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**Abstract**

We consider the interplay between income distribution and optimal commodity taxation, linking equity issues to optimal taxes through the effect of income distribution on market demand and its price elasticity. We find conditions to conciliate the equity and efficiency tradeoff and to assess the impact of inequality changes on the optimal taxation of necessity and luxury goods. We show that the regressivity or progressivity of the tax system is determined by the distribution of luxuries and necessities in the economy. If the tax system is regressive (progressive), a decrease (increase) of income inequality leads to an average decrease of the optimal tax rates, achieving welfare gains for society. Our analysis provides a framework to investigate the linkages between direct and indirect taxation.

**Keywords**: Optimal Commodity Taxation; Efficiency; Equity; Income Distribution.

**JEL classification**: H21; D63; D11.

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# 1 Introduction

Commodity taxation is one of the main sources of government revenue. According to OECD (2014) and European Commission (2014), taxation of goods and services in 2012 accounted for 34% of the tax revenue in Europe (11% of GDP), and 17.9% of the US tax yield (4.4% of GDP). The most important form of commodity taxation in Europe, i.e. VAT, collected 922 billion euros in 2012, amounting to 7.1% of EU-26 GDP, despite the 177 billion tax flow lost in informal or black markets (Barone *et al.*, 2014). Given the huge amount of financial resources absorbed by such taxes, the design and implementation of an optimal commodity taxation system is an important choice for the public sector, as well as for the wellbeing of a society.

Optimal commodity taxation has been strictly related to second-best policies since the seminal paper by Ramsey (1927), who simply ruled out the possibility of using lump sum taxes in a single agent economy. Since then, the main theoretical body on the issue has been developed in the 70s and 80s – the focus being on the features and properties of "second-best", fair and efficient consumption taxes, under different conditions in terms of the number and types of goods, individual heterogeneity, partial or general equilibrium, presence of market failures, compliance costs.[1] Arguably, the equity-efficiency tradeoff represents the most influential result for its public policy implications: once agent heterogeneity is brought into the analysis, there arises a conflict between efficiency and equity reasons for second-best optimal consumption taxation.[2] Indeed, efficiency requires higher taxation on goods with lower price elasticity, which however are typically necessities

[1] As to types and number of commodities, Corlett and Hague (1953) consider two goods and leisure, introducing labour supply decision in the optimal commodity literature. This framework was further developed by Meade (1955), Lipsey and Lancaster (1956-57), Harberg (1971), Atkinson and Sadmo (1980), King (1980). As to individual heterogeneity see Diamond and Mirless (1971), Feldstein (1972), Diamond (1975); in particular, Diamond and Mirless (1971) was the seminal paper including production into the analysis. The presence of market failures was addressed, *inter alios*, by Stiglitz and Dasgupta (1971), Atkinson and Stern (1974), Sadmo (1975), Bovenberg and de Mooij (1994), Cremer, Gahvari and Ladoux (1998), Bovenberg and Goulder (2002), Fullerton (1996). On compliance costs see, e.g., Cremer and Gahvari (1993). Boadway (2012) provides a recent general survey.

[2] The second best scenario is particularly suitable with agent heterogeneity, since lump sum person-specific taxes are unfeasible, and difficult to administer in a multi consumer setting.

representing a big share of expenditure for low-income individuals; this would negatively affect the wealth and welfare of people at the bottom of the income distribution, and increase inequality. As a consequence, an inequality-averse government would tax more heavily goods with higher price elasticity, which implies an efficiency loss.

It is in general well known that modeling aggregate demand as depending on aggregate (or mean) income may lead to misleading results, since it overlooks the effect of individual heterogeneity on aggregate behaviour. In studying commodity taxation, the analysis of the effect of the income distribution on market demand and its price elasticity becomes quite relevant, as the sensitivity of demand to price changes is the main factor driving the conflict between equity and efficiency. This paper focuses on the interplay between income distribution and optimal commodity taxation, in order to evaluate how income distribution affects second-best optimal commodity taxes. We find sufficient statistics, which in principle may be useful for policy reforms, to test whether inequality changes should lead to lower (higher) taxation of necessities and luxuries. Actually, incorporating equity issues *via* income distribution within optimal tax formulas, allows us to find situations where the tradeoff between equity and efficiency can be reconciled.

We model changes in income distribution as second-order stochastic-dominance shifts, such that a change of an exogenous parameter of the income density function standardly identifies an increase in inequality for a given mean. Although in our context income is exogenous, it can be looked at as the outcome of "deep primitives" driving individual choices in terms of labour supply, savings, wealth accumulation (e.g.: Chetty, 2009a). That is, we apply the concept of (net) taxable income, formalised in an elegant way by Feldstein (1999), as quite a useful notion in this framework, since it allows to avoid the "structural approach" of modelling labour supply decisions. In other words, we use a "reduced form" concept of income which implicitly incorporates complex decisions in terms of time, effort, skills, and so on.[3]

Moreover, we do not specify the source of the inequality shock which hits

[3] More precisely, the exogenous individual income ($y$) is related to individual taxable income ($I$) as $y = (1 - T)I + e$, where $e$ is the individual tax shelters, and $T$ is the income tax, which is not a control variable in our setting. See Feldstein (1999), Chetty (2009a,b), or Saez (2012) for a critical survey on taxable income. Since $y$ is exogenous (and so also the income tax), our setting is equivalent to a scenario in which one of the goods is leisure that will not be the object of optimal commodity taxation, and can be choosen as a *numéraire* (e.g., Atkinson and Stiglitz 1976, p. 59-63).

the income distribution: this can also be caused by government interventions on fiscal policies, such as income taxation, rebates, tax credits and other forms of taxation. In this sense, our framework can suggest a way to look at the interplay of direct and indirect taxation different from the existing literature, where the linkages between optimal commodity and income taxations are usually addressed using the Atkinson and Stiglitz (1976) approach. The latter has generated a great deal of research on second best interventions using linear-nonlinear income taxes and uniform *vs* differentiated commodity taxation;[4] however, their "structural approach" to modelling labour supply is characterized by an intrinsic complexity, which makes it quite difficult to obtain clear cut results without assuming strong restrictions on the functional form of utility. Our framework overcomes such a complexity, in that we investigate directly the impact of shocks to the income distribution on optimal commodity taxes, independently of the source of such shocks: by doing so, even if the shock is not driven by an optimal income taxation policy, one can in principle assess how (non optimal) changes in the income taxation affect the optimal taxation of necessities and luxuries, without looking at the complementarity or substitutability of such goods with leisure, which are difficult to test empirically.

In our model, we start by identifying the set of efficient taxes, which allows to interpret the Lagrangian multiplier as a measure of progressivity or regressivity of the tax system. In order to evaluate the regressive (or progressive) effects of the commodity tax burden, we extend the notion of regressivity embodied in the literature on optimal commodity taxation, which is usually limited to the observation of individual consumption shares decreasing with income. To this aim, we apply the notion of liability progression, tradition-

[4] The well known Atkinsons-Stiglitz (AS) theorem states sufficient conditions on the utility function for differentiated commodity taxation to be replaced by non-linear income taxation for second-best policy (or alternatively by a mix of uniform sales tax and non-linear income tax). Stronger conditions are required for differentiated commody taxation to be replaced by linear income taxation (i.e., linear Engel curve uniform across individual besides weak separability and uniform sub-utility across individuals: see Deaton, 1979). Boadway and Pestieau (2003), and Boadway (2012, pp.57-85) provide excellent discussion of the stream of literature generated by the AS framework to include, e.g, heterogeneous preferences (Saez, 2002), different individual needs (Cremer *et al.*, 2001), non optimal income taxes (Konishi,1995; Laroque, 2005, Kaplow, 2006), externalities (Cremer *et al.*, 1998), luxury and necessity (Boadway and Pestieau, 2012), various uses of time (Boadway *et al.*, 1994; Boadway and Gahvari, 2006), and other issues. Piketty and Saez (2013) discuss a more direct proof of AS theorem due to Laroque (2005) and Kaplow (2006).

ally used to assess the progressivity of income taxation. We show that this index depends on the distribution of necessities and luxuries in the economy, and is incorporated in the Lagrangian multiplier.

Following a change in income distribution, two effects emerge: a direct effect, which is driven by the impact of income distribution on aggregate demand; and an indirect effect, due to the individual behavioral response to tax (price) changes: the former is a sort of "mechanical effect" and the latter of "behavioural effect" (Saez, 2002). In order to discriminate between these two effects for a given commodity, we find a simple statistics which depends on the properties of both the Engel curve of that commodity, and (the mean and variance of) the income distribution. When the direct effect dominates, tax adjustments are determined by the effect of the distributional change on market demand, which allows to recover the classical equity-efficiency trade-off in the no-cross-effect scenario: higher equality may make the demand for a necessity more rigid, which leads to efficiency calling for its heavier taxation. By contrast, however, higher inequality can lead to the demand for a luxury (necessity) being more rigid (elastic), implying its higher (lower) taxation – that is, there arise situations where the conflict between efficient and fair taxation can be overcome. Finally, we look at some aggregate implications: we show that if the system is sufficiently progressive (regressive), an increase (decrease) of income inequality should lead to an average decrease of consumption taxation. This certainly occurs if luxuries dominate necessities in the individual tax liability – in which case, the taxation adjustment will lead to a welfare gain for society and again no trade-off presents itself.

The paper is organised as follows. After a presentation of the general framework in Section 2, we analyse how the progressivity of the tax system can be linked to the Lagrangian multiplier. We then assess in Section 3 the effects on consumption taxation of changes in income distribution, and take up the related implications both on the single commodity and in the aggregate. In Section 4 we revisit the traditional trade-off between equity and efficiency, which we consider in terms of welfare in Section 5. Concluding remarks are gathered in Section 6.

# 2 Basic definitions and general framework

We consider consumers' heterogeneity as solely due to income differences. Income $y \in \mathcal{Y}$ is continuously distributed over some positive support $[y_{\min}, y_{\max}]$

$= \mathcal{Y}$, with $F : \mathcal{Y} \times \Theta \rightarrow [0,1]$ the associated distribution; $\Theta \subset \mathbb{R}$ is some parameter space: $\theta \in \Theta$ is a distribution parameter which – as will be clear below – we use to measure inequality; we denote the income density by $f(y,\theta) = \frac{\partial F}{\partial y}$, so that $\mu = \int_{\mathcal{Y}} y f(y,\theta) dy > 0$ is aggregate (mean) income. We use $\sigma^2 = \int_{\mathcal{Y}} (y-\mu)^2 f(y,\theta) dy$ to denote income variance.

There are $i = 1,...,n$ commodities, such that the Marshallian demand for commodity $i$ by a consumer whose income is $y$, is $q_i(p,y)$, $p = (p_1,...,p_n)$ being the price vector. Market demand for that commodity is accordingly

$$Q_i(p,\theta) = \int_{\mathcal{Y}} q_i(p,y) f(y,\theta) dy$$

We assume throughout that for all $i = 1,...,n$, $q_i$ is a normal good, i.e. $\frac{\partial q_i}{\partial y} \geq 0$ for all $y \in \mathcal{Y}$.

We now proceed to formulate the standard problem of optimal commodity taxation in terms of the indirect utility $v(p,y) = u(q(p,y))$, where $q = (q_1,...,q_n)$ is a consumption vector. By defining the maximization problem in terms of aggregate utility (a utilitarian social welfare function), we have

$$\max_t V(p,\theta) = \max_t \int_{\mathcal{Y}} v(p,y) f(y,\theta) dy \tag{1a}$$

while the revenue constraint will be

$$\sum_i t_i Q_i(p,\theta) = R \tag{1b}$$

where $p = \widetilde{p} + t$, with $\widetilde{p}$ the given vector of net prices $\widetilde{p}_i$, $i = 1,...,n$. The Lagrangian becomes

$$\mathcal{L} = \int_{\mathcal{Y}} v(p,y) f(y,\theta) dy - \lambda \left( R - \sum_i t_i Q_i(p,\theta) \right)$$

so that:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial t_i} &= \int_{\mathcal{Y}} \frac{\partial v}{\partial p_i} f(y,\theta) dy + \lambda \left( Q_i + \sum_j t_j \frac{\partial Q_j}{\partial p_i} \right) = 0, \quad i = 1,...,n \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= R - \sum_i t_i Q_i(p,\theta) = 0 \end{aligned}$$

From Roy's identity, $\frac{\partial v}{\partial p_i} = -q_i(p,y)\frac{\partial v}{\partial y}$, hence the first $n$ equations can be written as

$$\sum_j t_j \frac{\partial Q_j}{\partial p_i} + Q_i = \tfrac{1}{\lambda} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta) dy, \qquad i = 1, ..., n \tag{2}$$

where $\int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta) dy > 0$ is the average marginal utility of income, weighted by the quantity of commodity $i$.

The solution of problem (1a,b) gives a tax *system* $t = (t_1, ..., t_n)$, i.e. a set of efficient tax rates. As a final step before proceeding to our analysis, a characterisation of the progressivity of such a system is useful.

## 2.1 The $\lambda$ multiplier and "structural" progressivity

As is well known, the multiplier $\lambda$ represents the shadow cost in terms of social welfare of increasing tax revenue by an additional unit. However, when income distribution is specifically considered, $\lambda$ can be seen to reflect the degree of income progressivity (or regressivity) of the tax system. This is not surprising in itself, as the shadow cost of a unit increase of tax revenue is bound to depend on how sensitive the tax revenue is to variations in income.

Consider the optimality condition (2) for commodity $i$,

$$\sum_j t_j \frac{\partial Q_j}{\partial p_i} + Q_i = \tfrac{1}{\lambda} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta) dy$$

and sum both sides over $i$ after multiplication by $p_i$. One gets

$$\sum_i p_i \sum_j t_j \frac{\partial Q_j}{\partial p_i} + \mu = \tfrac{1}{\lambda} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} y f(y,\theta) dy$$

Now observe that $\sum_i p_i \sum_j t_j \frac{\partial Q_j}{\partial p_i} = \sum_j t_j \sum_i p_i \frac{\partial Q_j}{\partial p_i}$, and that by the homogeneity of the individual demand curve

$$\sum_i p_i \frac{\partial q_j}{\partial p_i} + y \frac{\partial q_j}{\partial y} = 0$$

Then it must be true that

$$-\int_{\mathcal{Y}} \sum_j t_j y \frac{\partial q_j}{\partial y} f(y,\theta) dy + \mu = \tfrac{1}{\lambda} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} y f(y,\theta) dy$$

so that

$$\lambda = \frac{1}{\mu - R'} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} y f(y, \theta) dy \tag{3}$$

where $R' = \int_{\mathcal{Y}} \sum_j t_j \frac{\partial q_j}{\partial y} y f(y, \theta) dy$ and $\mu = \sum_i p_i Q_i$ by the (aggregate) budget constraint. Since the numerator is surely positive, the sign of $\lambda$ is given by the denominator: the latter is

$$\mu - R' = \int_{\mathcal{Y}} y \left( 1 - \sum_j t_j \frac{\partial q_j}{\partial y} \right) f(y, \theta) dy$$

From the individual budget constraint, we get $\sum_j p_j \frac{\partial q_j}{\partial y} = \sum_j (\widetilde{p}_j + t_j) \frac{\partial q_j}{\partial y} = 1$, which implies $1 - \sum_j t_j \frac{\partial q_j}{\partial y} = \sum_j \widetilde{p}_j \frac{\partial q_j}{\partial y}$: thus if the goods are normal, $\mu - R' > 0$ and $\lambda > 0$.

We are now ready to connect $\lambda$ to the overall progressivity (or regressivity) of the tax system. By (3),

$$\lambda = \tfrac{\mu - R}{\mu - R'} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} \tfrac{y f(y,\theta)}{\mu - R} dy = \alpha \int_{\mathcal{Y}} \frac{\partial v}{\partial y} \tfrac{y f(y,\theta)}{\mu - R} dy$$

where $\int_{\mathcal{Y}} \frac{\partial v}{\partial y} \frac{y f(y,\theta)}{\mu - R}$ depends on the tax structure only *via* marginal utility $\frac{\partial v}{\partial y}$, while $\alpha = (\mu - R) / (\mu - R')$ can be looked at as an aggregate index of "progressivity". To see this, consider the liability progression index $\ell(t, y)$ defined as

$$\ell(t, y) = \frac{y \sum_i t_i \frac{\partial q_i(p,y)}{\partial y}}{\sum_i t_i q_i(p, y)} \tag{4}$$

i.e., $\ell$ is the income elasticity of the individual tax burden $\tau(t, y) = \sum_i t_i q_i(p, y)$. If $\ell(t, y) > 1$ indirect taxation is "structurally" progressive, in the sense that it is characterized by a more-than-proportional rise in the commodity tax liability, relative to the increase in income (by the same token, $\ell < 1$ identifies a regressive tax structure). The corresponding aggregate expression would be

$$L(t, \theta) = \frac{\int_{\mathcal{Y}} y \sum_i t_i \frac{\partial q_i(p,y)}{\partial y} f(y, \theta) dy}{\int_{\mathcal{Y}} \sum_i t_i q_i(p, y) f(y, \theta) dy} = \frac{R'}{R}$$

so that a natural definition of the progressivity or regressivity of the tax system $t = (t_1, ..., t_n)$ should be the following:

**Definition** The tax system $t$ is progressive if $L(t, \theta) > 1$; it is regressive if $L(t, \theta) < 1$.

As is well known, using $\ell$ as a measure of tax liability progression has been generally related to income taxation (e.g., Jakobsson, 1976; Tanzi, 1976; Fries *et al.*, 1982; Hutton and Lambert, 1982; Lambert, 1993). Here we use this notion to describe how the individual and aggregate burden of consumption taxation react to income changes. It is thus a "structural" progression measure of the commodity tax system, since it gives information on how consumption tax revenue reacts to changes in income.[5] Clearly, given a set of tax rates $t$, the way consumption tax revenue will react to changes in income depends on the convexity or concavity of Engel curves and the way they are aggregated in the individual tax burden $\tau$. Indeed, the system turns out to

[5] One way to see in what sense $\ell$ is a measure of progressivity when applied to indirect taxation, is by referring to concentration curves. Following Kakwani (1977), let a concentration curve for commodity $i$ be defined as

$$C_i(a, \theta) = \frac{1}{Q_i(p, \theta)} \int_{y_m}^{y(a,\theta)} q_i(p, y) f(y, \theta) dy$$

where $y(a, \theta)$ is the percentile function such that $F(y(a, \theta), \theta) = a$. Dividing through and multiplying by $t_i > 0$, the same concentration curve can be read as a concentration curve for the tax liability $t_i q_i(p, y)$ associated to commodity $i$, defined as

$$C_i^\tau(a, \theta) = \frac{1}{t_i Q_i(p, \theta)} \int_{y_m}^{y(a,\theta)} t_i q_i(p, y) f(y, \theta) dy = C_i(a, \theta)$$

from which one can derive

$$\sum_i \omega_i C_i^\tau(a, \theta) = \tfrac{1}{R} \int_{y_m}^{y(a,\theta)} \tau(t, y) f(y, \theta) dy = L^\tau(a, \theta; t)$$

where $L^\tau(a, \theta; t)$ is the concentration curve of the overall tax burden, and $\omega_i = t_i Q_i(p, \theta)/R$ is the share of commodity $i$'s taxation on overall revenue. Take now two tax vectors $t$ and $t'$, such that $\ell \neq \ell'$: one way of saying that $\ell$ is a measure of progressivity is by noting that if $\ell > \ell'$ for all $y$'s, the tax burden is shared *less* equally under $\ell$ (i.e., the rich pay more) i.e. $L^\tau(a, \theta; t) < L^\tau(a, \theta; t')$: which is surely the case under Kakwani's Theorem 1 (1977, p.720), since the difference in the relevant elasticities is indeed $\ell - \ell' > 0$ (see also Benassi and Chirco, 2006, example 1).

be progressive (regressive) if the individual tax burden is a convex (concave) function of income, i.e. luxuries weigh more (less) than necessities:[6]

**Proposition 1** *If luxuries weigh more (less) than necessities in the individual tax liability $\tau$, the tax system is progressive (regressive).*

**Proof.** The budget constraint obviously implies that $\sum_i p_i \frac{\partial q_i}{\partial y} = 1$, which in turn implies $\sum_i p_i \frac{\partial^2 q_i}{\partial y^2} = 0$, meaning that (barring the case where *all* Engel curves are linear) some have to be convex and some to be concave: given some $t = (t_1, ..., t_n)$, this will determine the sign of $\frac{\partial^2 \tau}{\partial y^2} = \sum_i t_i \frac{\partial^2 q_i}{\partial y^2}$. If $\frac{\partial^2 \tau}{\partial y^2} > 0$, $\ell(t, y) > 1$, that is, $y \sum_i t_i \frac{\partial q_i(p,y)}{\partial y} > \sum_i t_i q_i(p, y)$, for all $y \in \mathcal{Y}$.[7] There follows that $\int_{\mathcal{Y}} y \sum_i t_i \frac{\partial q_i(p,y)}{\partial y} f(y, \theta) dy > \int_{\mathcal{Y}} \sum_i t_i q_i(p, y) f(y, \theta) dy$ and $R' > R$, i.e. $L(t, \theta) > 1$. Clearly, $R' < R$ and $L(t, \theta) < 1$ would follow from assuming $\frac{\partial^2 \tau}{\partial y^2} < 0$. ■

By Proposition 1, the multiplier $\lambda$ is associated with a measure of progressivity, as $\alpha$ greater (or lower) than one will signal a tax-burden-to-income ratio rising (or falling) with income. It should be noticed that the income convexity (or concavity) of the individual tax liability $\tau$ gives a sufficient condition for aggregate progressivity or regressivity of the tax system, *given some vector $t$*: different arrays of tax rates may deliver $R'$ larger or smaller than $R$, for the same given income distribution.

The idea of progressivity being somehow hidden inside the Lagrangian multiplier may be interesting, when connected with the social marginal utility of income (Diamond, 1975), which in our framework is

$$\gamma(t, y) = \frac{\partial v}{\partial y} + \lambda \frac{\partial \tau}{\partial y} \tag{5}$$

As is well known, this gives the marginal effect on utility and tax revenue of a lump sum transfer in favour of an individual with income $y$. It is generally assumed that the social marginal utility of income is decreasing in income, due to an inequality averse government giving more social weight to the poor

[6] With a slight *abus de language*, here we identifiy luxuries and necessities with reference to the convexity or concavity of Engel curves.

[7] We assume that the sign of $\partial^2 q_i / \partial y^2$ does not change with $y$. Convexity of $\tau(t, y)$ wrt $y$ implies progressivity, i.e. that $\tau(t, y)/y$ is increasing in $y$ , if $\tau(t, 0) = 0$ (e.g., Lambert, 2001, p.193).

(e.g, Feldstein, 1972).[8] However, the role of the individual as a tax payer should also be considered: if the system is sufficiently progressive then $\gamma$ may be increasing in income, and conversely if $\gamma$ is decreasing in income, there is a bound on the progressivity of the tax system given by the behaviour of the marginal utility of income.[9]

Clearly, Diamond's result that the multiplier $\lambda$ can be looked at as the expected value of $\gamma$ holds good in our framework, where one can easily see that[10]

$$\lambda = \int_{\mathcal{Y}} \gamma(t, y) \tfrac{yf(y,\theta)}{\mu} dy \tag{6}$$

where $yf(y, \theta)/\mu$ is the density which is relevant for the problem at hand, i.e. that describing the distribution of income "by income classes" (the ratio being between the sum total of income accruing to all consumers whose income is $y$, and total income).

In order to conclude this descriptive section on the progressivity of the tax system, some observation on the neutrality of taxation are in order. If $R' = R$, $\alpha = 1$ and the system will be "neutral": indeed, $L = 1$ means that $\int_{\mathcal{Y}} \sum_i t_i \left( y \frac{\partial q_i(p,y)}{\partial y} - q_i(p, y) \right) f(y, \theta) dy = 0$, so that in the aggregate we get the same result one would have with (zero intercept) linear individual Engel curves. Of course, the latter will indeed be linear if preferences were homothetic and marginal utility a constant – which would yield $\lambda = \frac{\partial v}{\partial y} \frac{\mu}{\mu - R}$, independent of the distribution of income.

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[8] In Diamond's (discrete framework) formulation, the social marginal utility of income would be

$$\gamma(t, y) = \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial y^h} + \lambda \frac{\partial \tau}{\partial y^h}$$

In our case, where the social welfare function $W$ is strictly utilitarian, the social planner's higher weight for the poor would obviously be the result of the marginal utility of income being decreasing – an assumption we shall make in the following sections (apparently supported by the empirical evidence: eg, Layard *et al.*, 2008) .

[9] This can be see by using definition (4) for $\ell$, and noting that $\partial^2 \tau / \partial y^2 = \lambda \tau \ell (\ell + \varepsilon - 1) / y^2$, where $\varepsilon$ is the income elasticity of $\ell$: thus $\partial^2 \gamma / \partial y^2 < 0$ implies $\tau \ell (\ell + \varepsilon - 1) / y^2 < -\partial^2 \gamma / \partial y^2$.

[10] See Appendix A.

# 3 Optimal commodity taxation and income distribution

In the previous section we have presented a version of the standard optimal taxation framework, where income distribution can be explicitly considered. In this section we consider how a change in income distribution, and in particular a change in the degree of inequality, is likely to affect optimal commodity taxation. To do so we need a precise measure of inequality, which we identify by the standard notion of second order stochastic dominance: i.e., we assume that an increase in $\theta$ signals an increase in inequality for given mean $\mu$, as we take $\theta$ to be a (inverse) parameter of second order stochastic dominance.[11] Under this stipulation, in the following sections we first consider the impact of an inequality increase in the single-commodity, single tax case; then we turn to aggregate implications.

## 3.1 Implications on the single commodity

Recall the optimality condition (2) for some commodity $i$

$$\sum_j t_j \frac{\partial Q_j}{\partial p_i} + Q_i = \tfrac{1}{\lambda} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta) dy$$

This can be set in terms of compensated demand by using Slutsky equation:[12]

$$\sum_j t_j \int_{\mathcal{Y}} \frac{\partial h_j}{\partial p_i} f(y,\theta) dy = -Q_i + \tfrac{1}{\lambda} \int_{\mathcal{Y}} \gamma\left(t,y\right) q_i(p,y) f(y,\theta) dy \tag{8}$$

where $\gamma\left(t,y\right)$ is the social marginal utility of income defined by (6). Equation (8), similarly to Diamond (1975, p.338) or Auerbach (1981, p. 107), connects

[11] An increase in $\theta$ signals an increase of inequality in the second order sense if $\int_{y_m}^{y} \left(\partial F/\partial\theta\right) dz \geq 0$ for all $y \in \mathcal{Y}$. As is well known, under this definition lower $\theta$ means a less unequal distribution – an inequality averse social planner would prefer it to a higher $\theta$ distribution; if the further restriction is added that $\int_{y_m}^{y_M} \left(\partial F/\partial\theta\right) dz = 0$ (i.e., mean income is not altered by changes in $\theta$), $\theta$ ranks equal mean distributions by their Lorenz curve (Atkinson, 1970). See Lambert (2001, ch.3) for an overall assessment of the welfare-theoretic foundations of inequality measures.

[12] We substitute for $\frac{\partial q_j}{\partial p_i} = \frac{\partial h_j}{\partial p_i} - \frac{\partial q_j}{\partial y} q_i(p,y)$, where $h_j = h_j\left(p, v\left(p,y\right)\right)$.

the marginal change in the compensated aggregate demand for good $i$ given on the LHS, $H_i$ say,[13] with an aggregate expression on the RHS in terms of consumption of commodity $i$ and the social marginal utility of income.

The effect of a change of $\theta$ on $H_i$ is given by

$$\begin{aligned}\frac{\partial H_i}{\partial \theta} &= -\int_{\mathcal{Y}} q_i(p,y) f_\theta(y,\theta) dy + \tfrac{\partial}{\partial \theta}\tfrac{1}{\lambda} \int_{\mathcal{Y}} \gamma(t,y)\, q_i(p,y) f(y,\theta) dy \\ &= -\int_{\mathcal{Y}} q_i(p,y) f_\theta(y,\theta) dy + \tfrac{1}{\lambda} \int_{\mathcal{Y}} (q_i - k_i)\, \gamma(t,y)\, f_\theta(y,\theta) dy \qquad (9)\end{aligned}$$

where $k_i$ is independent of $y$.[14] This expressions allows to distinguish two effects of an increase in inequality on optimal taxation for commodity $i$. The first term is the **direct effect** of the shock on aggregate consumption, which directly affects the excess burden: it is the change in consumption caused by the shift in the income distribution. The second effect can be seen as an **indirect effect**, driven by the structure of the fiscal system: it represents the changes in individual consumption patterns (and hence in revenue), brought about by the relative tax (price) adjstments. In a way, the direct effect can be looked at as a sort of distribution-induced income effect on consumption, while the latter is a sort of substitution effect, caused by the relative changes in commodity tax rates and prices.[15]

Given these definitions, we now take up the two effects separately. We first have:

**Proposition 2** *Let an increase in $\theta$ signals an increase in inequality for given mean. If the direct effect dominates the indirect effect, the optimal tax adjustments require an increase in taxation for luxuries (i.e. a compensated reduction in their aggregate demand), and a decrease in taxation for necessities.*

**Proof.** If the direct effect dominates, then $\frac{\partial H_i}{\partial \theta} > 0$ if $\int_{\mathcal{Y}} q_i(p,y) f_\theta(y,\theta) dy < 0$, and viceversa if $\int_{\mathcal{Y}} q_i(p,y) f_\theta(y,\theta) dy > 0$. If good $i$ is a luxury, then $(\partial^2 q_i/$

[13] Using the symmetry of the Slutsky matrix, $H_i = \sum_j t_j \int_{\mathcal{Y}} \frac{\partial h_j}{\partial p_i} f(y,\theta) dy$ is the change in the compensated aggregate demand for good $i$.

[14] We derive the second line of equation (9) in Appendix B. From now on we use subscripts to denote derivatives whenever convenient.

[15] This distinction is somewhat similar to that between the "mechanical" and the "behavioral" effects put forward by Saez (2002).

$\partial^2 y) > 0$ and $\int_{\mathcal{Y}} q_i(p,y) f_\theta(y,\theta)dy > 0$ by the properties of the second order stochastic dominance. ■

The proof is quite straightforward, as it plays exclusively on the standard properties of stochastic dominance; however, it brings forward a noteworthy implication: if the direct effect dominates, an increase in *equality* should lead to a decrease in the taxation of luxuries and an increase in that of necessities. We shall go back to this issue in Section 4, when we take up the tradeoff between equity and efficiency.

We now consider the **indirect effect** of an increase in inequality on the compensated aggregate demand of good $i$, which from (9) is given by

$$\frac{\partial}{\partial\theta}\left(\tfrac{1}{\lambda}\int_{\mathcal{Y}} \gamma\left(t,y\right) q_i(p,y) f(y,\theta)dy\right) = \tfrac{1}{\lambda}\int_{\mathcal{Y}} \left(q_i - k_i\right) \gamma\left(t,y\right) f_\theta(y,\theta)dy$$

In order to assess the sign of this effect, we assume that the social marginal utility $\gamma$ is linear in income, i.e. $\partial^2\gamma/\partial y^2 = 0$. This assumption, which is clearly quite convenient in analytical terms, seems to us an acceptable one, since the second derivative of $\gamma$ involves third order derivatives, which at a first approximation can arguably be considered negligible.

Let us now define $\eta_q^i$ and $\eta_\gamma$ as the (positive) elasticity of the slope of the Engel curve for commodity $i$, and the elasticity of the social marginal utility of income, respectively:

$$\eta_q = \left|\frac{\partial^2 q_i}{\partial y^2}\right| \frac{y}{\frac{\partial q_i}{\partial y}},\ \eta_\gamma = \frac{\partial\gamma}{\partial y}\frac{y}{\gamma}$$

These definitions allow a neater expressions of the following result:

**Proposition 3** *Assume* $\frac{\partial^2\gamma}{\partial y^2} = 0$ *and let an increase in* $\theta$ *signal an increase in inequality for given mean; suppose the indirect effect dominates the direct effect. Then*
*(a) if the social marginal utility of income is increasing in income, optimal tax adjustments require a decrease in the taxation of luxuries; taxation on necessities should decrease if* $\eta_q < 2\eta_\gamma$ *and it should increase if* $\eta_q > 2\eta_\gamma$*;*
*(b) if the social marginal utility of income is decreasing in income, optimal tax adjustments require an increase in the taxation of necessities; taxation on luxuries should increase if* $\eta_q < 2\left|\eta_\gamma\right|$ *and it should decrease if* $\eta_q > 2\left|\eta_\gamma\right|$.

**Proof.** If the indirect effect dominates, then the sign of $\partial H_i/\partial\theta$ is given by the sign of

$$\tfrac{1}{\lambda}\int_{\mathcal{Y}}\left[q_i(p,y)-k_i\right]\gamma\left(t,y\right)f_\theta(y,\theta)dy=A$$

and, from the properties of second order stochastic dominance, this expression will be positive or negative, according as $\frac{\partial^2}{\partial y^2}\left\{\left[q_i(p,y)-k\right]\gamma\left(y,t\right)\right\}=\gamma\frac{\partial^2 q_i}{\partial y^2}+2\frac{\partial\gamma}{\partial y}\frac{\partial q_i}{\partial y}+\left(q_i-k_i\right)\frac{\partial^2\gamma}{\partial y^2}$ is positive or negative. Consider now case (a) and assume $\partial^2\gamma/\partial y^2=0<\partial\gamma/\partial y$: then, (i) if $\frac{\partial^2 q_i}{\partial y^2}>0$, $A>0$; (ii) if $\frac{\partial^2 q_i}{\partial y^2}<0$ and $\left|\eta_q\right|<2\eta_\gamma$ then $A>0$; (iii) if $\frac{\partial^2 q_i}{\partial y^2}<0$ and $\left|\eta_q\right|>2\left|\eta_\gamma\right|$ then $A<0$. As to case (b), assume $\partial^2\gamma/\partial y^2=0>\partial\gamma/\partial y$: then (i) if $\frac{\partial^2 q_i}{\partial y^2}<0$, $A>0$; (ii) if $\frac{\partial^2 q_i}{\partial y^2}>0$ and $\eta_q>2\left|\eta_\gamma\right|$ then $A>0$; (iii) if $\frac{\partial^2 q_i}{\partial y^2}>0$ and $\eta_q<2\left|\eta_\gamma\right|$ then $A<0$.[16] ■

As we have seen, the adjustments of commodity taxation when the direct effect dominates are simply driven by the effect of income distribution on demand: e.g., following an increase in inequality, taxation on necessities decreases simply because the aggregate quantity has been reduced by the distributional shift. However, a markedly different – and somewhat more interesting – pattern occurs if the indirect effect dominates. In such a case, results differ according to the nature of the goods and the behaviour of the social marginal utility of income. If the social marginal utility of income is decreasing, meaning that the overall tax system is regressive (or not very progressive), the individual marginal tax yield does not increase enough with individual income: then an increase in inequality will virtually decrease overall revenue since it will depress the tax liability of some individuals (presumably the poor), without making good this loss with a sufficient increase the tax liability of the rich. In order to satisfy the revenue constraint, the government should increase taxation more on necessities than on luxuries. Only if the elasticity of the marginal consumption of luxury is higher that the elasticity of their social value, does taxation on luxuries decrease. If instead, the social marginal utility of income is increasing with income, then the taxation system is progressive enough to make $\gamma$ increasing with income. In such a case,

[16] Since $\gamma$ is linear in income one can write $\gamma\left(t,y\right)=\alpha(t)+\beta\left(t\right)y$ so that the condition $\eta_q\lesseqgtr 2\eta_\gamma$ can be written as $\eta_q\lesseqgtr 2\beta\left(t\right)y/(\alpha(t)+\beta\left(t\right)y)$, or equivalently $\frac{\partial^2 q_i}{\partial y^2}/\frac{\partial q_i}{\partial y}\lesseqgtr 2\beta\left(t\right)/(\alpha(t)+\beta\left(t\right)y)$.

the revenue boosted by the consumption of the rich is sufficient not only to compensate for the loss in revenue due to the decrease of consumption of the poor, but also to reduce taxation. Only if the reduction in consumption of necessities is higher than the reduction in their social value, should taxation increase for such goods.

Clearly, at this point we should ask what the sign of the overall effect is going to be. Ideally, one would like to connect the latter to the curvature of the demand function. Our main result under this respect is the following:

**Proposition 4** *Assume* $\frac{\partial^2 \gamma}{\partial y^2} = 0$ *and let an increase in* $\theta$ *signal an increase in inequality for given mean. If the social marginal utility of income is decreasing (increasing) in income, optimal tax adjustments on a given commodity should increase (decrease), i.e.* $sign\{\frac{\partial H_i}{\partial \theta}\} = sign\left\{\frac{\partial \gamma}{\partial y}\right\}$, *provided that (a)* $q_i$ *is convex in income and such that* $\frac{\partial^2 q_i}{\partial y^2} / \frac{\partial q_i}{\partial y} < \frac{2\mu}{\mu^2 + \sigma^2 - \mu y_{\min}}$, *or (b)* $q_i$ *is concave in income and such that* $\frac{\partial^2 q_i}{\partial y^2} / \frac{\partial q_i}{\partial y} > -\frac{2\mu}{\mu y_{\max} - (\mu^2 + \sigma^2)}$.

**Proof.** See Appendix C ■

The main implications of Proposition 4 are that if the social marginal utility of income is decreasing (increasing) in income, the optimal reaction to an increase in inequality should be that of raising (lowering) taxation on a given commodity, if the corresponding Engel curve is not too convex or too concave. The economics behind this is best seen with reference to Proposition 3: it is then readily seen that under conditions (a) and (b) the indirect effect dominates, which is due to the bound on the slope of the Engel curve. Take e.g. the case of luxuries with increasing $\gamma$: the direct effect – which depends *only* on the convexity of the Engel curve – would call for an increase in taxation (lower $H_i$) as aggregate quantity increases, while the upper bound on convexity plus the weight of the increasing social marginal utility of income allows the indirect effect to dominate and yield a tax decrease. Indeed, the total effect trades off the mere quantity effect against the adjustment required by the behaviour of the social marginal utility of income: if $\gamma$ is increasing, the marginal contribution to revenue of higher taxation for a unit increase in income is itself increasing. Given the income-convexity of $q_i$, higher inequality drives a higher weight of (the consumption of) high income consumers, which works directly through the direct effect, and *via* the weight of the increasing marginal social utility of income through the indirect effect. The strength of the latter *vis à vis* the former is thus given by how fast the social marginal utility of income rises relative to its average ($\lambda$), compounded with how fast the quantity $q_i$ rises with income: indeed, (9) reads as

$$\Lambda_\theta = \tfrac{\partial}{\partial\theta} \int_{\mathcal{Y}} \tfrac{\gamma(t,y)}{\lambda} q_i(p,y) f(y,\theta) dy$$

With linear $\gamma$, its average value (that is, $\lambda$) will be increasing in income variance:[17] a lower variance means a higher marginal contribution of $\gamma$ relative to $\lambda$, and hence looser bounds on the curvature of the Engel curve for the indirect effect to dominate. In this sense, the indirect effect is more likely to dominate the direct one, the lower income variance, as this increases the relative weight of the social marginal utility of income. In Section 4 we discuss the implications of such results in terms of the equity *vs* efficiency tradeoff.

## 3.2 Aggregate implications

Let us define $T(t,\theta) = \sum_i t_i Q_i(\widetilde{p}+t,\theta)$, and let $t$ be the vector of optimal tax rates solving problem (1), such that, for all $i = 1,...,n$, condition (2) holds:

$$\sum_j t_j \frac{\partial Q_j}{\partial p_i} + Q_i = \tfrac{1}{\lambda} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta) dy \tag{2}$$

while the revenure constraint is given by $T(t,\theta) = R$. Implicit differentiation of the latter yields:

$$\sum_i \frac{\partial T}{\partial t_i} \frac{dt_i}{d\theta} = -\frac{\partial T}{\partial \theta} \tag{10}$$

Under optimality, a shock to the income distribution should be such that (2) still holds: thus (10) becomes

$$\sum_i k_i(p,\theta) \frac{dt_i}{d\theta} = -\frac{\partial T/d\theta}{\mu - R'} \tag{11}$$

where $k_i(p,\theta) = \int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta) dy / \int_{\mathcal{Y}} \frac{\partial v}{\partial y} y f(y,\theta) dy > 0$ for all $i = 1,...,n$. This weighted average of tax adjustments is related to the concavity or convexity in income of the individual tax burden.

[17] Under linearity, $\gamma = \alpha + \beta y$, so that, using (6), $\lambda = \alpha + \beta\left(\mu + \sigma^2/\mu\right)$.

**Proposition 5** *Let an increase in $\theta$ signal an increase in inequality for given mean. Then, if luxuries weigh more (less) in the individual tax liability $\tau$ than necessities, i.e. commodity taxation is progressive (regressive), optimality requires an average decrease (increase) in commodity taxation, i.e.* $\sum_i k_i(p,\theta)\frac{dt_i}{d\theta} < (>)0$.

**Proof.** First notice that $\mu - R' > 0$, so the sign of $\sum_i k_i(p,\theta)\frac{dt_i}{d\theta}$ depends on the numerator. The latter can be written as $\frac{\partial}{\partial\theta}\int_{\mathcal{Y}}\tau(t,y)f(y,\theta)dy$, and since an increase in $\theta$ amounts to a second order stochastic dominance shift of the distribution, it follows from the general properties of stochastic dominance that $\frac{\partial}{\partial\theta}\int_{\mathcal{Y}}\tau(t,y)f(y,\theta)dy > 0$ if $\tau(y,t)$ is convex in $y$, i.e. if luxuries weigh more in the individual tax liability than necessities: hence $\sum_i k_i(p,\theta)\frac{dt_i}{d\theta} < 0$. By the same token, if $\tau(t,y)$ is concave in $y$, i.e. if necessities weigh more than luxuries, $\sum_i k_i(p,\theta)\frac{dt_i}{d\theta} > 0$. ■

Clearly, the opposite conclusion would be reached by changing the relevant signs when a variation of $\theta$ signals an increase in equality: if luxury goods weigh more (less) in the tax liability than necessities, optimality will then require an average increase (decrease) in commodities taxation for all goods.

These results rest on an aggregate constraint, which dictates the sign of the adjustment: if *ceteris paribus* a change in inequality would *virtually* increase overall revenue (which occurs when either inequality increases and luxuries dominate, or inequality decreases and necessities dominate), optimality requires that taxation be decreased on average, in order to keep the revenue constant. Thus, e.g., if inequality goes up and luxuries are the main source of revenue for the government, the distributional change will benefit the consumption of some households (presumably the richest ones) and boost overall revenue, in such a way as to more than compensate for the loss of revenue due to the reduction of consumption of other (presumably poorer) households: as a result, it is efficient to reduce taxation on average to balance the government revenue constraint. The same type of reasoning obviously applies to the other cases, and all can be naturally connected with the characterization of the tax system given by Proposition 1: indeed, one might rephrase Proposition 5 by saying that if the tax system is progressive (or regressive) in the sense of Proposition 1, then an increase in inequality should lead to an average decrease (or an increase) of tax rates.

These aggregate implications can be cast in terms of aggregate compensated demand by a direct aggregation of (9). Indeed, if one sums over $i$ both sides of (9) after multiplication by $p_i$, one obtains

$$\begin{aligned}\sum_i p_i \frac{\partial H_i}{\partial \theta} &= -\int_{\mathcal{Y}} \sum_i p_i q_i f_\theta(y,\theta) dy \\ &\quad + \tfrac{1}{\lambda} \int_{\mathcal{Y}} \left( \sum_i p_i q_i(p,y) - \sum_i p_i k_i \right) \gamma(t,y) f_\theta(y,\theta) dy \\ &= \tfrac{1}{\lambda} \int_{\mathcal{Y}} (y-1)\gamma(t,y) f_\theta(y,\theta) dy \qquad (12)\end{aligned}$$

where $\int_{\mathcal{Y}} \sum_i p_i q_i f_\theta(y,\theta)dy = \int_{\mathcal{Y}} y f_\theta(y,\theta)dy = 0$ since $\theta$ does not affect mean income.[18] The following can then be easily shown by assuming the linearity of the social marginal utility of income:

**Proposition 6** *Suppose* $\frac{\partial^2 \gamma}{\partial y^2} = 0$ *and let an increase in* $\theta$ *signal an increase in inequality for given mean: then if the social marginal utility is increasing (decreasing) optimality requires an overall decrease (increase) of taxation, i.e* $sign\left\{\sum_i p_i \frac{\partial H_i}{\partial \theta}\right\} = sign\left\{\frac{\partial \gamma}{\partial y}\right\}$.

**Proof.** By the standard properties of stochastic dominance, $\text{sign}\left\{\sum_i p_i \frac{\partial H_i}{\partial \theta}\right\} = \text{sign}\left\{\frac{\partial^2}{\partial y^2}\left[(y-1)\gamma(t,y)\right]\right\}$, while the linearity of $\gamma$ implies $\frac{\partial^2}{\partial y^2}\left[(y-1)\gamma(t,y)\right] = \frac{2}{\lambda}\frac{\partial \gamma}{\partial y}$. ■

The connection between Propositions 5 and 6 lies in the behaviour of the social marginal utility of income. Say the latter is increasing: by Proposition 6 an increase in inequality makes for a positive marginal value of the overall compensated demands, which implies an overall decrease in taxation. But the social marginal utility of income will be increasing when the tax system is progressive enough to overcome the (decreasing, say) marginal utility of consumption:[19] in such a case, an inequality shock boosts the consumption of the rich and thus the tax revenue (virtually beyond the government's revenue constraint), so that overall taxation should decrease to meet the

[18] Since $\sum_i p_i \int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta)dy = \int_{\mathcal{Y}} \frac{\partial v}{\partial y} y f(y,\theta)dy$, the individual budget constraint implies $\sum_i p_i k_i = 1$.

[19] Notice that $\partial^2 \tau / \partial y^2 > 0$ if $\ell > 1$. If the marginal utility of income is not decreasing, $\gamma$ wil be increasing in income provided $\tau \ell (\ell + \varepsilon - 1)/y^2 > -\partial^2 \gamma / \partial y^2$, (see note 10).

required revenue. This is consistent with the intuition underlying Proposition 5, as we know that the tax system will indeed be progressive if luxuries weigh more than necessities. While both propositions rely on the effects of the progressivity or regressivity of the tax system on overall tax revenue, the focus on compensated demands in Proposition 6 means that a milder condition on progressivity/regressivity is required in this case, *viz.*, that the system be progressive enough to make the social marginal utility increasing with income.

# 4 The equity *vs* efficiency tradeoff

In this section we look at the time honoured issue of the equity-efficiency trade-off, from the perspective of the interaction between an inequality increasing change in income distribution, and the reaction this implies in the tax structure if optimal taxation is to be implemented. In principle there is a well known conflict between efficiency and equity reasons concerning the optimal taxation of luxuries or necessities: efficiency requires a higher taxation on necessities, which however represent a great share of expenditure for low-income individuals; so efficient taxation is likely to be regressive and, in order to avoid these regressive effects, equity is traded off against efficiency by levying higher taxation on luxuries. The underlying rationale for this conclusion is that the price elasticity of necessities is typically low, which minimises the distortionary effects of taxation.

This intuition can be recovered also in our framework, where however the price elasticity of demand is itself affected by the income distribution. The own price elasticity of market demand for good $i$ is:

$$\eta_i^Q(p,\theta) = \int_{\mathcal{Y}} \eta_i(p,y)\varphi_i(p,y,\theta)$$

where

$$\eta_i(p,y) = \frac{-(\partial q_i/\partial p_i)p_i}{q_i}, \quad \varphi_i(y;p,\theta) = \frac{q_i\,(p,y)\,f(y,\theta)}{Q_i\,(p,\theta)}$$

the former being the (positive) individual price elasticity of demand, and the latter the density of the distribution of demand by income classes, such that the corresponding distribution is

$$\Phi_i(y,p,\theta) = \int_{y_{\min}}^{y} \varphi_i(x;p,\theta)dx$$

It can be easily shown that:

$$\frac{\partial \eta_i^Q}{\partial \theta} = \frac{p_i}{Q_i}\left\{\frac{\partial Q_i/\partial \theta}{Q_i(p,\theta)}\int_{\mathcal{Y}} \frac{\partial q_i}{\partial p_i} f(y,\theta)dy - \int_{\mathcal{Y}} \frac{\partial q_i}{\partial p_i} f_\theta(y,\theta)dy)\right\} \tag{13}$$

It is clear from (13) that the income distribution affects market demand elasticity. More precisely, it is possible to show that:[20]

**Proposition 7** *If the individual demand curve is convex (concave) in income and $\partial^3 q_i/\partial p_i \partial y^2$ is nonnegative (nonpositive) then a more unequal income distribution results in a lower (higher) price elasticity of market demand.*

**Proof.** If the Engel curve is convex, $\partial Q_i/\partial \theta > 0$: hence the first term is negative (since $\partial q_i/\partial p_i < 0$). By the same token, if the price derivative of the individual demand curve is convex also the second term is negative. Of course, the opposite results emerge if we are dealing with concave functions. ■

This simple result can be linked to the distinction between the direct and indirect effects of income distribution on aggregate demand and taxation discussed in Section 3, and shows how this distinction can enrich the analysis of the tradeoff between equity and efficiency. Indeed, it is easily seen that in the classical case of no cross effect, if the direct effect dominates the indirect effect, then there may be a reconciliation between equity and efficiency issues. To see this, consider the optimality condition (2) in the absence of cross effects

$$t_i \frac{\partial Q_i}{\partial p_i} + Q_i = \tfrac{1}{\lambda} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta)dy$$

which can be solved for $t_i$ in terms of own market price elasticity to get

$$t_i = \frac{1}{\eta_i^Q}\left(-p_i + \frac{1}{\lambda Q_i}\int_{\mathcal{Y}} \frac{\partial v}{\partial y} p_i q_i(p,y) f(y,\theta)dy\right)$$

[20] Within a discrete framework, Ibragimov and Ibragimov (2007) reach similar results using the Schur concavity (or convexity) properties of cross-price elasticities.

In this formulation, the direct effect boils down to

$$\frac{\partial}{\partial\theta}\frac{-p_i}{\eta_i^Q} = \frac{p_i}{\left(\eta_i^Q\right)^2}\frac{\partial\eta_i^Q}{\partial\theta}$$

so that the direct effect has the same pattern as the elasticity of demand, implying that (in the case considered in Proposition 7) luxuries should be taxed more since the inequality shock makes their demand more rigid; and by the same token, taxation on necessities should be reduced since their demand becomes more elastic after the inequality shock. Hence, if the direct effect dominates the indirect effect, there is in this case a reconciliation between equity and efficiency issues. Clearly, if we have an opposite second order distributional shock causing an increase in equality and the direct effect dominates, efficiency will call for an increase in taxation for necessities and a decrease for luxuries – in which case we recover the idea that it is efficient to tax more necessities, as the traditionally more rigid commodities.

## 4.1 Equity and efficiency: the tradeoff in terms of welfare

Let us define the social welfare function at the optimum as:

$$W(R,\theta) = V\left(\widetilde{p} + t(R,\theta),\theta\right) \tag{14}$$

which gives overall welfare once the tax burden has been efficiently allocated across commodities (and hence across tax-payers *via* their budget constraint).

An (exogenous) increase in inequality yields a change in welfare (after the optimal tax adjustment) given by

$$\frac{dW}{d\theta} = \sum_i \frac{\partial V}{\partial p_i}\frac{dt_i}{d\theta} + \frac{\partial V}{\partial\theta} \tag{15}$$

Now notice that (a) indirect utility is always non increasing in prices, i.e. $\frac{\partial v}{\partial p_i} \leq 0$, so $\frac{\partial V}{\partial p_i} < 0$; (b) if the marginal utility of income is decreasing, $v(p,y)$ is concave in income so that $\frac{\partial V}{\partial\theta} < 0$. Substituting in the revenue constraint, we get:

$$\sum_i \frac{\partial T}{\partial t_i}\frac{dt_i}{d\theta} = -\frac{\partial T}{\partial\theta}$$

From the first order conditions of problem (1) we have $\frac{\partial V}{\partial p_i} = -\lambda \frac{\partial T}{\partial t_i}$, so that

$$\begin{aligned}
\frac{dW}{d\theta} &= \sum_i \frac{\partial V}{\partial p_i}\frac{dt_i}{d\theta} + \frac{\partial V}{\partial \theta} \\
&= -\lambda \sum_i \frac{\partial T}{\partial t_i}\frac{dt_i}{d\theta} + \frac{\partial V}{\partial \theta} \\
&= \lambda \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta}
\end{aligned}$$

So an increase in inequality has an indirect effect on welfare, due to the efficient tax adjustments on consumption ($\lambda\frac{\partial T}{\partial \theta}$), and a direct effect due instead to the consumption changes directly driven by variation in individual income ($\frac{\partial V}{\partial \theta}$). The latter will be always negative if the marginal utility of income is decreasing in income, since then the social planner is inequality averse. However, the former will depend on the distribution of necessity vs luxury goods:

**Proposition 8** *Suppose an increase in $\theta$ signals an increase in inequality for given mean income. Then (a) if luxuries dominate in the individual tax liability, there is a welfare gain in the tax adjustments; (b) if necessities dominate, there is a welfare-efficiency tradeoff.*

Both results follow trivially from Proposition 2. On the one hand, if luxuries weigh more in the individual tax liability than necessities, an increase in inequality leads to an average decrease in taxation ($\frac{\partial T}{\partial \theta} > 0$), so that if the tax system is sufficiently progressive ( $\lambda\frac{\partial T}{\partial \theta} > \frac{\partial V}{\partial \theta}$) there will be a net increase in welfare; on the other hand, if necessities weigh more in the individual tax liability than luxuries, $\lambda\frac{\partial T}{\partial \theta} < 0$ and the tax adjustment will further decrease the welfare beyond the direct effect ($\frac{\partial V}{\partial \theta}$) – and a decrease in inequality will make for a welfare gain in the case of dominating necessities, and for a welfare-efficiency trade-off to occur in the case of a dominance of luxuries. By way of conclusion, one can connect this result with the distinction between the direct and the indirect effect: an increase in inequality can be met by optimal tax adjustments without trading off efficiency against equity, if luxuries dominate in a way strong enough (i.e., their Engel curves are sufficiently convex) to make the direct overcome the indirect effect. Indeed, in such a case, necessities should benefit from an efficient relative reduction in taxation, beyond the aggregate average reduction supported by the dominance of luxuries in the individual tax liability.

# 5 Concluding remarks

We provide a general framework to study the relationship between income distribution and optimal commodity taxation, and show that distributional shocks have two effects on optimal consumption taxes: a direct effect driven by changes in aggregate demands, and an indirect effect due to behavioural adjustments. In our framework, this leads to clear cut policy prescriptions for the taxation of luxuries and necessities and for aggregate taxation, as well as conditions under which the conflict between equity and efficiency can be overcome. These results are linked to to the overall regressivity or progressivity properties of the tax system.

We think that this analysis provides an interesting framework which can be further developed in future research. Under this respect, a first natural extension seems to be towards including individual taste heterogeneity. Secondly, as our theoretical results provide sufficient statistics related to the curvature of the Engel curve of a single commodity and the mean/variance of the income distribution, this could in principle be tested empirically. Thirdly, the framework we have developed suggests that the optimal taxation cost of internalising externalities, or controlling "internalities", can be different, according to the income distribution of a society. Finally, we believe this framework to be potentially fit for studying the linkages of direct and indirect taxation with different market power scenarios. We leave these suggestions for future research.

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# Appendix

**A. The expected value of $\gamma(t, y)$**

Given that $\gamma(t,y) = \frac{\partial v}{\partial y} + \lambda \frac{\partial \tau}{\partial y}$, and that from (3) and

$$\begin{aligned}\lambda &= \frac{1}{\mu - R'} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} y f(y,\theta) dy \\ R' &= \int_{\mathcal{Y}} \sum_j t_j \frac{\partial q_j}{\partial y} y f(y,\theta) dy = \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} y f(y,\theta) dy\end{aligned}$$

we have

$$\int_{\mathcal{Y}} \frac{\partial v}{\partial y} y f(y,\theta) dy = \lambda \left( \mu - \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} y f(y,\theta) dy \right)$$

so that

$$\begin{aligned}\int_{\mathcal{Y}} \gamma(t,y) \frac{y f(y,\theta)}{\mu} dy &= \int \left( \frac{\partial v}{\partial y} + \lambda \frac{\partial \tau}{\partial y} \right) \frac{y f(y,\theta)}{\mu} dy \\ &= \int \frac{\partial v}{\partial y} \frac{y f(y,\theta)}{\mu} dy + \lambda \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} \frac{y f(y,\theta)}{\mu} dy \\ &= \lambda \frac{\mu - \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} y f(y,\theta) dy}{\mu} + \lambda \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} \frac{y f(y,\theta)}{\mu} dy = \lambda\end{aligned}$$

**B. Derivation of equation (9)**

The effect of a change of $\theta$ on $H_i$ is given by

$$\frac{\partial H_i}{\partial \theta} = -\int_{\mathcal{Y}} q_i(p,y) f_\theta(y,\theta) dy + \frac{\partial}{\partial \theta} \left( \tfrac{1}{\lambda} \int_{\mathcal{Y}} \gamma(t,y)\, q_i(p,y) f(y,\theta) dy \right)$$

Letting $\frac{\partial}{\partial \theta} \left( \frac{1}{\lambda} \int_{\mathcal{Y}} \gamma(t,y)\, q_i(p,y) f(y,\theta) dy \right) = \Lambda_\theta$ to ease notation, we have to show that

$$\Lambda_\theta = \tfrac{1}{\lambda} \int_{\mathcal{Y}} \left[ q_i(p,y) - k_i \right] \gamma(t,y)\, f_\theta(y,\theta) dy \tag{B.1}$$

By differentiating $\frac{1}{\lambda} \int_{\mathcal{Y}} \gamma(t,y)\, q_i(p,y) f(y,\theta) dy$ wrt $\theta$ one gets

$$\begin{aligned}\Lambda_\theta &= -\tfrac{\lambda_\theta}{\lambda^2} \int_{\mathcal{Y}} \gamma(t,y)\, q_i(p,y) f(y,\theta) dy + \tfrac{1}{\lambda} \int_{\mathcal{Y}} \gamma_\theta(t,y)\, q_i(p,y) f(y,\theta) dy \\ &\quad + \tfrac{1}{\lambda} \int_{\mathcal{Y}} \gamma(t,y)\, q_i(p,y) f_\theta(y,\theta) dy.\end{aligned}$$

where to ease notation we let subscritps denote derivatives. We know that $\gamma(t,y) = \frac{\partial v}{\partial y} + \lambda \frac{\partial \tau}{\partial y}$, so that $\gamma_\theta = \lambda_\theta \frac{\partial \tau}{\partial y}$: hence, by substituting in the above we get

$$\begin{aligned}
\Lambda_\theta &= -\tfrac{\lambda_\theta}{\lambda} \int_{\mathcal{Y}} \left( \frac{\gamma(t,y)}{\lambda} - \frac{\partial \tau}{\partial y} \right) q_i(p,y) f(y,\theta) dy + \tfrac{1}{\lambda} \int_{\mathcal{Y}} \gamma(t,y)\, q_i(p,y) f_\theta(y,\theta) dy \\
&= -\tfrac{\lambda_\theta}{\lambda^2} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta) dy + \tfrac{1}{\lambda} \int_{\mathcal{Y}} \gamma(t,y)\, q_i(p,y) f_\theta(y,\theta) dy \qquad \text{(B.2)}
\end{aligned}$$

From $\lambda = \int_{\mathcal{Y}} \gamma(t,y) \frac{yf(y,\theta)}{\mu} = \int_{\mathcal{Y}} \frac{\partial v}{\partial y} \frac{yf(y,\theta)}{\mu} dy + \lambda \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} \frac{yf(y,\theta)}{\mu} dy$, one has

$$\lambda = \frac{\int_{\mathcal{Y}} \frac{\partial v}{\partial y} \frac{yf(y,\theta)}{\mu} dy}{1 - \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} \frac{yf(y,\theta)}{\mu} dy} \qquad \text{(B.3)}$$

so that

$$\begin{aligned}
\lambda_\theta &= \frac{\frac{\partial}{\partial \theta} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} \frac{yf(y,\theta)}{\mu} dy}{1 - \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} \frac{yf(y,\theta)}{\mu} dy} + \frac{\lambda \frac{\partial}{\partial \theta} \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} \frac{yf(y,\theta)}{\mu} dy}{1 - \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} \frac{yf(y,\theta)}{\mu} dy} \\
&= \frac{\frac{1}{\mu} \int_{\mathcal{Y}} \frac{\partial v}{\partial y} y f_\theta(y,\theta) dy + \frac{\lambda}{\mu} \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} y f_\theta(y,\theta) dy}{1 - \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} \frac{yf(y,\theta)}{\mu} dy} \\
&= \frac{\frac{1}{\mu} \int_{\mathcal{Y}} \gamma(t,y)\, y f_\theta(y,\theta) dy}{1 - \int_{\mathcal{Y}} \frac{\partial \tau}{\partial y} \frac{yf(y,\theta)}{\mu} dy}
\end{aligned}$$

One can now substituting the former in (B.2), to get:

$$
\begin{aligned}
\Lambda_\theta &= -\tfrac{\lambda_\theta}{\lambda^2}\int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta)dy + \tfrac{1}{\lambda}\int_{\mathcal{Y}} \gamma\left(t,y\right) q_i(p,y) f_\theta(y,\theta)dy \\
&= -\tfrac{1}{\lambda^2}\frac{\frac{1}{\mu}\int_{\mathcal{Y}} \gamma\left(t,y\right) y f_\theta(y,\theta)dy}{1-\int_{\mathcal{Y}} \frac{\partial \tau}{\partial y}\frac{yf(y,\theta)}{\mu}dy}\int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta)dy \\
&\quad +\tfrac{1}{\lambda}\int_{\mathcal{Y}} \gamma\left(t,y\right) q_i(p,y) f_\theta(y,\theta)dy \\
&= -\tfrac{1}{\lambda^2}\frac{\int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta)dy}{1-\int_{\mathcal{Y}} \frac{\partial \tau}{\partial y}\frac{yf(y,\theta)}{\mu}dy}\frac{1}{\mu}\int_{\mathcal{Y}} \gamma\left(t,y\right) y f_\theta(y,\theta)dy \\
&\quad +\tfrac{1}{\lambda}\int_{\mathcal{Y}} \gamma\left(t,y\right) q_i(p,y) f_\theta(y,\theta)dy \\
&= \tfrac{1}{\lambda}\int_{\mathcal{Y}} \left[q_i(p,y) - \tfrac{1}{\lambda\mu}\frac{\int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta)dy}{1-\int_{\mathcal{Y}} \frac{\partial \tau}{\partial y}\frac{yf(y,\theta)}{\mu}dy}\right]\gamma\left(t,y\right) f_\theta(y,\theta)dy \\
&= \tfrac{1}{\lambda}\int_{\mathcal{Y}} \left[q_i(p,y) - k_i\right]\gamma\left(t,y\right) f_\theta(y,\theta)dy
\end{aligned}
$$

where by using (B.3)

$$
\begin{aligned}
k_i &= \tfrac{1}{\lambda\mu}\frac{\int_{\mathcal{Y}} \frac{\partial v}{\partial y} q_i(p,y) f(y,\theta)dy}{1-\int_{\mathcal{Y}} \frac{\partial \tau}{\partial y}\frac{yf(y,\theta)}{\mu}dy} \\
&= \frac{\int_{\mathcal{Y}} \frac{\partial v}{\partial y}\frac{q_i(p,y)}{y}\frac{yf(y,\theta)}{\mu}dy}{\int_{\mathcal{Y}} \frac{\partial v}{\partial y}\frac{yf(y,\theta)}{\mu}dy}
\end{aligned}
$$

which is independent on $y$.

## C. Proof of Proposition 5

By the standard properties of second order stochastic dominance for given mean, it is true from (8) that

$$
\operatorname{sign}\left\{\frac{\partial H_i}{\partial \theta}\right\} = \operatorname{sign}\left\{\left[\gamma\left(t,y\right)-\lambda\right]\frac{\partial^2 q_i}{\partial y^2} + 2\frac{\partial q_i}{\partial y}\frac{\partial \gamma}{\partial y}\right\}
$$

Consider now case (a). Notice that $\operatorname{sign}\left\{\frac{\partial \gamma}{\partial y}\right\} = \operatorname{sign}\{\gamma\left(t,y\right)-\gamma\left(t,y_{\min}\right)\}$, and, since $\lambda$ is the expected value of $\gamma$, $\operatorname{sign}\left\{\frac{\partial \gamma}{\partial y}\right\} = \operatorname{sign}\{\lambda-\gamma\left(t,y_{\min}\right)\}$, so that

$$\frac{2\frac{\partial\gamma}{\partial y}}{\lambda-\gamma\left(t,y_{\min}\right)}>0$$

Let now $B=\left[\gamma\left(t,y\right)-\lambda\right]\frac{\partial^2 q_i}{\partial y^2}+2\frac{\partial q_i}{\partial y}\frac{\partial\gamma}{\partial y}$ and $B_{\min}=\left[\gamma\left(t,y_{\min}\right)-\lambda\right]\frac{\partial^2 q_i}{\partial y^2}+2\frac{\partial q_i}{\partial y}\frac{\partial\gamma}{\partial y}$: then either $\frac{\partial\gamma}{\partial y}<0$ and $B<B_{\min}$ or $\frac{\partial\gamma}{\partial y}>0$ and $B>B_{\min}$. The condition

$$\frac{\frac{\partial^2 q_i}{\partial y^2}}{\frac{\partial q_i}{\partial y}}<\frac{2\frac{\partial\gamma}{\partial y}}{\lambda-\gamma\left(t,y_{\min}\right)}$$

ensures $B_{\min}<0$ in the former case and $B_{\min}>0$ in the latter, and hence it provides an upper bound on (relative) convexity such that $\text{sign}\left\{\frac{\partial\gamma}{\partial y}\right\}=\text{sign}\left\{\frac{\partial H_i}{\partial\theta}\right\}$. If $\gamma$ is linear in income, $\frac{\partial\gamma}{\partial y}=\beta\left(t\right)$ is a constant such that $\gamma$ can be written as $\gamma\left(t,y\right)=\alpha(t)+\beta\left(t\right)y$; hence, using (6)

$$\gamma\left(t,y_{\min}\right)-\lambda=\beta\left[y_{\min}-\frac{\mu^2+\sigma^2}{\mu}\right]$$

so that

$$\frac{2\frac{\partial\gamma}{\partial y}}{\lambda-\gamma\left(t,y_{\min}\right)}=\frac{2\mu}{\mu^2+\sigma^2-\mu y_{\min}}$$

where we use the fact that $\int_{\mathcal{Y}}y^2 f(y,\theta)dy=\mu^2+\sigma^2$. Similarly for case (b), $\text{sign}\left\{\frac{\partial\gamma}{\partial y}\right\}=\text{sign}\{\gamma\left(t,y_{\max}\right)-\gamma\left(t,y\right)\}$, and, since $\lambda$ is the expected value of $\gamma$, $\text{sign}\left\{\frac{\partial\gamma}{\partial y}\right\}=\text{sign}\{\gamma\left(t,y_{\max}\right)-\lambda\}$, so that

$$\frac{2\frac{\partial\gamma}{\partial y}}{\lambda-\gamma\left(t,y_{\max}\right)}<0$$

Let $B_{\max}=\left[\gamma\left(t,y_{\max}\right)-\lambda\right]\frac{\partial^2 q_i}{\partial y^2}+2\frac{\partial q_i}{\partial y}\frac{\partial\gamma}{\partial y}$: then either $\frac{\partial\gamma}{\partial y}<0$ and $B<B_{\max}$ or $\frac{\partial\gamma}{\partial y}>0$ and $B>B_{\max}$. The condition

$$\frac{\frac{\partial^2 q_i}{\partial y^2}}{\frac{\partial q_i}{\partial y}}>\frac{2\frac{\partial\gamma}{\partial y}}{\lambda-\gamma\left(t,y_{\max}\right)}$$

ensures $B_{\max} < 0$ in the former case and $B_{\max} > 0$ in the latter, and hence it provides a lower bound on (relative) concavity such that $\operatorname{sign}\left\{\frac{\partial\gamma}{\partial y}\right\} = \operatorname{sign}\left\{\frac{\partial H_i}{\partial\theta}\right\}$. Again, one can write

$$\frac{2\frac{\partial\gamma}{\partial y}}{\lambda - \gamma\left(t, y_{\max}\right)} = -\frac{2\mu}{\mu y_{\max} - \left(\mu^2 + \sigma^2\right)}$$

by using the fact that under linearity $\gamma\left(t, y\right) = \alpha(t) + \beta\left(t\right) y$, and $\int_{\mathcal{Y}} y^2 f(y, \theta) dy = \mu^2 + \sigma^2$.

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