f(p) - g(p)|$.*

In order to prove this stability theorem, we consider \mathcal{F} endowed with the C^2 topology, which may be defined as follows. Let $\{U_\alpha\}$ be a coordinate covering of \mathcal{M} with coordinate maps $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^2$, and let $\{C_\alpha\}$ be a compact refinement of $\{U_\alpha\}$. For every positive constant $\delta > 0$ and for every $f \in \mathcal{F}$, define $N(f, \delta)$ as the subset of \mathcal{F} consisting of all maps g such that, denoting $f_\alpha = f \circ \varphi_\alpha^{-1}$ and $g_\alpha = g \circ \varphi_\alpha^{-1}$, it holds that $\max_{i+j \leq 2} |\frac{\partial^{i+j}}{\partial x^i \partial y^j} (f_\alpha - g_\alpha)| < \delta$, at all points of $\varphi_\alpha(C_\alpha)$. The C^2 topology is the topology obtained by taking the sets $N(f, \delta)$ as a base of neighborhoods.

Next we consider the strata \mathcal{F}^0 and \mathcal{F}^1 of the *natural stratification* of \mathcal{F} , as presented by Cerf in [4].

- The stratum \mathcal{F}^0 is the set of simple Morse functions.
- The stratum \mathcal{F}^1 is the disjoint union of two sets \mathcal{F}_α^1 and \mathcal{F}_β^1 , where
 - \mathcal{F}_α^1 is the set of functions whose critical levels contain exactly one critical point, and the critical points are all non-degenerate, except exactly one.
 - \mathcal{F}_β^1 is the set of Morse functions whose critical levels contain at most one critical point, except for one level containing exactly two critical points.

\mathcal{F}^1 is a sub-manifold of co-dimension 1 of $\mathcal{F}^0 \cup \mathcal{F}^1$, and the complement of $\mathcal{F}^0 \cup \mathcal{F}^1$ in \mathcal{F} is of co-dimension greater than 1. Hence, given two functions $f, g \in \mathcal{F}^0$, we can always find $\widehat{f}, \widehat{g} \in \mathcal{F}^0$ arbitrarily near to f, g , respectively, for which

- \widehat{f}, \widehat{g} are RL-equivalent to f, g , respectively,

and the path $h(\lambda) = (1 - \lambda)\widehat{f} + \lambda\widehat{g}$, with $\lambda \in [0, 1]$, is such that

- $h(\lambda)$ belongs to $\mathcal{F}^0 \cup \mathcal{F}^1$ for every $\lambda \in [0, 1]$;
- $h(\lambda)$ is transversal to \mathcal{F}^1 .

As a consequence, $h(\lambda)$ belongs to \mathcal{F}^1 for at most a finite collection of values λ , and does not traverse strata of co-dimension greater than 1 (see, e.g., [7]).

With these preliminaries set, the stability theorem will be proven by considering a path that connects f to g via \widehat{f} , $h(\lambda)$, and \widehat{g} as aforementioned. This path can be split into a number of segments whose endpoints are such that the stability theorem holds on them, as shown in some preliminary lemmas. In conclusion Theorem 4.1 will be proven by applying the triangle inequality of the editing distance.

In the following preliminary lemmas, f and g belong to \mathcal{F}^0 and $h : [0, 1] \rightarrow \mathcal{F}$ denotes their convex linear combination: $h(\lambda) = (1 - \lambda)f + \lambda g$.

Lemma 4.2. $\|h(\lambda') - h(\lambda'')\|_{C^0} = |\lambda' - \lambda''| \cdot \|f - g\|_{C^0}$ for every $\lambda', \lambda'' \in [0, 1]$.

Proof.

$$\begin{aligned} \|h(\lambda') - h(\lambda'')\|_{C^0} &= \|(1 - \lambda')f + \lambda'g - (1 - \lambda'')f - \lambda''g\|_{C^0} \\ &= \|(\lambda'' - \lambda')f - (\lambda'' - \lambda')g\|_{C^0} = |\lambda' - \lambda''| \cdot \|f - g\|_{C^0}. \end{aligned}$$

□

Lemma 4.3. *If $h(\lambda) \in \mathcal{F}^0$ for every $\lambda \in [0, 1]$, then $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \|f - g\|_{C^0}$.*

Proof. The statement can be proved in the same way as [5, Prop. 4.4].

□

Lemma 4.4. *Let $h(\lambda)$ intersect \mathcal{F}^1 transversely at $h(\bar{\lambda})$, $0 < \bar{\lambda} < 1$, and nowhere else. Then, for every constant value $\delta > 0$, there exist two real numbers λ', λ'' with $0 < \lambda' < \bar{\lambda} < \lambda'' < 1$, such that*

$$d((\Gamma_{h(\lambda')}, \ell_{h(\lambda')}), (\Gamma_{h(\lambda'')}, \ell_{h(\lambda'')})) \leq \delta.$$

Proof. In this proof we use the notion of universal deformation. More details on universal deformations may be found in [4, 14, 18].

In particular, since $h(\lambda)$ intersects \mathcal{F}^1 transversely at $h(\bar{\lambda})$, we can consider two different universal deformations of $\bar{h} = h(\bar{\lambda})$: $F(s, p) = \bar{h}(p) + s \cdot (g - f)(p)$, and $G(s, p)$ whose construction depends on whether \bar{h} belongs to \mathcal{F}_α^1 or \mathcal{F}_β^1 .

Let us assume for a moment that G has been defined.

We want to exploit the fact that, being two universal deformations of $\bar{h} \in \mathcal{F}^1$, F and G are equivalent. This means that there exist a diffeomorphism $\eta(s)$ with $\eta(0) = 0$, and a local diffeomorphism $\phi(s, (x, y))$, with $\phi(s, (x, y)) = (\eta(s), \psi(\eta(s), (x, y)))$ and $\phi(0, (x, y)) = (0, (x, y))$, such that $F = (\eta^*G) \circ \phi$. Hence, the Reeb graphs of $F(s, \cdot)$ and $G(\eta(s), \cdot)$ are isomorphic. Moreover, the difference in the values of the labels of corresponding vertices in the Reeb graphs of $F(s, \cdot)$ and $G(\eta(s), \cdot)$ continuously depends on s , and is 0 for $s = 0$. Therefore, for every $\delta > 0$, taking $|s|$ sufficiently small, it is possible to transform the labeled Reeb graph of $F(s, \cdot)$ into that of $G(\eta(s), \cdot)$, or viceversa, by a deformation of type (R) whose cost is not greater than $\delta/3$. Moreover, as equalities (4.1)-(4.2) will show, for every $\delta > 0$, $|s|$ can be taken sufficiently small that the distance between the labeled Reeb graphs of $G(\eta(s), \cdot)$ and $G(\eta(-s), \cdot)$ is not greater than $\delta/3$.

Applying the triangle inequality, we deduce that, for every $\delta > 0$, there exists a sufficiently small $s > 0$ such that the distance between the labeled Reeb graphs of $F(s, \cdot)$ and $F(-s, \cdot)$ is not greater than δ . Thus the claim follows taking $\lambda' = \bar{\lambda} - s$ and $\lambda'' = \bar{\lambda} + s$.

Let us construct the universal deformation G .

If $\bar{h} \in \mathcal{F}_\alpha^1$, let \bar{p} be the sole degenerate critical point for \bar{h} . Let (x, y) be a suitable local coordinate system around \bar{p} in which the canonical expression of \bar{h} is $\bar{h}(x, y) = \bar{h}(\bar{p}) \pm x^2 + y^3$. Let $\omega : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth function equal to 1 in a neighborhood of \bar{p} , which decreases moving from \bar{p} , and whose support is contained in the chosen coordinate chart around \bar{p} . Finally, let $G(s, (x, y)) = \bar{h}(x, y) + s \cdot \omega(x, y) \cdot y$.

For $s < 0$, $G(s, \cdot)$ has no critical points in the support of ω and is equal to \bar{h} everywhere else, while, for $s > 0$, $G(s, \cdot)$ has exactly two critical points in the support of ω , precisely $p_1 = (0, -\sqrt{\frac{s}{3}})$ and $p_2 = (0, \sqrt{\frac{s}{3}})$, and is equal to \bar{h} everywhere else (see Figure 5). Therefore, for every $s > 0$ sufficiently small, the labeled Reeb graph of $G(-s, \cdot)$ can be transformed into that of $G(s, \cdot)$ by an elementary deformation of type (B). By Definition 3.1, a direct computation shows that its cost is

$$(4.1) \quad \frac{\left| \bar{h}(\bar{p}) + \left(\sqrt{\frac{s}{3}}\right)^3 + s \cdot \sqrt{\frac{s}{3}} - \left(\bar{h}(\bar{p}) - \left(\sqrt{\frac{s}{3}}\right)^3 - s \cdot \sqrt{\frac{s}{3}} \right) \right|}{2} = 4 \cdot \left(\frac{s}{3}\right)^{3/2}.$$

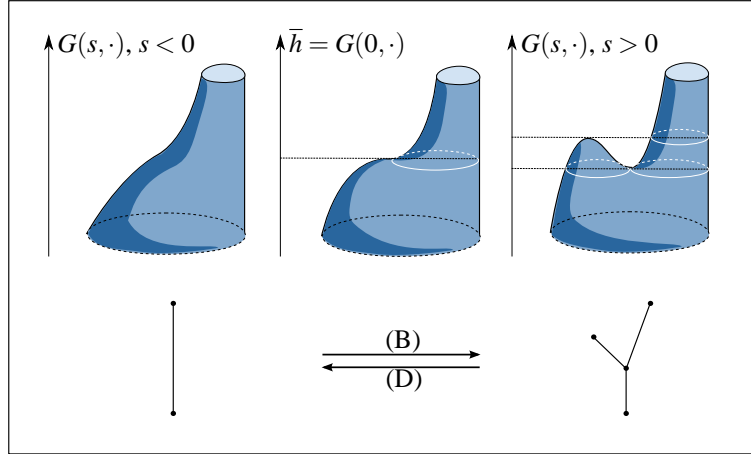


FIGURE 5. A function $\bar{h} \in \mathcal{F}_\alpha^1$ with one degenerate critical point (center) admits a universal deformation $G(s, \cdot)$ in which, for $s < 0$, the degenerate critical point disappears (left), while, for $s > 0$, the degenerate critical point is split into non-degenerate singularities (right). The labeled Reeb graphs associated with $G(s, \cdot)$ for $s < 0$ and $s > 0$ can be obtained one from the other through an elementary deformation of type (B) or (D).

Obviously, in the case $s < 0$, the deformation we consider is of type (D), and its cost is the same because of Proposition 3.2.

Let us now assume that $\bar{h} \in \mathcal{F}_\beta^1$, and let \bar{p} and \bar{q} be the critical points of \bar{h} such that $\bar{h}(\bar{p}) = \bar{h}(\bar{q})$.

We distinguish the following two situations:

- (1) The points \bar{p} and \bar{q} belong to two different connected components of $\bar{h}^{-1}(\bar{h}(\bar{p}))$ (see Figure 6).

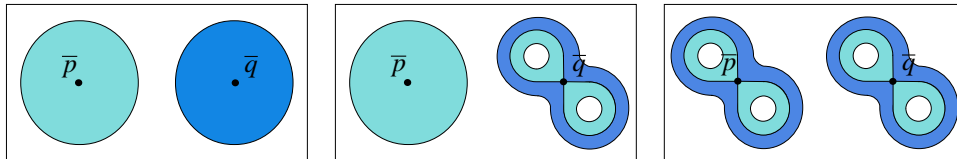


FIGURE 6. Two critical points belonging to different connected components of the same critical level. The dark (light, respectively) regions correspond to points under (over, respectively) the considered critical level. Possibly inverting the colors of one or both the components, we obtain all the possible cases.

- (2) The points \bar{p} and \bar{q} belong to the same connected component of $\bar{h}^{-1}(\bar{h}(\bar{p}))$ (see Figure 7).

In both the cases (1) and (2), since \bar{p} is non-degenerate, there exists a suitable local coordinate system (x, y) around \bar{p} in which the canonical expression of \bar{h} is $\bar{h}(x, y) = \bar{h}(\bar{p}) + x^2 + y^2$ if \bar{p} is a minimum, or $\bar{h}(x, y) = \bar{h}(\bar{p}) - x^2 - y^2$ if \bar{p} is a maximum, or $\bar{h}(x, y) = \bar{h}(\bar{p}) \pm x^2 \mp y^2$ if \bar{p} is a saddle point.

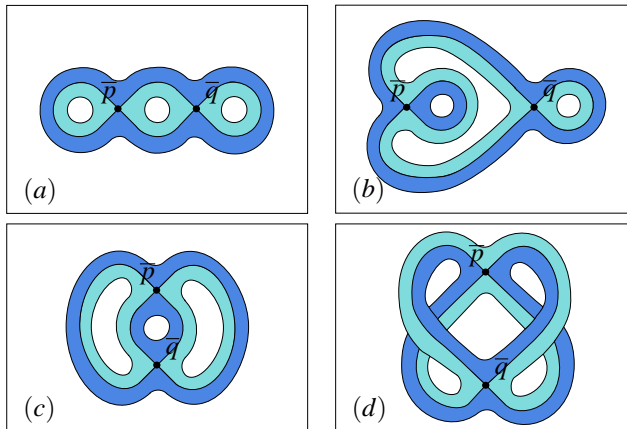


FIGURE 7. Two critical points belonging to the same connected component of the same critical level. The dark (light, respectively) regions correspond to points under (over, respectively) the considered critical level. Possibly inverting the colors of this component, we obtain all the possible cases.

Let $\omega : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth function equal to 1 in a neighborhood of \bar{p} , which decreases moving from \bar{p} , and whose support is contained in the coordinate chart around \bar{p} in which \bar{h} has one of the above expressions. Finally, let $G(s, (x, y)) = \bar{h}(x, y) + s \cdot \omega(x, y)$.

For every s sufficiently small, $G(s, \cdot)$ has the same critical points, with the same indices, as \bar{h} . As for critical values, they are the same as well, apart from the value taken at \bar{p} : $G(s, \bar{p}) = \bar{h}(\bar{p}) + s$. Therefore, for every s sufficiently small, the labeled Reeb graph of $G(s, \cdot)$, with $s < 0$, can be transformed into that of $G(s, \cdot)$, with $s > 0$, by one of the following elementary deformations.

In all the cases (1), for every $s > 0$ sufficiently small, the deformation T which takes the labeled Reeb graph of $G(-s, \cdot)$ to the labeled Reeb graph of $G(s, \cdot)$ is of type (R).

As for the cases (2), the following deformations shall be considered:

- If \bar{p} and \bar{q} are as in Figure 7 (a), for every $s > 0$ sufficiently small, the deformation T which takes the labeled Reeb graph of $G(-s, \cdot)$ to the labeled Reeb graph of $G(s, \cdot)$ is of type (K_1) (see e.g. the example in Figure 8).
- If \bar{p} and \bar{q} are as in Figure 7 (b), for every $s > 0$ sufficiently small, the deformation T which takes the labeled Reeb graph of $G(-s, \cdot)$ to the labeled Reeb graph of $G(s, \cdot)$ is of type (K_3) , while the deformation which takes the labeled Reeb graph of $G(s, \cdot)$ to the labeled Reeb graph of $G(-s, \cdot)$ is of type (K_2) (see e.g. the example in Figure 9).
- If \bar{p} and \bar{q} are as in Figure 7 (c) or (d), for every $s > 0$ sufficiently small, the deformation T which takes the labeled Reeb graph of $G(-s, \cdot)$ to the labeled Reeb graph of $G(s, \cdot)$ is of type (R) (see e.g. the examples in Figures 10-11).

In all the cases, the cost of the considered deformation T is:

$$(4.2) \quad c(T) = |\bar{h}(\bar{p}) - s - (\bar{h}(\bar{p}) + s)| = 2s.$$

□

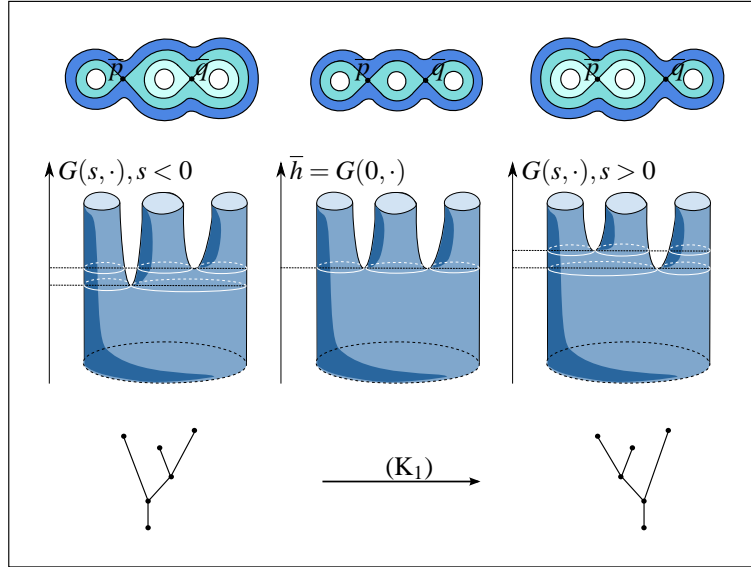


FIGURE 8. A function $\bar{h} \in \mathcal{F}_\beta^1$ with two saddles as in Figure 7 (a) (center) admits a universal deformation $G(s, \cdot)$ in which these two critical points belong to different critical levels (left-right). The labeled Reeb graphs associated with $G(s, \cdot)$ for $s < 0$ and $s > 0$ can be obtained one from the other through an elementary deformation of type (K_1) .

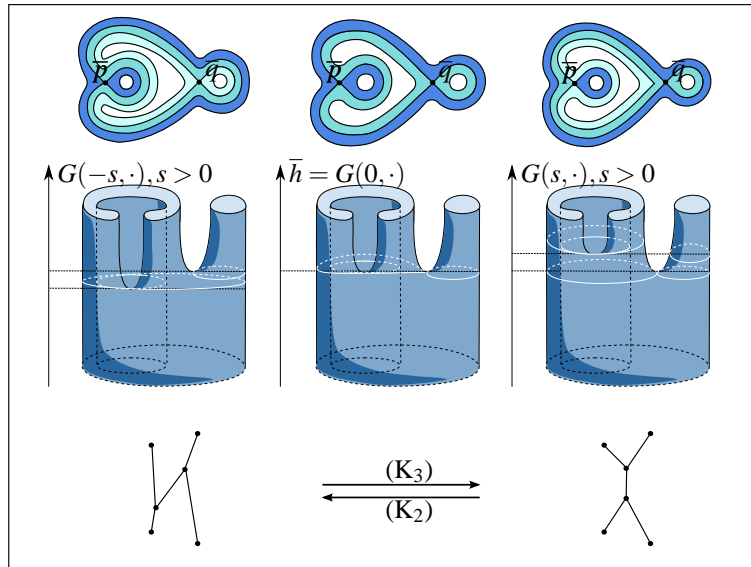


FIGURE 9. A function $\bar{h} \in \mathcal{F}_\beta^1$ with two saddles as in Figure 7 (b) (center) admits a universal deformation $G(s, \cdot)$ in which these two critical points belong to different critical levels (left-right). The labeled Reeb graphs associated with $G(s, \cdot)$ for $s < 0$ and $s > 0$ can be obtained one from the other through an elementary deformation of type (K_3) or its inverse (K_2) .

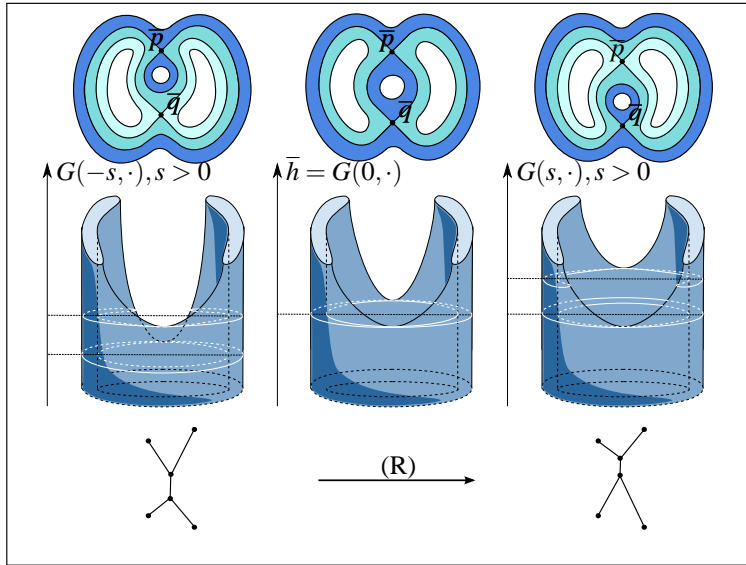


FIGURE 10. A function $\bar{h} \in \mathcal{F}_\beta^1$ with two saddles as in Figure 7 (c) (center) admits a universal deformation $G(s, \cdot)$ in which these two critical points belong to different critical levels (left-right). The labeled Reeb graphs associated with $G(s, \cdot)$ for $s < 0$ and $s > 0$ can be obtained one from the other through an elementary deformation of type (R).

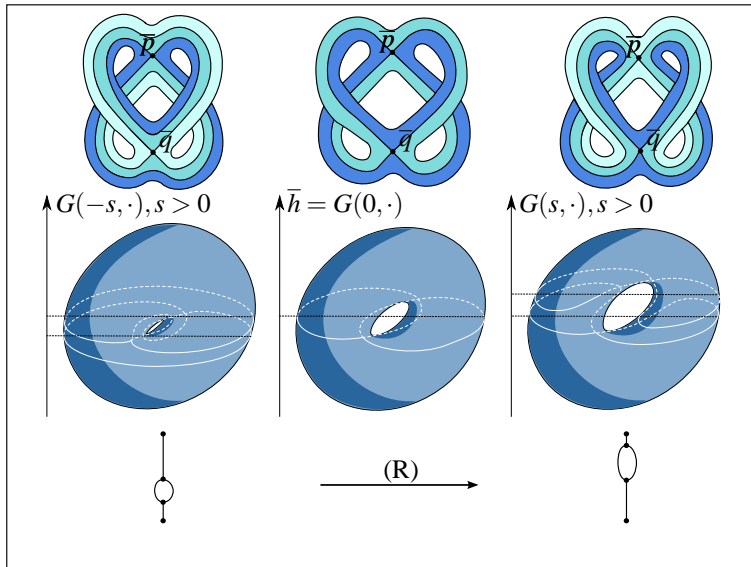


FIGURE 11. A function $\bar{h} \in \mathcal{F}_\beta^1$ with two saddles as in Figure 7 (d) (center) admits a universal deformation $G(s, \cdot)$ in which these two critical points belong to different critical levels (left-right). The labeled Reeb graphs associated with $G(s, \cdot)$ for $s < 0$ and $s > 0$ can be obtained one from the other through an elementary deformation of type (R).

Lemma 4.5. *If $h(\lambda)$ belongs to \mathcal{F}^0 for every $\lambda \in [0, 1]$ apart from one value $0 < \bar{\lambda} < 1$ at which h transversely intersects \mathcal{F}^1 , then $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \|f - g\|_{C^0}$.*

Proof. Let $\bar{h} = h(\bar{\lambda})$. By Lemma 4.4, for every real number $\delta > 0$, we can find two values $0 < \lambda' < \bar{\lambda} < \lambda'' < 1$ such that $d((\Gamma_{h(\lambda')}, \ell_{h(\lambda')}), (\Gamma_{h(\lambda'')}, \ell_{h(\lambda'')})) \leq \delta$.

Applying the triangle inequality, we have:

$$d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq d((\Gamma_f, \ell_f), (\Gamma_{h(\lambda')}, h(\lambda')|_{\cdot})) + d((\Gamma_{h(\lambda')}, h(\lambda')|_{\cdot}), (\Gamma_{h(\lambda'')}, h(\lambda'')|_{\cdot})) \\ + d((\Gamma_{h(\lambda'')}, h(\lambda'')|_{\cdot}), (\Gamma_g, \ell_g)).$$

From Lemma 4.3, and from Lemma 4.2 with $f = h(0)$, $g = h(1)$, we get

$$d((\Gamma_f, \ell_f), (\Gamma_{h(\lambda')}, h(\lambda')|_{\cdot})) \leq \|f - h(\lambda')\|_{C^0} = \lambda' \cdot \|f - g\|_{C^0},$$

and

$$d((\Gamma_{h(\lambda'')}, h(\lambda'')|_{\cdot}), (\Gamma_g, \ell_g)) \leq \|h(\lambda'') - g\|_{C^0} = (1 - \lambda'') \cdot \|f - g\|_{C^0}.$$

Hence,

$$d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq (1 + \lambda' - \lambda'') \cdot \|f - g\|_{C^0} + \delta.$$

In conclusion, given that $0 < \lambda' < \lambda''$, the inequality $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \|f - g\|_{C^0} + \delta$ holds. This yields the claim by the arbitrariness of δ . \square

We are now ready to prove the stability Theorem 4.1.

Proof of Theorem 4.1. Recall from [9] that \mathcal{F}^0 is open in \mathcal{F} endowed with the C^2 topology. Then, for every sufficiently small real number $\delta > 0$, the neighborhoods $N(f, \delta)$ and $N(g, \delta)$ are contained in \mathcal{F}^0 . Take $\hat{f} \in N(f, \delta)$ and $\hat{g} \in N(g, \delta)$ such that the path $h : [0, 1] \rightarrow \mathcal{F}$, with $h(\lambda) = (1 - \lambda)\hat{f} + \lambda\hat{g}$, belongs to \mathcal{F}^0 for every $\lambda \in [0, 1]$, except for at most a finite number n of values at which h transversely intersects \mathcal{F}^1 .

We begin by proving our statement for \hat{f} and \hat{g} , and then show its validity for f and g .

We proceed by induction on n .

If $n = 0$ or $n = 1$, the inequality $d((\Gamma_{\hat{f}}, \ell_{\hat{f}}), (\Gamma_{\hat{g}}, \ell_{\hat{g}})) \leq \|\hat{f} - \hat{g}\|_{C^0}$ holds because of Lemma 4.3 or 4.5, respectively.

Let us assume the claim is true for $n \geq 1$, and prove it for $n + 1$.

Let $0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \dots < \mu_n < \lambda_n < \mu_{n+1} < 1$, with $h(0) = \hat{f}$, $h(1) = \hat{g}$, $h(\mu_i) \in \mathcal{F}^1$, for every $i = 1, \dots, n + 1$, and $h(\lambda_j) \in \mathcal{F}^0$, for every $j = 1, \dots, n$. We consider h as the concatenation of the paths $h^1, h^2 : [0, 1] \rightarrow \mathcal{F}$, defined, respectively, as $h^1(\lambda) = (1 - \lambda)\hat{f} + \lambda h(\lambda_n)$, and $h^2(\lambda) = (1 - \lambda)h(\lambda_n) + \lambda\hat{g}$. The path h^1 transversally intersect \mathcal{F}^1 at n values μ_1, \dots, μ_n . Hence, by inductive hypothesis, we have $d((\Gamma_{\hat{f}}, \ell_{\hat{f}}), (\Gamma_{h(\lambda_n)}, \ell_{h(\lambda_n)})) \leq \|\hat{f} - h(\lambda_n)\|_{C^0}$. Moreover, the path h^2 transversally intersect \mathcal{F}^1 only at the value μ_{n+1} . Consequently, by Lemma 4.5, we have $d((\Gamma_{h(\lambda_n)}, \ell_{h(\lambda_n)}), (\Gamma_{\hat{g}}, \ell_{\hat{g}})) \leq \|h(\lambda_n) - \hat{g}\|_{C^0}$. Using the triangle inequality and Lemma 4.2, we can conclude that:

$$(4.3) \quad d((\Gamma_{\hat{f}}, \ell_{\hat{f}}), (\Gamma_{\hat{g}}, \ell_{\hat{g}})) \leq d((\Gamma_{\hat{f}}, \ell_{\hat{f}}), (\Gamma_{h(\lambda_n)}, \ell_{h(\lambda_n)})) + d((\Gamma_{h(\lambda_n)}, \ell_{h(\lambda_n)}), (\Gamma_{\hat{g}}, \ell_{\hat{g}})) \\ \leq \lambda_n \|\hat{f} - \hat{g}\|_{C^0} + (1 - \lambda_n) \|\hat{f} - \hat{g}\|_{C^0} = \|\hat{f} - \hat{g}\|_{C^0}.$$

Let us now estimate $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$. By triangle inequality, we have:

$$d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq d((\Gamma_f, \ell_f), (\Gamma_{\hat{f}}, \ell_{\hat{f}})) + d((\Gamma_{\hat{f}}, \ell_{\hat{f}}), (\Gamma_{\hat{g}}, \ell_{\hat{g}})) + d((\Gamma_{\hat{g}}, \ell_{\hat{g}}), (\Gamma_g, \ell_g)).$$

Since $\hat{f} \in N(f, \delta) \subset \mathcal{F}^0$ and $\hat{g} \in N(g, \delta) \subset \mathcal{F}^0$, the following facts hold: (a) for every $\lambda \in [0, 1]$, $(1 - \lambda)f + \lambda\hat{f}, (1 - \lambda)g + \lambda\hat{g} \in \mathcal{F}^0$; (b) $\|f - \hat{f}\|_{C^0} \leq \delta$ and $\|g - \hat{g}\|_{C^0} \leq \delta$. Hence,

from (a), we have $d((\Gamma_f, \ell_f), (\Gamma_{\hat{f}}, \ell_{\hat{f}})) \leq \|f - \hat{f}\|_{C^0}$, and $d((\Gamma_g, \ell_g), (\Gamma_{\hat{g}}, \ell_{\hat{g}})) \leq \|\hat{g} - g\|_{C^0}$ because of Lemma 4.3. Using also (b), and inequality 4, we deduce that $d((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \leq \|f - g\|_{C^0} + 2\delta$. This yields the conclusion by the arbitrariness of δ . \square

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