THE EDIT DISTANCE FOR REEB GRAPHS OF SURFACES

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ABSTRACT. Reeb graphs are structural descriptors that capture shape properties of a topological space from the perspective of a chosen function. In this work we define a combinatorial metric for Reeb graphs of orientable surfaces in terms of the cost necessary to transform one graph into another by edit operations. The main contributions of this paper are the stability property and the optimality of this edit distance. More precisely, the stability result states that changes in the functions, measured by the maximum norm, imply not greater changes in the corresponding Reeb graphs, measured by the edit distance. The optimality result states that our edit distance discriminates Reeb graphs better than any other metric for Reeb graphs of surfaces satisfying the stability property.

INTRODUCTION

In shape comparison, a widely used scheme is to measure the dissimilarity between descriptors associated with each shape rather than to match shapes directly. Reeb graphs describe shapes from topological and geometrical perspectives. In this framework, a shape is modeled as a topological space X endowed with a scalar function $f : X \to \mathbb{R}$. The role of f is to explore geometrical properties of the space X. The Reeb graph of f is obtained by shrinking each connected component of a level set of f to a single point [20]. Reeb graphs have been used as an effective tool for shape analysis and description tasks since [24, 23].

One of the most important questions is whether Reeb graphs are robust against perturbations that may occur because of noise and approximation errors in the data acquisition process. Whereas in the past researchers dealt with this problem developing heuristics so that Reeb graphs would be resistant to connectivity changes caused by simplification, subdivision and remesh, and robust against noise and certain changes due to deformation [11, 2], in the last years the question of Reeb graph stability has been investigated from the theoretical point of view. In [7] an edit distance between Reeb graphs of curves endowed with Morse functions is introduced and shown to yield stability. Importantly, despite the combinatorial nature of this distance, it coincides with the natural pseudo-distance between shapes [8], thus showing the maximal discriminative power for this sort of distances. Very recently a functional distortion distance between Reeb graphs has been proposed in [1], with proven stable and discriminative properties. The functional distortion distance is based on continuous maps between the topological spaces realizing the Reeb graphs, so that it is not combinatorial in its definition. Noticeably, it allows for comparison of non-homeomorphic spaces meaning that it can be used to deal also with artifacts that change the homotopy type of the space, although as a consequence it cannot fully discriminate shapes and stability is not proven in that case.

In this paper we deal with the comparison problem for Reeb graphs of surfaces. Indeed the case of surfaces seems to us the most interesting area of application of the Reeb graph as a shape descriptor. As a tradeoff between generality and simplicity, we confine ourselves

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to the case of smooth compact orientable surfaces without boundary endowed with simple Morse functions.

The basic properties we consider important for a metric between Reeb graphs are: the robustness to perturbations of the input functions; the ability to discriminate functions on the same manifold; the deployment of the combinatorial nature of graphs. For this reason, we apply to the case of surfaces the same underlying ideas as used in [7] for curves. Starting from Reeb graphs labeled on the vertices by the function values, the following steps are carried out: first, a set of admissible edit operations is detected to transform a labeled Reeb graph into another; then a suitable cost is associated to each edit operation; finally, a combinatorial dissimilarity measure between labeled Reeb graphs, called an *edit distance*, is defined in terms of the least cost necessary to transform one graph into another by edit operations. However, the passage from curves to surfaces is not automatic since Reeb graphs of surfaces are structurally different from those of curves. For example, the degree of vertices is different for Reeb graphs of curves and surfaces. Therefore, the set of edit operations as well as their costs cannot be directly imported from the case of curves but need to be suitably defined. In conclusion, our edit distance between Reeb graphs belongs to the family of Graph Edit Distances [10], widely used in pattern analysis.

Our first main result is that changes in the functions, measured by the maximum norm, imply not greater changes in this edit distance, yielding the stability property under function perturbations. To prove this result, we track the changes in the Reeb graphs as the function varies along a linear path avoiding degeneracies. From the stability property, we deduce that the edit distance between the Reeb graphs of two functions f and g defined on a surface is a lower bound for the natural pseudo-distance between f and g obtained by minimizing the change in the functions due to the application of a self-diffeomorphism of the manifold, with respect to the maximum norm. The natural pseudo-distance can be thought as a way to compare f and g directly, while the edit distance provides an indirect comparison between f and g through their Reeb graphs. Thus, by virtue of the stability result, the edit distance provides a combinatorial tool to estimate the natural pseudo-distance.

Our second contribution is the proof that the edit distance between Reeb graphs of surfaces actually coincides with the natural pseudo-distance. This is proved by showing that for every edit operation on a Reeb graph there is a self-homeomorphism of the surface whose cost is not greater than that of the considered edit operation. This result implies that the edit distance is actually a metric and not only a pseudo-metric. Morever it shows that the edit distance is an optimal distance for Reeb graphs of surfaces in that it has the maximum discriminative power among all the distances between Reeb graphs of surfaces with the stability property.

In conclusion, this paper shows that the results of [7] for curves also hold in the more interesting case of surfaces.

The paper is organized as follows. In Section 1 we recall the basic properties of labeled Reeb graphs of orientable surfaces. In Section 2 we define the edit deformations between labeled Reeb graphs, and show that through a finite sequence of these deformations we can always transform a Reeb graph into another. In Section 3 we define the cost associated with each type of edit deformation and the edit distance in terms of this cost. Section 4 illustrates the robustness of Reeb graphs with respect to the edit distance. Eventually, Section 5 provides relationships between our edit distance and other stable metrics: the natural pseudo-distance, the bottleneck distance and the functional distortion distance.

A number of questions remain open and are not treated in this paper. The most important one is how to compute the edit distance. Indeed, whereas in some particular cases we can deduce the value of the edit distance from the lower bounds provided by the bottleneck distance of persistence diagrams or the functional distortion distance of Reeb graphs, in general we do not know how to compute it. A second open problem is to which extent the theory developed in this paper for the smooth category can be transported into the piecewise linear category. A third question that would deserve investigation is how to generalize the edit distance to compare functions on non-homeomorphic surfaces as well, and the relationship with the functional distortion distance in that case.

1. LABELED REEB GRAPHS OF ORIENTABLE SURFACES

Hereafter, $\mathscr{M}$ denotes a connected, closed (i.e. compact and without boundary), orientable, smooth surface of genus $\mathfrak{g}$, and $\mathscr{F}(\mathscr{M})$ the set of C^∞ real functions on $\mathscr{M}$.

For $f \in \mathscr{F}(\mathscr{M})$, we denote by K_f the set of its critical points. If $p \in K_f$, then the real number $f(p)$ is called a *critical value* of f, and the set $\{q \in \mathscr{M} : q \in f^{-1}(f(p))\}$ is called a *critical level* of f. Moreover, a critical point p is called *non-degenerate* if the Hessian matrix of f at p is non-singular. The *index* of a non-degenerate critical point p of f is the dimension of the largest subspace of the tangent space to $\mathscr{M}$ at p on which the Hessian matrix is negative definite. In particular, the index of a point $p \in K_f$ is equal to 0,1, or 2 depending on whether p is a minimum, a saddle, or a maximum point of f.

A function $f \in \mathscr{F}(\mathscr{M})$ is called a *Morse function* if all its critical points are non-degenerate. Besides, a Morse function is said to be *simple* if each critical level contains exactly one critical point. The set of simple Morse functions will be denoted by $\mathscr{F}^0(\mathscr{M})$, as a reminder that it is a sub-manifold of $\mathscr{F}(\mathscr{M})$ of co-dimension 0 (see also Section 4).

Definition 1.1. Let $f \in \mathscr{F}^0(\mathscr{M})$, and define on $\mathscr{M}$ the following equivalence relation: for every $p,q \in \mathscr{M}$, $p \sim_f q$ whenever p,q belong to the same connected component of $f^{-1}(f(p))$. The quotient space $\mathscr{M}/\sim_f$ is the *Reeb graph* associated with f.

Our assumption that f is a simple Morse function allows us to consider the space $\mathscr{M}/\sim_f$ as a graph whose vertices correspond bijectively to the critical points of f. For this reason, in the following, we will often identify vertices with the corresponding critical points.

Proposition 1.2. *([20]) The Reeb graph Γ_f associated with $f \in \mathscr{F}^0(\mathscr{M})$ is a finite and connected simplicial complex of dimension 1. A vertex of Γ_f has degree equal to 1 if it corresponds to a critical point of f of index 0 or 2, while it has degree equal to 2,3, or 4 if it corresponds to a critical point of f of index 1.*

Throughout this paper, Reeb graphs are regarded as combinatorial graphs and not as topological spaces. The vertex set of Γ_f will be denoted by $V(\Gamma_f)$, and its edge set by $E(\Gamma_f)$. Moreover, if $v_1, v_2 \in V(\Gamma_f)$ are adjacent vertices, i.e., connected by an edge, we will write $e(v_1,v_2) \in E(\Gamma_f)$.

Our assumptions that $\mathscr{M}$ is orientable, compact and without boundary ensure that there are no vertices of degree 2 or 4. Moreover, if p,q,r denote the number of minima, maxima, and saddle points of f, from the relationships between the Euler characteristic of $\mathscr{M}$, $\chi(\mathscr{M})$, and p,q,r, i.e. $\chi(\mathscr{M}) = p+q-r$, and between $\chi(\mathscr{M})$ and the genus $\mathfrak{g}$ of $\mathscr{M}$, i.e. $\chi(\mathscr{M}) = 2-2\mathfrak{g}$, it follows that the cardinality of $V(\Gamma_f)$, which is $p+q+r$, is also equal to $2(p+q+\mathfrak{g}-1)$, i.e. is even in number. The minimum number of vertices of a Reeb graph is achieved whenever $p=q=1$, and consequently $r=2\mathfrak{g}$. In this case the cardinality of $V(\Gamma_f)$ is equal to $2\mathfrak{g}+2$. In general, if $\mathscr{M}$ has genus $\mathfrak{g}$ then Γ_f has exactly $\mathfrak{g}$ linearly independent cycles. We will call a cycle of length m in the graph an *m-cycle*. These observations motivate the following definition.

Definition 1.3. We shall call *minimal* the Reeb graph Γ_f of a function f having $p = q = 1$. Moreover, we say that Γ_f is *canonical* if it is minimal and all its cycles, if any, are 2-cycles.

We underline that our definition of canonical Reeb graph is slightly different from the one in [13]. This choice has been done to simplify the proof of Proposition 2.7.

Examples of minimal and canonical Reeb graphs are displayed in Figure 1.

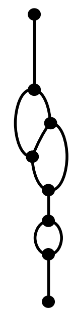

FIGURE 1. Examples of minimal Reeb graphs. The graph on the right is also canonical.

In what follows, we label the vertices of Γ_f by equipping each of them with the value of f at the corresponding critical point. We denote such a labeled graph by (Γ_f, ℓ_f), where $\ell_f : V(\Gamma_f) \to \mathbb{R}$ is the restriction of $f : \mathscr{M} \to \mathbb{R}$ to K_f. In a labeled Reeb graph, each vertex v of degree 3 has at least two of its adjacent vertices, say v_1, v_2, such that $\ell_f(v_1) < \ell_f(v) < \ell_f(v_2)$. An example is displayed in Figure 2.

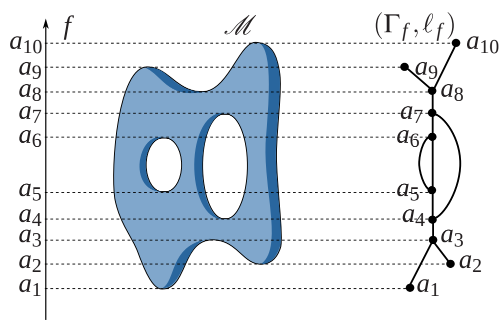

FIGURE 2. Left: the height function $f : \mathscr{M} \to \mathbb{R}$; center: the surface $\mathscr{M}$ of genus $\mathfrak{g} = 2$; right: the associated labeled Reeb graph (Γ_f, ℓ_f).

Let us consider the realization problem, i.e. the problem of constructing a smooth surface and a simple Morse function on it from a graph on an even number of vertices, all of which are of degree 1 or 3, appropriately labeled. We need the following definition.

Definition 1.4. We shall say that two labeled Reeb graphs $(\Gamma_f, \ell_f), (\Gamma_g, \ell_g)$ are *isomorphic*, and we write $(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, if there exists a graph isomorphism $\Phi : V(\Gamma_f) \to V(\Gamma_g)$ such that, for every $v \in V(\Gamma_f)$, $\ell_f(v) = \ell_g(\Phi(v))$ (i.e. Φ preserves edges and vertex labels).

Proposition 1.5 (Realization Theorem)**.** *Let (Γ,ℓ) be a labeled graph, where Γ is a graph with m linearly independent cycles, on an even number of vertices, all of which are of degree 1 or 3, and $\ell: V(\Gamma)\to\mathbb{R}$ is an injective function such that, for any vertex v of degree 3, at least two among its adjacent vertices, say w,w', are such that $\ell(w)<\ell(v)<\ell(w')$. Then an orientable closed surface $\mathscr{M}$ of genus $\mathfrak{g}=m$, and a simple Morse function $f:\mathscr{M}\to\mathbb{R}$ exist such that $(\Gamma_f,\ell_f)\cong(\Gamma,\ell)$.*

Proof. Under our assumption on the degree of vertices of Γ, $\mathscr{M}$ and f can be constructed as in the proof of Thm. 2.1 in [17]. □

We now deal with the uniqueness problem, up to isomorphism of labeled Reeb graphs. First of all we consider the following two equivalence relations on $\mathscr{F}^0(\mathscr{M})$.

Definition 1.6. Let $\mathscr{D}(\mathscr{M})$ be the set of self-diffeomorphisms of $\mathscr{M}$. Two functions $f,g\in\mathscr{F}^0(\mathscr{M})$ are called *right-equivalent* (briefly, *R-equivalent*) if there exists $\xi\in\mathscr{D}(\mathscr{M})$ such that $f=g\circ\xi$. Moreover, f,g are called *right-left equivalent* (briefly, *RL-equivalent*) if there exist $\xi\in\mathscr{D}(\mathscr{M})$ and an orientation preserving self-diffeomorphism η of $\mathbb{R}$ such that $f=\eta\circ g\circ\xi$.

These equivalence relations on functions are mirrored by Reeb graphs isomorphisms.

Proposition 1.7 (Uniqueness Theorem)**.** *If $f,g\in\mathscr{F}^0(\mathscr{M})$, then*

(1) f and g are RL-equivalent if and only if their Reeb graphs Γ_f and Γ_g are isomorphic by an isomorphism $\Phi: V(\Gamma_f)\to V(\Gamma_g)$ that preserves the vertex order, i.e., for every $v,w\in V(\Gamma_f)$, $\ell_f(v)<\ell_f(w)$ if and only if $\ell_g(\Phi(v))<\ell_g(\Phi(w))$;

(2) f and g are R-equivalent if and only if their labeled Reeb graphs (Γ_f,ℓ_f) and (Γ_g,ℓ_g) are isomorphic.

Proof. For the proof of statement (1) see [15, 22]. As for statement (2), we note that two *R*-equivalent functions are, in particular, *RL*-equivalent. Hence, by statement (1), their Reeb graphs are isomorphic by an isomorphism that preserves the vertex order. Since f and g necessarily have the same critical values, this isomorphism also preserves labels. Vice-versa, if (Γ_f,ℓ_f) and (Γ_g,ℓ_g) are isomorphic, then f and g have the same critical values. Moreover, by statement (1), there exist $\xi\in\mathscr{D}(\mathscr{M})$ and an orientation preserving self-diffeomorphism η of $\mathbb{R}$ such that $f=\eta\circ g\circ\xi$. Let us set $h=g\circ\xi$. The function h belongs to $\mathscr{F}^0(\mathscr{M})$ and has the same critical points with the same indexes as f, and the same critical values as g and hence as f. Thus, we can apply [14, Lemma 1] to f and h and deduce the existence of a self-diffeomorphism ξ' of $\mathscr{M}$ such that $f=h\circ\xi'$. Thus $f=g\circ\xi\circ\xi'$, yielding that f and g are *R*-equivalent. A direct proof of the *R*-equivalence of functions with isomorphic labeled Reeb graphs is also obtainable from Lemma 5.3. □

2. EDIT DEFORMATIONS BETWEEN LABELED REEB GRAPHS

In this section we list the edit deformations admissible to transform labeled Reeb graphs into one another when different simple Morse functions on the same surface are considered. We introduce at first elementary deformations, then, by virtue of the Realization Theorem (Proposition 1.5), the deformations obtained by their composition.

Elementary deformations allow us to insert or delete a vertex of degree 1 together with an adjacent vertex of degree 3 (deformations of *birth* type (B) and *death* type (D)), maintain the same vertices and edges while changing the vertex labels (deformations of *relabeling* type (R)), or change some vertices adjacencies and labels (deformations of type (K_1), (K_2), (K_3) introduced by Kudryavtseva in [13]). A sketch of these elementary deformations can be found in Table 1. The formal definition is as follows.

Definition 2.1. With the convention of denoting the open interval with endpoints $a,b \in \mathbb{R}$ by $]a,b[$, the elementary deformations of type (B), (D), (R), (K$_i$), $i = 1,2,3$, are defined as follows.

(B) T is an *elementary deformation* of (Γ_f, ℓ_f) *of type* (B) if, for a fixed edge $e(v_1,v_2) \in E(\Gamma_f)$, with $\ell_f(v_1) < \ell_f(v_2)$, $T(\Gamma_f,\ell_f)$ is a labeled graph (Γ,ℓ) such that
  - $V(\Gamma) = V(\Gamma_f) \cup \{u_1,u_2\}$;
  - $E(\Gamma) = \big(E(\Gamma_f) - \{e(v_1,v_2)\}\big) \cup \{e(v_1,u_1), e(u_1,u_2), e(u_1,v_2)\}$;
  - $\ell_{|V(\Gamma_f)} = \ell_f$ and $\ell_f(v_1) < \ell(u_i) < \ell(u_j) < \ell_f(v_2)$, with $\ell^{-1}(]\ell(u_i),\ell(u_j)[) = \emptyset$, $i,j \in \{1,2\}, i \neq j$.

(D) T is an *elementary deformation* of (Γ_f,ℓ_f) *of type* (D) if, for fixed edges $e(v_1,u_1)$, $e(u_1,u_2)$, $e(u_1,v_2) \in E(\Gamma_f)$, u_2 being of degree 1, such that $\ell_f(v_1) < \ell_f(u_i) < \ell_f(u_j) < \ell_f(v_2)$, with $\ell_f^{-1}(]\ell_f(u_i),\ell_f(u_j)[) = \emptyset$, $i,j \in \{1,2\}, i \neq j$, $T(\Gamma_f,\ell_f)$ is a labeled graph (Γ,ℓ) such that
  - $V(\Gamma) = V(\Gamma_f) - \{u_1,u_2\}$;
  - $E(\Gamma) = \big(E(\Gamma_f) - \{e(v_1,u_1), e(u_1,u_2), e(u_1,v_2)\}\big) \cup \{e(v_1,v_2)\}$;
  - $\ell = \ell_{f|V(\Gamma_f)-\{u_1,u_2\}}$.

(R) T is an *elementary deformation* of (Γ_f,ℓ_f) *of type* (R) if $T(\Gamma_f,\ell_f)$ is a labeled graph (Γ,ℓ) such that
  - $\Gamma = \Gamma_f$;
  - $\ell : V(\Gamma) \to \mathbb{R}$ induces the same vertex-order as ℓ_f except for at most two non-adjacent vertices, say u_1,u_2, with $\ell_f(u_1) < \ell_f(u_2)$ and $\ell_f^{-1}(]\ell_f(u_1),\ell_f(u_2)[) = \emptyset$, for which $\ell(u_1) > \ell(u_2)$ and $\ell^{-1}(]\ell(u_2),\ell(u_1)[) = \emptyset$.

(K$_1$) T is an *elementary deformation* of (Γ_f,ℓ_f) *of type* (K$_1$) if, for fixed edges $e(v_1,u_1)$, $e(u_1,u_2)$, $e(u_1,v_4)$, $e(u_2,v_2)$, $e(u_2,v_3) \in E(\Gamma_f)$, with two among v_2,v_3,v_4 possibly coincident, $\ell_f(v_1) < \ell_f(u_1) < \ell_f(u_2) < \ell_f(v_2), \ell_f(v_3), \ell_f(v_4)$, and $\ell_f^{-1}(]\ell_f(u_1),\ell_f(u_2)[) = \emptyset$ (resp. $\ell_f(v_2), \ell_f(v_3), \ell_f(v_4) < \ell_f(u_2) < \ell_f(u_1) < \ell_f(v_1)$, and $\ell_f^{-1}(]\ell_f(u_2),\ell_f(u_1)[) = \emptyset$), $T(\Gamma_f,\ell_f)$ is a labeled graph (Γ,ℓ) such that:
  - $V(\Gamma) = V(\Gamma_f)$;
  - $E(\Gamma) = \big(E(\Gamma_f) - \{e(v_1,u_1), e(u_2,v_2)\}\big) \cup \{e(v_1,u_2), e(u_1,v_2)\}$;
  - $\ell_{|V(\Gamma)-\{u_1,u_2\}} = \ell_f$ and $\ell_f(v_1) < \ell(u_2) < \ell(u_1) < \ell_f(v_2)$, $\ell_f(v_3)$, $\ell_f(v_4)$, with $\ell^{-1}(]\ell(u_2),\ell(u_1)[) = \emptyset$ (resp. $\ell_f(v_2)$, $\ell_f(v_3)$, $\ell_f(v_4) < \ell(u_1) < \ell(u_2) < \ell_f(v_1)$, with $\ell^{-1}(]\ell(u_1),\ell(u_2)[) = \emptyset$).

(K$_2$) T is an *elementary deformation* of (Γ_f,ℓ_f) *of type* (K$_2$) if, for fixed edges $e(v_1,u_1)$, $e(v_2,u_1)$, $e(u_1,u_2)$, $e(u_2,v_3)$, $e(u_2,v_4) \in E(\Gamma_f)$, with u_1,u_2 of degree 3, v_2,v_3 possibly coincident with v_1,v_4, respectively, and $\ell_f(v_1), \ell_f(v_2) < \ell_f(u_1) < \ell_f(u_2) < \ell_f(v_3), \ell_f(v_4)$, with $\ell_f^{-1}(]\ell_f(u_1),\ell_f(u_2)[) = \emptyset$, $T(\Gamma_f,\ell_f)$ is a labeled graph (Γ,ℓ) such that:
  - $V(\Gamma) = V(\Gamma_f)$;
  - $E(\Gamma) = \big(E(\Gamma_f) - \{e(v_1,u_1), e(u_2,v_3)\}\big) \cup \{e(v_1,u_2), e(u_1,v_3)\}$;
  - $\ell_{|V(\Gamma)-\{u_1,u_2\}} = \ell_f$ and $\ell_f(v_1)$, $\ell_f(v_2) < \ell(u_2) < \ell(u_1) < \ell_f(v_3)$, $\ell_f(v_4)$, with $\ell^{-1}(]\ell(u_2),\ell(u_1)[) = \emptyset$.

(K$_3$) T is an *elementary deformation* of (Γ_f,ℓ_f) *of type* (K$_3$) if, for fixed edges $e(v_1,u_2)$, $e(u_1,u_2)$, $e(v_2,u_1)$, $e(u_1,v_3)$, $e(u_2,v_4) \in E(\Gamma_f)$, with u_1,u_2 of degree 3, v_2,v_3 possibly coincident with v_1,v_4, respectively, and $\ell_f(v_1), \ell_f(v_2) < \ell_f(u_2) < \ell_f(u_1) < \ell_f(v_3), \ell_f(v_4)$, with $\ell_f^{-1}(]\ell_f(u_2),\ell_f(u_1)[) = \emptyset$, $T(\Gamma_f,\ell_f)$ is a labeled graph (Γ,ℓ) such that:
  - $V(\Gamma) = V(\Gamma_f)$;

– $E(\Gamma) = \big(E(\Gamma_f) - \{e(v_1,u_2), e(u_1,v_3)\}\big) \cup \{e(v_1,u_1), e(u_2,v_3)\}$;
– $\ell_{|V(\Gamma)-\{u_1,u_2\}} = \ell_f$ and $\ell_f(v_1),\ \ell_f(v_2) < \ell(u_1) < \ell(u_2) < \ell_f(v_3),\ \ell_f(v_4)$, with $\ell^{-1}(]\ell(u_1), \ell(u_2)[) = \emptyset$.

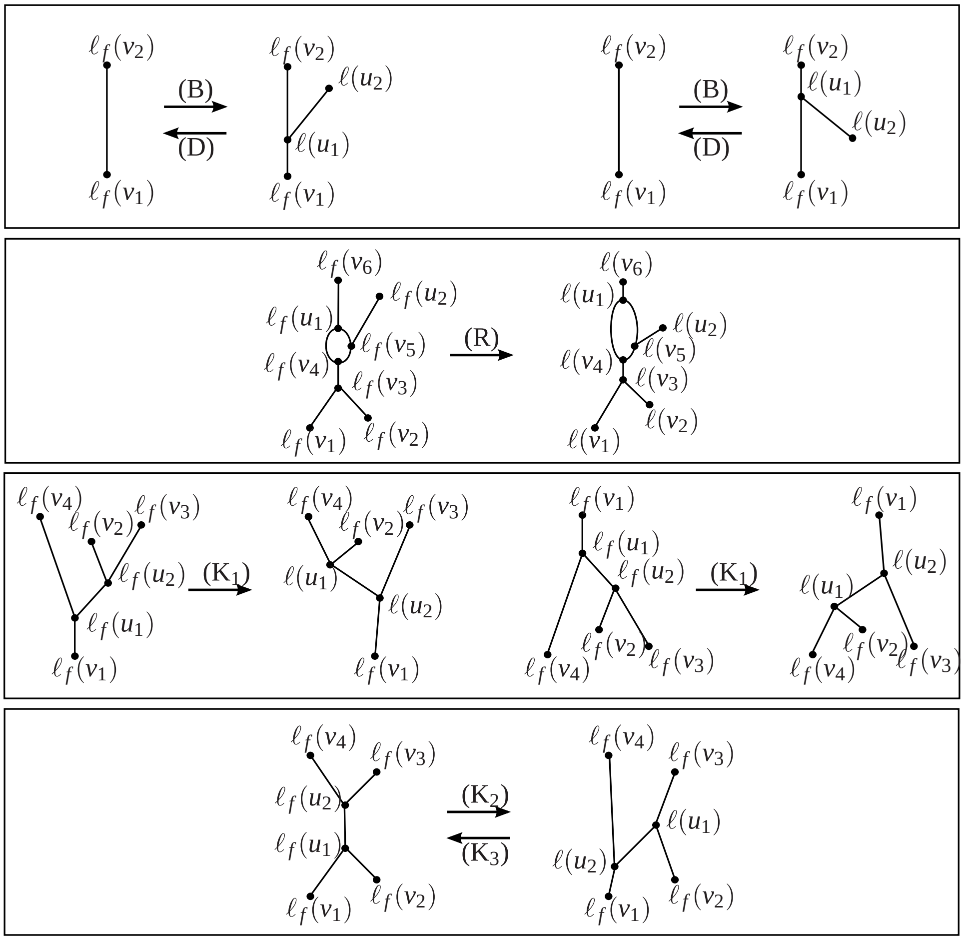

TABLE 1. A schematization of the elementary deformations of a labeled Reeb graph provided by Definition 2.1.

We underline that the definition of the deformations of type (B), (D) and (R) is essentially different from the definition of analogous deformations in the case of Reeb graphs of curves as given in [7], even if the associated cost will be the same (see Section 3). This is because the degree of the involved vertices is 2 for Reeb graphs of closed curves, whereas it is 1 or 3 for Reeb graphs of surfaces.

Proposition 2.2. *Let $f \in \mathscr{F}^0(\mathscr{M})$ and $(\Gamma, \ell) = T(\Gamma_f, \ell_f)$ for some elementary deformation T. Then there exists $g \in \mathscr{F}^0(\mathscr{M})$ such that $(\Gamma_g, \ell_g) \cong (\Gamma, \ell)$.*

Proof. The claim follows from Proposition 1.5. □

As a consequence of Proposition 2.2, we can apply elementary deformations iteratively. This fact is used in the next Definition 2.3. Given an elementary deformation T of (Γ_f, ℓ_f)

and an elementary deformation S of $T(\Gamma_f,\ell_f)$, the juxtaposition ST means applying first T and then S.

Definition 2.3. We shall call *deformation* of (Γ_f,ℓ_f) any finite ordered sequence $T=(T_1,T_2,\ldots,T_r)$ of elementary deformations such that T_1 is an elementary deformation of (Γ_f,ℓ_f), T_2 is an elementary deformation of $T_1(\Gamma_f,\ell_f)$, ..., T_r is an elementary deformation of $T_{r-1}T_{r-2}\cdots T_1(\Gamma_f,\ell_f)$. We shall denote by $T(\Gamma_f,\ell_f)$ the result of the deformation $T_rT_{r-1}\cdots T_1$ applied to (Γ_f,ℓ_f).

In the rest of the paper we write $\mathscr{T}((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$ to denote the set of deformations turning (Γ_f,ℓ_f) into (Γ_g,ℓ_g) up to isomorphism:

$$\mathscr{T}((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))=\{T=(T_1,\ldots,T_n),n\geq 1: T(\Gamma_f,\ell_f)\cong(\Gamma_g,\ell_g)\}.$$

We now introduce the concept of inverse deformation.

Definition 2.4. Let $T\in\mathscr{T}((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$ and let Φ be the labeled graph isomorphism between $T(\Gamma_f,\ell_f)$ and (Γ_g,ℓ_g). We denote by T^{-1}, and call it the *inverse* of T in $\mathscr{T}((\Gamma_g,\ell_g),(\Gamma_f,\ell_f))$, the deformation that acts on the vertices, edges, and labels of (Γ_g,ℓ_g) as follows: identifying $T(\Gamma_f,\ell_f)$ with (Γ_g,ℓ_g) via Φ,

- if T is an elementary deformation of type (D) deleting two vertices, then T^{-1} is of type (B) inserting the same vertices, with the same labels, and viceversa;
- if T is an elementary deformation of type (R) relabeling vertices of $V(\Gamma_f)$, then T^{-1} is again of type (R) relabeling these vertices in the inverse way;
- if T is an elementary deformation of type (K_1) relabeling two vertices, then T^{-1} is again of type (K_1) relabeling the same vertices in the inverse way;
- if T is an elementary deformation of type (K_2) relabeling two vertices, then T^{-1} is of type (K_3) relabeling the same vertices in the inverse way, and viceversa;
- if $T=(T_1,\ldots,T_r)$, then $T^{-1}=(T_r^{-1},\ldots,T_1^{-1})$.

From the fact that $T^{-1}T(\Gamma_f,\ell_f)\cong(\Gamma_f,\ell_f)$ it follows that the set $\mathscr{T}((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$, when non-empty, always contains infinitely many deformations. We end the section proving that for $f,g\in\mathscr{F}^0(\mathscr{M})$ it is always non-empty. We first need two lemmas which are widely inspired by [13, Lemma 1 and Thm. 1], respectively.

Lemma 2.5. *Let (Γ_f,ℓ_f) be a labeled Reeb graph. The following statements hold:*

(i) *For any $u,v\in V(\Gamma_f)$ corresponding to two minima or two maxima of f, there exists a deformation T such that u and v are adjacent to the same vertex w in $T(\Gamma_f,\ell_f)$.*

(ii) *For any m-cycle C in Γ_f, $m\geq 2$, there exists a deformation T such that C is a 2-cycle in $T(\Gamma_f,\ell_f)$.*

Proof. Let us prove statement (i) assuming that in (Γ_f,ℓ_f) there exist two vertices u,v corresponding to two minima of f. The case of maxima is analogous.

Let us consider a path γ on Γ_f having u,v as endpoints, whose length is $m\geq 2$, and the finite sequence of vertices through which it passes is $(w_0,w_1,\ldots,w_m)$, with $w_0=u,w_m=v$, and $w_i\neq w_j$ for $i\neq j$. We aim at showing that there exists a deformation T such that in $T(\Gamma_f,\ell_f)$ the vertices u,v are adjacent to the same vertex w, with $w\in\{w_1,\ldots,w_{m-1}\}$, and thus the path γ is transformed by T into a path γ' which is of length 2 and passes through the vertices u,w,v.

If $m=2$, then it is sufficient to take T as the deformation of type (R) such that $T(\Gamma_f,\ell_f)=(\Gamma_f,\ell_f)$ since γ already coincides with γ'. If $m>2$, let $w_i=\operatorname{argmax}\{\ell_f(w_j): w_j \text{ with } 0\leq j\leq m\}$. It holds that $w_i\neq u,v$ because u,v are minima of f and is unique because f

is simple. It is easy to observe that, in a neighborhood of w_i, possibly after a finite sequence of deformations of type (R), the graph gets one of the configurations shown in Figure 3 $(a)-(e)$ (left).

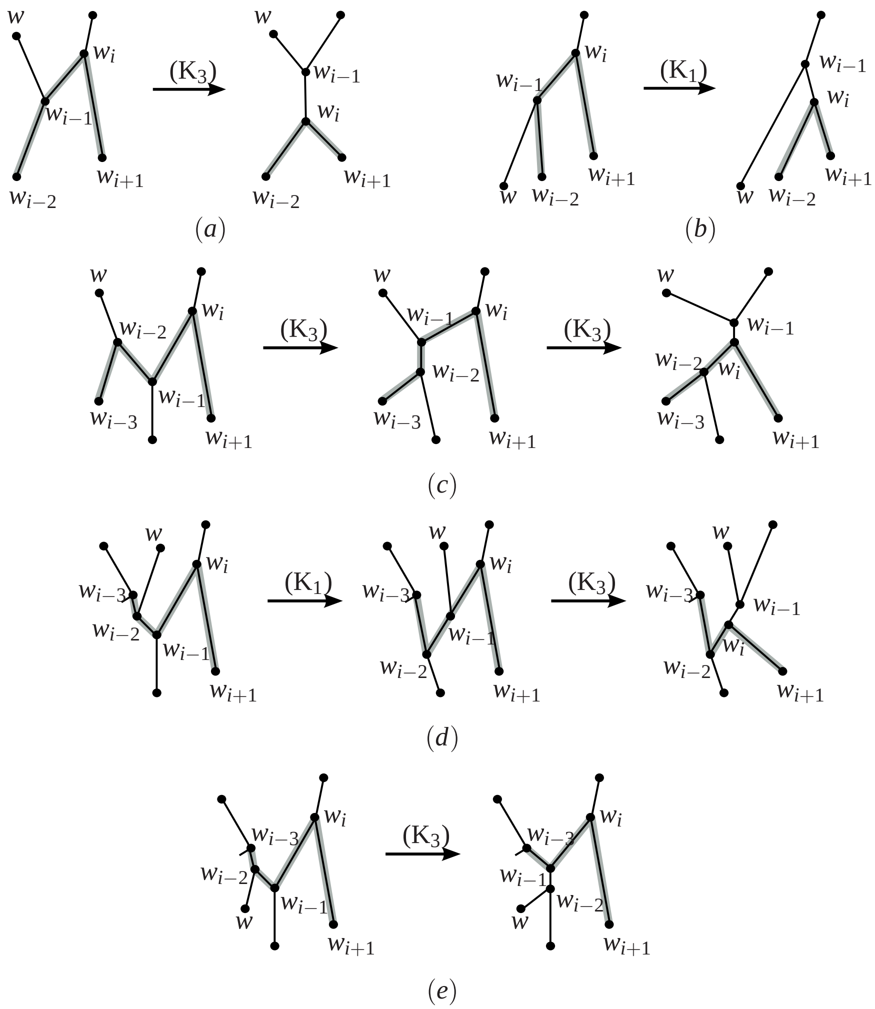

FIGURE 3. Possible configurations of a simple path on a labeled Reeb graph in a neighborhood of its maximum point, and elementary deformations which reduce its length. To facilitate the reader, f has been represented as the height function, so that $\ell_f(w_a) < \ell_f(w_b)$ if and only if w_a is lower than w_b in the pictures.

The same figure shows that a finite sequence of deformations of type (K$_1$), (K$_3$), and, possibly, (R) transforms the simple path γ, which has length m, into a simple path of length $m-1$. Iterating this procedure, we deduce the desired claim.

The proof of statement (ii) is analogous to that of statement (i), provided that γ is taken to be an m-cycle with $u \equiv v$ of degree 3, and $u = \operatorname{argmin}\{\ell_f(w_j) : w_j \text{ with } 0 \le j \le m-1\}$. $\square$

Lemma 2.6. *Every labeled Reeb graph (Γ_f, ℓ_f) can be transformed into a canonical one through a finite sequence of elementary deformations.*

Proof. Our proof is in two steps: first we show how to transform an arbitrary Reeb graph into a minimal one; then how to reduce a minimal Reeb graph to the canonical form.

The first step is by induction on $s = p+q$, with p and q denoting the number of minima and maxima of f. If $s=2$, then Γ_f is already minimal (see Definition 1.3). Let us assume that any Reeb graph with $s \geq 2$ vertices of degree 1 can be transformed into a minimal one through a certain deformation. Let Γ_f have $s+1$ vertices of degree 1. Thus, at least one between p and q is greater than one. Let $p > 1$ (the case $q > 1$ is analogous). By Lemma 2.5 (i), if u, v correspond to two minima of f, we can construct a deformation T such that in $T(\Gamma_f, \ell_f)$ these vertices are both adjacent to a certain vertex w of degree 3. Let $T(\Gamma_f, \ell_f) = (\Gamma, \ell)$, with $\ell(u) < \ell(v) < \ell(w)$. If there exists a vertex $w' \in \ell^{-1}(]\ell(v), \ell(w)[)$, since v, w' cannot be adjacent, we can apply a deformation of type (R) relabeling only v, and get a new labeling ℓ' such that $\ell'(w')$ is not contained in $]\ell'(v), \ell'(w)[$. Possibly repeating this procedure finitely many times, we get a new labeling, that for simplicity we still denote by ℓ, such that $\ell^{-1}(]\ell(v), \ell(w)[) = \emptyset$. Hence, through a deformation of type (D) deleting v, w, the resulting labeled Reeb graph has s vertices of degree 1. By the inductive hypothesis, it can be transformed into a minimal Reeb graph.

Now we prove the second step. Let Γ_f be a minimal Reeb Graph, i.e. $p = q = 1$. The total number of splitting saddles (i.e. vertices of degree 3 for which there are two higher adjacent vertices) of Γ_f is $\mathfrak{g}$. If $\mathfrak{g} = 0$, then Γ_f is already canonical. Let us consider the case $\mathfrak{g} \geq 1$. Let $v \in V(\Gamma_f)$ be a splitting saddle such that, for every cycle C containing v, $\ell_f(v) = \min_{w \in C}\{\ell_f(w)\}$, and let C be one of these cycles. By Lemma 2.5 (ii), there exists a deformation T that transforms C into a 2-cycle, still having v as the lowest vertex. Let v' be the highest vertex in this 2-cycle. We observe that no other cycles of $T(\Gamma_f, \ell_f)$ contain v and v', otherwise the initial assumption on $\ell_f(v)$ would be contradicted. Hence v, v' and the edges adjacent to them are not touched when applying again Lemma 2.5 (ii) to reduce the length of another cycle. Therefore, iterating the same argument on a different splitting saddle, after at most $\mathfrak{g}$ iterations (actually at most $\mathfrak{g} - 1$ would suffice) Γ_f is transformed into a canonical Reeb graph. □

Proposition 2.7. *Let $f, g \in \mathscr{F}^0(\mathscr{M})$. The set $\mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ is non-empty.*

Proof. By Lemma 2.6 we can find two deformations T_f and T_g transforming (Γ_f, ℓ_f) and (Γ_g, ℓ_g), respectively, into canonical Reeb graphs. Apart from the labels, Γ_f and Γ_g are isomorphic because associated with the same surface $\mathscr{M}$. Hence, $T_f(\Gamma_f, \ell_f)$ can be transformed into a graph isomorphic to $T_g(\Gamma_g, \ell_g)$ through an elementary deformation of type (R), say T_{R}. Thus $(\Gamma_g, \ell_g) \cong T_g^{-1} T_{\mathrm{R}} T_f(\Gamma_f, \ell_f)$, i.e. $T_g^{-1} T_{\mathrm{R}} T_f \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$. □

A simple example illustrating the above proof is given in Figure 4.

3. EDIT DISTANCE BETWEEN LABELED REEB GRAPHS

In this section we introduce an edit distance between labeled Reeb graphs, in terms of the cost necessary to transform one graph into another.

We begin by defining the cost of a deformation. For the sake of simplicity, in view of Proposition 2.2, whenever $(\Gamma_g, \ell_g) \cong (\Gamma, \ell)$, we identify $V(\Gamma_g)$ with $V(\Gamma)$, and ℓ_g with ℓ. For all the notation referring to the elementary deformations, see Definition 2.1.

Definition 3.1. Let $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ be a deformation.

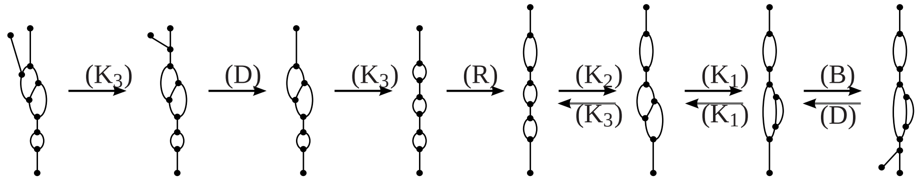

FIGURE 4. Using the procedure followed in the proof of Proposition 2.7, the leftmost labeled Reeb graph is transformed into the rightmost one applying first the deformation which reduces the former into its canonical form, then an elementary deformation of type (R), and eventually the inverse of the deformation which reduces the latter into its canonical form.

- For T elementary of type (B), inserting the vertices $u_1, u_2 \in V(\Gamma_g)$, the associated cost is
$$c(T) = \frac{|\ell_g(u_1) - \ell_g(u_2)|}{2}.$$
- For T elementary of type (D), deleting the vertices $u_1, u_2 \in V(\Gamma_f)$, the associated cost is
$$c(T) = \frac{|\ell_f(u_1) - \ell_f(u_2)|}{2}.$$
- For T elementary of type (R), relabeling the vertices $v \in V(\Gamma_f) = V(\Gamma_g)$, the associated cost is
$$c(T) = \max_{v \in V(\Gamma_f)} |\ell_f(v) - \ell_g(v)|.$$
- For T elementary of type (K_i), with $i = 1,2,3$, relabeling the vertices $u_1, u_2 \in V(\Gamma_f)$, the associated cost is
$$c(T) = \max\{|\ell_f(u_1) - \ell_g(u_1)|, |\ell_f(u_2) - \ell_g(u_2)|\}.$$
- For $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, with $T = (T_1, \dots, T_r)$, the associated cost is
$$c(T) = \sum_{i=1}^{r} c(T_i).$$

Proposition 3.2. *For every deformation* $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, $c(T^{-1}) = c(T)$.

Proof. It is sufficient to observe that, for every deformation $T = (T_1, \dots, T_r)$ such that $T(\Gamma_f, \ell_f) \cong (\Gamma_g, \ell_g)$, Definitions 3.1 and 2.4 imply the following equalities:
$$c(T) = \sum_{i=1}^{r} c(T_i) = \sum_{i=1}^{r} c(T_i^{-1}) = c(T^{-1}).$$
□

Theorem 3.3. *For every* $f, g \in \mathscr{F}^0(\mathscr{M})$, *we set*
$$d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) = \inf_{T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))} c(T).$$
It holds that d_E *is a pseudo-metric on isomorphism classes of labeled Reeb graphs.*

Proof. By Proposition 2.7, d_E is a real number. The coincidence property can be verified by observing that the deformation of type (R) such that $T(\Gamma_f, \ell_f) = (\Gamma_f, \ell_f)$ has a cost equal to 0; the symmetry property is a consequence of Proposition 3.2; the triangle inequality can be proved in the standard way. □

In order to say that d_E is actually a metric, we need to prove that if $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$ vanishes then $(\Gamma_f,\ell_f)\cong(\Gamma_g,\ell_g)$. This will be done in Section 5. Nevertheless, for simplicity, we already refer to d_E as to the *edit distance*.

The following proposition shows that when a labeled Reeb graph can be transformed into another one through a finite sequence of deformations of type (D), the same transformation can be realized also through a cheaper deformation which involves a relabeling of vertices. Analogous propositions can be given for other types of deformations. These results yield, in some cases, sharper estimates of the edit distance between labeled Reeb graphs.

Proposition 3.4. *For $T\in\mathscr{T}((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$, if $T=(T_1,\dots,T_n)$ and T_i is an elementary deformation of type (D) for each $i=1,\dots,n$, then there exists a deformation $S\in\mathscr{T}((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$, with $S=(S_0,S_1\dots,S_n)$ such that S_0 is an elementary deformation of type (R), $S_1,\dots,S_n$ are elementary deformations of type (D), and $c(S)=\max_{i=1,\dots,n}c(T_i)$. Hence $c(S)<c(T)$ when $n>1$.*

Proof. Let $T=(T_1,\dots,T_n)$, with each T_i of type (D), and let v_i,w_i be the vertices of Γ_f deleted by T_i. It is not restrictive to assume that $\ell_f(v_i)<\ell_f(w_i)$. For $n=1$, it is sufficient to take S_0 as the elementary deformation of type (R) such that $S_0(\Gamma_f,\ell_f)=(\Gamma_f,\ell_f)$ and $S_1=T_1$. For $n>1$, for every i,j with $1\le i,j\le n$, let us set $T_i\preceq T_j$ if and only if $[\ell_f(v_i),\ell_f(w_i)]\subseteq[\ell_f(v_j),\ell_f(w_j)]$. Let us denote by $T_{r_1},\dots,T_{r_m}$ the maximal elements of the poset $(\{T_1,\dots,T_n\},\preceq)$.

We observe that, for $1\le i\le n$, there exists exactly one value k, with $1\le k\le m$, for which $[\ell_f(v_i),\ell_f(w_i)]\subseteq[\ell_f(v_{r_k}),\ell_f(w_{r_k})]$. Moreover, $[\ell_f(v_i),\ell_f(w_i)]\cap[\ell_f(v_{r_h}),\ell_f(w_{r_h})]=\emptyset$ for every $h\ne k$ because T_i is an elementary deformation of type (D).

To define S_0, we take $\ell:V(\Gamma_f)\to\mathbb{R}$ as follows. Let $0<\varepsilon<\min_{k=1,\dots,m}\dfrac{\ell_f(w_{r_k})-\ell_f(v_{r_k})}{2}$. For $1\le k\le m$, we set $\ell(v_{r_k})=\dfrac{\ell_f(w_{r_k})+\ell_f(v_{r_k})}{2}-\varepsilon$ and $\ell(w_{r_k})=\dfrac{\ell_f(w_{r_k})+\ell_f(v_{r_k})}{2}+\varepsilon$. Next, for $1\le i\le n$, assuming $[\ell_f(v_i),\ell_f(w_i)]\subseteq[\ell_f(v_{r_k}),\ell_f(w_{r_k})]$, we let $\lambda_i,\mu_i\in[0,1]$ be the unique values for which $\ell_f(v_i)=(1-\lambda_i)\ell_f(v_{r_k})+\lambda_i\ell_f(w_{r_k})$ and $\ell_f(w_i)=(1-\mu_i)\ell_f(v_{r_k})+\mu_i\ell_f(w_{r_k})$, and we set $\ell(v_i)=(1-\lambda_i)\ell(v_{r_k})+\lambda_i\ell(w_{r_k})$ and $\ell(w_i)=(1-\mu_i)\ell(v_{r_k})+\mu_i\ell(w_{r_k})$. We observe that ℓ preserves the vertex order induced by ℓ_f and, therefore, S_0 defined by setting $S_0(\Gamma_f,\ell_f)=(\Gamma_f,\ell)$ is an elementary deformation of type (R). By Definition 3.1, the cost of S_0 is

$$c(S_0)\;=\;\max_{i=1,\dots,n}\left\{\max\left\{\left|\ell_f(v_i)-\ell(v_i)\right|,\left|\ell_f(w_i)-\ell(w_i)\right|\right\}\right\}.$$

A direct computation shows that $\ell(v_i)-\ell_f(v_i)\le\ell(v_{r_k})-\ell_f(v_{r_k})$ and $\ell_f(v_i)-\ell(v_i)\le\ell_f(w_{r_k})-\ell(w_{r_k})$. Analogously, $\ell(w_i)-\ell_f(w_i)\le\ell(v_{r_k})-\ell_f(v_{r_k})$ and $\ell_f(w_i)-\ell(w_i)\le\ell_f(w_{r_k})-\ell(w_{r_k})$. Hence

$$\begin{aligned}c(S_0)\;&=\;\max_{k=1,\dots,m}\left\{\max\left\{\ell(v_{r_k})-\ell_f(v_{r_k}),\ell_f(w_{r_k})-\ell(w_{r_k})\right\}\right\}\\ &=\;\max_{k=1,\dots,m}\frac{\ell_f(w_{r_k})-\ell_f(v_{r_k})}{2}-\varepsilon=\max_{k=1,\dots,m}c(T_{r_k})-\varepsilon.\end{aligned}\tag{3.1}$$

Now we set S_i, for $i=1,\dots,n$, to be the elementary deformation of type (D) that deletes the vertices v_i,w_i from $S_0(\Gamma_f,\ell_f)$. If $[\ell_f(v_i),\ell_f(w_i)]\subseteq[\ell_f(v_{r_k}),\ell_f(w_{r_k})]$, then

$$c(S_i)\;=\;\frac{\ell(w_i)-\ell(v_i)}{2}\le\frac{\ell(w_{r_k})-\ell(v_{r_k})}{2}=\varepsilon.\tag{3.2}$$

Setting $S=(S_0,S_1,\dots,S_n)$, we have $S\in\mathscr{T}((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$, and by formulas (3.1) and (3.2):

$$c(S)\;=\;c(S_0)+\sum_{i=1}^{n}c(S_i)\le\max_{k=1,\dots,m}c(T_{r_k})-\varepsilon+n\cdot\varepsilon.$$

Therefore, $\max_{k=1,\dots,m}c(T_{r_k})-\varepsilon\le c(S)\le\max_{k=1,\dots,m}c(T_{r_k})+(n-1)\varepsilon$. By the arbitrariness of ε, we get $c(S)=\max_{k=1,\dots,m}c(T_{r_k})$, yielding the claim.

□

4. Stability

This section is devoted to proving that Reeb graphs of orientable surfaces are stable under function perturbations. More precisely, it will be shown that arbitrary changes in simple Morse functions with respect to the C^0-norm imply not greater changes in the edit distance between the associated labeled Reeb graphs. Formally:

Theorem 4.1. *For every $f,g\in\mathscr{F}^0(\mathscr{M})$, letting $\|f-g\|_{C^0}=\max_{p\in\mathscr{M}}|f(p)-g(p)|$, we have*

$$d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))\le\|f-g\|_{C^0}.$$

We observe that such a result is strictly related the way the cost of an elementary deformation of type (R) was defined as the following Example 1 shows.

Example 1. Let $f,g:\mathscr{M}\to\mathbb{R}$ with $f,g\in\mathscr{F}^0(\mathscr{M})$ as illustrated in Figure 5.

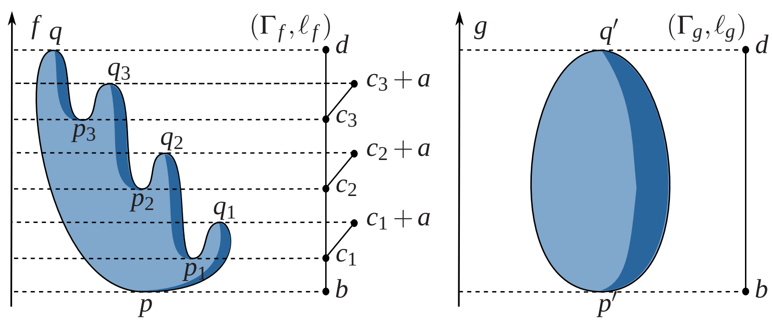

FIGURE 5. The functions $f,g\in\mathscr{F}^0(\mathscr{M})$ considered in Example 1.

Let $f(q_i)-f(p_i)=a$, $i=1,2,3$. Up to re-parameterization of $\mathscr{M}$, we have $\|f-g\|_{C^0}=\frac{a}{2}$. The deformation T that deletes the three edges $e(p_i,q_i)\in E(\Gamma_f)$ has cost $c(T)=3\cdot\frac{a}{2}$, implying $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))\le 3\cdot\|f-g\|_{C^0}$. On the other hand, applying Proposition 3.4 we see that actually $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))\le\|f-g\|_{C^0}$. Indeed, for every $0<\varepsilon<\frac{a}{2}$, we can apply to (Γ_f,ℓ_f) a deformation of type (R), that relabels the vertices p_i,q_i, $i=1,2,3$, in such a way that $\ell_f(p_i)$ is increased by $\frac{a}{2}-\varepsilon$, and $\ell_f(q_i)$ is decreased by $\frac{a}{2}-\varepsilon$, composed with three deformations of type (D) that delete p_i, q_i and the edge $e(p_i,q_i)$, for $i=1,2,3$ respectively.

In order to prove Theorem 4.1, we consider the set $\mathscr{F}(\mathscr{M})$ of smooth real-valued functions on $\mathscr{M}$ endowed with the C^2 topology, which may be defined as follows. Let $\{U_\alpha\}$ be a coordinate covering of $\mathscr{M}$ with coordinate maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$, and let $\{C_\alpha\}$ be a compact refinement of $\{U_\alpha\}$. For every positive constant $\delta > 0$ and every $f \in \mathscr{F}(\mathscr{M})$, define $N(f,\delta)$ as the subset of $\mathscr{F}(\mathscr{M})$ consisting of all maps g such that, denoting $f_\alpha = f \circ \varphi_\alpha^{-1}$ and $g_\alpha = g \circ \varphi_\alpha^{-1}$, it holds that $\max_{i+j\le 2}\left|\frac{\partial^{i+j}}{\partial x^i \partial y^j}(f_\alpha - g_\alpha)\right| < \delta$ at all points of $\varphi_\alpha(C_\alpha)$. The C^2 topology is the topology obtained by taking the sets $N(f,\delta)$ as a base of neighborhoods.

Next we consider the strata $\mathscr{F}^0(\mathscr{M})$ and $\mathscr{F}^1(\mathscr{M})$ of the *natural stratification* of $\mathscr{F}(\mathscr{M})$, as presented by Cerf in [4].

- The stratum $\mathscr{F}^0(\mathscr{M})$ is the set of simple Morse functions.
- The stratum $\mathscr{F}^1(\mathscr{M})$ is the disjoint union of two sets $\mathscr{F}^1_\alpha(\mathscr{M})$ and $\mathscr{F}^1_\beta(\mathscr{M})$, where
  - $\mathscr{F}^1_\alpha(\mathscr{M})$ is the set of functions whose critical levels contain exactly one critical point, and the critical points are all non-degenerate, except exactly one.
  - $\mathscr{F}^1_\beta(\mathscr{M})$ is the set of Morse functions whose critical levels contain at most one critical point, except for one level containing exactly two critical points.

$\mathscr{F}^1(\mathscr{M})$ is a sub-manifold of co-dimension 1 of $\mathscr{F}^0(\mathscr{M}) \cup \mathscr{F}^1(\mathscr{M})$, and the complement of $\mathscr{F}^0(\mathscr{M}) \cup \mathscr{F}^1(\mathscr{M})$ in $\mathscr{F}(\mathscr{M})$ is of co-dimension greater than 1. Hence, given two functions $f,g \in \mathscr{F}^0(\mathscr{M})$, we can always find $\widehat{f},\widehat{g} \in \mathscr{F}^0(\mathscr{M})$ arbitrarily near to f,g, respectively, for which

- $\widehat{f}$, $\widehat{g}$ are RL-equivalent to f, g, respectively,

and the path $h(\lambda) = (1-\lambda)\widehat{f} + \lambda\widehat{g}$, with $\lambda \in [0,1]$, is such that

- $h(\lambda)$ belongs to $\mathscr{F}^0(\mathscr{M}) \cup \mathscr{F}^1(\mathscr{M})$ for every $\lambda \in [0,1]$;
- $h(\lambda)$ is transversal to $\mathscr{F}^1(\mathscr{M})$.

As a consequence, $h(\lambda)$ belongs to $\mathscr{F}^1(\mathscr{M})$ for at most a finite collection of values λ, and does not traverse strata of co-dimension greater than 1 (see, e.g., [9]).

With these preliminaries set, the stability theorem will be proven by considering a path that connects f to g via $\widehat{f}$, $h(\lambda)$, and $\widehat{g}$ as aforementioned. This path can be split into a finite number of linear sub-paths whose endpoints are such that the stability theorem holds on them, as will be shown in some preliminary lemmas. In conclusion, Theorem 4.1 will be proven by applying the triangle inequality of the edit distance.

In the following preliminary lemmas, f and g belong to $\mathscr{F}^0(\mathscr{M})$ and $h : [0,1] \to \mathscr{F}(\mathscr{M})$ denotes their convex linear combination: $h(\lambda) = (1-\lambda)f + \lambda g$.

Lemma 4.2. $\|h(\lambda') - h(\lambda'')\|_{C^0} = |\lambda' - \lambda''| \cdot \|f-g\|_{C^0}$ *for every* $\lambda', \lambda'' \in [0,1]$.

Proof.

$$\begin{aligned}\|h(\lambda') - h(\lambda'')\|_{C^0} &= \|(1-\lambda')f + \lambda' g - (1-\lambda'')f - \lambda'' g\|_{C^0} \\ &= \|(\lambda''-\lambda')f - (\lambda''-\lambda')g\|_{C^0} = |\lambda'-\lambda''| \cdot \|f-g\|_{C^0}.\end{aligned}$$

□

Lemma 4.3. *If* $h(\lambda) \in \mathscr{F}^0(\mathscr{M})$ *for every* $\lambda \in [0,1]$, *then* $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \le \|f-g\|_{C^0}$.

Proof. The statement can be proved in the same way as [7, Prop. 5.4]. □

Lemma 4.4. *Let* $h(\lambda)$ *intersect* $\mathscr{F}^1(\mathscr{M})$ *transversely at* $h(\overline{\lambda})$, $0 < \overline{\lambda} < 1$, *and nowhere else. Then, for every constant value* $\delta > 0$, *there exist two real numbers* λ', λ'' *with* $0 < \lambda' < \overline{\lambda} < \lambda'' < 1$, *such that*

$$d_E((\Gamma_{h(\lambda')},\ell_{h(\lambda')}),(\Gamma_{h(\lambda'')},\ell_{h(\lambda'')})) \le \delta.$$

Proof. In this proof we use the notion of universal deformation of a function. More details on universal deformations may be found in [16, 21]. In particular, we will consider two different universal deformations F and G of $\overline{h} = h(\overline{\lambda})$. Firstly we show how F and G yield the claim, and then construct them.

We use the fact that, being two universal deformations of $\overline{h} \in \mathscr{F}^1(\mathscr{M})$, F and G are equivalent. This means that there exist a diffeomorphism $\eta(s)$ of $\mathbb{R}$ with $\eta(0) = 0$, and a local diffeomorphism $\phi(s,(x,y))$, with $\phi(s,(x,y)) = (\eta(s), \psi(\eta(s),(x,y)))$ and $\phi(0,(x,y)) = (0,(x,y))$, such that $F = (\eta^* G) \circ \phi$. Hence, apart from the labels, the Reeb graphs of $F(s,\cdot)$ and $G(\eta(s),\cdot)$ are isomorphic. Moreover, the difference the labels at corresponding vertices in the Reeb graphs of $F(s,\cdot)$ and $G(\eta(s),\cdot)$ continuously depends on s, and is 0 for $s = 0$. Therefore, for every $\delta > 0$, taking $|s|$ sufficiently small, it is possible to transform the labeled Reeb graph of $F(s,\cdot)$ into that of $G(\eta(s),\cdot)$, or viceversa, by a deformation of type (R) whose cost is not greater than $\delta/3$. Moreover, as equality (4.3) will show, for every $\delta > 0$, $|s|$ can be taken sufficiently small that the distance between the labeled Reeb graphs of $G(\eta(s),\cdot)$ and $G(\eta(-s),\cdot)$ is not greater that $\delta/3$. Thus, applying the triangle inequality, we deduce that, for every $\delta > 0$, there exists a sufficiently small $s > 0$ such that the distance between the labeled Reeb graphs of $F(s,\cdot)$ and $F(-s,\cdot)$ is not greater than δ. The claim follows taking $\lambda' = \overline{\lambda} - s$ and $\lambda'' = \overline{\lambda} + s$.

We now construct the universal deformations $F(s,p)$ and $G(s,p)$, with $s \in \mathbb{R}$ and $p \in \mathscr{M}$. We define F by setting $F(s,p) = \overline{h}(p) + s \cdot (g - f)(p)$. This deformation is universal because $h(\lambda)$ intersects $\mathscr{F}^1(\mathscr{M})$ transversely at $h(\overline{\lambda})$. In order to construct G, let us consider separately the two cases $\mathscr{F}^1_\alpha(\mathscr{M})$ and $\mathscr{F}^1_\beta(\mathscr{M})$.

Case $\overline{h} \in \mathscr{F}^1_\alpha(\mathscr{M})$*:* Let $\overline{p}$ be the sole degenerate critical point of $\overline{h}$. Let (x,y) be a suitable local coordinate system around $\overline{p}$ in which the canonical expression of $\overline{h}$ is $\overline{h}(x,y) = \overline{h}(\overline{p}) \pm x^2 + y^3$. Let $\omega : \mathscr{M} \to \mathbb{R}$ be a smooth function equal to 1 in a neighbor-

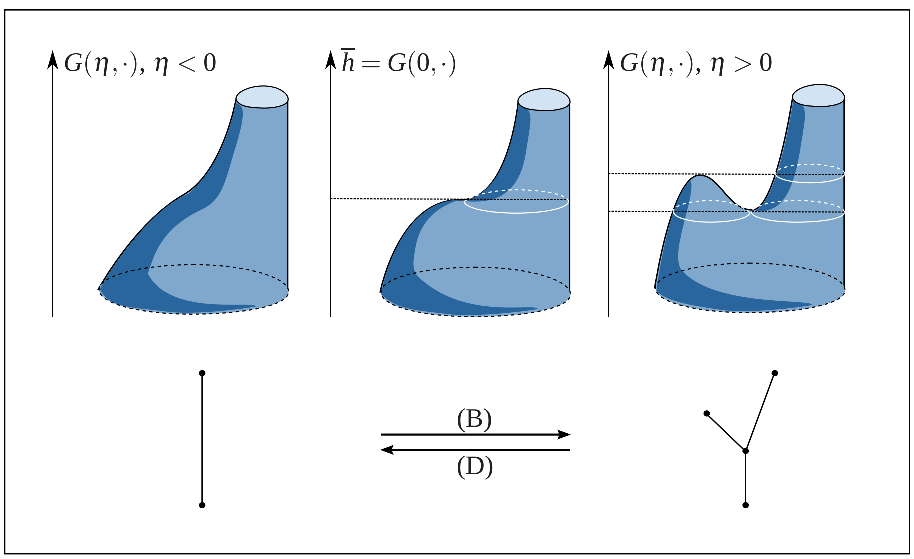

FIGURE 6. Center: A function $\overline{h} \in \mathscr{F}^1_\alpha(\mathscr{M})$; left-right: The universal deformation $G(\eta,\cdot)$ with the associated labeled Reeb graphs for $\eta < 0$ and $\eta > 0$.

hood of $\overline{p}$, which decreases moving from $\overline{p}$, and whose support is contained in the chosen coordinate chart around $\overline{p}$. Finally, let $G(\eta,(x,y)) = \overline{h}(x,y) - \eta\cdot\omega(x,y)\cdot y$, where $\eta\in\mathbb{R}$. For $\eta<0$, $G(\eta,\cdot)$ has no critical points in the support of ω and is equal to $\overline{h}$ everywhere else, while, for $\eta>0$, $G(\eta,\cdot)$ has exactly two critical points in the support of ω, precisely $p_1 = \left(0,-\sqrt{\frac{\eta}{3}}\right)$ and $p_2 = \left(0,\sqrt{\frac{\eta}{3}}\right)$, and is equal to $\overline{h}$ everywhere else (see Figure 6). Therefore, for every $\eta>0$ sufficiently small, the labeled Reeb graph of $G(-\eta,\cdot)$ can be transformed into that of $G(\eta,\cdot)$ by an elementary deformation T of type (B). Obviously, in the case $\eta<0$, the deformation we consider is of type (D).

By Definition 3.1 and Proposition 3.2, a direct computation shows that the cost of T is

$$c(T) \;=\; 2\cdot\left(\frac{|\eta|}{3}\right)^{3/2}. \tag{4.1}$$

Case $\overline{h}\in\mathscr{F}^1_\beta(\mathscr{M})$*:* Let $\overline{p}$ and $\overline{q}$ be the critical points of $\overline{h}$ such that $\overline{h}(\overline{p}) = \overline{h}(\overline{q})$. Since $\overline{p}$ is non-degenerate, there exists a suitable local coordinate system (x,y) around $\overline{p}$ in which the canonical expression of $\overline{h}$ is $\overline{h}(x,y) = \overline{h}(\overline{p}) + x^2 + y^2$ if $\overline{p}$ is a minimum, or $\overline{h}(x,y) = \overline{h}(\overline{p}) - x^2 - y^2$ if $\overline{p}$ is a maximum, or $\overline{h}(x,y) = \overline{h}(\overline{p}) \pm x^2 \mp y^2$ if $\overline{p}$ is a saddle point. Let $\omega:\mathscr{M}\to\mathbb{R}$ be a smooth function equal to 1 in a neighborhood of $\overline{p}$, which decreases moving from $\overline{p}$, and whose support is contained in the coordinate chart around $\overline{p}$ in which $\overline{h}$ has one of the above expressions. Finally, let $G(\eta,(x,y)) = \overline{h}(x,y) + \eta\cdot\omega(x,y)$, where $\eta = \eta(s)$, $s\in\mathbb{R}$. For every $\eta\in\mathbb{R}$, with $|\eta|$ sufficiently small, $G(\eta,\cdot)$ has the same critical points, with the same indices, as $\overline{h}$. As for critical values, they are the same as well, apart from the value taken at $\overline{p}$: $G(\eta,\overline{p}) = \overline{h}(\overline{p}) + \eta$.

We distinguish the following two situations illustrated in Figures 7 and 8:

(1) the points $\overline{p}$ and $\overline{q}$ belong to two different connected components of $\overline{h}^{-1}(\overline{h}(\overline{p}))$;

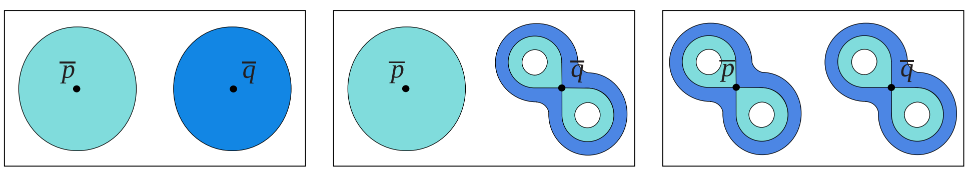

FIGURE 7. Two critical points in different connected components of the same critical level. The dark (resp. light) regions correspond to points below (resp. above) this critical level. Possibly inverting the colors of one or both the components, we have all the possible cases.

(2) the points $\overline{p}$ and $\overline{q}$ belong to the same connected component of $\overline{h}^{-1}(\overline{h}(\overline{p}))$.

In the situation (1), for every $\eta>0$ sufficiently small, the labeled Reeb graphs of $G(-\eta,\cdot)$ and $G(\eta,\cdot)$ can be obtained one from the other through an elementary deformation T of type (R) (see, e.g., Figure 9).

In the situation (2), the following elementary deformations need to be considered:

- If $\overline{p}$ and $\overline{q}$ are as in Figure 8 (a), then, for every $\eta>0$ sufficiently small, the labeled Reeb graphs of $G(-\eta,\cdot)$ and $G(\eta,\cdot)$ can be obtained one from the other through an elementary deformation T of type (K_1) (see, e.g., Figure 10).
- If $\overline{p}$ and $\overline{q}$ are as in Figure 8 (b), then, for every $\eta>0$ sufficiently small, the labeled Reeb graphs of $G(-\eta,\cdot)$ and $G(\eta,\cdot)$ can be obtained one from the other through an elementary deformation T of type (K_3) or (K_2) (see, e.g., Figure 11).

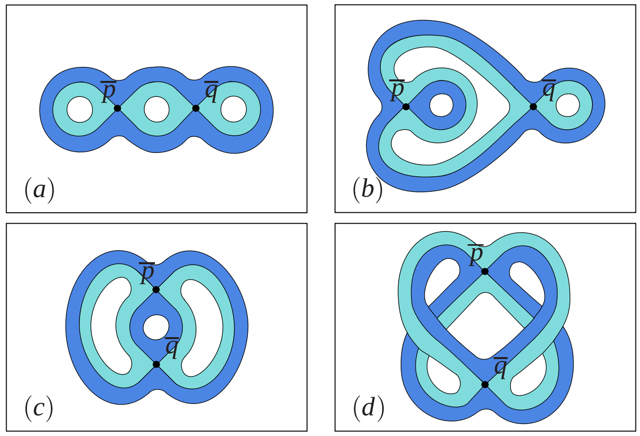

FIGURE 8. Two critical points in the same connected component of the same critical level. The dark (resp. light) regions correspond to points below (resp. above) this critical level. Possibly inverting the colors of this component, we have all the possible cases.

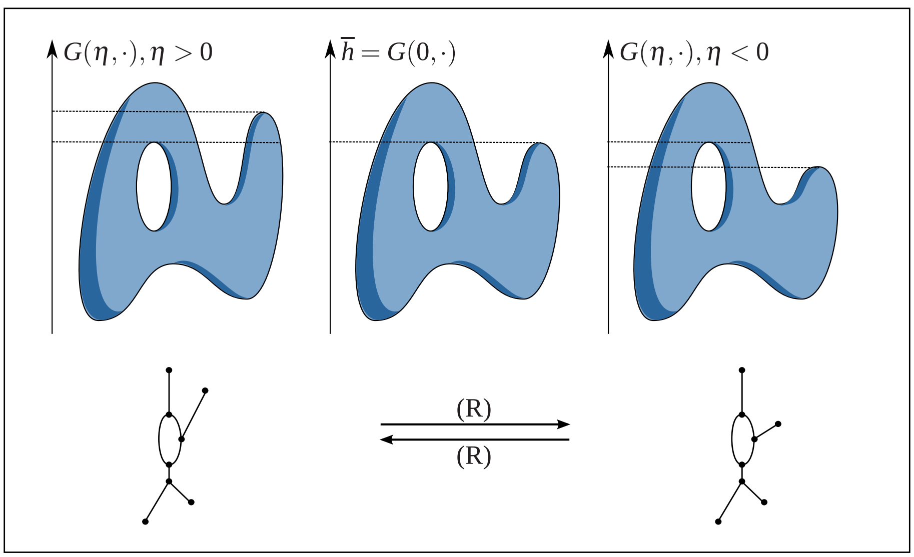

FIGURE 9. Center: A function $\overline{h} \in \mathscr{F}^1_\beta(\mathscr{M})$ as in case (1); left-right: The universal deformation $G(\eta,\cdot)$ with the associated labeled Reeb graphs for $\eta < 0$ and $\eta > 0$.

- If $\overline{p}$ and $\overline{q}$ are as in Figure 8 (c) or (d), then, for every $\eta > 0$ sufficiently small, the labeled Reeb graphs of $G(-\eta,\cdot)$ and $G(\eta,\cdot)$ can be obtained one from the other through an elementary deformation T of type (R) (see, e.g., Figures 12-13).

In all the cases, for every $\eta > 0$ sufficiently small, the cost of the considered deformation T is:

$$c(T) \;=\; |\overline{h}(\overline{p}) - \eta - (\overline{h}(\overline{p}) + \eta)| = 2\eta. \tag{4.2}$$

In conclusion, from equalities (4.1) and (4.2), for every $\eta > 0$ sufficiently small, we get

$$d_E((\Gamma_{G(-\eta,\cdot)},\ell_{G(-\eta,\cdot)}),(\Gamma_{G(\eta,\cdot)},\ell_{G(\eta,\cdot)})) \le \max\left\{2\cdot\left(\frac{\eta}{3}\right)^{3/2},2\eta\right\}.$$

Thus, for every $\delta > 0$, we can always take a value $|s|$ sufficiently small that $|\eta(s)|$ results small enough to imply the following inequality:

$$d_E((\Gamma_{G(-\eta(s),\cdot)},\ell_{G(-\eta(s),\cdot)}),(\Gamma_{G(\eta(s),\cdot)},\ell_{G(\eta(s),\cdot)})) \le \delta/3. \tag{4.3}$$

□

Lemma 4.5. *If $h(\lambda)$ belongs to $\mathscr{F}^0(\mathscr{M})$ for every $\lambda \in [0,1]$ apart from one value $0 < \overline{\lambda} < 1$ at which h transversely intersects $\mathscr{F}^1(\mathscr{M})$, then $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \le \|f-g\|_{C^0}$.*

Proof. Let $\overline{h} = h(\overline{\lambda})$. By Lemma 4.4, for every real number $\delta > 0$,we can find two values $0 < \lambda' < \overline{\lambda} < \lambda'' < 1$ such that $d_E((\Gamma_{h(\lambda')},\ell_{h(\lambda')}),(\Gamma_{h(\lambda'')},\ell_{h(\lambda'')})) \le \delta$.

Applying the triangle inequality, we have:

$$\begin{aligned} d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \;\le\; & d_E((\Gamma_f,\ell_f),(\Gamma_{h(\lambda')},\ell_{h(\lambda')}))+d_E((\Gamma_{h(\lambda')},\ell_{h(\lambda')}),(\Gamma_{h(\lambda'')},\ell_{h(\lambda'')})) \\ & +d_E((\Gamma_{h(\lambda'')},\ell_{h(\lambda'')}),(\Gamma_g,\ell_g)). \end{aligned}$$

Moreover, we get

$$d_E((\Gamma_f,\ell_f),(\Gamma_{h(\lambda')},\ell_{h(\lambda')})) \le \|f-h(\lambda')\|_{C^0} = \lambda'\cdot\|f-g\|_{C^0},$$

and

$$d_E((\Gamma_{h(\lambda'')},\ell_{h(\lambda'')}),(\Gamma_g,\ell_g)) \le \|h(\lambda'')-g\|_{C^0} = (1-\lambda'')\cdot\|f-g\|_{C^0},$$

where the inequalities follow from Lemma 4.3, and equalities from Lemma 4.2 with $f = h(0), g = h(1)$. Hence,

$$d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \le (1+\lambda'-\lambda'')\cdot\|f-g\|_{C^0}+\delta.$$

In conclusion, given that $0 < \lambda' < \lambda''$, the inequality $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \le \|f-g\|_{C^0}+\delta$ holds. This yields the claim by the arbitrariness of δ. □

We are now ready to prove the stability Theorem 4.1.

Proof of Theorem 4.1. Recall from [12] that $\mathscr{F}^0(\mathscr{M})$ is open in $\mathscr{F}(\mathscr{M})$ endowed with the C^2 topology. Thus, for every sufficiently small real number $\delta > 0$, the neighborhoods $N(f,\delta)$ and $N(g,\delta)$ are contained in $\mathscr{F}^0(\mathscr{M})$. Take $\widehat{f} \in N(f,\delta)$ and $\widehat{g} \in N(g,\delta)$ such that the path $h : [0,1] \to \mathscr{F}(\mathscr{M})$, with $h(\lambda) = (1-\lambda)\widehat{f}+\lambda\widehat{g}$, belongs to $\mathscr{F}^0(\mathscr{M})$ for every $\lambda \in [0,1]$, except for at most a finite number n of values, $\mu_1,\mu_2,\ldots,\mu_n$, at which h transversely intersects $\mathscr{F}^1(\mathscr{M})$. We begin by proving our statement for $\widehat{f}$ and $\widehat{g}$, and then show its validity for f and g. We proceed by induction on n. If $n = 0$ or $n = 1$, the inequality $d_E((\Gamma_{\widehat{f}},\ell_{\widehat{f}}),(\Gamma_{\widehat{g}},\ell_{\widehat{g}})) \le \|\widehat{f}-\widehat{g}\|_{C^0}$ holds because of Lemma 4.3 or 4.5, respectively. Let us assume the claim is true for $n \ge 1$, and prove it for $n+1$. Let $0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots < \mu_n < \lambda_n < \mu_{n+1} < 1$, with $h(0) = \widehat{f}$, $h(1) = \widehat{g}$, $h(\mu_i) \in \mathscr{F}^1(\mathscr{M})$, for every $i = 1,\ldots,n+1$, and $h(\lambda_j) \in \mathscr{F}^0(\mathscr{M})$, for every $j = 1,\ldots,n$. We consider h as the concatenation of the paths $h^1,h^2 : [0,1] \to \mathscr{F}(\mathscr{M})$, defined, respectively, as $h^1(\lambda) = (1-\lambda)\widehat{f}+\lambda h(\lambda_n)$, and $h^2(\lambda) = (1-\lambda)h(\lambda_n)+\lambda\widehat{g}$. The path h^1 transversally intersect $\mathscr{F}^1(\mathscr{M})$ at n values $\mu_1,\ldots,\mu_n$. Hence, by the inductive hypothesis, we have $d_E((\Gamma_{\widehat{f}},\ell_{\widehat{f}}),(\Gamma_{h(\lambda_n)},\ell_{h(\lambda_n)})) \le \|\widehat{f}-h(\lambda_n)\|_{C^0}$. Moreover, the path h^2 transversally intersect $\mathscr{F}^1(\mathscr{M})$ only at the value

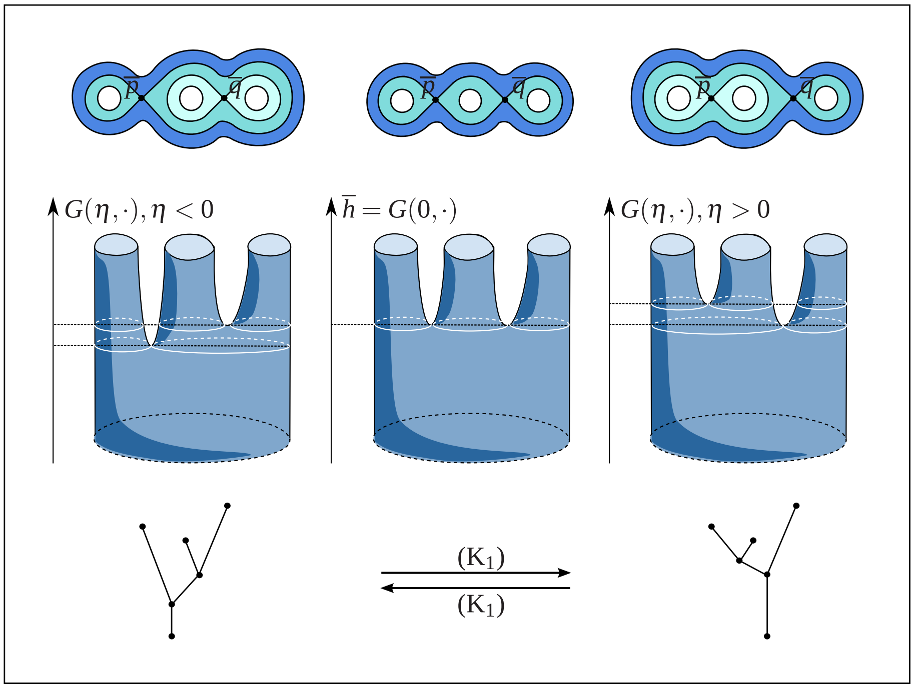

FIGURE 10. Center: A function $\overline{h} \in \mathscr{F}^1_\beta(\mathscr{M})$ as in case (2) with $\overline{p}, \overline{q}$ as in Figure 8 (a); left-right: The universal deformation $G(\eta,\cdot)$ with the associated labeled Reeb graphs for $\eta < 0$ and $\eta > 0$.

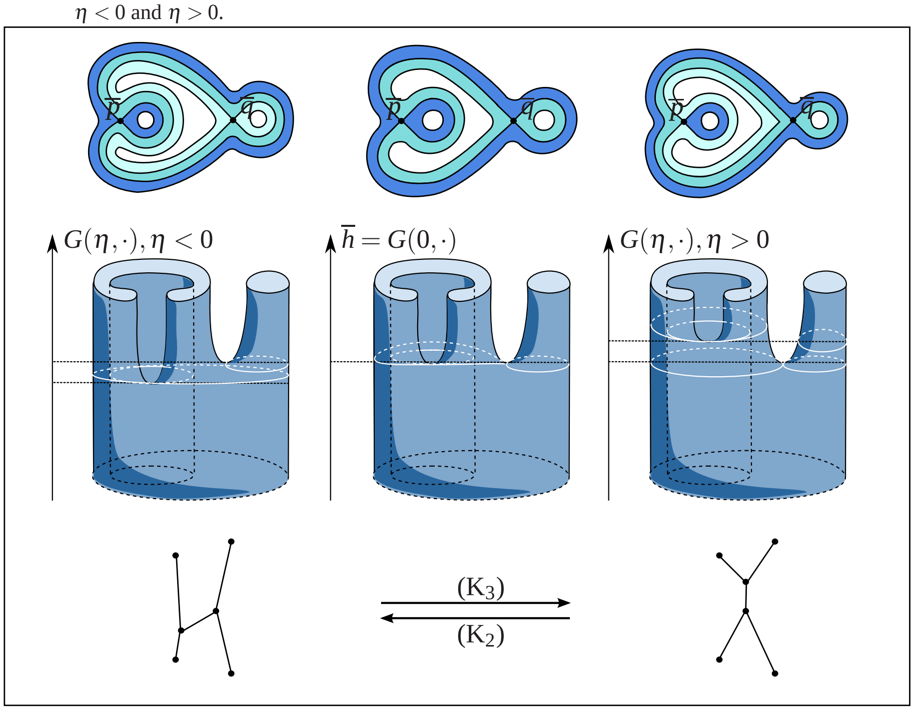

FIGURE 11. Center: A function $\overline{h} \in \mathscr{F}^1_\beta(\mathscr{M})$ as in case (2) with $\overline{p}, \overline{q}$ as in Figure 8 (b); left-right: The universal deformation $G(\eta,\cdot)$ with the associated labeled Reeb graphs for $\eta < 0$ and $\eta > 0$.

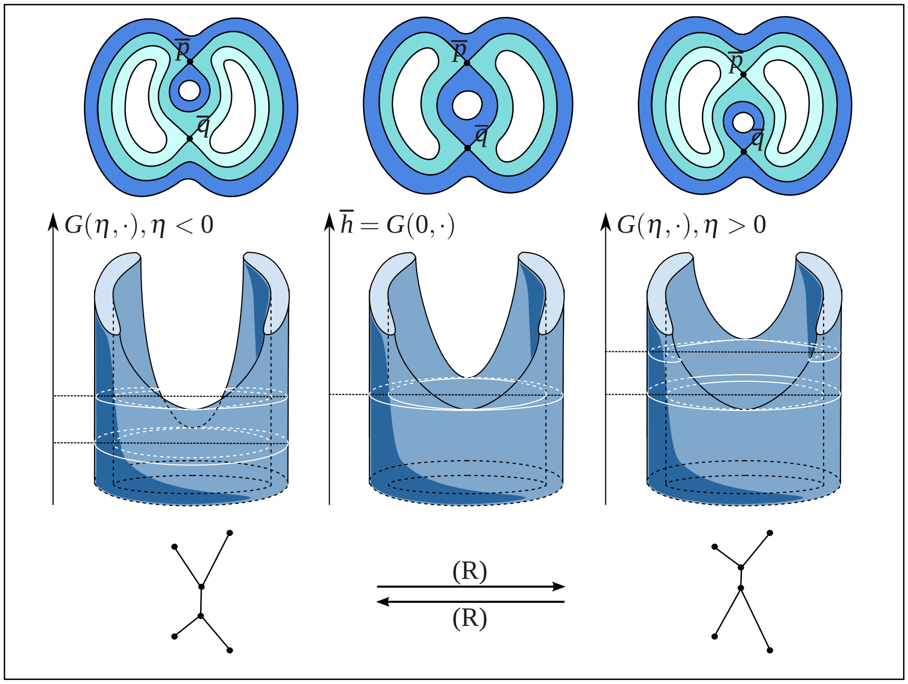

FIGURE 12. Center: A function $\overline{h}\in\mathscr{F}^1_\beta(\mathscr{M})$ as in case (2) with $\overline{p},\overline{q}$ as in Figure 8 (c); left-right: The universal deformation $G(\eta,\cdot)$ with the associated labeled Reeb graphs for $\eta<0$ and $\eta>0$.

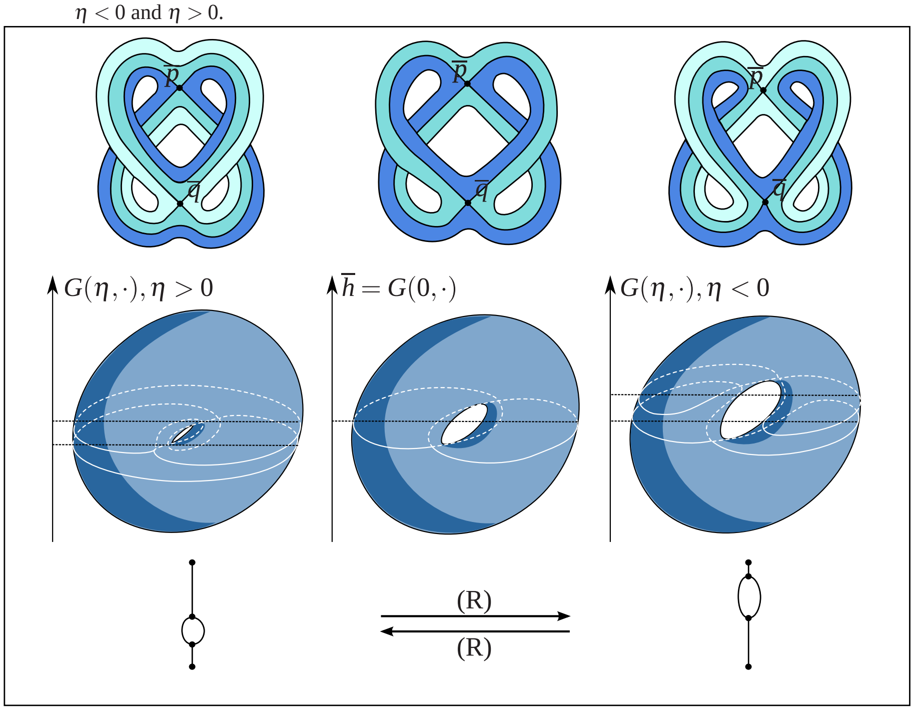

FIGURE 13. Center: A function $\overline{h}\in\mathscr{F}^1_\beta(\mathscr{M})$ as in case (2) with $\overline{p},\overline{q}$ as in Figure 8 (d); left-right: The universal deformation $G(\eta,\cdot)$ with the associated labeled Reeb graphs for $\eta<0$ and $\eta>0$.

μ_{n+1}. Consequently, by Lemma 4.5, we have $d_E((\Gamma_{h(\lambda_n)}, \ell_{h(\lambda_n)}), (\Gamma_{\widehat{g}}, \ell_{\widehat{g}})) \le \|h(\lambda_n) - \widehat{g}\|_{C^0}$. Using the triangle inequality and Lemma 4.2, we can conclude that:

$$\begin{aligned} d_E((\Gamma_{\widehat{f}}, \ell_{\widehat{f}}), (\Gamma_{\widehat{g}}, \ell_{\widehat{g}})) &\le d_E((\Gamma_{\widehat{f}}, \ell_{\widehat{f}}), (\Gamma_{h(\lambda_n)}, \ell_{h(\lambda_n)})) + d_E((\Gamma_{h(\lambda_n)}, \ell_{h(\lambda_n)}), (\Gamma_{\widehat{g}}, \ell_{\widehat{g}})) \\ &\le \lambda_n \|\widehat{f} - \widehat{g}\|_{C^0} + (1-\lambda_n)\|\widehat{f} - \widehat{g}\|_{C^0} = \|\widehat{f} - \widehat{g}\|_{C^0}. \end{aligned} \tag{4.4}$$

Let us now estimate $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$. By the triangle inequality, we have:

$$d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \le d_E((\Gamma_f, \ell_f), (\Gamma_{\widehat{f}}, \ell_{\widehat{f}})) + d_E((\Gamma_{\widehat{f}}, \ell_{\widehat{f}}), (\Gamma_{\widehat{g}}, \ell_{\widehat{g}})) + d_E((\Gamma_{\widehat{g}}, \ell_{\widehat{g}}), (\Gamma_g, \ell_g)).$$

Since $\widehat{f} \in N(f, \delta) \subset \mathscr{F}^0(\mathscr{M})$ and $\widehat{g} \in N(g, \delta) \subset \mathscr{F}^0(\mathscr{M})$, the following facts hold: (a) for every $\lambda \in [0,1]$, $(1-\lambda)f + \lambda \widehat{f}, (1-\lambda)g + \lambda \widehat{g} \in \mathscr{F}^0(\mathscr{M})$; (b) $\|f - \widehat{f}\|_{C^0} \le \delta$ and $\|\widehat{g} - g\|_{C^0} \le \delta$. Hence, from (a) and Lemma 4.3, we get $d_E((\Gamma_f, \ell_f), (\Gamma_{\widehat{f}}, \ell_{\widehat{f}})) \le \|f - \widehat{f}\|_{C^0}$, and $d_E((\Gamma_g, \ell_g), (\Gamma_{\widehat{g}}, \ell_{\widehat{g}})) \le \|\widehat{g} - g\|_{C^0}$. Using inequality (4.4) and the triangle inequality of $\|\cdot\|_{C^0}$, we deduce that

$$\begin{aligned} d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) &\le \|f - \widehat{f}\|_{C^0} + \|\widehat{f} - \widehat{g}\|_{C^0} + \|\widehat{g} - g\|_{C^0} \\ &\le \|f - g\|_{C^0} + 2(\|f - \widehat{f}\|_{C^0} + \|\widehat{g} - g\|_{C^0}). \end{aligned}$$

Hence, from (b), we have $d_E((\Gamma_f, \ell_f), (\Gamma_g, \ell_g)) \le \|f - g\|_{C^0} + 4\delta$. This yields the conclusion by the arbitrariness of δ. □

5. Relationships with other stable metrics

In this section, we consider relationships between the edit distance and other metrics for shape comparison: the natural pseudo-distance between functions [8], the functional distortion distance between Reeb graphs [1], and the bottleneck distance between persistence diagrams [6]. More precisely, the main result we are going to show states that the natural pseudo-distance between two simple Morse functions f and g and the edit distance between the corresponding Reeb graphs actually coincide (Theorem 5.6). As a consequence, we deduce that the edit distance is a metric (Corollary 5.7), and that it is more discriminative than the bottleneck distance between persistence diagrams (Corollary 5.8) and the functional distortion distance between Reeb graphs (Corollary 5.9).

The natural pseudo-distance is a dissimilarity measure between any two functions defined on the same compact manifold obtained by minimizing the difference in the functions via a re-parameterization of the manifold [8]. In general, the natural pseudo-distance is only a pseudo-metric. However it turns out to be a metric in some particular cases such as the case of simple Morse functions on a smooth closed connected surface, considered up to R-equivalence, as proved in [3]. We give the definition in this context.

Definition 5.1. The *natural pseudo-distance* between R-equivalence classes of simple Morse functions f, g on the same surface $\mathscr{M}$ is defined as

$$d_N([f],[g]) = \inf_{\xi \in \mathscr{D}(\mathscr{M})} \|f - g \circ \xi\|_{C^0},$$

where $\mathscr{D}(\mathscr{M})$ is the set of self-diffeomorphisms on $\mathscr{M}$.

In order to study d_N, it is often useful to consider the following fact.

Proposition 5.2. *Letting $\mathscr{H}(\mathscr{M})$ be the set of self-homeomorphisms on $\mathscr{M}$, it holds that* $d_N([f],[g]) = \inf_{\xi \in \mathscr{H}(\mathscr{M})} \|f - g \circ \xi\|_{C^0}$.

Proof. Let $d = d_N([f],[g])$. Clearly $\inf_{\xi \in \mathscr{H}(\mathscr{M})} \|f - g\circ\xi\|_{C^0} \le d$. By contradiction, assuming that $\inf_{\xi \in \mathscr{H}(\mathscr{M})} \|f - g\circ\xi\|_{C^0} < d$, there exists a homeomorphism $\overline{\xi}$ such that $\|f - g\circ\overline{\xi}\|_{C^0} < d$. On the other hand, by [25, Cor. 1.18], for every metric δ on $\mathscr{M}$ and for every $n \in \mathbb{N}$, there exists a diffeomorphism $\xi_n : \mathscr{M} \to \mathscr{M}$ such that $\delta(\xi_n(p), \overline{\xi}(p)) < 1/n$, for every $p \in \mathscr{M}$. Hence, by the continuity of g, and applying the reverse triangle inequality, we deduce that

$$\lim_{n\to\infty} \left| \|f - g\circ\overline{\xi}\|_{C^0} - \|f - g\circ\xi_n\|_{C^0} \right| \le \lim_{n\to\infty} \|g\circ\xi_n - g\circ\overline{\xi}\|_{C^0} = 0.$$

Therefore, for n sufficiently large, there exists a diffeomorphism ξ_n such that $\|f - g\circ\xi_n\|_{C^0} < d$, yielding a contradiction. □

The following Lemmas 5.3-5.5 state that the cost of each elementary deformation upper-bounds the natural pseudo-distance. Their proofs deploy the concepts of elementary cobordism and rearrangement, whose detailed treatment can be found in [18].

Lemma 5.3. *For every elementary deformation $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ of type (R), $c(T) \ge d_N([f],[g])$.*

Proof. Since T is of type (R), there exists an edge preserving bijection $\Phi : V(\Gamma_f) \to V(\Gamma_g)$. Hence, f and g have the same number of critical points of the same type: $K_f = \{p_1, \ldots, p_n\}$, $K_g = \{p'_1, \ldots, p'_n\}$, with $\Phi(p_i) = p'_i$, and p_i, p'_i both being of index 0,1, or 2.

Let $c_i = f(p_i)$ and $c'_i = g(p'_i)$ for $i = 1, \ldots, n$. We shall construct a homeomorphism $\xi : \mathscr{M} \to \mathscr{M}$ such that $\xi_{|K_f} = \Phi$ and $\|f - g\circ\xi\|_{C^0} = \max_{i=1,\ldots,n} |c_i - c'_i| = c(T)$. By Proposition 5.2, this will yield the claim.

Let us endow $\mathscr{M}$ with a Riemannian metric, and consider the smooth vector field $X = -\frac{\nabla f}{\|\nabla f\|^2}$ on $\mathscr{M} \setminus K_f$, and the smooth vector field $Y = \frac{\nabla g}{\|\nabla g\|^2}$ on $\mathscr{M} \setminus K_g$. Let us denote by $\varphi_t(p)$ and $\psi_t(p)$ the flow lines defined by X and Y, on $\mathscr{M} \setminus K_f$ and $\mathscr{M} \setminus K_g$, respectively. We observe that f strictly decreases along X-trajectories, while g strictly increases along Y-trajectories. Moreover, no two X-trajectories (resp. Y-trajectories) pass through the same p. Hence, $\varphi_t(p)$ and $\psi_t(p)$ are injective functions of t and p, separately. By [19, Prop. 1.3], φ and ψ are continuous in t and p when restricted to compact submanifolds of $\mathscr{M} \setminus K_f$ and $\mathscr{M} \setminus K_g$, respectively.

Let us fix a real number $\varepsilon > 0$ sufficiently small so that, for $i = 1, \ldots, n$, $f^{-1}([c_i - \varepsilon, c_i + \varepsilon]) \cap K_f = \{p_i\}$ and $g^{-1}([c'_i - \varepsilon, c'_i + \varepsilon]) \cap K_g = \{p'_i\}$.

In order to construct the desired homeomorphism ξ on $\mathscr{M}$, the main idea is to cut $\mathscr{M}$ into cobordisms and define suitable homeomorphisms on each of these cobordisms that can be glued together to obtain ξ. The fact that ξ is not required to be differentiable but only continuous facilitates the gluing process.

Let us consider the cobordisms obtained cutting $\mathscr{M}$ along the level curves $f^{-1}(c_i \pm \varepsilon)$ and $g^{-1}(c'_i \pm \varepsilon)$ for $i = 1, \ldots, n$. According to whether these cobordisms contain points of maximum, minimum, saddle points, or no critical points at all, we treat the cases differently.

Case 1: Let p_i, p'_i be points of maximum or minimum of f and g, respectively. Let $D = D_i$ (resp. $D' = D'_i$,) be the connected component of $f^{-1}([c_i - \varepsilon, c_i + \varepsilon])$ (resp. $g^{-1}([c'_i - \varepsilon, c'_i + \varepsilon])$) that contains p_i (resp. p'_i). D and D' are topolological disks. Let $\sigma^D : \partial D \to \partial D'$ be a given homeomorphism between the boundaries of D and D'.

Claim 1. There exists a homeomorphism $\xi^D : D \to D'$ such that:

(a_1) $\xi^D|_{\partial D} = \sigma^D$;
(b_1) $\max_{p\in D}|f(p) - g\circ\xi^D(p)| = |c_i - c'_i|$.

Proof of Claim 1. We first prove Claim 1 for maxima. We set $\xi^D(p_i) = p'_i$, and, for every $p \in D\setminus\{p_i\}$, $\xi^D(p) = p'$, where $p' = \psi_{f(p)-c_i+\varepsilon}\circ\sigma^D\circ\varphi_{f(p)-c_i+\varepsilon}(p)$. In plain words, for each $p \in D$ we follow the X-flow downwards until the intersection with $f^{-1}(c_i-\varepsilon)$; then we apply the homeomorphism σ^D to go from $f^{-1}(c_i-\varepsilon)$ to $g^{-1}(c'_i-\varepsilon)$; finally, we follow the Y-flow upwards.

The function ξ^D is injective as can be seen using the aforementioned injectivity property of φ and ψ. Moreover, ξ^D is surjective because, for every $p\in D\setminus\{p_i\}$, there exists a flow line passing for p. Furthermore, ξ^D is continuous on $D\setminus\{p_i\}$ because composition of continuous functions. The continuity can be extended to the whole D as can be seen taking a sequence (q_j) in $D\setminus\{p_i\}$ converging to p_i. Since $\lim_j f(q_j) = c_i$, by construction of ξ^D it holds that $\lim_j g(\xi^D(q_j)) = c'_i$. We see that $\lim_j \xi^D(q_j) = p'_i$ because p'_i is the only point of D' where g takes value equal to c'_i,. Therefore ξ^D is continuous on D. Moreover, since ξ^D is a continuous bijection from a compact space to a Hausdorff space, it is a homeomorphism. Finally, property (a_1) holds by construction and property (b_1) holds because, for every $p\in D$, $g(\xi^D(p)) = f(p)+c'_i-c_i$.

To prove Claim 1 when p_i, p'_i are minimum points of f and g, it is sufficient to replace $\varphi_{f(p)-c_i+\varepsilon}(p)$ and $\psi_{f(p)-c_i+\varepsilon}(p)$ by $\varphi_{f(p)-c_i-\varepsilon}(p)$ and $\psi_{f(p)-c_i-\varepsilon}(p)$, respectively.

Case 2: Let p_i, p'_i be two splitting saddle points or two joining saddle points of f and g, respectively, and let P and P' be the connected component of $f^{-1}([c_i-\varepsilon, c_i+\varepsilon])$ and $g^{-1}([c'_i-\varepsilon, c'_i+\varepsilon])$, respectively, that contain p_i and p'_i. Let $\sigma^P: \partial^- P \to \partial^- P'$ be a given homeomorphism between the lower boundaries of P and P'.
Claim 2. There exists a homeomorphism $\xi^P: P\to P'$ such that:

(a_2) $\xi^P|_{\partial^- P} = \sigma^P$;
(b_2) $\max_{p\in P}|f(p) - g\circ\xi^P(p)| = |c_i - c'_i|$.

Proof of Claim 2. Let us consider the case p_i, p'_i are two splitting saddle points of f and g respectively, so that P and P' are two upside-down pairs of pants. We let p_a, p_b be the only two points of intersection of $f^{-1}(c_i-\varepsilon/2)$ with the trajectories of the gradient vector field X coming from p_i. Analogously, we let p'_a, p'_b be the only two points of intersection of $g^{-1}(c'_i-\varepsilon/2)$ with the trajectories of the gradient vector field Y leading to p'_i.

The pair of pants P can be decomposed into $P = M\cup N\cup O$ with $M = \{p\in P: f(p)\in [c_i-\varepsilon, c_i-\varepsilon/2]\}$, $N = \{p\in P: f(p)\in[c_i-\varepsilon/2, c_i]\}$ and $O = \{p\in P: f(p)\in[c_i, c_i+\varepsilon]\}$. Analogously, the pair of pants P' can be decomposed into $P' = M'\cup N'\cup O'$ with $M' = \{p'\in P': g(p')\in[c'_i-\varepsilon, c'_i-\varepsilon/2]\}$, $N' = \{p'\in P': g(p')\in[c'_i-\varepsilon/2, c'_i]\}$, and $O' = \{p'\in P': g(p')\in[c'_i, c'_i+\varepsilon]\}$.

The construction of ξ^P is based on gluing three homeomorphisms $\xi^M: M\to M'$, $\xi^N: N\to N'$, $\xi^O: O\to O'$ together.

First, we observe that, M and M' being cylinders, it is possible to construct a homeomorphism ξ^M that extends σ^P to M in such a way that $\xi^M(p_a) = p'_a$ and $\xi^M(p_b) = p'_b$, also sending the level-sets of f into those of g. In this way $\max_{p\in M}|f(p)-g\circ\xi^M(p)| = |c_i-c'_i|$.

Next, we define ξ^N by setting $\xi^N(p_i) = p'_i$, and, for every $p\neq p_i$, $\xi^N(p) = p'$, where $p' = \psi_{f(p)-c_i+\varepsilon/2}\circ\xi^M\circ\varphi_{f(p)-c_i+\varepsilon/2}(p)$. It agrees with ξ^M on $\partial M\cap\partial N$ and $\max_{p\in N}|f(p) - g\circ\xi^N(p)| = |c_i-c'_i|$. Moreover, ξ^N is bijective and continuous on $N\setminus\{p_i\}$ by arguments similar to those used in the proof of Claim 1. To see that continuity extends to p_i, let (q_j) be a sequence converging to p_i. The sequence $(\varphi_{f(p)-c_i+\varepsilon/2}(q_j))$ has at most two accumulating points, precisely the points p_a, and p_b. By the construction of ξ^M, $\xi^M(p_a) = p'_a$ and $\xi^M(p_b) = p'_b$, hence the sequence $(\xi^N(q_j))$ converges to p'_i. In conclusion, ξ^N is bijective and continuous, therefore it is a homeomorphism.

Finally, we construct ξ^O by using again the trajectories of X and Y: for each $p \in O$ we follow the flow of X downwards until the intersection with $f^{-1}(c_i)$. If the intersection point q is different from p_i, we set $\xi^O(p)$ equal to the point p' on the trajectory of $\xi^N(q)$ such that $p' = \psi_{f(p)-c_i}(\xi^N(q))$. Otherwise, if $q = p_i$, we consider a sequence (r_j) of points in the same connected component of $O \setminus \{p_i\}$ as p and converging to p. The intersection of $f^{-1}(c_i)$ with the downward flow through r_j, $j \in \mathbb{N}$, gives a sequence (q_j) converging to p_i and belonging to one and the same component of $f^{-1}(c_i) \setminus \{p_i\}$ as p. By the continuity of ξ^N the sequence $(\xi^N(q_j))$ converges to p'_i and its points belong to one and the same component of $g^{-1}(c'_i) \setminus \{p'_i\}$. Hence the sequence $(\psi_{f(r_j)-c_i}(\xi^N(q_j)))$ converges to a point p'. We set $\xi^O(p) = p'$. By the continuity of φ and ψ, this definition does not depend on the choice of the sequence (r_j). By construction, ξ^O is continuous and the proof that it is a homeomorphism can be handled by arguments similar to those used for ξ^N. Moreover, it agrees with ξ^N on $\partial N \cap \partial O$ and $\max_{p \in N} |f(p) - g \circ \xi^N(p)| = |c_i - c'_i|$.

In conclusion, ξ^P can be constructed by gluing the homeomorphisms ξ^M, ξ^N, ξ^O together and the properties (a_2) and (b_2) hold by construction.

The case when p_i, p'_i are two joining saddle points of f and g, respectively, can be treated analogously. We have only to take into account that P is now a pairs of pants, and hence $M = \{p \in P : f(p) \in [c_i - \varepsilon, c_i - \varepsilon/2]\}$ is a pair of cylinders each containing one point of intersection between $f^{-1}(c_i - \varepsilon/2)$ and the trajectories of the gradient vector field X coming from p_i. Similarly for P'.

Case 3: Let p_i, p_j (resp. p'_i, p'_j) be critical points connected by an edge in the Reeb graph of f (resp. g), and assume $c_i < c_j$ (resp. $c'_i < c'_j$). Let $C = \{p \in \mathscr{M} : [p] \in e(p_i, p_j),\ c_i + \varepsilon \le f(p) \le c_j - \varepsilon\}$ and $C' = \{p \in \mathscr{M} : [p] \in e(p'_i, p'_j),\ c'_i + \varepsilon \le g(p) \le c'_j - \varepsilon\}$. C and C' are two topological cylinders. Let $\sigma^C : \partial^- C \to \partial^- C'$ be a given homeomorphism between the lower boundaries of C and C'.

Claim 3. There exists a homeomorphism $\xi^C : C \to C'$ such that:

(a_3) $\xi^C|_{\partial^- C} = \sigma^C$;
(b_3) $\max_{p \in C} |f(p) - g \circ \xi^C(p)| = \max\{|c_i - c'_i|, |c_j - c'_j|\}$.

Proof of Claim 3. To prove Claim 3, for every $p \in C$, we set λ_p equal to the only value in $[0,1]$ for which $f(p) = (1-\lambda_p)(c_i + \varepsilon) + \lambda_p(c_j - \varepsilon)$, and define $\xi^C(p) = p'$, with $p' = \psi_{\lambda_p(c'_j - c'_i - 2\varepsilon)} \circ \sigma^C \circ \varphi_{\lambda_p(c_j - c_i - 2\varepsilon)}(p)$.

By the same arguments as used to prove the previous Claims 1 and 2, ξ^C is a homeomorphism. It satisfies property (a_3) by construction. To prove (b_3), it is sufficient to observe that, for every $p \in C$,

$$\begin{aligned} |f(p) - g(\xi^C(p))| &= |(1-\lambda_p)(c_i + \varepsilon) + \lambda_p(c_j - \varepsilon) - (c'_i + \varepsilon + \lambda_p(c'_j - c'_i - 2\varepsilon))| \\ &= |(1-\lambda_p)(c_i - c'_i) + \lambda_p(c_j - c'_j)|. \end{aligned}$$

Let us now construct the desired homeomorphism $\xi : \mathscr{M} \to \mathscr{M}$. Let $\{p_1, \ldots, p_s\} \subseteq K_f$, $s \le n$, be the set of critical points of f of index 0 or 2, and, for $i = 1, \ldots, s$, let D_i, D_i' be as in Claim 1.

The spaces $W = \overline{\mathscr{M} \setminus \bigcup_{i=1}^s D_i}$ and $W' = \overline{\mathscr{M} \setminus \bigcup_{i=1}^s D_i'}$ can be decomposed into the union of cobordisms containing either no critical points or exactly one critical point of index 1. By Claims 2 and 3, it is possible to extend a given homeomorphism $\sigma^W : \partial^- W \to \partial^- W'$ defined between the lower boundaries of W and W' to a homeomorphism $\xi^W : W \to W'$ by gluing all the homeomorphisms on cobordisms along their boundary components in the direction of the increasing of the functions f and g. Next, by Claim 1, we can glue this homeomorphism ξ^W along each boundary component of W to a homeomorphism ξ^{D_i} : $D_i \to D_i'$, for $i = 1, \ldots, s$. As a result, we get the desired self-homeomorphism ξ of $\mathscr{M}$ such that $\max_{p \in \mathscr{M}} |f(p) - g \circ \xi(p)| = \max_{i=1,\ldots,n} |c_i - c_i'|$. □

Lemma 5.4. *For every elementary deformation $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ of type* (B) *or* (D)*, $c(T) \ge d_N([f], [g])$.*

Proof. We prove the assertion only for the case when T is of type (D), because the other case will then follow from $c(T^{-1}) = c(T)$ and the symmetry property of d_N.

By definition of elementary deformation of type (D), T transforms (Γ_f, ℓ_f) into a labeled Reeb graph that differs from (Γ_f, ℓ_f) in that two vertices, say $p_1, p_2 \in K_f$, have been deleted together with their connecting edges. Otherwise vertices, adjacencies and labels are the same. Assuming $f(p_1) = c_1, f(p_2) = c_2$, with $c_1 < c_2$, we have $c(T) = \frac{c_2 - c_1}{2}$. We recall that $f^{-1}([c_1, c_2]) \cap K_f = \{p_1, p_2\}$. By Proposition 3.4, there exists a deformation $S = (S_0, S_1) \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$, S_0 being of type (R), S_1 of type (D), such that $c(S) = c(T)$. In particular, as shown in the proof of the same proposition (formulas (3.1) and (3.2)), for every $\varepsilon > 0$ sufficiently small, S_0 and S_1 can be built so that $c(S_0) = \frac{c_2 - c_1}{2} - \varepsilon$ and $c(S_1) = \varepsilon$.

For any h_ε for which $S_0(\Gamma_f, \ell_f) \cong (\Gamma_{h_\varepsilon}, \ell_{h_\varepsilon})$, by Lemma 5.3 we have $d_N([f], [h_\varepsilon]) \le c(S_0) = \frac{c_2 - c_1}{2} - \varepsilon$. Thus,

$$d_N([f], [g]) \le d_N([f], [h_\varepsilon]) + d_N([h_\varepsilon], [g]) \le \frac{c_2 - c_1}{2} - \varepsilon + d_N([h_\varepsilon], [g]).$$

Therefore, proving that $d_N([h_\varepsilon], [g]) \le 4\varepsilon$ will yield the claim, by the arbitrariness of $\varepsilon > 0$.

Let W_ε be the connected component of $h_\varepsilon^{-1}([\frac{c_1+c_2}{2} - 2\varepsilon, \frac{c_1+c_2}{2} + 2\varepsilon])$ containing p_1, p_2, and let us assume that ε is so small that $h_\varepsilon^{-1}([\frac{c_1+c_2}{2} - 2\varepsilon, \frac{c_1+c_2}{2} + 2\varepsilon])$ does not contain other critical points of h_ε. By the Cancellation Theorem in [18, Sect. 5], it is possible to define a new simple Morse function $h_\varepsilon' : \mathscr{M} \to \mathbb{R}$ which coincides with h_ε on $\mathscr{M} \setminus W_\varepsilon$, and has no critical points in W_ε. In particular, $(\Gamma_{h_\varepsilon'}, \ell_{h_\varepsilon'}) \cong (\Gamma_g, \ell_g)$, implying that h_ε' and g are R-equivalent. It necessarily holds that

$$d_N([h_\varepsilon], [h_\varepsilon']) \le \max_{p \in \mathscr{M}} |h_\varepsilon(p) - h_\varepsilon'(p)| = \max_{p \in W_\varepsilon} |h_\varepsilon(p) - h_\varepsilon'(p)| \le 4\varepsilon.$$

Moreover, by the R-equivalence of h_ε' and g, we have $d_N([h_\varepsilon'], [g]) = 0$, so that $d_N([h_\varepsilon], [g]) \le 4\varepsilon$ by the triangle inequality property of d_N. □

Lemma 5.5. *For every elementary deformation $T \in \mathscr{T}((\Gamma_f, \ell_f), (\Gamma_g, \ell_g))$ of type (K$_i$), $i = 1, 2, 3$, $c(T) \ge d_N([f], [g])$.*

Proof. For an elementary deformation T of type (K$_i$), $i = 1, 2, 3$, the sets K_f and K_g have the same cardinality, and all but at most two of the critical values of f and g coincide. Let $K_f = \{p_1, \ldots, p_n\}$ and $K_g = \{p_1', \ldots, p_n'\}$, with $f(p_k) = c_k$, $g(p_k') = c_k'$ for every $k =$

$1,\ldots,n$. Assuming that the points p_1, p_2 correspond to the vertices u_1, u_2 of Γ_f shown in Table 1, rows 3-4, it holds that $c_1 < c_2$, $c'_1 > c'_2$, and $c_k = c'_k$ for $k = 3,\ldots,n$. Moreover, $K_f \cap f^{-1}([c_1,c_2]) = \{p_1,p_2\}$ and $K_g \cap g^{-1}([c'_2,c'_1]) = \{p'_1,p'_2\}$. Since $f,g \in \mathscr{F}^0(\mathscr{M})$, there exist $a,b \in \mathbb{R}$, with $a<b$, such that c_1,c_2 and c'_1,c'_2 are the sole critical values of f and g, respectively, that belong to the interval $[a,b]$. Let us denote by W the connected component of $f^{-1}([a,b])$ containing p_1,p_2. Under our assumptions, we can apply the Preliminary Rearrangement Theorem [18, Thm 4.1], and deduce that, for some choice of a gradient-like vector field X for f, there exists a Morse function $h: W \to \mathbb{R}$ that has the same gradient-like vector field as f, coincides with $f_{|W}$ near ∂W and is equal to f plus a constant in some neighborhood of p_1 and in some neighborhood of p_2. Moreover, $K_h = K_{f_{|W}}$, $h(p_1) = c'_1$, $h(p_2) = c'_2$. We can extend h to the whole surface by defining

$$\widehat{h}(p) = \begin{cases} f(p), & \text{if } p \in \mathscr{M} \setminus W, \\ h & \text{if } p \in W. \end{cases}$$

Hence, $\widehat{h} \in \mathscr{F}^0(\mathscr{M})$ and $(\Gamma_{\widehat{h}}, \ell_{\widehat{h}}) \cong T(\Gamma_f, \ell_f)$, implying that $\widehat{h}$ is R-equivalent to g. Therefore, by Definition 1.6, $d_N([f],[g]) = d_N([f],[\widehat{h}])$.

Let us prove that $d_N([f],[\widehat{h}]) \le c(T)$. We observe that, by the definitions of d_N and $\widehat{h}$, we get:

$$d_N([f],[\widehat{h}]) \le \|f - \widehat{h}\|_{C^0} = \max_{p\in\mathscr{M}} |f(p) - \widehat{h}(p)| = \max_{p\in W} |f(p) - h(p)|. \tag{5.1}$$

To estimate the value of $\max\limits_{p\in W} |f(p)-h(p)|$, we review the construction of the function h, as given in [18]. Let $\mu : W \to [a,b]$ be a smooth function that is constant on each trajectory of X, zero near the set of points on trajectories going to or from p_1, and one near the set of points on trajectories going to or from p_2. Then the function h can be defined as $h(p) = G(f(p),\mu(p))$, where $G: [a,b]\times[0,1] \to [a,b]$ is a smooth function defined as $G(x,t) = (1-t)\cdot G(x,0) + t\cdot G(x,1)$, with the following properties (see also Figure 14):

- $\frac{\partial G}{\partial x}(x,0) = 1$ for x in a neighborhood of c_1 (in particular $G(x,0) = x + c'_1 - c_1$ for x in a neighborhood of c_1),
  $\frac{\partial G}{\partial x}(x,1) = 1$ for x in a neighborhood of c_2 (in particular $G(x,1) = x + c'_2 - c_2$ for x in a neighborhood of c_2);
- For all x and t, $G(x,t)$ monotonically increases from a to b as x increases from a to b;
- $G(x,t) = x$ for x near to a or b and for every $t \in [0,1]$.

By the construction of h and the inequality (5.1), we have:

$$\begin{aligned} d_N([f],[\widehat{h}]) &\le \max_{p\in W} |f(p) - G(f(p),\mu(p))| = \max\{|f(p) - G(f(p),0)|, |f(p) - G(f(p),1)|\} \\ &= \max\{|c_1 - c'_1|, |c_2 - c'_2|\} = c(T). \end{aligned}$$

□

Theorem 5.6. *Let $f,g \in \mathscr{F}^0(\mathscr{M})$, and (Γ_f,ℓ_f), (Γ_g,ℓ_g) be the associated labeled Reeb graphs. Then $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) = d_N([f],[g])$.*

Proof. The inequality $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \ge d_N([f],[g])$ holds because, for every deformation $T \in \mathscr{T}((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$, $c(T) \ge d_N([f],[g])$. To see this, let $T = (T_1,\ldots,T_n)$, and set $T_i \cdots T_1(\Gamma_f,\ell_f) \cong (\Gamma_{f^{(i)}}, \ell_{f^{(i)}})$, $f = f^{(0)}$, $g = f^{(n)}$. From Lemmas 5.3-5.5 and the

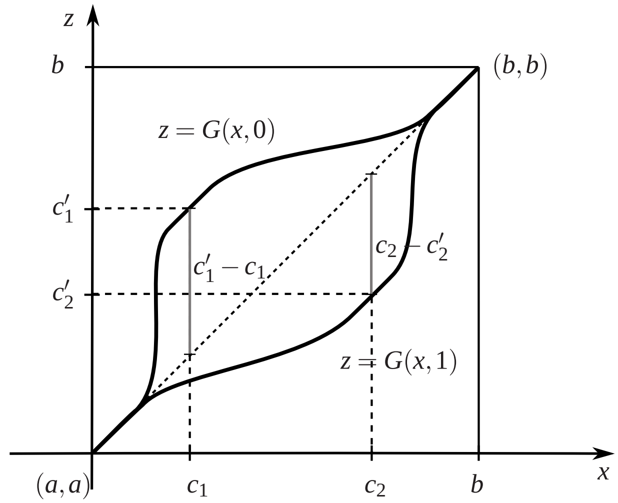

FIGURE 14. The function G introduced in [18] and used in the proof of Lemma 5.5.

triangle inequality property of d_N, we get

$$c(T) = \sum_{i=1}^{n} c(T_i) \geq \sum_{i=1}^{n} d_N([f^{(i-1)}],[f^{(i)}]) \geq d_N([f],[g]).$$

Conversely, by Theorem 4.1, $d_E((\Gamma_f,\ell_f),(\Gamma_{g\circ\xi},\ell_{g\circ\xi})) \leq \|f - g\circ\xi\|_{C^0}$, for every $\xi \in \mathscr{D}(\mathscr{M})$. Therefore $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \leq \inf_{\xi\in\mathscr{D}(\mathscr{M})} \|f - g\circ\xi\|_{C^0} = d_N([f],[g])$ because $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) = d_E((\Gamma_f,\ell_f),(\Gamma_{g\circ\xi},\ell_{g\circ\xi}))$. □

Corollary 5.7. *For every $f,g \in \mathscr{F}^0(\mathscr{M})$, the edit distance between the associated labeled Reeb graphs is a metric on isomorphism classes of labeled Reeb graphs.*

Proof. The claim is an immediate consequence of Theorem 5.6 together with [3, Thm. 4.2], which states that the natural pseudo-distance is actually a metric on the space $\mathscr{F}^0(\mathscr{M})$. □

Corollary 5.8. *For every $f,g \in \mathscr{F}^0(\mathscr{M})$, $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \geq d_B(D_f,D_g)$, where d_B denotes the bottleneck distance between the persistence diagrams D_f and D_g of f and g. In some cases this inequality is strict.*

Proof. The inequality $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \geq d_B(D_f,D_g)$ holds because of Theorem 5.6 and the fact that the bottleneck distance is a lower bound for the natural pseudo-distance (cf. [5]).

As for the second statement, an example showing that the edit distance between the labeled Reeb graphs of two functions $f,g \in \mathscr{F}^0(\mathscr{M})$ can be strictly greater than the bottleneck distance between the persistence diagrams of f and g is displayed in Figure 15. Indeed, f and g have the same persistence diagrams for any homology degree implying that $d_B(D_f,D_g) = 0$, whereas the labeled Reeb graphs are not isomorphic, implying that $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) > 0$. □

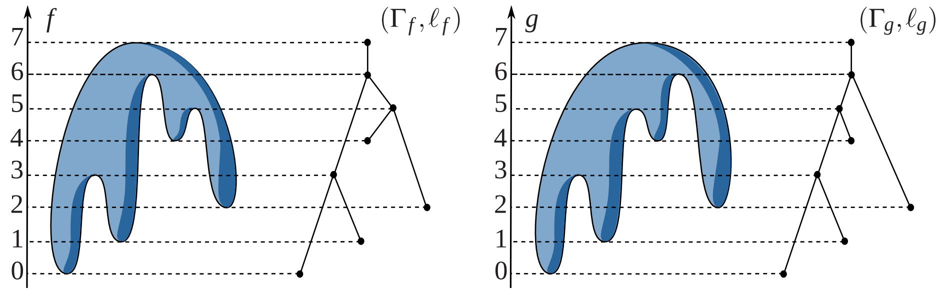

FIGURE 15. The example used in the proof of Corollary 5.8 to show that the edit distance between labeled Reeb graphs can be more discriminative than the bottleneck distance between persistence diagrams whenever the same functions are considered.

Corollary 5.9. *For every* $f,g \in \mathscr{F}^0(\mathscr{M})$, $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \geq d_{FD}(R_f,R_g)$, *where* d_{FD} *denotes the functional distortion distance between the spaces* $R_f = \mathscr{M}/\sim_f$ *and* $R_g = \mathscr{M}/\sim_g$. *In some cases this inequality is strict.*

Proof. The inequality $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) \geq d_{FD}(R_f,R_g)$ is a consequence of the stability of Reeb graphs with respect to d_{FD} [1, Thm. 4.1], and can be seen in the same way as the second inequality shown in the proof of Theorem 5.6.

As for the second statement, an example showing that, for two functions $f,g \in \mathscr{F}^0(\mathscr{M})$, $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g))$ can be strictly greater than $d_{FD}(R_f,R_g)$ is displayed in Figure 16. In

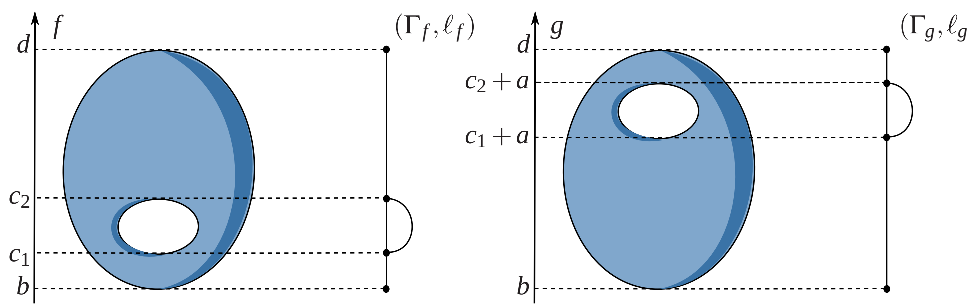

FIGURE 16. The example used in the proof of Corollary 5.9 to show that the edit distance between labeled Reeb graphs can be more discriminative than the functional distortion distance between Reeb graphs whenever the same functions are considered.

this case, $d_E((\Gamma_f,\ell_f),(\Gamma_g,\ell_g)) = a$, because a is both the cost of the deformation T of type (R) that changes the vertex label c_i into $c_i + a$, $i = 1,2$, and the value of the bottleneck distance between the 1st homology degree (ordinary) persistence diagrams of f and g. On the other hand, $d_{FD}(R_f,R_g) \leq (c_2 - c_1)/4$ as can be seen by considering any continuous map $\Phi : R_f \to R_g$ that takes each point of R_f to a point of R_g with the same function value, together with any continuous map $\Psi : R_g \to R_f$ that takes each point of R_g to a point of R_f with the same function value. □

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