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Corrado Benassi
Massimiliano Castellani
Maurizio Mussoni

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C. Benassi, M. Castellani*, M. Mussoni
Alma Mater Studiorum - Università di Bologna
Dipartimento di Scienze Economiche
Piazza Scaravilli 2, 40125 Bologna, Italy

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Abstract

In the framework of a vertically differentiated mixed duopoly, with uncovered market and costless quality choice, we study the existence of a price equilibrium when a welfare-maximizing public firm producing low quality goods competes against a profit-maximizing private firm producing high quality goods. We show that a price equilibrium exists if the quality spectrum is wide enough *vis à vis* a measure of the convexity of the distribution of the consumers' willingness to pay, and that such equilibrium is unique if this sufficient condition is tightened. Log-concavity of the income distribution is inconsistent with the existence of equilibrium.

JEL classification: D43, L13, L51.

Keywords: price equilibrium, vertical differentiation, mixed duopoly

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*Corresponding author: Tel: +390512098020 Fax: +39051221968
E-mail: m.castellani@unibo.it

1 Introduction

Mixed oligopolies can be observed in many countries and industries. In mixed industries (e.g. public utilities, transportation, telecommunication, energy, postal services, education, health care, etc.) public firms compete with private firms in price, quantity and the quality of goods. It is frequently argued that public firms supply goods or services, the quality of which is lower than that provided by private firms: e.g., such is allegedly the case in many countries for education and health care, or in transportation and postal services (Ishibashi & Kaneko 2008). A number of papers address the question of why this should be so, in the framework of a welfare-maximizing public firm competing with a profit-maximizing private one.[1] However, the answer they provide is usually sought by assuming away any role for the distribution of the willingness to pay across consumers, either because the crucial feature of uncovered market is ruled out, or because – while allowing for uncovered markets – the standard, uniform-distribution model of vertical differentiation is used.[2] This is somewhat surprising on at least two counts: at a very general level, most informal arguments justifying the very existence of public firms competing with private firms rely on distributional concerns

[1]For the standard theory of mixed oligopoly see, e.g., Harris & Wiens (1980), De Fraja & Delbono (1989), Grilo (1994), Barros & Martinez-Giralt (2002), Cantos-S. & Moner-Colonques (2006), Maldonado & Cremer (2013). For alternative theories of mixed oligopoly with non-welfare-maximizing behavior, see Fershtman (1990), Cremer et al. (1991), Barros (1995) and Estrin & De Meza (1995). In particular, to analyze mixed oligopoly equilibria when the firms' objectives are endogenous, De Donder & Roemer (2009) study a vertically differentiated mixed market where one firm is profit-maximizing while the other maximizes revenues, but one firm becomes welfare-maximizing when the government takes a participation in it.

[2]Thus, e.g., Ishibashi & Kaneko (2008) use the Hotelling model to argue that in a duopoly equilibrium the public firm would supply the lower quality, and the private firm the higher (in fact, higher than efficient) quality level. On the other hand, Delbono et al. (1996) use the standard uncovered market model to show that an equilibrium where the public (private) firm chooses the low (high) quality exists, though an equilibrium with inverted quality allocations also exists, and market segmentation is exogenous (also, this is a framework where it is problematic to find analytical solutions). For an overview of this issue, see Fraja & Delbono (1990)

about inequality and providing the poor with access to goods and services; and, more to the point at the analytical level, it is in general well known that the distribution of the willingness to pay affects the firms' equilibrium choices and can in principle affect the very existence of equilibria (Grandmont 1993, Anderson et al. 1997).

In this paper we focus on the existence of a price equilibrium in a vertically differentiated mixed duopoly with uncovered market, to confirm that the distribution of the willingness to pay affects equilibria. We assume costless quality choice, which allows us to concentrate upon the relevant features of demand and hence the distribution of the willingness to pay, and we model a mixed duopoly as a case where a welfare-maximizing, low-quality producing public firm competes against a profit-maximizing, high-quality producing private firm. In this framework we show that for a price equilibrium to exist the distribution of the willingness to pay cannot be logconcave, and that sufficient conditions for existence and uniqueness place a lower bound on the (given) quality spectrum – a lower bound which is higher, the higher the given convexity bound on the income distribution.

The paper is organized as follows. Section 2 presents the model and the general framework of mixed duopoly with vertical differentiation; Section 3 gives the solution for a market price equilibrium and discusses existence and uniqueness; Section 4 presents an example where the consumers' willingness to pay is supposed to be distributed as a Pareto distribution, while some concluding remarks are gathered in Section 5.

2 The model

We start from a standard model of duopoly competition with vertical differentiation, uncovered market and costless quality choice, as developed by

Mussa & Rosen (1978), Shaked & Sutton (1982) and (Tirole 1988, chap. 7.5). There are two competing firms, $i = H, L$, playing a non-cooperative game on price. Each firm i produces a good of quality $s_i \in \{s_H, s_L\}$, where $0 < s_L < s_H < \infty$ and $\Delta = s_H - s_L > 0$ denotes the quality differential. We crucially assume that L is a a welfare-maximizing public firm producing low quality goods, while H is a profit-maximizing private firm producing high quality goods; production costs are normalized to zero.[3] The firms' profits are $\Pi_i = p_i D_i$, where p_i and D_i, $i = H, L$, denote prices and demands: higher quality s_H sells at a price p_H, and lower quality s_L at a price p_L.

Each consumer is identified by her marginal willingness to pay for quality, θ, and has a utility $U_i(\theta) = \theta s_i - p_i$ if she buys a unit of good from firm i, and 0 otherwise. The marginal consumer, who is indifferent between buying the high and the low quality, has utility $U_H(\theta) = U_L(\theta)$, and is accordingly identified by $\theta_H = (p_H - p_L)/\Delta$; the marginal consumer who is indifferent between purchasing the low quality commodity and nothing at all has utility $U_L(\theta) = 0$, and is identified by $\theta_L = p_L/s_L$.[4] Clearly, θ_L and θ_H denote the positions of these marginal consumers along the 'income' scale: for later reference, it is useful to derive the price elasticities of θ_L and θ_H, which are given by $\varepsilon_H = \frac{\partial \theta_H}{\partial p_H}\frac{p_H}{\theta_H} = \frac{p_H}{p_H - p_L} > 1$ and $\varepsilon_L = \frac{\partial \theta_H}{\partial p_L}\frac{p_L}{\theta_H} = \frac{-p_L}{p_H - p_L} < 0$, such that $\varepsilon_H + \varepsilon_L = 1$.

Normalizing the consumers' population to 1 and assuming that the willingness to pay θ is continuously distributed over some nonnegative support $\Theta \subseteq \mathbb{R}_+$, we define the density function $f(\theta)$ such that the implied cumula-

[3] This is clearly equivalent to costs being fixed and independent of quality. Although obviously questionable on a number of grounds, such an assumption allows to focus on the firms' strategic choices as driven by demand, and hence to bring out the role of the distribution of the consumers' willingness to pay, e.g., Tirole (1988, p. 147) and Wauthy (1996).

[4] These are the basic features of the standard vertical differentiation model (Mussa & Rosen 1978); as is well known, the marginal willingness to pay θ can be looked at as a proxy for income (Gabszewicz & Thisse 1979).

tive distribution is $F : \Theta \to [0,1]$. Using primes to denote derivatives, it is convenient for our purposes to define also the following elasticities:

$$\eta(\theta) = \frac{\theta f(\theta)}{1 - F(\theta)}, \tag{1}$$

$$\pi(\theta) = \lim_{h \to 0} \frac{d \log \left(\frac{1}{\mu} \int_y^{\theta + h} x f(x) dx \right)}{d \log \theta} = 1 + \frac{\theta f'(\theta)}{f(\theta)}, \tag{2}$$

where definition (1) is the (positive) elasticity of $1 - F(\theta)$ and definition (2) is the Esteban elasticity of the density $f(\theta)$.[5] We use these definitions to gather our basic assumptions on F in the following assumption.

Assumption 1 *The distribution F is such that:*

(a) the lower bound of the support Θ ($\theta_{\min}$ say) obeys $\theta_{\min} = 0 = \eta(\theta_{\min})$, and the upper bound ($\theta_{\max}$ say) is such that $\lim_{\theta \to \theta_{\max}} \eta(\theta) > 1$;

(b) there exists some $\alpha \in (0,1]$ such that, $\forall \tilde{\alpha} \in [\alpha, 1]$, $(1 + \tilde{\alpha}) \eta(\theta) + \pi(\theta) - 1 > 0$, $\forall\ \theta \in \Theta$;

(c) let $\tilde{\theta}$ be the smallest value such that $\eta(\cdot) = 1$: then there exists a (unique) value θ°, $0 < \theta^{\circ} \leq \tilde{\theta}$ such that $\pi(\theta^{\circ}) = 0$, and such that $\pi(\theta^{\circ}) > 0$ for $\theta < \theta^{\circ}$ and $\pi(\theta^{\circ}) < 0$ for $\theta > \theta^{\circ}$.

Assumption 1(a) implies that at equilibrium the market cannot be completely covered, and that the value of θ at which $\eta(\theta) = 1$ (which is pivotal in what follows) lies strictly within Θ. Assumption 1(b) implies that $(1 - F)^{-\alpha}$ is a convex function, which in turn limits in some way the convexity of the

[5] Esteban (1986) defines the function $\pi(\cdot)$ as per our definition (2) and shows that it stands in a one-to-one relationship with the underlying density $f(\cdot)$: accordingly, it gives an alternative representation of the density itself, which in some circumstances may be useful, especially so as some regularity features are apparently supported by empirical evidence. See (Benassi & Chirco 2006) for the relationship between the Esteban elasticity and stochastic dominance, and Majumder & Chakravarty (1990) for some related empirical evidence.

relationship between the size of the covered market, $1 - F(\cdot)$, and the consumers' willingness to pay θ.[6] It should be stressed that by excluding the extreme value $\alpha = 0$ we are ruling out log-concavity, while α being finite rules out the uniform distribution, which one would get as $\alpha \to -\infty$, e.g., Caplin & Nalebuff (1991, p. 3). The same assumption also implies that:

$$\eta(\theta) + \pi(\theta) > 1 - \alpha\eta(\theta), \tag{3}$$

which in turn means that η is monotonically increasing for $\eta(\theta) \leq 1$, i.e. over $\left[0, \tilde{\theta}\right]$. Also, this places a restriction on the function $\pi(\cdot)$ to the effect that, for $\eta(\theta) \leq 1$, i.e. over $\left[0, \tilde{\theta}\right]$,

$$\pi(\theta) > -\alpha \geq -1. \tag{4}$$

All this should clarify Assumption 1(c), which rules that the function π (surely positive by condition (3) as θ nears zero) changes sign only once within $\left[0, \tilde{\theta}\right]$.[7]

Since we look for the Nash equilibrium of the game, we first have to determine the demand functions faced by firms L and H: $D_H = 1 - F(\theta_H)$, $D_L = F(\theta_H) - F(\theta_L)$, where $F(\theta_j)$ represents the fraction of consumers with a taste parameter less than θ_j, $j = L, H$. The corresponding profit functions are given by $\Pi_H = p_H D_H$, $\Pi_L = p_L D_L$. Finally, we define the social welfare function as the sum of the firms' and the consumers' surplus: $W = s_H \int_{\theta_H}^{\infty} \theta f(\theta)\, d\theta + s_L \int_{\theta_L}^{\theta_H} \theta f(\theta)\, d\theta$, and crucially assume that the public

[6] Indeed, it is easily seen that $\frac{d^2}{d\theta^2}(1-F)^{-\alpha} = \frac{\alpha}{\theta} \frac{f(\theta)}{(1-F(\theta))^{\alpha+1}} \left[(1+\alpha)\eta(\theta) + \pi(\theta) - 1\right] > 0$. Following Caplin & Nalebuff (1991), we can say that function $(1-F)$ is ρ-concave (with $\rho = -\alpha < 0$), which is equivalent to saying that $-(1-F)^{\alpha}$ is concave. Moreover, a ρ-concave function is also $\tilde{\rho}$-concave for all $\tilde{\rho} < \rho$, which means that what is true for a given $\alpha \in (0, 1]$ is also true for every $\tilde{\alpha} > \alpha$ included in the same interval.

[7] This is the case with many widely used distributions, such as Gamma and Pareto distributions.

firm sets the price of low quality goods p_L to maximize the social welfare function W.

3 Price equilibrium

In this Section, we take up Nash equilibria in prices: we first study existence, and then enquire about uniqueness.[8]

3.1 Existence of the price equilibrium

Given the price p_L set by the public firm on the 'low-quality' goods, p_H is charged by firm H maximizing its profit Π_H. The corresponding first order conditions (FOCs) in terms of elasticity are given by:

$$\eta\left(\theta_H\right)\varepsilon_H = 1, \tag{5}$$

which implies $\eta\left(\theta_H\right) < 1$. The second order conditions (SOCs) can be similarly characterized in elasticity terms as:

$$2\eta\left(\theta_H\right) + \pi\left(\theta_H\right) > 0, \tag{6}$$

which implies, given that $\eta\left(\theta_H\right) < 1$ by condition (5), the necessary condition $\pi\left(\theta_H\right) > -1$, consistently with (4).[9]

To set the price p_L, the public firm maximizes the social welfare W. The

[8]Given a quality pair (s_H, s_L), existence and uniqueness can clearly be established with reference to a (or the) pair of marginal consumers along the 'income' scale, (θ_H^*, θ_L^*), as it will be $p_H^* = \theta_H^*\left(s_H - s_L\right) + \theta_L^* s_L$, and $p_L = \theta_L^* s_L$.

[9]The FOCs and SOCs for firm H can be written out as: $\frac{\partial \Pi_H}{\partial p_H} = 1 - F\left(\theta_H\right) - \frac{p_H}{\Delta} f\left(\theta_H\right) = 0$; $\frac{\partial^2 \Pi_H}{\partial p_H^2} = -2f\left(\theta_H\right) - \frac{p_H}{\Delta} f'\left(\theta_H\right) < 0$, from which (5) and (6) can easily be derived by using definitions (1) and (2).

corresponding FOCs are:

$$\frac{\eta(\theta_H)}{\eta(\theta_L)} = \frac{1 - F(\theta_L)}{1 - F(\theta_H)} > 1, \tag{7}$$

from which $\theta_H > \theta_L$ implies $\eta(\theta_H) > \eta(\theta_L)$. The SOCs are given by:

$$(1 - \eta(\theta_H))\,\pi(\theta_H) + \eta(\theta_H)\,\pi(\theta_L) > 0, \tag{8}$$

which again are set in elasticity terms.[10]

As a result, at a price equilibrium for given s_H and s_L, p_H and p_L are identified by the twin FOCs (5) and (7), such that the twin SOCs (6) and (8) hold.

Before enquiring about existence of equilibrium, it is worth stressing that – irrespective of our assumptions on the distribution of the willingness to pay and indeed justifying them – the basic framework we are using (though indeed quite standard) is inconsistent with a logconcave distribution of the consumers' willingness to pay – including the limit case of the uniform distribution. Intuitively, this is so because of the way a marginal change in prices affects the positions of the marginal consumers. An increase in p_L pushes the marginal consumers nearer each other, by shifting linearly one to the right (θ_L) and the other to the left (θ_H) – that is, the set of middle-class consumers patronizing low-quality gets smaller, and that of the high-income consumers patronizing high quality gets larger. Since this has opposite effects on overall welfare, the latter is maximized when the marginal contribution to welfare of enlarging the set of high-quality consumers is equal to the marginal cost

[10] The FOCs and SOCs for the public firm are respectively $\frac{\partial W}{\partial p_L} = \theta_H f(\theta_H) - \theta_L f(\theta_L) = 0$, and $\frac{\partial^2 W}{\partial p_L^2} = -\frac{f(\theta_H)}{\Delta}\pi(\theta_H) - \frac{f(\theta_L)}{s_L}\pi(\theta_L) < 0$. In equilibrium, the latter is equivalent to condition (8), as can be seen by multiplying through by $p_L > 0$, substituting for $\varepsilon_L = 1 - \varepsilon_H$, and taking advantage of the FOCs (5) and (7).

of pricing out the poor. This, however, (a) requires that the income density falls sharply enough as we move from θ_L to θ_H, and (b) has to be consistent with the high-quality firm maximizing its profits. The latter obviously calls for the price elasticity of demand for high-quality be one: given the structure of preferences (such that a small increase in p_H has a big effect on the location of the high-quality marginal consumer: $\varepsilon_H > 1$), this in turn dictates that $\eta(\theta_H) < 1$ as from (5). Logconcavity, by constraining the relationship between $\eta(\cdot)$ and $\pi(\cdot)$ as defined in (1) and (2), is inconsistent with both requirements holding at once: if the distribution is logconcave, high-quality demand being sufficiently rigid is inconsistent with the density falling rapidly enough around θ_H, which under logconcavity would mean high demand elasticity from the marginal high-quality consumer.[11]

We can now state the following proposition on the existence of a price equilibrium.

Proposition 1 *Let* (s_H, s_L) *be a given pair of qualities, such that* $0 < s_L < s_H < \infty$*, and let* $k = s_L/\Delta$ *such that* $\eta(\theta^\circ) < \frac{1}{1+k}$*. Then under Assumption 1 a price equilibrium exists.*

Proof. See Appendix 6A ■

Proposition (1) establishes that a price equilibrium exists, if some constraints are satisfied concerning the distribution of the willingness to pay *vis à vis* the quality spectrum being offered. First notice that welfare maximization by firm L leads to θ_H lying on a downward portion of the density $f(\theta)$.

[11] Log-concavity amounts to the constraint $\pi(\theta) > 1 - \eta(\theta)$ for all θ, such that $\eta(\theta_H) < 1$ is inconsistent with $\pi(\theta_H) < 0$. On the other hand, $\pi(\theta_H)$ has to be negative, if welfare has to be maximized. This condition, which dictates that the density should be sufficiently (and negatively) steep around θ_H, can be seen by observing that $\theta f(\theta)$ is the marginal contribution to social welfare of the consumers whose willingness to pay is θ, and that its derivative is $f(\theta)\pi(\theta)$: the former cannot be increasing around θ_H if FOCs (7) are to be satisfied, i.e. if $\theta_H f(\theta_H) = \theta_L f(\theta_L)$ (see also footnote 10).

Indeed, the FOCs (7) boil down to $\theta_L f(\theta_L) = \theta_H f(\theta_H)$: analytically, this is inconsistent with both marginal consumers being on an upward sloping portion of the density itself, while economically it amounts to the marginal gain in welfare due to a marginal increase in p_L being nil. In other words, welfare maximization leads to an 'aggressive' behaviour by the low quality (public) firm which expands output, driving the 'high-quality' indifferent consumer (identified by θ_H) towards the right tail of the distribution.[12] This in turn accounts for our Assumption 1(a) ruling out complete market coverage, as this would imply $\theta_H f(\theta_H) = 0$, which is inconsistent with firm H maximizing its profits.[13] It also accounts for Assumption 1(b), which rules out log-concavity: as already remarked, log-concave distributions (as well as the uniform distribution) are inconsistent with a price equilibrium of this game.

Secondly, the condition $\eta(\theta^\circ) < \frac{1}{1+k}$, together with Assumption 1(b), implies $k < \alpha$, i.e.

$$\frac{s_H}{s_L} > 1 + \frac{1}{\alpha}, \tag{9}$$

which again is consistent with ruling out log-concavity ($\alpha = 0$), and links the width of the admissible quality spectrum to the degree of convexity of the distribution of the consumers' willingness to pay. In fact, the lower α, the higher the lower bound on the (given) quality differential consistent with the existence of a price equilibrium, while the upper limit case where $\alpha = 1$ yields the constraint $s_H > 2s_L$.[14] Intuitively, this happens because *ceteris paribus* the width of the quality spectrum affects the concavity of the firm's payoff: if the two products are close substitutes, the demand for

[12] Notice that, in the 'ordinary' case of both firms being profit-maximizers, both marginal consumers will be on the left of the mode when the density is symmetric and unimodal. See, e.g., Benassi et al. (2006).

[13] Given $p_L = 0$, firm H would set a price p_H such that $\eta(p_H/\Delta) = 1$ so that $\theta_H f(\theta_H) > 0$.

[14] In this case $[1 - F(\theta)]^{-1}$ would be a convex function. Notice that if $1 - F(\theta)$ is log-concave, $[1 - F(\theta)]^{-1}$ is convex, but not viceversa.

(say) firm L's product will be very elastic, and indeed too much for firm's L payoff (welfare) function to be well behaved.[15] A minimum quality spread ensures that vertical product differentiation survives in equilibrium, and that welfare is not maximized by setting the price equal to the marginal cost: the welfare gain associated to complete market coverage is less than the welfare loss associated with lower profits for both the high and the low quality firms.[16]

Finally, from the existence proof reported in Appendix 6A, it turns out that a *necessary* condition for existence is that $\theta_L^* < \theta^\circ < \theta_H^*$: i.e., along the distribution of the willingness to pay, θ° is a sort of pivotal point around which the positions of the marginal consumers arrange themselves. This in turn implies that at equilibrium one necessarily has:

$$\eta(\theta_H) - \eta(\theta_L) > \frac{1}{\frac{s_H}{s_L}\left(\frac{s_H}{s_L} - 1\right)}, \tag{10}$$

which means that the minimum (elasticity) distance between the two marginal consumers (and hence the market for the low quality commodity) is higher, the lower the quality ratio.[17] In some sense there is a trade-off between how steeply demand rises with the willingness to pay going from θ_L to θ_H, and the quality differential: if the latter is low, equilibrium with vertical differentiation requires that 'middle-class' consumers are very willing to pay

[15]Take, e.g., the SOCs for firm L in footnote (10): since $\pi(\theta_H)$ will be negative at equilibrium, this expression cannot be negative if Δ is too small, and more generally, if products were too close welfare would be a convex function of p_L.

[16]Under this respect, Assumption 1(b) plays a key role, as it amounts *ceteris paribus* to a lower boundary on $\eta(\cdot)$: if the covered market is sufficiently elastic with respect to the consumers' willingness to pay, the marginal gain in welfare from a price reduction will be low.

[17]Under our assumptions $\eta(\theta)$ is monotonically increasing in the relevant interval: equation (10) then follows by noting that in equilibrium it must be $\eta(\theta_L) < 1/(1+k)$ and substituting for the definition of k. The income level θ° is such that $\pi(\theta^\circ) = 0$, i.e. the elasticity of the density equals -1.

for even a modest quality premium.[18]

While Assumption 1 and the condition $\eta(\theta^{\circ}) < \frac{1}{1+k}$ set out in Proposition 1 are sufficient to ensure the existence of a price equilibrium, one is naturally interested in looking at the circumstances under which such an equilibrium is unique. Indeed, since we have to rule out log-concavity, we cannot use the well-known result by Caplin & Nalebuff (1991) to the effect that log-concavity implies uniqueness. This is the issue we take up next.

3.2 Uniqueness of the price equilibrium

Our main result on uniqueness is the following:

Proposition 2 *Let the assumptions of Proposition 1 hold, and assume further that:*

(a) $k \leq \frac{\alpha^2}{1-\alpha^2}$, *and*

(b) $\alpha \leq 1/2$,

then the price equilibrium is unique.

Proof. See Appendix 7B ■

Both sufficient conditions can be read as strengthening the looser conditions which ensure existence. Indeed, it is easily seen that condition (a) in Proposition 2 amounts to:

$$\frac{s_H}{s_L} \geq \frac{1}{\alpha^2}, \tag{11}$$

and that, comparing this with constraint (9), $1/\alpha^2 > 1 + 1/\alpha$ for $\alpha \leq 1/2$, i.e. condition (b) in Proposition 2. In this sense uniqueness is delivered when the lower bound on the quality differential is higher than that which

[18]Notice that, in the case where the public firm's objective was justified in terms of the median voter theorem, if the income distribution is asymmetric and unimodal, the policy makers may look at the marginal willingness to pay of the median consumer, instead of that of the average consumer (as required by social welfare maximization).

is sufficient to ensure existence, so that (broadly speaking) for the price equilibrium to be unique, the quality levels should be sufficiently far apart, by an amount which is determined by the concavity of the distribution.

4 An example: the Pareto distribution

In this Section we apply our model to the case of a Pareto distribution. While clearly limited to a specific case, we believe that this example can serve as an illustration of the way the distribution of the willingness to pay affects equilibrium outcomes. Suppose then that the consumers' willingness to pay is distributed as a Pareto distribution of the second kind (Johnson et al. 1995), so that the density and the cumulative distributions are respectively $f(\theta,\gamma) = \gamma\,(1+\theta)^{-(1+\gamma)}$ and $F(\theta,\gamma) = 1 - \frac{1}{(1+\theta)^{\gamma}}$, defined over the support $\Theta = [0,\infty)$, where $\gamma > 1$ is a given parameter. It is then easily seen that $\pi(\theta,\gamma) = \frac{1-\gamma\theta}{1+\theta}$ and $\eta(\theta,\gamma) = \frac{\gamma\theta}{1+\theta}$ such that Assumption 1 is satisfied. In particular, Assumption 1(b) holds for any $\alpha = 1/\gamma$, so that $1 - F(\theta)$ is ρ-concave with $\rho = -1/\gamma$; notice also that in this case we have $\tilde{\theta} = \frac{1}{\gamma-1} > \theta^{\circ} = \frac{1}{\gamma}$ such that $\eta(\tilde{\theta}) = 1$ and $\pi\,(\theta^{\circ}) = 0$. In addition, $\eta\,(\theta^{\circ}) = \gamma/\,(1+\gamma)$, such that the condition $\eta\,(\theta^{\circ}) < 1/\,(1+k)$ set out in Proposition 1 reduces to $k < 1/\gamma$. It should be remarked that in this framework γ is an inverse parameter of first order stochastic dominance, so that higher values of γ support lower mean values of the consumers' willingness to pay.[19]

We now perform a numerical simulation with different values of γ, say between $\gamma = 2$ and $\gamma = 3$, to see the way a shift on the distribution of the willingness to pay affects the equilibrium prices. To do so we set $k = 1/8 < 1/\gamma$, which is equivalent to $s_H/s_L = 9$. According to Proposition 2 (and condition (11)), this quality ratio delivers a unique equilibrium for $\alpha \geq 1/3$,

[19]Mean willingness to pay is $\mu = 1/(\gamma - 1)$.

which is verified as $\alpha = 1/\gamma \geq 1/3$, while $\alpha \leq 1/2$ as required by sufficient condition (b) of the same Proposition. Within this framework, we perform three simulations for $\gamma = 2$, $\gamma = 5/2$ and $\gamma = 3$. Table 1 reports equilibrium values of the positions of the marginal consumers, θ_H^* and θ_L^*, obtained for these different values of γ.

	$\gamma = 2$	$\gamma = 5/2$	$\gamma = 3$
θ_H^*	0.93629	0.61392	0.45498
θ_L^*	0.25484	0.25318	0.24011

Table 1: Equilibrium marginal values of willingness to pay

From Table 1, we see that in the case of a Pareto distribution, an increase in γ (i.e. lower mean income) leads to a leftward shift of both marginal consumers, together with a decrease in the distance between them. This would point to decreasing income leading to stiffer competition, which is confirmed by Table 2 below, such that relative prices decrease unambiguously with higher values of γ.

	$\gamma = 2$	$\gamma = 5/2$	$\gamma = 3$
$\frac{p_H^*}{p_L^*}$	30.392	20.399	16.159

Table 2: Equilibrium relative prices

In Table 3 we report the behaviour of hedonic prices.

	$\gamma = 2$	$\gamma = 5/2$	$\gamma = 3$
$\frac{p_H^*}{s_H^*}$	0.86057	0.57384	0.43111
$\frac{p_L^*}{s_L^*}$	0.25484	0.25318	0.24011

Table 3: Equilibrium hedonic prices

Table 3 suggests that lower average income – at least in this example – puts a downward pressure on the price per 'unit of quality', which appears to be stronger for the (profit-maximizing) high quality firm. Both relationships (Tables 2 and 3) are apparently monotone, suggesting that the competition of the public sector (or of the regulated industry) will be more intense in case of lower average income (i.e., the larger the parameter γ).

5 Conclusions

Starting by informal arguments that public firms competing with private firms rely on distributional concerns about inequality, and by formal reasoning that the distribution of the willingness to pay affects the firms' equilibrium choices, in this paper we show that a price equilibrium in a vertically differentiated mixed duopoly with uncovered market exists, if the quality spectrum is wide enough *vis à vis* a measure of the convexity of the distribution of the consumers' willingness to pay, and that such equilibrium is unique if this sufficient condition is tightened. In particular, we show that for a price equilibrium to exist the distribution of the willingness to pay cannot be logconcave, and that sufficient conditions for existence and uniqueness place on the quality spectrum a lower bound, which is higher, the higher the given convexity bound on the income distribution.

By way of example, we apply our model to a Pareto distribution, and find that a decrease of average income is (broadly speaking) associated with higher competitive pressure from the public firm, as signaled, e.g., by a decrease of the distance between the marginal willingness to pay for high *vs* low quality goods, and of relative prices – the price of high quality decreases relative to that of low quality; also, the decrease in hedonic prices appears to be stronger for the high quality goods. Though obviously constrained by the specific form of this example, these results confirm that assumptions about the distribution of the consumers' willingness to pay do play a key role in assessing the working of vertically differentiated markets.

6 Appendix A: Proof of Proposition 1

First notice that Assumption 1(c) implies:

$$\frac{1}{1+\alpha} < \eta\left(\theta^{\circ}\right) < \frac{1}{1+k} < 1,$$

so that $k < \alpha \leq 1$. Let now $\tilde{\theta} > \theta^{\circ}$ be defined by the condition $\eta\left(\bar{\theta}\right) = \frac{1}{1+k}$, and $\tilde{\theta} > \bar{\theta}$ be defined by $\eta\left(\tilde{\theta}\right) = 1$: since by Assumption 1(b) $\eta(\cdot)$ is monotonically increasing over $\left[0, \tilde{\theta}\right]$, both are uniquely identified. Let now $A = \left[\bar{\theta}, \tilde{\theta}\right]$, and define the function $\sigma : A \to \mathbb{R}_+$ such that:

$$\sigma\left(\theta_H\right) = \frac{\theta_H}{k}\left(\frac{1}{\eta\left(\theta_H\right)} - 1\right),$$

which associates to any given θ_H the values of $\theta_L = \sigma$ which are consistent with profit maximization by firm H, as from (5), since $\varepsilon_H = 1 + \frac{\theta_L}{\theta_H}\frac{1}{k}$. Now observe that, since $\theta^{\circ} < \bar{\theta}$, one has $\pi\left(\bar{\theta}\right) < 0$, and $\pi\left(\sigma\left(\theta_H\right)\right) < 0$ for $\theta_H \in A$; also, $\sigma'\left(\theta_H\right) < 0$, so that $\sigma\left(\bar{\theta}\right) = \bar{\theta} > \sigma\left(\tilde{\theta}\right) = 0$. Indeed:

$$\sigma'\left(\theta_H\right) = -\frac{2\eta_H + \pi\left(\theta_H\right) - 1}{k\eta\left(\theta_H\right)} < 0,$$

by Assumption 1(b). Let now define the function $\lambda : A \to \mathbb{R}$, such that:

$$\lambda\left(\theta_H\right) = \eta\left(\theta_H\right)\left[1 - F\left(\theta_H\right)\right] - \left[1 - F\left(\sigma\left(\theta_H\right)\right)\right]\eta\left(\sigma\left(\theta_H\right)\right),$$

and notice that $\lambda\left(\bar{\theta}\right) = 0$, $\lambda\left(\tilde{\theta}\right) = 1 - F\left(\tilde{\theta}\right) > 0$, while $\lambda'\left(\bar{\theta}\right) = f\left(\bar{\theta}\right)\pi\left(\bar{\theta}\right)\left[1 - \sigma'\left(\bar{\theta}\right)\right] < 0$, as $\sigma'\left(\bar{\theta}\right) < 0$ and $\pi\left(\bar{\theta}\right) < 0$. Then, by continuity, there exists a value θ_H^* such that $\lambda\left(\theta_H^*\right) = 0 < \lambda'\left(\theta_H^*\right)$.

We claim that the pair $\left(\theta_H^*, \theta_L^*\right) = \left(\theta_H^*, \sigma\left(\theta_H^*\right)\right)$ identifies an equilibrium. Indeed, by condition (7), at $\lambda\left(\theta_H^*\right) = 0$ the FOCs of the public firm L are satisfied, while $\sigma(\theta_H^*) = \theta_L^*$ is such that $\eta_H(\theta_H^*)\varepsilon_H = 1$, so that both FOCs (5) and (7) are satisfied. Note that, as σ is decreasing, $\sigma(\theta_H^*) = \theta_L^* < \sigma(\bar{\theta}) = \bar{\theta}$, which means that $\pi_H(\theta_H^*) < 0$ (as $\theta_H^* > \bar{\theta} > \theta^\circ$), while $\pi(\theta_L^*)$ will be positive if $\theta_L^* < \theta^\circ$. Notice also that $\eta(\theta_L^*) = \eta(\sigma(\theta_H^*)) < \eta(\bar{\theta}) = 1/(1+k)$, so that $\lambda(\theta_H^*) = 0$ implies $\theta_L^* < \theta^\circ < \theta_H^*$. As to SOCs, firm H's are satisfied as by Assumption 1(b) $2\eta(\theta_H^*) + \pi(\theta_H^*) - 1 > (1+\alpha)\eta(\theta_H^*) + \pi(\theta_H^*) - 1 > 0$. As to firm L, recall from condition (8) that one should have $(1-\eta(\theta_H^*))\pi(\theta_H^*) + \eta(\theta_H^*)\pi(\theta_L^*) > 0$. To see this is so, observe that at our (candidate) equilibrium $\lambda(\theta_H^*) = 0 < \lambda'(\theta_H^*)$. In particular, we have:

$$\lambda'(\theta_H^*) = f(\theta_H)\left[\pi(\theta_H) + \pi(\sigma)\frac{f(\theta_L)}{f(\theta_H)}\frac{2\eta(\theta_H) + \pi(\theta_H) - 1}{k\eta(\theta_H)}\right] > 0,$$

where we substitute for:

$$\sigma'(\theta_H^*) = -\frac{2\eta(\theta_H^*) + \pi(\theta_H^*) - 1}{k\eta(\theta_H^*)} < 0.$$

As a result,

$$\pi(\theta_H^*) + \pi(\theta_L^*)\frac{f(\theta_L^*)}{f(\theta_H^*)}\frac{2\eta(\theta_H^*) + \pi(\theta_H^*) - 1}{k\eta(\theta_H^*)} > 0,$$

which, since $\lambda = 0$ implies $f(\theta_L^*)\theta_L^* = f(\theta_H^*)\theta_H^*$, gives:

$$k\eta(\theta_H^*)\pi(\theta_H^*) + \frac{\theta_H^*}{\theta_L^*}\left[2\eta(\theta_H^*) + \pi(\theta_H^*) - 1\right]\pi(\theta_L^*) > 0,$$

where we recall that $\pi(\theta_H^*) < 0$. From the definition of $\sigma = \theta_L$ we have $k\eta(\theta_H^*) = \frac{\theta_H^*}{\theta_L^*}\left[1 - \eta(\theta_H^*)\right]$. So if $\lambda' > 0 = \lambda$ we have:

$$\frac{\theta_H^*}{\theta_L^*}\left\{\left[1 - \eta(\theta_H^*)\right]\pi(\theta_H^*) + \left[2\eta(\theta_H^*) + \pi(\theta_H^*) - 1\right]\pi(\theta_L^*)\right\} > 0,$$

that is, $\{[1-\eta(\theta_H^*)]\pi(\theta_H^*)+[2\eta(\theta_H^*)+\pi(\theta_H^*)-1]\pi(\theta_L^*)\}>0$, which implies $\pi(\theta_L^*)>0$ as $\pi(\theta_H^*)<0$. On the other hand, $0<2\eta(\theta_H^*)+\pi(\theta_H^*)-1<2\eta(\theta_H^*)-1$, hence $\eta(\theta_H^*)>\frac{1}{2}$. Since $\pi(\theta_L^*)>0$ we can write:

$$\begin{aligned} 0 &< [1-\eta(\theta_H^*)]\pi(\theta_H^*)+[2\eta(\theta_H^*)+\pi(\theta_H^*)-1]\pi(\theta_L^*)< \\ & [1-\eta(\theta_H^*)]\pi(\theta_H^*)+[2\eta(\theta_H^*)-1]\pi(\theta_L^*), \end{aligned}$$

where the last term is smaller than $[1-\eta(\theta_H^*)]\pi(\theta_H^*)+\eta(\theta_H^*)\pi(\theta_L^*)$. So this is positive and $\lambda'(\theta_H^*)>0$, implies that the SOCs for firm L are verified.■

7 Appendix B: Proof of Proposition 2

We show that if $k\leq\frac{\alpha^2}{1-\alpha^2}$ and $\alpha\leq 1/2$, $\lambda(\theta_H^*)=0$ implies $\lambda'(\theta_H^*)>0$: this proves uniqueness. To do so we proceed into two steps.

7.1 Step 1

We show that if $\lambda(\theta_H^*)=0<\lambda'(\theta_H^*)$, then $\eta(\theta_L^*)<1-\alpha$, where $\theta_L^*=\sigma(\theta_H^*)$. Note that by Proposition 1 there certanly exists one such θ_H^*, and moreover there certanly exists one $\theta_H^m<\theta_H^*$ such that $\lambda'(\theta_H^m)=0>\lambda(\theta_H^m)$. By the definition of $\lambda'(\cdot)$, θ_H^m satisfies:

$$\pi(\theta_H^m)+\pi(\theta_L^m)\frac{f(\theta_L^m)}{f(\theta_H^m)}\frac{2\eta(\theta_H^m)+\pi(\theta_H^m)-1}{k\eta(\theta_H^m)}=0,$$

where $\theta_L^m=\sigma(\theta_H^m)$. Since, by the definition of $\sigma(\cdot)$, $k\eta(\theta_H^m)=\frac{\theta_H^m}{\theta_L^m}[1-\eta(\theta_H^m)]$, we have:

$$\pi(\theta_H^m)+\pi(\theta_L^m)\frac{\theta_L^m f(\theta_L)}{\theta_H^m f(\theta_H^m)}\frac{2\eta(\theta_H^m)+\pi(\theta_H^m)-1}{1-\eta(\theta_H^m)}=0, \tag{B.1}$$

where $\frac{\theta_L^m f(\theta_L)}{\theta_H^m f(\theta_H^m)} = \frac{\eta(\theta_L^m)[1-F(\theta_L^m)]}{\eta(\theta_H^m)[1-F(\theta_H^m)]} > 1$ as $\lambda(\theta_H^m) < 1$, and hence $x(\theta_H^m) = \frac{\theta_L^m f(\theta_L)}{\theta_H^m f(\theta_H^m)} \frac{2\eta(\theta_H^m)+\pi(\theta_H^m)-1}{1-\eta(\theta_H^m)} > 1$, since $\frac{2\eta(\theta_H^m)+\pi(\theta_H^m)-1}{1-\eta(\theta_H^m)} > 1$. That the latter is true can be seen by observing that:

$$\frac{2\eta(\theta_H^m) + \pi(\theta_H^m) - 1}{1 - \eta(\theta_H^m)} > \frac{(1-\alpha)\eta(\theta_H^m)}{1 - \eta(\theta_H^m)},$$

due to Assumption 1(b), and that $\eta(\theta_H^*) > 1/(1+k)$, so that $(1-\alpha)\eta(\theta_H^*) > (1-\alpha)/(1+k)$, while $1 - \eta(\theta_H^*) < k/(1+k)$. Hence,

$$\frac{2\eta(\theta_H) + \pi(\theta_H) - 1}{1 - \eta(\theta_H)} > \frac{1-\alpha}{k} > 1,$$

where the last inequality stems from $\alpha \leq 1/2$ and $k < \alpha$ implying $\alpha + k < 1$, i.e. $1 - \alpha > k$. Since $x(\theta_H^m) > 1$ and as $\pi(\theta_H^m) > -\alpha$, we have:

$$\pi(\theta_L^m) = -\frac{\pi(\theta_H^m)}{x(\theta_H^m)} < \frac{\alpha}{x(\theta_H^m)} < \alpha. \tag{B.2}$$

From equation (B.1) we have:

$$\eta(\theta_L^m)\pi(\theta_L^m) = -\pi(\theta_H^m)\eta(\theta_H^m)\frac{1 - F(\theta_H^m)}{1 - F(\theta_L^m)} \frac{1 - \eta(\theta_H^m)}{2\eta(\theta_H^m) + \pi(\theta_H^m) - 1},$$

where $\frac{1-F(\theta_H^m)}{1-F(\theta_L^m)} < 1$. From (B.2):

$$\eta(\theta_L^m)\pi(\theta_L^m) < \frac{\alpha\eta(\theta_H^m)(1 - \eta(\theta_L^m))}{2\eta(\theta_H^m) + \pi(\theta_H^m) - 1},$$

where the last inequality comes from $\eta(\theta_H^m) > \eta(\theta_L^m)$. Hence:

$$\eta(\theta_L^m)\left[\pi(\theta_L^m) + \frac{\alpha\eta(\theta_H^m)}{2\eta(\theta_H^m) + \pi(\theta_H^m) - 1}\right] < \frac{\alpha\eta(\theta_H^m)}{2\eta(\theta_H^m) + \pi(\theta_H^m) - 1}.$$

Since $\pi\left(\theta_L^m\right) < \alpha$,

$$\eta\left(\theta_L^m\right)\left[\pi\left(\theta_L^m\right) + \frac{\alpha\eta\left(\theta_H^m\right)}{2\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1}\right] < \alpha\eta\left(\theta_L^m\right)\frac{3\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1}{2\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1},$$

while $2\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1 < 2\eta\left(\theta_H^m\right)$, as $\pi\left(\theta_H^m\right) - 1 < 0$, so that:

$$\frac{\alpha\eta\left(\theta_H^m\right)}{2\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1} > \frac{\alpha\eta\left(\theta_H^m\right)}{2\eta\left(\theta_H^m\right)} = \frac{\alpha}{2}.$$

There follows that:

$$\eta\left(\theta_L^m\right)\frac{3\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1}{2\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1} < \frac{1}{2},$$

i.e.:

$$\eta\left(\theta_L^m\right) < \frac{1}{2}\frac{2\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1}{3\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1}.$$

So $\eta\left(\theta_L^m\right) + \alpha < 1$ will hold true if:

$$\eta\left(\theta_L^m\right) + \alpha < \frac{1}{2}\frac{2\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1}{3\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1} + \alpha < 1,$$

that is:

$$\frac{\eta\left(\theta_H^m\right)}{3\eta\left(\theta_H^m\right) + \pi\left(\theta_H^m\right) - 1} > 2\alpha - 1,$$

which is certainly true for $\alpha \leq \frac{1}{2}$, as the LHS is surely positive: hence $\eta\left(\theta_L^m\right) < 1-\alpha$. Since at $\theta_H^* > \theta_H^m$ and $\sigma'\left(\cdot\right) < 0$, $\theta_L^* = \sigma\left(\theta_H^*\right) < \sigma\left(\theta_H^m\right) = \theta_L^m$, and since $\eta\left(\cdot\right)$ is monotonically increasing $\eta\left(\theta_L^*\right) < \eta\left(\theta_L^m\right) < 1-\alpha$.

7.2 Step 2

We show that equilibrium is unique. Since $\eta(\theta_L^*) < 1 - \alpha$, we can use Assumption 1(b) to get:

$$\pi(\theta_L^*) > 1 - (1+\alpha)\,\eta(\theta_L^*) > 1 - (1+\alpha)(1-\alpha) = \alpha^2.$$

Equilibrium is unique if $\lambda = 0$ implies $\lambda' > 0$. Since we know that $\lambda = 0$ implies $\pi(\theta_L^*) > \alpha^2$, it is true that:

$$\begin{aligned} & [1-\eta(\theta_H^*)]\,\pi(\theta_H^*) + [2\eta(\theta_H^*) + \pi(\theta_H^*) - 1]\,\pi(\theta_L^*) \\ > \; & [1-\eta(\theta_H^*)]\,\pi(\theta_H^*) + [2\eta(\theta_H^*) + \pi(\theta_H^*) - 1]\,\alpha^2. \end{aligned}$$

At $\lambda = 0$ the sign of λ' is given by that of:

$$[1-\eta(\theta_H^*)]\,\pi(\theta_H^*) + [2\eta(\theta_H^*) + \pi(\theta_H^*) - 1]\,\pi(\theta_L^*),$$

so uniqueness follows if parameters are so arranged that $[1-\eta(\theta_H^*)]\,\pi(\theta_H^*) + [2\eta(\theta_H^*) + \pi(\theta_H^*) - 1]\,\alpha^2 > 0$, i.e:

$$\frac{2\eta(\theta_H^*) + \pi(\theta_H^*) - 1}{1-\eta\left(\theta_H^*\right)} > \frac{-\pi(\theta_H^*)}{\alpha^2},$$

which is equivalent to:

$$-\pi(\theta_H^*)\left(\frac{1}{1-\eta\left(\theta_H^*\right)} + \frac{1}{\alpha^2}\right) < \frac{2\eta(\theta_H^*) - 1}{1-\eta\left(\theta_H^*\right)}.$$

Since $-\pi(\theta_H^*) < -1 + (1+\alpha)\,\eta(\theta_H^*)$, this will be true if:

$$[-1 + (1+\alpha)\,\eta(\theta_H^*)]\left(\frac{1}{1-\eta\left(\theta_H^*\right)} + \frac{1}{\alpha^2}\right) < \frac{2\eta(\theta_H^*) - 1}{1-\eta\left(\theta_H^*\right)},$$

yielding:

$$(1+\alpha)\,\eta\left(\theta_H^*\right)\left(\frac{1}{1-\eta\left(\theta_H^*\right)}+\frac{1}{\alpha^2}\right)<\frac{2\eta\left(\theta_H^*\right)-1}{1-\eta\left(\theta_H^*\right)}+\frac{1}{1-\eta\left(\theta_H^*\right)}+\frac{1}{\alpha^2},$$

from which, after rearrangement, we obtain:

$$\frac{(1-\alpha)\,\eta\left(\theta_H^*\right)}{1-\eta\left(\theta_H^*\right)}+\frac{1}{\alpha^2}>\frac{(1+\alpha)\,\eta\left(\theta_H^*\right)}{\alpha^2}. \tag{B.3}$$

We now invoke the condition $k \leq \alpha^2/\left(1-\alpha^2\right)$, which is consistent with $k < \alpha$, as $\alpha^2/\left(1-\alpha^2\right) < \alpha$ for $\alpha \leq 1/2$. Under this assumption, $1-\eta\left(\theta_H^*\right) < \alpha^2$: indeed, this is equivalent to $\eta\left(\theta_H^*\right) > 1-\alpha^2$, which is true as $\eta\left(\theta_H^*\right) > 1/\left(1+k\right)$ and:

$$\frac{1}{1+k}-\left(1-\alpha^2\right)=\frac{-k+\alpha^2+\alpha^2 k}{1+k}>0.$$

There follows that $\frac{(1-\alpha)\eta\left(\theta_H^*\right)}{1-\eta\left(\theta_H^*\right)} > \frac{(1-\alpha)\eta\left(\theta_H^*\right)}{\alpha^2}$, so that (B.3) holds if:

$$\frac{(1-\alpha)\,\eta\left(\theta_H^*\right)}{\alpha^2}+\frac{1}{\alpha^2}>\frac{(1+\alpha)\,\eta\left(\theta_H^*\right)}{\alpha^2},$$

i.e.:

$$1 > 2\eta\left(\theta_H^*\right)\alpha,$$

which is surely true for $\alpha \leq 1/2$, as $2\eta\left(\theta_H^*\right)\alpha < 1$.■

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Alma Mater Studiorum - Università di Bologna
DEPARTMENT OF ECONOMICS

Strada Maggiore 45
40125 Bologna - Italy
Tel. +39 051 2092604
Fax +39 051 2092664
http://www.dse.unibo.it