

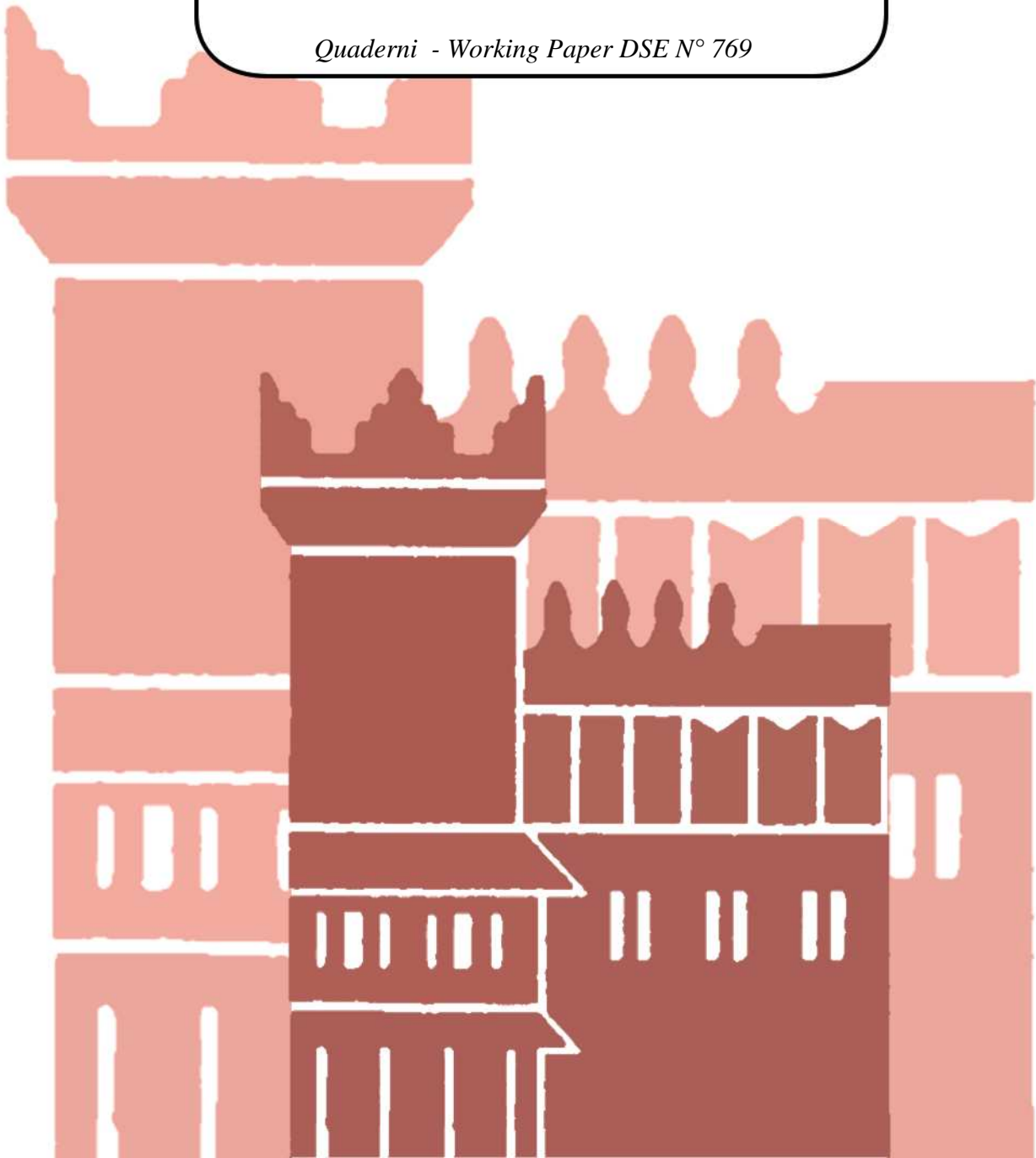


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**Equilibrium Selection in a Cashless
Economy with Transaction Frictions in
the Bond Market**

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Equilibrium Selection in a Cashless Economy with Transaction Frictions in the Bond Market

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Abstract

The present paper introduces two bonds in a standard New-Keynesian model to study the role of segmentation in bond markets for the determinacy of rational expectations equilibria. We use a strongly-separable utility function to model ‘liquid’ bonds that provide transaction services for the purchase of consumption goods. ‘Illiquid’ bonds, instead, provide the standard services of store of value. We interpret liquid bonds as mimicking short-term instruments, and illiquid bonds to represent long-dated instruments. In this simple setting, the expectation hypothesis holds after log-linearizing the model and after pricing the bonds according to an affine scheme. We assume that monetary policy follows a standard Taylor rule. In this context, the inflation targeting parameter should be higher than one for determinacy of rational expectations equilibria to be achieved. We compute an analytical solution for the bond pricing kernel. We also show that the possibility of obtaining this analytical solution depends on the type of utility function. When utility is weakly separable between consumption and liquid bonds, the Taylor principle holds conditional to the output and inflation coefficients in the Taylor rule. Achieving solution determinacy requires constraining these coefficients within bounds that depend on the structural parameters of the model, like the intertemporal elasticity of consumption substitution.

JEL Classification No.: E43, E63.

Keywords: term structure, determinacy, pricing kernel, fiscal and monetary policy.

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1 Introduction

The Taylor principle has become one of the pillars of modern normative analysis of monetary policy. In a nutshell, it prescribes that the central bank should adjust the nominal rate of interest more than one-for-one as a response to changes in the inflation rate. In the standard New-Keynesian models, the Taylor principle alone pins down the equilibrium inflation rate. Yet, the recent experience of the reaction implemented by several central banks in the middle of the financial turmoil has stressed the role that the presence of assets with different characteristics, such as maturity and liquidity profile, cover in the monetary transmission mechanism.

In this paper we study the role of endogenous spreads between government bonds with different maturities in defining the conditions of determinacy of a rational expectation equilibrium (REE) induced by monetary policy rules *à la* Taylor. Our goal is to investigate if the Taylor principle still holds in an economy where government bonds with different characteristics coexist.

We consider a New Keynesian model where bonds provide liquidity services in place of money. The model includes two types of bonds: liquid and illiquid. Liquid bonds are directly providers of liquidity services. They play a role similar to that of real money balances in the usual New Keynesian monetary models. Under this view, these bonds make it easier for consumers to make transactions. Like the standard modelling approach used for real money balances, we insert the real quantity of liquid bonds directly into the utility function.¹ Illiquid bonds are instead used as an intertemporal store of value, that is as a mere financial asset. To make the model analytically tractable, we do not explicitly consider the role of transaction costs for illiquid bonds. In this sense, the model discussed here represents the simplest possible framework that allows for the simultaneous presence of interest rates on two different government bonds in a general equilibrium framework with both nominal and real frictions. Assuming that assets are held until maturity, we interpret liquid bonds as mimicking short-dated instruments, and illiquid bonds as representing long-term assets.

This paper is close to [Canzoneri and Diba \(2005\)](#) and [Canzoneri, Cumby, Diba, and López-Salido \(2011\)](#) who study the interactions between monetary and fiscal policy when bonds and money are imperfect substitutes in providing transaction services for the purchase of consumption goods. We fill a gap in the current literature on determinacy and monetary policy initiated by [Bullard and Mitra \(2002\)](#), by inserting an explicit role for the term structure of interest rates and solving for an endogenous term spread. [McCough, Rudebusch, and Williams \(2005\)](#) have addressed a similar question, but their model is not solved analytically for an endogenous term spread in the standard New Keynesian model.

¹An alternative to this formulation is provided by an explicit modelling of transaction costs in the budget constraint of the representative agent, like in [Sims \(1994\)](#).

Our main result is that, even with the term structure embedded in a simple model with nominal rigidities, the Taylor principle is verified if liquidity services provided by short term bonds arise from a strongly separable argument of the utility function. If not, the results are no longer clear-cut and the range of values for the inflation targeting parameter for which we get determinacy of a REE becomes non-linear and strongly dependent on output targeting coefficient.

In the present framework, the term structure emerges as an affine representation where the expectation hypothesis (EH) holds in log-linear approximation. If liquid bonds enter into the instantaneous utility function in a strongly separable way, as in the standard neo-keynesian model described in [Woodford \(2003\)](#), the parameters for monetary policy rules should lie in the same region required in models without the term structure for determinate equilibria to emerge. In other words, regardless of the vehicle providing liquidity services - either liquid bonds or money -, what really matters for determinacy is whether the liquid demand is linear with respect to consumption. Instead, if liquidity services enter in a weakly separable way, the standard Taylor principle does not hold any longer. In this case, the size of the inflation targeting coefficient strongly depends on the size of the output targeting coefficient in a non-linear way. Under this perspective, modelling the term structure with strongly separable liquidity services does not determine an important change in determinacy conditions.

Our model achieves determinacy of the REE because of the joint role of fiscal and monetary policy. Fiscal policy rules are a key ingredient since they allow to widen the range of parameters for which determinacy exists. In particular, following the jargon of [Leeper \(1991\)](#) and [Sims \(1994\)](#), we find that determinacy is obtained either by considering active-monetary with passive-fiscal or, alternatively, by passive-monetary with active-fiscal. Passive fiscal policy is defined by setting tax revenue to react with respect to the outstanding real level of debt formed by both liquid and illiquid bonds. The intensity of the reaction of fiscal policy can be changed in order to properly pin down the equilibrium.

The intuition for our results has to do with the modelling assumptions that lead two interest rates to coexist in general equilibrium model. Including the term structure does not matter for the determinacy condition of the model with strongly separable utility, since it does not change the parameter setting for monetary policy rules. This observation does not hold any longer with weakly separable utility where the traditional arguments for determinacy induced by Taylor-type rules are very different and strongly dependent on the core parameters of the model.

The remainder of the paper is organized as follows. The following section introduces the modelling framework. Section 3 describes the calibration of the benchmark model and the impulse responses to specific macroeconomic shocks. Section 4 outlines the solution

of the model and the resulting asset pricing kernel. Section 5 presents the main results about equilibrium determinacy. Section 6 generalizes these findings to the case with a utility function that is weakly separable between consumption and liquid bonds. Section 7 concludes. An additional appendix contains the detailed computations for the model solution and the entire set of proofs.²

2 The Model

The general feature of our model is to consider the explicit role of bonds as providers of transactional services. We assume the existence of two types of bonds, namely liquid and illiquid bonds. For their intrinsic nature, liquid bonds are assumed to be a proxy for money holdings. They are held by the representative agent in order to facilitate the transactions due to the purchase of consumption goods. Illiquid bonds are held for financial investment purposes. Both types of bonds pay an interest rate which differ in equilibrium because of two elements, namely the explicit role of transaction services and the endogenously determined term premia.

2.1 Households

We assume the existence of an infinite number of heterogeneous agents indexed on the real line between 0 and 1. Each i -th agent maximizes the following utility function:

$$U_t = \sum_{t=0}^{\infty} \beta^t u \left(C_{it}, \frac{B_{i1t}}{P_t}, L_{it} \right) \quad (1)$$

where the instantaneous utility function $u(\dots)$ is defined as:

$$u \left(C_{it}, \frac{B_{i1t}}{P_t}, L_{it} \right) = \frac{C_{it}^{(1-\frac{1}{\sigma})}}{(1-\frac{1}{\sigma})} + \chi \frac{\left(\frac{B_{i1t}}{P_t} \right)^{(1-\frac{1}{\sigma})}}{(1-\frac{1}{\sigma})} - \frac{L_{it}^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \quad (2)$$

where C_{it} indicates the amount of consumption expressed by each i -th agent, B_{i1t} indicates the amount of nominal liquid bond holdings (here indexed with 1). The general price level is given by P_t . Instantaneous utility depends positively from C_{it} and B_{i1t} , while negatively from labor supply L_{it} . In (2), σ indicates the intertemporal elasticity of substitution, χ is a scale parameter and η is the Frisch labor elasticity. The utility function here represented is strongly separable between all its arguments, as in the traditional New Keynesian microfounded model where money is inserted in (2) in place of liquid bonds B_{i1t} .

²Further analytical proofs can be obtained from the authors upon request.

Given standard assumptions, the demand side of the model economy boils down to a representative agent that solves an intertemporal portfolio allocation problem. This consists in the maximization of the utility function subject to the following budget constraint:

$$\frac{B_{i1t}}{R_{1t}P_t} + \frac{B_{i2t}}{R_{2t}P_t} = \frac{B_{i1t-1}}{P_t} + \frac{B_{i2t-1}}{R_{1t}P_t} + \frac{W_t L_{it}}{P_t} - C_{it} - T_{it} \quad (3)$$

where B_{i2t} indicates the stock of illiquid nominal bonds. Moreover, W_t is the nominal wage, T_{it} is the tax collected, assumed to be lump sum. Moreover, R_{1t} , R_{2t} indicate, respectively, the rate of return paid by liquid and illiquid bonds.

The first order conditions with respect to C_{it} , L_{it} , B_{i1t} and B_{i2t} , are given, respectively, by:

$$C_{it}^{-\frac{1}{\sigma}} = \lambda_t \quad (4)$$

$$L_{it}^{\frac{1}{\eta}} = \lambda_t W_t \quad (5)$$

$$\chi b_{i1t}^{\chi(1-\frac{1}{\sigma})-1} + \beta E_t \frac{\lambda_{t+1}}{\pi_{t+1}} = \frac{\lambda_t}{R_{1t}} \quad (6)$$

$$\beta E_t \frac{\lambda_{t+1}}{\pi_{t+1} R_{1t+1}} = \frac{\lambda_t}{R_{2t}} \quad (7)$$

Equation (4) indicates the first order with respect to consumption; equation (5) defines the optimal labor supply choice and equates the disutility from work effort to the real wage weighted by the marginal utility of consumption; equation (6) is the optimal intertemporal allocation of liquid bonds B_{i1t} and (7) is the result of the optimal choice of illiquid bonds B_{i2t} ; λ_t is the Lagrange multiplier. In particular, we express the demand for bonds in real terms, after having defined $b_{i1t} = B_{i1t}/P_t$, $b_{i2t} = B_{i2t}/P_t$.

By mixing up both (6) - (7) we get the following expression for liquid bond demand:

$$\chi b_{i1t}^{\chi(1-\frac{1}{\sigma})-1} = \left[\frac{R_{2t} - R_{1t} E_t R_{1t+1}}{R_{1t} R_{2t}} \right] C_t^{-\frac{1}{\sigma}} \quad (8)$$

It is immediate to verify that demand for liquid bonds (after dropping subscript index i) satisfies the following properties, provided that $\chi(1 - \frac{1}{\sigma}) < 1$:

$$\begin{aligned} \frac{\partial b_{1t}}{\partial C_t} &> 0 & \frac{\partial b_{1t}}{\partial R_{1t}} &> 0 \\ \frac{\partial b_{1t}}{\partial R_{1t+1}} &> 0 & \frac{\partial b_{1t}}{\partial R_{2t}} &< 0 \end{aligned}$$

The intuition goes as follows: the increase in consumption increases the demand for liquid bonds, since they are employed for transaction. On the other hand, the increase in current,

R_{1t} , and expected rate of return on liquid bonds, R_{1t+1} , increases the demand for liquid bonds, since investors tend to favor investment with higher return, given the same level of risk. In the same guise, the increase in the return of illiquid bonds, R_{2t} , depresses the demand for liquid bonds.

To complete the outline of the demand side, we assume the existence of a large number of differentiated goods indexed over the real line between 0 and 1. This allows each firm to have a control of the price of her final good to be sold, since output becomes demand determined. Following the approach by [Dixit and Stiglitz \(1977\)](#), we assume that the consumption bundle C_{it} demanded by each agent $i \in [0, 1]$ is a CES type aggregate of all the $j \in [0, 1]$ varieties of final goods produced in this economy, as described by:

$$C_{it} = \left[\int_0^1 c_t^i(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \quad (9)$$

where θ is the elasticity of substitution between different varieties of goods produced by each firm j . To guarantee the existence of an equilibrium, the elasticity θ is restricted to be bigger than one. Standard optimization problem for the choice of the optimal composition of bundle (9) lead to the following constant-elasticity inverse demand function:

$$\frac{c_t^i(j)}{C_{it}} = \left[\frac{p_t(j)}{P_t} \right]^{-\theta} \quad (10)$$

where $p_t(j)$ is the price of variety j and P_t is the general price index defined as:

$$P_t = \left[\int_0^1 p_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}} \quad (11)$$

As $\theta \rightarrow \infty$ demand function becomes perfectly elastic, and the differentiated goods are perfect substitutes. The aggregate price level P_t is beyond the control of each individual firm. Similar steps can be applied to public expenditure G_t , so that aggregate demand is defined as the sum of private and public consumption for each variety goods: $C_t(j) + G_t(j) = Y_t(j)$, which after aggregating over all varieties $j \in [0, 1]$ becomes: $C_t + G_t = Y_t$.

In order to simplify, we assume the existence of a perfectly symmetrical equilibrium where all agents make the same choice *ex-post*. Therefore, we can drop index i from all equations in the model.

2.2 The pricing kernel

The inclusion of bonds into the utility function makes the intertemporal pricing scheme of liquid bonds vs. illiquid bonds different, because of the presence of utility terms in equation (5). To appreciate this, for sake of simplicity, let us consider the case where there are no

liquid bonds in the utility function, obtained by setting $\chi = 0$. In this case, following [Ljungqvist and Sargent \(2004\)](#), we get:

$$\beta E_t \frac{\lambda_{t+1}}{\pi_{t+1}} = \frac{\lambda_t}{R_{1t}} \quad (12)$$

$$\beta E_t \frac{\lambda_{t+1}}{\pi_{t+1} R_{1t+1}} = \frac{\lambda_t}{R_{2t}} \quad (13)$$

The pricing kernel M_{t+1} is defined to be:

$$M_{t+1} = \beta E_t \frac{\lambda_{t+1}}{\lambda_t \pi_{t+1}} \quad (14)$$

As it is well known, the pricing kernel is the tool to price recursively the entire term structure, starting from the shortest maturity bond. In our case, it is obvious to see that the pricing of the two-periods bond is, after a recursive application of the kernel, given by:

$$\beta^2 E_t \frac{\lambda_{t+2}}{\pi_{t+1} \pi_{t+2}} = \frac{\lambda_t}{R_{2t}} \quad (15)$$

By the same sort of argument, if we generalize to j -th period bond, we get:

$$\beta^j E_t \frac{\lambda_{t+j}}{\lambda_{t+j-1} (\pi_{t+1} \cdots \pi_{t+j-1} \pi_{t+j})} = R_{jt}^{-1} \quad (16)$$

The classical approach to the term structure implies that expected future short term interest rates determine long-term interest rates. This defines the well known *expectations hypothesis* (EH, henceforth), which in our case can be simply stated as: $R_{2t} = R_t E_t R_{1t+1}$. From equation (13), we get:

$$R_{2t}^{-1} = \beta E_t \left[\frac{\lambda_{t+1}}{\pi_{t+1} \lambda_t} \right] E_t R_{1t+1}^{-1} + cov_t \left[\beta \frac{\lambda_{t+1}}{\pi_{t+1} \lambda_t}, R_{1t+1}^{-1} \right] \quad (17)$$

which, after using (12) becomes:

$$R_{2t}^{-1} = R_t^{-1} E_t R_{1t+1}^{-1} + cov_t \left[\beta \frac{\lambda_{t+1}}{\pi_{t+1} \lambda_t}, R_{1t+1}^{-1} \right] \quad (18)$$

From (18), we observe that the EH holds if and only if utility is linear in consumption, such that $\frac{\lambda_{t+1}}{\pi_{t+1} \lambda_t} = 1$, and when the stochastic process of π_t , so that the covariance term becomes zero. In the case under exam, instead, with the inclusion of bonds into the utility function, we observe that the pricing kernel is affected by utility terms. In fact, by taking

advantage of the first order conditions (6)-(7), we can rewrite (18) as follows:

$$R_{2t}^{-1} = \left[R_t^{-1} - \frac{\chi b_{1t}^{\chi(1-\frac{1}{\sigma})-1}}{\lambda_t} \right] E_t R_{1t+1}^{-1} + cov_t \left[\beta \frac{\lambda_{t+1}}{\pi_{t+1} \lambda_t}, R_{1t+1}^{-1} \right] \quad (19)$$

Thus, by setting $\chi = 0$ in (19) we get exactly the setting outlined in (18). From (19) we immediately obtain the kernel expression such that:

$$\begin{aligned} M_{t+1} &= \beta E_t \frac{\lambda_{t+1}}{\lambda_t \pi_{t+1}} = \\ &= \left[R_t^{-1} - \frac{\chi b_{1t}^{\chi(1-\frac{1}{\sigma})-1}}{\lambda_t} \right] \end{aligned} \quad (20)$$

Thus, after plugging (21) into (19), we can observe that the interaction between the preference structure and the pricing structure is also reflected in the second order terms. This shows that modelling bonds with a sort of ‘liquidity preference’ motivation delivers a non-standard representation of the stochastic pricing equations for the financial assets. Finally, taking advantage of (4) into (14), we have:

$$M_{t+1} = \beta E_t \left(\frac{C_t}{C_{t+1}} \right)^{\frac{1}{\sigma}} \frac{1}{\pi_{t+1}} \quad (21)$$

In order to have a version of the kernel (14) to be employed for the analysis, we need to have a solution for consumption and inflation as a function of all the shocks of the system.

2.3 Firms

We assume the presence of a continuum of monopolistically competitive firms distributed on the unit line $[0, 1]$, indexed by $j \in (0, 1)$. Each individual firm faces a downward sloped demand curve for her differentiated product $Y_t(j)$:

$$P_t(j) = \left[\frac{Y_t(j)}{Y_t} \right]^{-\frac{1}{\theta}} P_t \quad (22)$$

It is well known that demand function (22) can be directly derived by following the details from [Dixit and Stiglitz \(1977\)](#).

The production function of each variety j employs only labor as input and it is given by:

$$Y_t(j) = A_t L_t^\alpha(j) \quad (23)$$

Note that all firms producing j varieties are subjected to an homogenous technological shock A_t , for which we assume the following structure (in log-linear terms):

$$a_t = (1 - \rho_a) a + \rho_a a_{t-1} + a_t^{1/2} \sigma_a \epsilon_t^a \quad (24)$$

where ϵ_t^a is an innovation term distributed according to a standardized Normal distribution.

Price rigidities are modeled through Calvo (1983) method of price adjustment. Every period, each seller sets prices with probability $1 - \alpha$, with $\alpha \in (0, 1)$, independently from the time of the last change. The parameter α indicates the degree of price stickiness.

Let us define the evolution of the price level. Let P_t be the general price level index, and be \wp_t the new price chosen at date t , by all sellers. Thus the price level is given by:

$$P_t^{1-\theta} = \left[\int_0^1 p_t(i)^{1-\theta} di \right] = (1 - \alpha) \wp_t^{1-\theta} + \alpha \int_0^1 p_{t-1}(i)^{1-\theta} di \quad (25)$$

which is equivalent to write, given the definition of the general price level:

$$P_t^{1-\theta} = (1 - \alpha) \wp_t^{1-\theta} + \alpha P_{t-1}^{1-\theta} \quad (26)$$

To determine the price level we need the choice of \wp_t . It is interesting to note that the optimal choice of \wp_t depends only upon the current and the expected future evolution of the entire sequence of $\{P_t\}_{t=0}^{\infty}$, so there is no need to know other aspects of the price distribution.

Firms set their own price by maximizing the following profit function:

$$\Omega_t(j) = E_t \sum_{k=0}^{\infty} (\alpha\beta)^k \left[\lambda_{t+k}(j) p_t^{1-\theta}(j) P_{t+k}^{\theta} Y_{t+k} - \omega \left(p_t(j)^{-\theta} P_{t+k}^{\theta} Y_{t+k} \right) \right] \quad (27)$$

By taking the First Order Condition with respect to $p_t(j)$, we get:

$$E_t \sum_{k=0}^{\infty} (\alpha\beta)^k \left\{ (1 - \theta) \lambda_{t+k}(j) \left(\frac{p_t(j)}{P_{t+k}} \right)^{-\theta} Y_{t+k} + \omega'(\cdot) \theta p_t^{-1-\theta}(j) P_{t+k}^{\theta} Y_{t+k} \right\} = 0 \quad (28)$$

After simplifying we have:

$$E_t \sum_{k=0}^{\infty} (\alpha\beta)^k \left\{ \lambda_{t+k}(j) p_t(j) - \omega'(\cdot) \frac{\theta}{\theta - 1} \right\} = 0 \quad (29)$$

After imposing the ex-post homogeneity condition (assuming that the pricing problem solved by each firm is equal for all firms producing the i -th varieties), and the resource

constraint, we get:

$$E_t \sum_{k=0}^{\infty} (\alpha\beta)^k \left[u'(Y_{t+k}) \frac{p_t(j)}{P_{t+k}} - \left(\frac{\theta}{\theta-1} \right) \omega' \left(Y_{t+k} \left(\frac{p_t(j)}{P_{t+k}} \right)^{-\theta} \right) \right] = 0 \quad (30)$$

where $\omega(\cdot)$ is the utility function representing the preferences towards work vs. leisure, which in our case is given by:

$$\omega(\cdot) = \frac{L_{it}^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \quad (31)$$

In this paper we abstract from the explicit definition of price distortions induced by monopolistic competition and nominal price rigidities. For a more general treatment, we address the reader to [Woodford \(2003\)](#) and [Schmitt-Grohé and Uribe \(2005\)](#), [Schmitt-Grohé and Uribe \(2007\)](#). Taking advantage of the form of the production function, the first order condition on consumption given by, the first order condition of the pricing problem is given by:

$$\left[\frac{p_t(i)}{P_t} \right]^{1-\theta+\frac{\theta}{\alpha}(1+\frac{1}{\eta})} = \frac{\theta}{(\theta-1)\alpha} \left(1 + \frac{1}{\eta} \right) \frac{1}{\lambda_t} A_t^{-\frac{1}{\alpha}(1+\frac{1}{\eta})} Y_t^{\frac{1}{\alpha}(1+\frac{1}{\eta})-1} \quad (32)$$

After log-linearization, we get the following expression for the aggregate supply function:

$$\beta\pi_{t+1} = \pi_t - ky_t + \mu_a a_t + \mu_g g_t \quad (33)$$

where:

$$k = \frac{(1-\delta)(1-\delta\beta)}{\delta} \left\{ \frac{\sigma S_c [1 + \eta(1-\eta)] + \alpha\eta}{\sigma S_c [\alpha\eta(1-\theta) + \theta(1+\eta)]} \right\} \quad (34)$$

$$\eta_a = \frac{(1-\delta)(1-\delta\beta)}{\delta} \left[\frac{1+\eta}{\alpha\eta(1-\theta) + \theta(1+\eta)} \right] \quad (35)$$

$$\eta_g = \frac{(1-\delta)(1-\delta\beta)}{\delta} \left[\frac{\alpha\eta g}{S_c \sigma [\alpha\eta(1-\theta) + \theta(1+\eta)]} \right] \quad (36)$$

Interestingly, the aggregate supply function [\(33\)](#) depends on the exogenous shocks of the system, a_t , g_t .

A model like the one introduced here is often solved in terms of the output gap. This would allow to get rid of the explicit formulation of the shock from the AS curve, since the definition of the potential output is a linear combination of the shock. We keep the formulation in terms of output level, since it allows for a neat derivation of the kernel from the model solution in terms of the shocks of the system.

2.4 Fiscal policy

The government issues two types of bonds: liquid, B_{1t} , and illiquid, or long term maturity, B_{2t} . government Budget Constraint in nominal terms is given by:

$$\frac{B_{1t}}{R_{1t}P_t} + \frac{B_{2t}}{R_{2t}P_t} = \frac{B_{1t-1}}{P_t} + \frac{B_{2t-1}}{R_{1t}P_t} + G_t - T_t \quad (37)$$

where G_t indicates the government expenditure, net of interest expenses. We assume that bond demand expressed by each i -the agent matches the supply supplied by the government according to the following equilibrium conditions:

$$B_{1t} = \int_0^1 B_{i1t} di; \quad B_{2t} = \int_0^1 B_{i2t} di;$$

In the same fashion, the total amount of fiscal collection is equal to the sum of taxes paid at the level of each i -the agent:

$$T_t = \int_0^1 T_{it} di$$

government expenditure follows a stochastic process given by:

$$g_t = (1 - \rho_g) g + \rho_g g_{t-1} + g_t^{1/2} \sigma_g \epsilon_t^g \quad (38)$$

Equation (38) represents a policy shock included in the model. The term ϵ_t^g indicates an i.i.d. normally-distributed shock with zero mean and variance σ_g^2 .

Overall the solution of the pricing kernel and the term structure is a function of two shocks. There is a technological shock, representing business cycle fluctuations, and a policy shock, in the form of g_t in (38). As we will see in the following sections, this structure permits to investigate interesting questions about the response of both short and long rates with respect to policy changes and their feedback into the entire model economy.

According to the fiscal theory of price level determination (FTPL), the comparative evaluation of alternative monetary policy rules should not be thought in isolation from an explicit design of the fiscal policy stance. As described by [Leeper \(1991\)](#) and [Sims \(1994\)](#), to prevent a strong and prolonged increase of prices, we need to introduce a policy rule whereby taxes react to changes in the outstanding level of real public debt:

$$T_t = \psi_0 + \psi \frac{B_{1t-1}}{P_t} + \psi \frac{B_{2t-1}}{P_t} \quad (39)$$

According to (39), the level of fiscal revenue T_t is set in order to react with respect to the previous period debt: both one-period debt and two-periods debt, all expressed in real terms. As described by [Leeper \(1991\)](#), [Sims \(1994\)](#) and [Woodford \(2003\)](#), rule (39) is sufficiently

general to encompass a wide set of debt-targeting rules, including balance-budgeting rules. The parameter capturing the strength of tax response to debt fluctuations is given by ψ , which has been set to be equal for both short and long-term debt.

According to [Leeper \(1991\)](#) fiscal policy is defined to be ‘passive’ if the following conditions is respected:

$$|\beta^{-1} - \psi| < 1 \tag{40}$$

and active otherwise. A passive fiscal policy sets taxes to avoid an expansionary path for the public debt, which may lead to unwanted inflationary pressure. By changing ψ in (39) it is possible to individuate the impact on government’s solvency after an increase in the level of real debt.

The forthcoming discussion shows that, for liquid bonds, we have an explicit demand function depending on the preference structure of the investor. Illiquid bond are solved endogenously, representing a residual adjustment in the government budget constraint. This is meant to represent a condition where illiquid bonds (or long-term bonds) absorb all the bond-demand not satisfied by the supply conditions for short-term bonds. Implicitly, our model designs the behavior of a secondary market for government bonds by imagining a trading of illiquid bonds (with a two-period maturity) in each period.

2.5 Monetary policy

The question addressed in the present paper regards the role of monetary policy rules in tying down the determinacy properties of a rational expectations equilibrium. The types of monetary policy rules to be studied represent a variant of the basic interest rate pegging rule proposed by [Taylor \(1993\)](#), generally represented by:

$$R_{1t} = R_1 \left(\frac{\pi_{t+n_\pi}}{\pi} \right)^{\phi_\pi} \left(\frac{Y_{t+n_y}}{Y} \right)^{\phi_y} \left(\frac{R_{1t+n_R-1}}{i} \right)^{\phi_R} \tag{41}$$

where ϕ_π , ϕ_y , ϕ_R indicate the response of policy rate R_{1t} with respect to inflation, output and lagged R_{1t} itself, over different time horizon (n_π , n_y , n_i), and with different intensity captured by coefficient ϕ_π , ϕ_y , ϕ_i . We assume all parameters ϕ_π , ϕ_y , ϕ_R to be positive.

Given (41) the demand for short term bonds is fully determined. The general setting of monetary policy rule specified in (41) can be simplified to the following simpler Taylor rule:

$$R_{1t} = (1 - \rho) (\phi_\pi \pi_t + \phi_y y_t) + \rho R_{1t-1} \tag{42}$$

with $\rho = 0$ be the interest rate smoothing parameter.

3 Calibration and impulse responses

In this section we present the procedure adopted to calibrate the model together with a short impulse-response analysis, in order to discover the main dynamic properties of model economy under study. The scope of this section is to show the basic properties of the model conditional to the evolution of exogenous shocks. The model is calibrated by using quarterly data from US economy for the sample 1960:1-2010:3 and the calibrated parameter values are reported in Table 1. The annual inflation rate considered during the considered sample period is 4.09 per cent, while the steady state short term interest rate R_1 is 5.58 per cent obtained as the mean of 3-months Treasury Bill Rate during the considered sample period. To capture the steady state of the illiquid bonds, we take the mean of the 10-year return of government bond, given by 6.5 per cent. The steady state level of output has been obtained as the mean of quarterly GDP, constant prices, seasonally adjusted over the sample period 1960:1-2010:3. This number has been normalized by considering the civilian population considered over the same sample period, according to the methodology described by Kim (2000).

The intertemporal discount rate consistent with the figures outlined earlier is equal to 0.99, as it is standard in the current literature. We also assume that the inverse of risk aversion coefficient in the utility function σ has been set equal to 0.5, together with Frisch labor supply elasticity η equal to 1, and the scale parameter χ has been set equal to 0.3, as in Galí (2008). The labor share in the production function α is set equal to 0.67.

The share of consumption over GDP is set to be 0.57, implying a public expenditure to GDP ratio equal to 0.43, an high value if compared to the true data, given the absence of investments from the model. Price rigidity parameter δ is set equal to 2/3, implying an average price duration of three quarters, consistent with the empirical evidence. In the same way, the elasticity of substitution between differentiated goods is equal to 6, as commonly assumed in the traditional new-keynesian dynamic models. The parameter representing the response of fiscal revenue to outstanding short and long term debt is set to be 0.05.

The steady state level of labor supply is given by $L = 0.33$, implying a 1/3 ratio of working activities to non-working activities. The steady state level of total public debt is set to be 33 per cent over GDP, equal to the average of US Federal Public Debt to GDP ratio for the sample period considered. The short term debt has been left to be free: for the simulation reported, we set as a benchmark value 40 per cent of the total level of debt, implying a 60 per cent of the long term debt. Finally, the monetary policy assumed for impulse-response function is the standard Taylor rule with both contemporaneous inflation and output targeting, with $\phi_\pi = 1.5$ and $\phi_y = 0.5$ and $\rho = 0$. The autoregressive coefficient for the shocks are $\rho_A = 0.9$, $\rho_G = 0.5$. Standard deviation are, instead, set to $\sigma_A = 0.007$ and $\sigma_G = 0.01$. These figures are consistent with the values used in similar papers for the

U.S.

The model is solved up to first order and the impulse response functions conditional to one standard deviation technology and public expenditure shock have been reported in Figure 1 and 2, respectively. From Figure 1 we observe that technological shock expands output and consumption but reduces the labor effort. The increase in aggregate demand raises inflation rate, with a consequent increase of both short and long nominal interest rates. The level of both short and long term bonds in real terms decreases because of the increase of the inflation rate. This determines a reduction of tax revenue. Labor supply falls because the productivity shock is perceived as a windfall gain.

The results for an expansionary government expenditure shock are reported in Figure 2. Output and consumption increase, along with inflation and both the short and long term rates. Obviously, the surge in public debt (both short and long term in real terms) makes taxes to increase, too. Interestingly, the reaction of short term rate is stronger than long term one, highlighting a smooth-out effect, as it is customary for term structure models.

Overall, the impulse response functions show an pattern of the variables compatible with that of the standard new Keynesian model, with the additional feature arising from the interplay between short and long term interest rates. Under this perspective, the inclusion of two-bonds does not imply non-standard dynamic patterns of the variables to a major extent.

4 An analytic solution for the pricing kernel

4.1 Model reduction

The first step consists in log-linearizing the model around the deterministic steady state. This is now a standard procedure and we are not going to describe the full details of it. A technical appendix available upon request will report the full log-linearized version of model. The results from our reduction strategy can be collected in the following proposition:

Proposition 1 *The reduced form model can be represented as follows:*

$$\pi_{t+1} = \frac{1}{\beta}\pi_t - \frac{k}{\beta}\eta_{ya}a_t - \frac{k}{\beta}\eta_{yg}g_t \quad (43)$$

$$\begin{aligned} b_{2t+1} + \eta_{\pi 1}\pi_{t+1} + \eta_{ba2}a_{t+2} + \eta_{ba1}a_{t+1} + \eta_{bg2}g_{t+2} + \eta_{bg1}g_{t+1} = \\ = \eta_{b2}b_{2t} - \eta_{\pi}\pi_t - \eta_{ba}a_t - \eta_{bg}g_t \end{aligned} \quad (44)$$

where coefficients η_{ya} , η_{yg} , $\eta_{\pi 1}$, η_{ba2} , η_{ba1} , η_{bg2} , η_{bg1} , η_{b2} , η_{π} , η_{ba} , η_{bg} are given in Appendix 1.

Proof 1 See Appendix 2.

The functional form of the model described in (43) and (44) can be directly employed in the analytical solution of the kernel, which is explicitly discussed in the following proposition.

Proposition 2 *The analytical solution to the pricing kernel is:*

$$m_{t+1} = \lambda_0 + \lambda_1 a_t - \lambda_2 g_t - \eta_1 a_t^{1/2} \sigma_a \epsilon_{t+1}^a - \eta_2 g_t^{1/2} \sigma_g \epsilon_{t+1}^g \quad (45)$$

where coefficients are:

$$\begin{aligned} \lambda_0 &= \delta - f_\pi - (1 - \rho_a) a \left(\frac{\eta_{ca}}{\sigma} + \alpha_a \right) + (1 - \rho_g) g \left(\frac{\eta_{cg}}{\sigma} - \alpha_g \right) \\ \lambda_1 &= \frac{\eta_{ca} (1 - \rho_a)}{\sigma} - \alpha_a \rho_a \\ \lambda_2 &= \frac{\eta_{cg} (1 - \rho_g)}{\sigma} + \alpha_g \rho_g \\ \eta_1 &= \frac{\eta_{ca}}{\sigma} + \alpha_a \\ \eta_2 &= \frac{\eta_{cg}}{\sigma} + \alpha_g \end{aligned}$$

Proof 2 See Appendix 2.

The solution presented in the previous proposition depends on the assumptions introduced in the microfounded model. We should stress that we do not need a numerical solution of the model written in state-space form to price government bonds in our framework. Differently from most contributions in macro-finance, we obtain an analytical solution for the price kernel consistent with the model structure.

4.2 Bond pricing

Given the solution outlined in the previous section, the model has two state variables governing the dynamics of the pricing kernel: a_t and g_t . Since the pricing kernel is conditionally lognormal, the short term rate R_{1t} is given by:

$$R_{1t} = -\log E_t \exp(m_{t+1}) \quad (46)$$

which, given lognormality, becomes:

$$R_{1t} = -E_t m_{t+1} - \frac{1}{2} \text{var}_t(m_{t+1}) \quad (47)$$

Equation (47) clearly shows that fluctuations in the short rate are represented as a combination of changes of movements in the conditional mean and variance of the pricing

kernel. The conditional mean of the log of the pricing kernel is given by:

$$E_t m_{t+1} = \lambda_0 + \lambda_1 a_t - \lambda_2 g_t \quad (48)$$

while conditional variance is:

$$\text{var}_t(m_{t+1}) = \eta_1^2 a_t \sigma_a^2 + \eta_2^2 g_t \sigma_g^2 \quad (49)$$

Interestingly, this model allows for a time-varying structure for the conditional variance of the kernel, given the time-varying volatility induced by shocks. The type of shocks here included captures two types of shocks: a technological shock capturing business cycle patterns and a fiscal policy shock, capturing policy-related shocks. This allows to study the reaction of term structure with respect to fiscal policy shocks. To get a constant conditional variance, we need to set $\eta_1 = \eta_2 = 0$. This condition is fairly restrictive, since we have seen that coefficients η_1, η_2 are function of the core parameters of the model, so it will be only by chance that they may be equal to zero.

By combining (47) with (48) and (49), the solution of the short-rate interest rate can be written as:

$$R_{1t} = -\lambda_0 - \left(\lambda_1 + \frac{\eta_1^2}{2} \sigma_a^2 \right) a_t + \left(\lambda_2 - \frac{\eta_2^2}{2} \sigma_g^2 \right) g_t \quad (50)$$

We can now further generalize the previous argument by extending the pricing scheme to longer-term government bond. In what follows we are going to present a general formulation to price a generic k maturity bond and we will extend the analytics to the type of illiquid (or long term bonds) bonds.

Following [Atkeson and Kehoe \(2008\)](#), let us consider the price of a k -th period maturity bond p_t^k :

$$p_t^k = \log E_t \exp \left(m_{t+1} + p_{t+1}^{k-1} \right) \quad (51)$$

Our goal is to derive the affine recursive pricing formula. We set the price of a k -th period maturity bond as a function of the state variables a_t and g_t , as follows:

$$p_t^k = -A_k - B_k a_t - C_k g_t \quad (52)$$

The solution is collected in the following Proposition.

Proposition 3 *The affine recursive coefficients of k -th maturity bond prices are given by:*

$$A_k = -\lambda_0 + A_{k-1} + B_{k-1} (1 - \rho_a) a + C_{k-1} (1 - \rho_g) g \quad (53)$$

$$B_k = \rho_a B_{k-1} - \frac{\sigma_a^2}{2} (\eta_1^2 + B_{k-1}^2) - \lambda_1 \quad (54)$$

$$C_k = \lambda_2 + \rho_g C_{k-1} - \frac{\sigma_g^2}{2} (\eta_2^2 - C_{k-1}^2) \quad (55)$$

with $A_1 = \lambda_0$, $B_1 = \lambda_1 + \frac{\eta_1^2 \sigma_a^2}{2}$, $C_1 = \frac{\eta_2^2 \sigma_g^2}{2} - \lambda_2$.

Proof 3 See Appendix 2.

The yield R_{kt} on a k maturity bond can be expressed as:

$$R_{kt} = -\frac{p_t^k}{k} \quad (56)$$

which, by using (52), becomes:

$$R_{kt} = \frac{1}{k} (A_k + B_k a_t + C_k g_t) \quad (57)$$

We can now compute the term spread, i.e. the difference between long-term R_{kt} and short-term yield R_{1t} , which by using (50) and (57), becomes:

$$R_{kt} - R_{1t} = \left(\frac{A_k}{k} + \lambda_0 \right) + \left(\frac{B_k}{k} + \lambda_1 + \frac{\eta_1^2 \sigma_a^2}{2} \right) a_t + \left(\frac{C_k}{k} - \lambda_2 + \frac{\eta_2^2 \sigma_g^2}{2} \right) g_t \quad (58)$$

Differently from [Atkeson and Kehoe \(2008\)](#), our setting does not allow for a parallel shift of the term structure, since all yield change differently after a shock to a_t or g_t . This is due to the assumptions made for (24) (38), which are not random walk. This implies a general set of formula for recursive terms of the affine coefficients A_k and B_k which are non-linear, as proved in Proposition 3. On the other hand, equation (58) shows that the difference between a k -th maturity bond and the short rate bond is mainly due to exogenous shock fluctuations. Apart from this, the two yields differ for a constant term given by $\left(\frac{A_k}{k} + \lambda_0 \right)$.

We are now in the position of specifying the pattern for the long-term rate for $k = 2$, as assumed in the present setting. We can collect the results in the following Corollary:

Corollary 1 For a two-period illiquid bond, the yield and the term spread are respectively

given by:

$$R_{2t} = \frac{1}{2}(A_2 + B_2 a_t + C_2 g_t) \quad (59)$$

$$R_{2t} - R_t = \left(\frac{A_2}{2} + \lambda_0\right) + \left(\frac{B_2}{2} + \lambda_1 + \frac{\eta_1^2 \sigma_a^2}{2}\right) a_t + \left(\frac{C_2}{2} - \lambda_2 + \frac{\eta_2^2 \sigma_g^2}{2}\right) g_t \quad (60)$$

where the coefficients are:

$$A_2 = B_1(1 - \rho_a)a + C_1(1 - \rho_g)g \quad (61)$$

$$B_2 = \rho_a B_1 - \frac{\sigma_a^2}{2}(\eta_1^2 + B_1^2) - \lambda_1 \quad (62)$$

$$C_2 = \lambda_2 + \rho_g C_1 - \frac{\sigma_g^2}{2}(\eta_2^2 - C_1^2) \quad (63)$$

with B_1 and C_1 defined in Proposition 1

Proof 4 By setting $k = 2$ in (52), (53)-(55), (57) and (58), rearrange and simplify, it is immediate to get the results stated in the text.

We can rewrite equation (60) in a more suitable fashion, so that the link between the returns on long-term and short-term bonds can be expressed as:

$$R_{2t} = R_t + \eta_0 + \eta_a a_t + \eta_g g_t \quad (64)$$

where:

$$\eta_0 = \left(\frac{A_2}{2} + \lambda_0\right) \quad (65)$$

$$\eta_a = \left(\frac{B_2}{2} + \lambda_1 + \frac{\eta_1^2 \sigma_a^2}{2}\right) \quad (66)$$

$$\eta_g = \left(\frac{C_2}{2} - \lambda_2 + \frac{\eta_2^2 \sigma_g^2}{2}\right) \quad (67)$$

From (64), excluding the stochastic processes a_t and g_t , we can see that the two returns R_t and R_{2t} differ only for a constant term. This raises two observations. First, in general equilibrium, the coexistence of two interest rates can be consistent with a constant wedge between the two if there is no uncertainty. Second, the wedge fluctuates with a drift if uncertainty is added. We would get a very similar result if we imposed exogenously a

relation between liquid and illiquid bonds like (64). For example, this can be obtained by assuming that the relation between the two rates evolves according to:

$$R_{2t} = HR_t Z_t^\nu \quad (68)$$

with H is a constant, Z_t is stochastic term, for which we can assume an autoregressive structure. The log-linearized version of (68) is:

$$\tilde{R}_{2t} = \eta_0 + \tilde{R}_t + \nu \zeta_t \quad (69)$$

where $\eta_0 = \log H$, $\zeta_t = \log Z_t - \log Z$, and a letter without time subscript denote a steady state. If we express ζ_t as a linear combination of a_t , g_t we can immediately get a representation very similar to that reported in (64):

$$\tilde{R}_{2t} = \eta_0 + \tilde{R}_t + \nu (\xi a_t + (1 - \xi) g_t) \quad (70)$$

The representation of equation (70) is qualitatively similar to (68). In fact, if the focus of the analysis is on evaluating the effect on determinacy of a rational expectations equilibrium induced by term structure, the two representations under (68)-(70) do not imply any differences in the results about determinacy. In other words, it does not matter in terms of the analysis about determinacy whether the link between liquid and illiquid bond rate of return is imposed, as occurs with (70), or whether it is explicitly derived by following the procedure previously described earlier. The advantage of the endogenous derivation of the link between R_{2t} and R_t consists in the fact that the resulting coefficients are functions of all the core parameters of the model.

The representation outlined in equations (68)-(70) has been considered in [Marzo and Zagaglia \(2008\)](#). The different results obtained in [Marzo and Zagaglia \(2008\)](#) are mainly due to the different modelling strategy adopted to include liquid and illiquid bond: in fact, in that paper illiquid bond are subjected to transaction costs in the representative agent's budget constraint. This delivers a different functional form for the aggregate supply function that depends explicitly on the short term interest rate (the yield paid to liquid bond, equal to the policy rate), creating a different transmission channel of short term rate to term structure. In the present paper, instead, bonds are included directly in the utility function, and there are no transaction costs in the household's budget constraint.

A final remark about the expectations hypothesis is in order. Taking advantage of the log-linearized reduced form of the model, we can derive the following relation (expressed in log-linear terms) between long and short rate:

$$R_{2t} = R_{1t} + R_{1t+1} \quad (71)$$

Equation (71) is the log-linear representation of the expectation hypothesis. In other words, the inclusion of bonds in the utility in a weakly separable way, does not induce a violation of the EH. To get a model setting where the EH appears to be violated, it would be necessary to include stronger frictions among different types of bonds or money, as outlined in a preliminary work done by [Marzo and Zagaglia \(2008\)](#).

5 Determinacy of rational expectations equilibria

In this section we are going to study the determinacy properties of the model. Each determinacy condition is derived conditional to a specific monetary policy rule. We are going to consider seven variants of the rules proposed in (41)-(42): a contemporaneous inflation-targeting rule, and a rule for backward-looking and for forward-looking inflation targeting. In addition, we will consider a policy rule for flexible inflation targeting, where target for output is also included.

After taking advantage of the log-linearized version of the model, we can reduce the system to the aggregate supply function (43), the Taylor rule (42), the government budget constraint (44) and the following version of the intertemporal IS equation:

$$E_t y_{t+1} - S_g E_t g_{t+1} + \sigma S_c E_t \pi_{t+1} = y_t - S_g g_t + \sigma S_c R_{1t} \quad (72)$$

where S_g and S_c indicate, respectively, the share of public expenditure and consumption over GDP. Another equation of the system is given by the aggregate supply curve given by (33). For what concerns determinacy analysis we can drop from the aforementioned equations all terms involving exogenous stochastic processes a_t and g_t , since they do not impact on the dynamic properties of the model. As a general remark, after using the Taylor rule to eliminate R_{1t} into equation (72) and in the government budget constraint (37), we can further reduce the model down to a three-equation system in the variables π_t, y_t and b_{2t} . As a result, we obtain the following matrix representation for the system:

$$AZ_{t+1} = BZ_t \quad (73)$$

where vector Z_t is given by $Z_t = [\pi_t, y_t, b_{2t}]'$, and matrices A and B are properly defined according the specific setting adopted. We can rewrite the system as follows:

$$Z_{t+1} = \Gamma Z_t \quad (74)$$

with: $\Gamma = A^{-1}B$. Matrix Γ includes the driving dynamic properties of the system. Thus, the determinacy analysis is focused entirely on it.

5.1 Pure inflation targeting

The monetary policy rule here studied are given by:

$$R_{1t} = \phi_\pi \pi_t \tag{75}$$

$$R_{1t} = \phi_\pi \pi_{t+1} \tag{76}$$

$$R_{1t} = \phi_\pi \pi_{t-1} \tag{77}$$

Rule (75) is a simple representation of a targeting regime for current inflation, while (76) indicates a pure expected inflation targeting, and (77) represents a lagged inflation targeting.³ After plugging equations (75)-(77) into (72) and (37) and rearranging, we obtain a three-equation system that can be represented as (74). Rules (75)-(77) are denoted as absolute inflation targeting to stress the absence of any other goal for monetary authority. In contrast, with flexible inflation targeting, output is also an argument of the monetary policy rule.

The determinacy conditions for a REE induced by the rules (75) and (76) are stated in Proposition 4. The backward-looking rule implies an upper bound for the coefficient ϕ_π that is discussed in Proposition 5.

Proposition 4 *Given $\phi_\pi > 0$, conditions for determinacy of a REE to be unique under a Taylor Rule of types (75)-(76) are given by:*

$$\phi_\pi > 1 \tag{78}$$

$$1 - \beta < \psi < 1 + \beta \tag{79}$$

Or, alternatively:

$$\phi_\pi < 1 \tag{80}$$

$$1 - \beta > \psi \quad \psi > 1 + \beta \tag{81}$$

Proof 5 *See Appendix 2.*

According to the results outlined in Proposition 4, the Taylor principle for a model with our term structure specification is complied upon fully with a pure inflation targeting rules. This is not the case for a model with backward-looking inflation targeting such as (77), as detailed in the following proposition.

Proposition 5 *Given $\phi_\pi > 0$, conditions for determinacy of a REE to be unique under a*

³These policy rules have also been considered by Bullard and Mitra (2002) and Lubik and Marzo (2007) in a model with one type of bond and money inserted in the utility function in a strongly separable way.

Taylor Rule (77) are given by:

$$1 < \phi_\pi < 1 + \frac{2(1+\beta)}{\sigma S_c k} \quad (82)$$

$$1 - \beta < \psi < 1 + \beta \quad (83)$$

Or, alternatively:

$$\phi_\pi < 1 \quad (84)$$

$$1 - \beta > \psi \quad \psi > 1 + \beta \quad (85)$$

Proof 6 See Appendix 2.

From this result, we observe that a backward inflation rule prescribes an upper bound that is a function of the inflation targeting coefficient. In this respect, the model partially confirms the results existing in the literature, with a model including the term structure. The results replicate the interaction between active-monetary and passive-fiscal outlined in [Leeper \(1991\)](#). A determinate equilibrium can be reached also with an inflation targeting coefficient lower than one (ϕ_π), provided that fiscal policy is set to be active, or non-respondent to the path of outstanding debt. The novelty presented here, consists in the presence of an upper bound for the backward-looking inflation targeting rule.

5.2 Flexible inflation targeting

In what follows, we consider a monetary policy rule with inflation and output stabilization. This is meant to mimic a regime of flexible inflation targeting according to [Svensson \(2003\)](#). We focus on two types of rules, namely a classical Taylor rule, with contemporaneous targeting of inflation and output, and a variant with expected inflation targeting together with current output targeting.

The results for the classical Taylor rule are collected in the [Proposition 6](#)

Proposition 6 *Under simple Taylor Rule with contemporaneous inflation and output targeting given by:*

$$R_{1t} = \phi_\pi \pi_t + \phi_y y_t \quad (86)$$

Provided that $\phi_\pi, \phi_y > 0$ conditions for determinacy of a REE to be unique are:

$$k\phi_\pi + \phi_y > \frac{1 - \beta}{\sigma S_c \beta} \quad (87)$$

and:

$$k(\phi_\pi - 1) + \phi_y(1 - \beta) > 0 \quad (88)$$

$$1 - \beta < \psi < 1 + \beta \quad (89)$$

Alternatively, the REE is determinate if either (87) or (88) or both have the reverted inequality and:

$$\phi_\pi < 1 \quad (90)$$

$$1 - \beta > \psi \quad \psi > 1 + \beta \quad (91)$$

Proof 7 See Appendix 2.

Conditions (87)-(89) highlight a tension between ϕ_π and ϕ_y , provided that $\phi_\pi > 1$. These results confirm the findings by Bullard and Mitra (2002) and Lubik and Marzo (2007).

A second type of Taylor rule is represented by an expected inflation targeting together with a current output targeting, obtained by setting $\rho = 0$ in (42) and considering the expected inflation rate in place of the current rate. The Taylor rule to be studied is now given by:

$$R_{1t} = \phi_\pi E_{t+1}\pi_{t+1} + \phi_y y_t \quad (92)$$

The results on the determinacy properties of the equilibrium induced by (92) are collected in Proposition 7.

Proposition 7 *Under expected inflation and current output targeting rule given by (92), provided that $\phi_\pi > 0, \phi_y > 0$, conditions for a REE to be unique are:*

$$1 < \phi_\pi < \frac{2(1+\beta)}{k\sigma S_c\sigma} + \phi_y \frac{2(1+\beta)}{k} + 1 \quad (93)$$

and:

$$1 - \beta < \psi < 1 + \beta \quad (94)$$

Alternatively, the REE is determinate if either (93) is not satisfied to get determinacy, condition (94) must be replaced by:

$$1 - \beta > \psi \quad \psi > 1 + \beta \quad (95)$$

Proof 8 See Appendix 2.

Even in this case, condition (93) identifies an upper bound for the inflation targeting coefficient conditional on the size of output targeting. Fiscal policy induces an additional degree of freedom, thus letting the policy maker choose between the combination *active*-monetary and *passive*-fiscal or vice-versa.

5.3 Interest-rate smoothing

By setting $\rho \neq 0$ in equation (42), we obtain a Taylor rule with interest rate smoothing. The results are collected in Proposition 8:

Proposition 8 *With rule (42), provided that $\phi_\pi, \phi_y, \rho > 0$, conditions for a REE to be unique are:*

$$k(\phi_\pi + \rho - 1) + (1 - \beta)\phi_y > 0 \quad (96)$$

$$\rho < \beta \quad (97)$$

$$1 - \beta < \psi < 1 + \beta \quad (98)$$

Alternatively, if either (96) or (97) are not satisfied, REE determinacy is obtained by replacing (98) by:

$$1 - \beta > \psi \quad \psi > 1 + \beta \quad (99)$$

Proof 9 See Appendix 2.

These results confirm the standard outcomes from a simple new Keynesian model without the term structure. From this perspective, the inclusion of the term structure appears irrelevant in terms of determinacy analysis as the Taylor principle is fully satisfied (see Bullard and Mitra, 2002). A possible interpretation of these results is related to the role of the expectations hypothesis, which holds perfectly in the log-linear version of the model. The next step is to verify if these results hold when a different assumption is made for modelling bonds.

6 Robustness

In this section, we investigate the role of the modelling assumption for the liquidity services provided by government bonds. We have assumed that the utility function is strongly separable between bonds, consumption and labor effort. In what follows, we consider a utility function where bonds and consumption are weakly substitutes:

$$u_t = \frac{\left[C_t^\gamma b_{1t}^{1-\gamma} \right]^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{L_t^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}} \quad (100)$$

with $b_{1t} = B_{1t}/P_t$. From (100), we observe that liquid bond b_{1t} are treated as if they were cash balances, since they directly provide utility to the representative agent with a direct interaction with consumption. The first order condition with respect to consumption (4) is now replaced by:

$$\gamma C_t^{\gamma(1-\frac{1}{\sigma})-1} b_{1t}^{(1-\gamma)(1-\frac{1}{\sigma})} = \lambda_t \quad (101)$$

Moreover, the first order condition with respect to liquid bonds b_{1t} is now given by:

$$(1-\gamma) C_t^{\gamma(1-\frac{1}{\sigma})} b_{1t}^{(1-\gamma)(1-\frac{1}{\sigma})-1} + \beta E_t \frac{\lambda_{t+1}}{\pi_{t+1}} = \frac{\lambda_t}{R_{1t}} \quad (102)$$

Thus, rearranging (101) and (102), the demand for liquid bonds takes the form:

$$b_{1t} = \frac{(1-\gamma)}{\gamma} C_t \frac{R_{1t} R_{2t}}{(R_{2t} - R_{1t+1} R_{1t})} \quad (103)$$

It is not difficult to check that liquid bond demand (103) still respects the usual properties: it is increasing in consumption C_t and R_{1t} , R_{1t+1} and decreasing with respect to R_{2t} .

In order to reduce the model, we can write the log-linearized version of the optimality condition (101) as follows:

$$\left[\gamma \left(1 - \frac{1}{\sigma} \right) - 1 \right] c_t + (1-\gamma) \left(1 - \frac{1}{\sigma} \right) b_{1t} = \lambda_t \quad (104)$$

while the log-linearized version of the liquid bond demand (103) is:

$$b_{1t} = c_t + \frac{1}{(1-\beta R_1)} R_{1t} - \frac{R_1 \beta}{(\pi - \beta R_1)} R_{2t} + \frac{R_1 \beta}{(\pi - \beta R_1)} R_{1t+1} \quad (105)$$

Equations (104)-(105) are the key ingredients of the model. In equation (104), the Lagrange multiplier depends on nominal rates through equation (105), differently from the standard case, where λ_t is a function of C_t only. This feature produces a strong impact on the resulting form of the intertemporal IS and aggregate supply equations. In fact, the IS

curve depends on both expected and current short term rate in log-linear terms:

$$y_{t+1} - \sigma S_c \alpha_{R1} R_{1t+1} + \sigma S_c \pi_{1t+1} + g \sigma g_{t+1} = y_t - \sigma S_c \alpha_{R1} R_{1t} + g \sigma g_t \quad (106)$$

while Aggregate Supply now becomes:

$$\beta \pi_{t+1} = \pi_t - k y_t + \eta_{as} a_t + \eta_{gs} g_t + \eta_{RS} R_{1t} \quad (107)$$

where all coefficients are reported in Appendix 1.

From equation (107), we note that the introduction of weak separability in the utility function modifies the functional form of the AS equation in a substantial way. The short term interest rate directly affects the expected inflation rate together with exogenous shocks. A similar result would have been obtained after the introduction of transaction costs in the representative agent's budget constraint. Intuitively, this means that monetary policy, by controlling the short term rate R_{1t} , affects directly the firm's costs and her ability to borrow from banks. An increase in short term rate R_{1t} has the effects of increasing the cost structure of firms, implying an increase of expected inflation as direct consequence.

Also in this formulation the expectations hypothesis holds in log-linear terms. Equation (71) applies to this context too. The model can be reduced through the same steps of the benchmark case. The system can be set in the form (73) with vector of variables still given by: $Z_t = [\pi_t, y_t, b_{2t}]'$. Matrices A and B from (73) are now defined as:

$$A = \begin{bmatrix} \beta & 0 & 0 \\ [\sigma S_c (1 - \alpha_R \phi_\pi)] & (1 - \sigma S_c \alpha_{R1} \phi_y) & 0 \\ \mu_{\pi 1} & \mu_{y1} & 1 \end{bmatrix} \quad (108)$$

$$B = \begin{bmatrix} (1 + \eta_{RS} \phi_\pi) & -(k - \eta_{RS} \phi_y) & 0 \\ \alpha_R \sigma S_c \phi_\pi & (1 - \sigma S_c \alpha_{R1} \phi_y) & 0 \\ \mu_\pi & \mu_y & \frac{(1-\psi)}{\beta} \end{bmatrix} \quad (109)$$

while matrix Γ becomes:

$$\Gamma = \begin{bmatrix} \frac{(1+\eta_{RS}\phi_\pi)}{\beta} & -\frac{(k-\eta_{RS}\phi_y)}{\beta} & 0 \\ \varphi_1 & \varphi_2 & 0 \\ \varphi_3 & \varphi_4 & \frac{(1-\psi)}{\beta} \end{bmatrix} \quad (110)$$

where all coefficients are reported in Appendix 1.

With the present setting at hands, the following result holds:

Proposition 9 *With rule (42), provided that $\phi_\pi, \phi_y, \rho > 0$, conditions for a REE to be*

unique are:

$$\operatorname{argmax} \{ \bar{\phi}_{\pi 2}, \bar{\phi}_{\pi 3} \} < \phi_{\pi} < \frac{k - \eta_{RS} \phi_y}{k \alpha_R (1 + \eta_{RS}) + \eta_{RS} \phi_y} \quad (111)$$

$$\frac{1}{\sigma S_c \alpha_{R1}} < \phi_y \quad (112)$$

with

$$\begin{aligned} \bar{\phi}_{\pi 2} &= \frac{\sigma S_c \phi_y (\alpha_R + \beta \alpha_{R1}) - (1 + \beta)}{\eta_{RS} (1 - k S_c \alpha_R)} \\ \bar{\phi}_{\pi 3} &= \frac{(1 - \beta) \sigma S_c \phi_y + \eta_{RS} \sigma S_c \phi_y - \sigma S_c k}{\eta_{RS} \sigma S_c \phi_y + k \alpha_R (\sigma S_c - \eta_{RS})} \end{aligned}$$

$$1 - \beta < \psi < 1 + \beta \quad (113)$$

Alternatively, if either (111) or (112) are not satisfied, REE determinacy is obtained by replacing (113) by:

$$1 - \beta > \psi \quad \psi > 1 + \beta \quad (114)$$

Proof 10 See Appendix 2.

These results highlight the presence of a set of non-linear bounds for monetary policy parameters ϕ_{π} and ϕ_y : this property is entirely dependent on the setting adopted for the bond modelling approach. In fact, under the assumption of weakly separable utility function between consumption and liquid bonds modifies both aggregate supply and intertemporal IS curve. This makes the Taylor principle no longer determined since the inflation targeting coefficient is now dependent on output targeting coefficient.

To provide a graphical representation of the implications of Proposition 9, we have simulated the evolution pattern of ϕ_{π} and ϕ_y conditional to two different values for the parameter representing the intertemporal elasticity of substitution in consumption σ . Figure 3 reports the numerical bounds for the regions of determinate model solution, where ϕ_y varies between 0 and 10. The parameters of the model are exactly the same as those described in Table 1, apart from γ , which has been set equal to 0.8, to assign a larger weight to consumption in utility. The top panel in Figure 3 is obtained for a value of σ equal to 2, while in the bottom panel we set $\sigma = 0.5$. In both pictures, the determinacy region is identified with a text label. Outside the bounds, the model is not characterized by unique solutions unless we change the parameter of the fiscal policy rule (114). Therefore, the picture reported in Figure 3 has been plotted with a setting consistent with the condition (113). From the top panel of Figure 3, we observe a determinacy region when $\sigma = 2$ only

when there is an almost one-to-one increase in both ϕ_π and ϕ_y .

Intuitively, with $\sigma > 1$, the marginal utility of consumption is decreasing with respect to b_{1t} . Hence, a negative shock to inflation implies an increase in demand for liquid bonds b_{1t} , a lower marginal utility of consumption, and a lower labor supply. On the other hand, if $\sigma < 1$, the marginal utility of consumption is increasing with respect to b_{1t} . In this case, a negative shock to inflation implies a decrease in demand for liquid bonds b_{1t} , together with an increase in marginal utility of consumption and higher labor supply. Therefore, the elasticity of intertemporal substitution is a crucial parameter for the determinacy region induced by a Taylor rule, differently from the case of strongly separable utility.

7 Concluding remarks

This paper studies the role of the term structure of interest rates within a simple new-Keynesian model. We investigate the determinacy conditions induced by Taylor-type monetary policy rules. We find that what really matters for determinacy is not just the inclusion of both long and short term bonds, but the way in which liquidity services are modelled. With liquid bonds entering the utility function in a weakly separable way with consumption, the requirement for the inflation coefficient in the monetary policy rule for the determination of equilibria is similar to what arises in models without bond market frictions. However, if transaction services enter in a weakly separable way, the bounds of the determinacy regions produced by the inflation coefficient of the Taylor rule becomes non-linear. In this case, the results are no longer clear-cut, since the bounds are strongly dependent on other policy parameters, as well as on the core parameters of the model.

In our model, the expectations hypothesis holds and the bond pricing scheme follows an affine structure. Under this perspective, the difference between long and short term interest rates in a log-linear approximation is due to exogenous shock hitting both the level and the slope of the resulting term structure.

Our results shed light on the role of alternative modelling frameworks for liquidity services in the definition of determinacy. The role of fiscal policy becomes also evident in affecting bond pricing through public spending shock. Considering the fiscal policy stance, as in the fiscal theory of the price level, allows a wider characterization of the equilibrium conditions in comparison with what has been proposed in the literature. In this sense, the model outlined here can be easily generalized to include money and an explicit set of transaction costs between money and multiple bonds.

A Coefficients

A.1 Model equations (sections 2-6)

$$\eta_{ca} \equiv \frac{\sigma \left(1 + \frac{1}{\eta}\right)}{S_c \sigma \left(\frac{1}{\eta} + 1 - \alpha\right) + \alpha}$$

$$\eta_{cg} \equiv \frac{S_g \sigma \left(1 + \frac{1}{\eta}\right) - \alpha}{S_c \sigma \left(\frac{1}{\eta} + 1 - \alpha\right) + \alpha}$$

$$\eta_{ya} \equiv \frac{\sigma \left(1 + \frac{1}{\eta}\right) - \alpha \eta_{ca}}{\sigma \left(1 + \frac{1}{\eta}\right)}$$

$$\eta_{yg} \equiv \frac{\alpha \eta_{cg}}{\sigma \left(1 + \frac{1}{\eta}\right)}$$

$$\eta_b = \left\{ \left[\chi \left(1 - \frac{1}{\sigma}\right) - 1 \right] \left(1 - \frac{\beta R_1}{\pi}\right) \right\}^{-1}$$

$$\eta_{\pi 1} = \frac{\phi_\pi}{\beta} + \frac{1}{\beta} \left(\frac{b \eta_b \beta}{\pi} - 1 \right) - \frac{b R_2 (1 + \eta_b) \phi_\pi}{R_1} + \frac{(1 - \psi)}{\pi} \left[b R_2 + \frac{R_2}{R_1} - b \eta_b \beta R_1 R_2 \right]$$

$$\eta_{ba2} \equiv \phi_y \eta_{ya} + \frac{b \eta_b \beta \eta_{ca}}{R_2 \sigma \pi} - \frac{\eta_{ca}}{\sigma}$$

$$\eta_{ba1} \equiv \frac{\eta_{ca}}{\sigma} - \frac{b \eta_b \eta_{ca} \pi}{\beta \sigma} + \frac{b (1 - \psi) \eta_b R_1^2 \eta_{ca}}{\sigma \pi} - \left(\frac{b \eta_b \beta}{\pi} - 1 \right) k \eta_{ya} - \phi_\pi k \eta_{ya} +$$

$$- \frac{b (1 + \eta_b) R_2 \phi_y \eta_{ya}}{R_1}$$

$$\eta_{bg2} \equiv \phi_y \eta_{yg} - \frac{b \eta_b \beta \eta_{cg}}{\sigma \pi} + \frac{\eta_{cg}}{\sigma}$$

$$\eta_{bg1} \equiv \frac{b \eta_b \eta_{cg} R_2}{\sigma R_1} - \frac{\eta_{cg}}{\sigma} + \frac{b (1 - \psi) \eta_b R_1^2 \eta_{cg}}{\sigma \pi} - \frac{g R_2}{b_2} - k \phi_\pi \eta_{yg} +$$

$$- \left(\frac{b \eta_b \beta}{\pi} - 1 \right) k \eta_{yg} - \frac{b (1 + \eta_b) R_2 \phi_y \eta_{yg}}{R_1}$$

$$\eta_{b2} \equiv \frac{R_2}{\pi} \left(\frac{1}{R_1} - \psi \right)$$

$$\eta_\pi \equiv \phi_\pi \left[\frac{b (1 - \psi)}{\pi} \eta_b R_2 + \frac{1}{\beta} \right]$$

$$\eta_{ba} \equiv \phi_y \eta_{ya} \left[\frac{b (1 - \psi)}{\pi} \eta_b R_2 + \frac{1}{\beta} \right] + \frac{b (1 - \psi) R_1 \eta_b \eta_{ca}}{\beta \sigma}$$

$$\eta_{bg} \equiv \frac{b(1-\psi)R_1\eta_b\eta_{cg}}{\beta\sigma} - \phi_y\eta_{yg} \left[\frac{b(1-\psi)}{\pi}\eta_b R_2 + \frac{1}{\beta} \right]$$

A.2 Variant discussed in Section 7

$$\begin{aligned} \alpha_{R1} &= (1-\gamma) \left(1 - \frac{1}{\sigma}\right) \eta_{bR} \\ \alpha_R &= (1-\gamma) \left(1 - \frac{1}{\sigma}\right) \eta_{bR} - 1 \\ \eta_{bR} &= \frac{(\pi - \beta R_1) - \beta R_1(1 - \beta R_1)}{(\pi - \beta R_1)(1 - \beta R_1)} \\ k &= \frac{(1-\delta)(1-\delta\beta)}{\delta} \left\{ \frac{\sigma S_c(1 + \eta(1-\alpha)) + \alpha\eta}{\sigma S_c h_s} \right\} \\ \eta_{as} &= \frac{(1+\eta)}{h_s} \left[\frac{(1-\delta)(1-\delta\beta)}{\delta} \right] \\ \eta_{gs} &= \frac{g\alpha\eta}{\sigma S_c h_s} \left[\frac{(1-\delta)(1-\delta\beta)}{\delta} \right] \\ \eta_{Rs} &= \frac{g\alpha\eta\alpha_{RS}}{h_s} \left[\frac{(1-\delta)(1-\delta\beta)}{\delta} \right] \\ h_s &= \alpha\eta(1-\theta) + \theta(1+\eta) \\ \mu_{\pi 1} &= \frac{R_2}{b_2} \left\{ \frac{b_1\eta_{bR}\phi_\pi}{R_1} - \frac{b_1\phi_\pi}{R_1} - \frac{b_2\phi_\pi}{R_2} - \frac{b_2\phi_\pi}{R_2\beta} (1 + \eta_{RS}\phi_\pi) - \frac{b_2\phi_y f_p}{R_2} \right\} \\ f_p &= \frac{1}{1 - \sigma S_c \alpha_{R1}} \left[\frac{\sigma S_c}{\beta} (\alpha_R \phi_\pi - 1) (1 + \eta_{RS} \phi_\pi) - \alpha_R \sigma S_c \phi_\pi \right] \\ f_y &= \frac{1}{1 - \sigma S_c \alpha_{R1}} \left[1 - \frac{\sigma S_c}{\beta} (\alpha_{R1} \phi_\pi - 1) (k - \eta_{RS} \phi_y) - \alpha_R \sigma S_c \phi_y \right] \\ \mu_{y1} &= \frac{R_2}{b_2} \left\{ \frac{b_1}{R_1 S_c} + \frac{b_1\eta_{bR}\phi_y}{R_1} - \frac{b_1\phi_y}{R_1} - \frac{b_2\phi_y}{R_2} - \frac{b_2\phi_\pi}{R_2\beta} (k - \eta_{RS}\phi_y) - \frac{b_2\phi_y f_y}{R_2} \right\} \\ \mu_\pi &= \frac{R_2}{b_2} \left\{ \frac{b_1(1-\psi)\eta_{bR}\phi_\pi}{\pi} - (1-\psi) \left(\frac{b_1}{\pi} + \frac{b_2}{\pi R_1} \right) - \frac{(1-\psi)b_2\phi_\pi}{\pi R_1} \right\} \\ \mu_y &= \frac{R_2}{b_2} \left\{ \frac{b_1(1-\psi)}{\pi S_c} + \frac{(1-\psi)b_1\eta_{bR}\phi_y}{\pi} - \frac{(1-\psi)b_2\phi_y}{\pi R_1} \right\} \end{aligned}$$

$$\begin{aligned}
\varphi_1 &= \frac{\sigma S_c \phi_\pi \eta_{RS} (\alpha_R \phi_\pi - 1) + \alpha_R \phi_\pi \sigma S_c (1 - \beta) - \sigma S_c}{\beta (1 - \sigma S_c \alpha_{R1} \phi_\pi)} \\
\varphi_2 &= \frac{(k - \eta_{RS} \phi_y) \sigma S_c (1 - \alpha_R \phi_\pi) + \beta (1 - \alpha_R \sigma S_c \phi_y)}{\beta (1 - \sigma S_c \alpha_{R1} \phi_\pi)} \\
\varphi_3 &= \alpha_{31} (1 + \phi_\pi \eta_{RS}) - \alpha_{32} \alpha_R \sigma S_c \phi_\pi + \mu_\pi \\
\varphi_4 &= (\eta_{RS} \phi_y - k) \alpha_{31} + \alpha_{32} (1 - \alpha_R \sigma S_c \phi_y) + \mu_y \\
\alpha_{31} &= \frac{\sigma S_c (1 - \alpha_R \phi_\pi) \mu_{y1} - (1 - \sigma S_c \alpha_{R1} \phi_y) \mu_{\pi 1}}{\beta (1 - \sigma S_c \alpha_{R1} \phi_y)} \\
\alpha_{32} &= -\frac{\mu_{y1}}{1 - \sigma S_c \alpha_{R1} \phi_y}
\end{aligned}$$

B Schur-Cohn criterion

B.1 2×2 matrix

The characteristic polynomial for a generic 2×2 matrix A is $x^2 - \text{tr}(A)x + \det(A) = 0$. From [La Salle \(1986\)](#), conditions for the two roots to lie outside the unitary circle are given by:

$$|\det(A)| > 1 \quad (115)$$

$$|\text{tr}(A)| < 1 + \det(A) \quad (116)$$

In particular, condition (116) can be split in the following two inequalities:

$$1 + \det(A) + \text{tr}(A) > 0 \quad (117)$$

$$1 + \det(A) - \text{tr}(A) > 0 \quad (118)$$

3×3 MATRIX

We collect in what follows the full set of conditions to be satisfied by a generic 3×3 matrix B to obtain one root inside and two roots outside the unit circle. The characteristic polynomial for a 3×3 matrix is:

$$P(\lambda) = \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0 \quad (119)$$

where $A_0 = -\det(B)$; $A_2 = -\text{tr}(B)$; $A_1 = -\text{tr}(B)$; $A_0 = (b_{11}b_{12} - b_{21}b_{12}) + (b_{22}b_{33} - b_{32}b_{23}) + (b_{11}b_{33} - b_{31}b_{13})$. Therefore, necessary and sufficient conditions are given by the following restrictions on the coefficients of the characteristic polynomial (119). Thus, either:

1. CASE 1

$$1 + A_2 + A_1 + A_0 < 0 \quad (120)$$

$$-1 + A_2 - A_1 + A_0 > 0 \quad (121)$$

or:

2. CASE 2

$$1 + A_2 + A_1 + A_0 > 0 \quad (122)$$

$$-1 + A_2 - A_1 + A_0 < 0 \quad (123)$$

$$A_0^2 - A_0 A_2 + A_1 - 1 > 0 \quad (124)$$

or:

3. CASE 3

$$1 + A_2 + A_1 + A_0 > 0 \quad (125)$$

$$-1 + A_2 - A_1 + A_0 < 0 \quad (126)$$

$$A_0^2 - A_0 A_2 + A_1 - 1 < 0 \quad (127)$$

$$|A_2| > 3 \quad (128)$$

C Proofs

C.1 Proof of proposition 1

From the log-linearization of the First Order Condition on labor (5) and the production function, we find:

$$y_t = \left(\frac{1 + \frac{1}{\eta}}{1 + \frac{1}{\eta} - \alpha} \right) a_t - \frac{\alpha}{\sigma \left(\frac{1}{\eta} + 1 - \alpha \right)} c_t \quad (129)$$

Given (129) and the resource constraint log-linearized (see the technical appendix for details), we obtain the following equations linking consumption to the core shock hitting the economy:

$$c_t = \eta_{ca} a_t - \eta_{cg} g_t \quad (130)$$

where coefficients were reported in Appendix 1. Taking advantage of (130) we can also define the output equation, as follows:

$$y_t = \eta_{ya} a_t - \eta_{yg} g_t \quad (131)$$

with the coefficients η_{ya} , η_{yg} in Appendix 1. The log-linearized equation for liquid bond is:

$$b_{1t} = \eta_b \lambda_t - \eta_b R_{1t} + \eta_b \frac{\beta R_1}{\pi} \pi_{t+1} - \eta_b \frac{\beta R_1}{\pi} \lambda_{t+1} \quad (132)$$

Moreover, the log-linearized version of the Taylor rule with contemporaneous inflation and output targeting is:

$$R_{1t} = \phi_\pi \pi_t + \phi_y \eta_{ya} a_t - \phi_y \eta_{yg} g_t \quad (133)$$

after having substituted out for (131). Moreover, after further substitutions, equation (132) can be rewritten as follows:

$$b_{1t} = -\frac{\eta_b \eta_{ca}}{\sigma} a_t + \frac{\eta_b \eta_{cg}}{\sigma} g_t - \eta_b R_{1t} + \frac{\eta_b \beta R_1}{\pi} \pi_{t+1} + \eta_b \frac{\beta R_1 \eta_{ca}}{\sigma \pi} a_{t+1} - \eta_b \frac{\beta R_1 \eta_{cg}}{\sigma \pi} g_{t+1} \quad (134)$$

From FOC with respect to b_{2t} , after rearrangement, we get the following expression for R_{2t} :

$$R_{2t} = -\frac{1}{\sigma} \eta_{ca} a_t + \frac{1}{\sigma} \eta_{cg} g_t + \frac{1}{\sigma} \eta_{ca} a_{t+1} - \frac{1}{\sigma} \eta_{cg} g_{t+1} + \pi_{t+1} - R_{1t+1} \quad (135)$$

From Aggregate Supply function, we have:

$$\pi_{t+1} = \frac{1}{\beta} \pi_t - \frac{k}{\beta} \eta_{ya} a_t - \frac{k}{\beta} \eta_{yg} g_t \quad (136)$$

Let the ratio between liquid and illiquid bonds to be: $b = b_1/b_2$. Taking advantage of both (43) and (133) together with their respective forward versions we obtain the following expression for the government budget constraint:

$$\begin{aligned} b_{2t+1} + \eta_{\pi 1} \pi_{t+1} + \eta_{ba2} a_{t+2} + \eta_{ba1} a_{t+1} + \eta_{bg2} g_{t+2} + \eta_{bg1} g_{t+1} = \\ = \eta_{b2} b_{2t} - \eta_\pi \pi_t - \eta_{ba} a_t - \eta_{bg} g_t \end{aligned} \quad (137)$$

Q.E.D.

C.2 Proof of proposition 2

In order to achieve a solution for the full model, we start by noting that equation (43) does not depend on b_{2t} . Therefore, we can solve for π_t from (43) and then substitute out into (44) to solve for b_{2t} . From (43), we note immediately that $\beta^{-1} > 1$. This implies an explosive root. Therefore, following Sargent (1979), we can solve (43) as follows:

$$\pi_{t+1} = \frac{-\left(\frac{1}{\mu_2}\right) L^{-1}}{\left(1 - \frac{1}{\mu_2} L^{-1}\right)} [\eta_{ya} a_t + \eta_{yg} g_t] \quad (138)$$

Where μ_2 is the explosive root of (43). By applying lag polynomial L to (138) we find:

$$\pi_{t+1} = \frac{-\left(\frac{1}{\mu_2}\right)}{\left(1 - \frac{1}{\mu_2} L^{-1}\right)} [\eta_{ya} a_{t+1} + \eta_{yg} g_{t+1}] \quad (139)$$

By using the definition of lag polynomial as in [Sargent \(1979\)](#):

$$\begin{aligned}\pi_{t+1} &= -\frac{k}{\mu_2} \sum_{i=0}^{\infty} \left(\frac{1}{\mu_2}\right)^i [\eta_{ya}a_{t+1} + \eta_{yg}g_{t+1}] = \\ &= \frac{k\eta_{ya}}{\mu_2} \sum_{i=0}^{\infty} \left(\frac{1}{\mu_2}\right)^i a_{t+i+1} + \frac{k\eta_{yg}}{\mu_2} \sum_{i=0}^{\infty} \left(\frac{1}{\mu_2}\right)^i g_{t+i+1}\end{aligned}\tag{140}$$

Applying the definition of stochastic processes for a_t and g_t given in [\(24\)](#) and [\(38\)](#) and developing the series in [\(140\)](#).

$$\begin{aligned}\pi_{t+1} &= \frac{k\eta_{ya}}{\mu_2} \left[\frac{(1-\rho_a)a}{1-\frac{1}{\mu_2}} + \sum_{i=0}^{\infty} \left(\frac{\rho_a}{\mu_2}\right)^i a_{t+1} \right] + \\ &+ \frac{k\eta_{yg}}{\mu_2} \left[\frac{(1-\rho_g)g}{1-\frac{1}{\mu_2}} + \sum_{i=0}^{\infty} \left(\frac{\rho_g}{\mu_2}\right)^i g_{t+1} \right]\end{aligned}\tag{141}$$

Thus, applying the formula to compute the sum of infinite terms:

$$\begin{aligned}\pi_{t+1} &= \frac{k\eta_{ya}}{\mu_2} \frac{(1-\rho_a)a\mu_2}{(\mu_2-1)} + \frac{k\eta_{ya}}{\mu_2} \left(\frac{\mu_2}{\mu_2-\rho_a}\right) a_{t+1} + \\ &+ \frac{k\eta_{yg}}{\mu_2} \frac{(1-\rho_g)g\mu_2}{(\mu_2-1)} + \frac{k\eta_{yg}}{\mu_2} \left(\frac{\mu_2}{\mu_2-\rho_g}\right) g_{t+1}\end{aligned}\tag{142}$$

Thus, simplifying and considering the definition of the root $\mu_2 = \beta^{-1}$, after rearranging, we get the following solution for π_t :

$$\pi_t = f_\pi + \alpha_a a_t + \alpha_g g_t\tag{143}$$

where:

$$\begin{aligned}f_\pi &= \frac{k\eta_{ya}(1-\rho_a)a\beta}{(1-\beta)} + \frac{k\eta_{yg}(1-\rho_g)g\beta}{(1-\beta)} \\ \alpha_a &= \frac{k\eta_{ya}\beta}{(1-\beta\rho_g)} \\ \alpha_g &= \frac{k\eta_{yg}\beta}{(1-\beta\rho_g)}\end{aligned}$$

To solve for b_{2t} , insert the solution for π_t from [\(143\)](#) together with the expression for [\(24\)](#) and [\(38\)](#) into [\(44\)](#). After rearranging, we get the following expression for b_{2t} ready to be solved:

$$b_{2t+1} = h + \eta_{b2}b_{2t} - \delta_a a_{t+1} - \delta_g g_{t+1}\tag{144}$$

where:

$$\begin{aligned}
\gamma_0 &\equiv \eta_{\pi 1} \alpha_a + \eta_{ba 2} (1 - \rho_a) a + \eta_{bg 2} (1 - \rho_g) g + \eta_{\pi} f_{\pi} \\
h &\equiv \frac{\gamma_a (1 - \rho_a) a}{\rho_a} + \frac{\gamma_g (1 - \rho_g) g}{\rho_g} - \gamma_0 \\
\delta_a &\equiv \eta_{\pi 1} \alpha_a + \eta_{ba 2} \rho_a + \eta_{ba 1} + \frac{\gamma_a}{\rho_a} \\
\delta_g &\equiv \eta_{\pi 1} \alpha_g + \eta_{ba 2} \rho_g + \eta_{bg 1} + \frac{\gamma_g}{\rho_g}
\end{aligned}$$

By applying standard methods, we can rewrite and solve (144) as follows (see [Sargent \(1979\)](#)):

$$\begin{aligned}
b_{2t+1} &= \frac{h}{1 - \eta_{b 2} L} - \frac{1}{1 - \eta_{b 2} L} (\delta_a a_{t+1} + \delta_g g_{t+1}) = \\
&= \frac{h}{1 - \eta_{b 2}} - \delta_a \sum_{i=0}^{\infty} \eta_{b 2}^i a_{t-i+1} - \delta_g \sum_{i=0}^{\infty} \eta_{b 2}^i g_{t-i+1} = \\
&= \frac{h}{1 - \eta_{b 2}} - \delta_a \sum_{i=0}^{\infty} (\rho_a \eta_{b 2})^i a_{t+1} - \delta_g \sum_{i=0}^{\infty} (\rho_g \eta_{b 2})^i g_{t+1}
\end{aligned} \tag{145}$$

Where in the last line we took advantage of (24) and (38). Finally, by applying the infinite sum of series, the solution to equation (145) can be rewritten as:

$$b_{2t+1} = \frac{h}{1 - \eta_{b 2}} - \frac{\delta_a}{1 - \rho_a \eta_{b 2}} a_t - \frac{\delta_g}{1 - \rho_g \eta_{b 2}} g_t \tag{146}$$

Therefore, the solution of system is fully captured by equations (143) and (146). We can now solve explicitly for the pricing kernel. Thus, taking advantage of the definitions of shocks, the first order conditions on consumption log-linearized, the resource constraint log-linearized and the definition of the stochastic processes, we get:

$$\begin{aligned}
c_{t+1} - c_t &= \eta_{ca} (a_{t+1} - a_t) - \eta_{cg} (g_{t+1} - g_t) = \\
&= \eta_{ca} (1 - \rho_a) a + \eta_{ca} (\rho_a - 1) a + \eta_{ca} a_t^{1/2} \sigma_a \epsilon_{t+1}^a - \eta_{cg} (1 - \rho_g) g + \\
&\quad - \eta_{cg} (\rho_g - 1) g - \eta_{cg} g_t^{1/2} \sigma_g
\end{aligned} \tag{147}$$

Using (147) and (143) lagged forward for π_{t+1} , after using again the shocks definition, we get the solution for the kernel equation (21) as a function of the exogenous forces of the system a_t, g_t reported in the text.

Q.E.D.

C.3 Proof of proposition 3

Let us start with $k = 1$. To find the price of liquid, short term bond, let us set $p_{t+1}^0 = 0$ in (52) to get:

$$\begin{aligned} p_t^1 &= \log E_t \exp(m_{t+1} + p_{t+1}^0) = \log E_t \exp(m_{t+1}) = \\ &= E_t m_{t+1} + \frac{1}{2} \text{var}_t(m_{t+1}) = \\ &= -A_1 - B_1 a_t - C_1 g_t \end{aligned} \quad (148)$$

Now, by using (50) together with (148):

$$-A_1 - B_1 a_t - C_1 g_t = -\lambda_0 - \left(\lambda_1 + \frac{\eta_1^2}{2} \sigma_a^2 \right) a_t + \left(\lambda_2 - \frac{\eta_2^2}{2} \sigma_g^2 \right) g_t \quad (149)$$

After equating coefficients it is immediate to get the definitions of A_1, B_1, C_1 , as stated in the text.

For $k > 1$, we need to write coefficients of k as functions of coefficients at $k - 1$. We can rewrite (52) as follows:

$$p_{t+1}^{k-1} = -A_{k-1} - B_{k-1} a_{t+1} - C_{k-1} g_{t+1} \quad (150)$$

Thus, by applying the definition of (24) and (38), we get:

$$\begin{aligned} p_{t+1}^{k-1} &= -A_{k-1} - B_{k-1} \left[(1 - \rho_a) a + \rho_a a_t + a_t^{1/2} \sigma_a \epsilon_{t+1}^a \right] - \\ &\quad + C_{k-1} \left[(1 - \rho_g) g + \rho_g g_t + g_t^{1/2} \sigma_g \epsilon_{t+1}^g \right] \end{aligned} \quad (151)$$

Therefore, by using (48)-(49) together with (151)

$$\begin{aligned} E_t \left(m_{t+1} + p_{t+1}^{k-1} \right) &= \lambda_0 + \lambda_1 a_t - \lambda_2 g_t - A_{k-1} - B_{k-1} (1 - \rho_a) a - B_{k-1} \rho_a a_t - \\ &\quad + C_{k-1} (1 - \rho_g) g - C_{k-1} \rho_g g_t \end{aligned} \quad (152)$$

$$\text{Var}_t \left(m_{t+1} + p_{t+1}^{k-1} \right) = \frac{1}{2} \left[\sigma_a^2 (\eta_1^2 + B_{k-1}^2) a_t + \sigma_g^2 (\eta_2^2 - C_{k-1}^2) g_t \right] \quad (153)$$

Therefore, given lognormality, we can combine bond pricing (51)-(52) with (152)-(153):

$$\begin{aligned} -A_k - B_k a_t - C_k g_t &= \lambda_0 + \lambda_1 a_t - \lambda_2 g_t - A_{k-1} - B_{k-1} (1 - \rho_a) a - \\ &\quad + B_{k-1} \rho_a a_t - C_{k-1} (1 - \rho_g) g - C_{k-1} \rho_g g_t + \\ &\quad + \frac{\sigma_a^2}{2} (\eta_1^2 + B_{k-1}^2) a_t + \frac{\sigma_g^2}{2} (\eta_2^2 - C_{k-1}^2) g_t \end{aligned} \quad (154)$$

By equating coefficients, we obtain the claim of the proposition.
Q.E.D.

C.4 Proof of proposition 4

We divide the proof in three parts, each relative to a specific monetary rule (75)- (77).

C.4.1 Current inflation rule (75)

With the inclusion of (75) in (43),(44) and (72) we can set the system in the form (74), where matrix Γ is defined as:

$$\Gamma = \begin{bmatrix} \beta^{-1} & -k\beta^{-1} & 0 \\ -(\sigma S_c \beta^{-1} + \sigma S_c \phi_\pi) & (1 + k\sigma S_c \beta^{-1}) & 0 \\ \alpha_{\gamma 1} & -\alpha_{\gamma 2} & \beta^{-1}(1 - \psi) \end{bmatrix} \quad (155)$$

Coefficients are given by:

$$\alpha_{\pi 1} = \frac{1}{\beta} \left(\frac{bR_1\eta_b}{\beta} - 1 \right) + \frac{\phi_\pi}{\beta} (1 - b\pi - bR_1\eta_b) + \\ + R_1(1 - \psi) \left[\frac{1}{\beta} \left(b + \frac{1}{R_1} \right) - \frac{bR_1\eta_b\eta_{cg}}{\pi} \right]$$

$$\alpha_y = \frac{k}{\beta} \left[1 - \frac{bR_1\eta_b}{\beta} - \phi_\pi \right]$$

$$\alpha_\pi = -\frac{(1 - \psi)\phi_\pi}{\beta}$$

$$\alpha_{\gamma 1} = \frac{1}{\beta} (\sigma S_c \alpha_y - \alpha_{\pi 1}) - \alpha_y \sigma S_c \phi_\pi - \alpha_\pi$$

$$\alpha_{\gamma 2} = \frac{k}{\beta} (\sigma S_c \alpha_y - \alpha_{\pi 1}) + \alpha_y$$

To get determinacy, we need two roots of matrix (155) to be outside the unit circle, and one inside since public debt b_{2t} is a predetermined variable. Since (155) is upper triangular, we can concentrate on the submatrix A_{11} given by:

$$A_{11} = \begin{bmatrix} \beta^{-1} & -k\beta^{-1} \\ -(\sigma S_c \beta^{-1} + \sigma S_c \phi_\pi) & (1 + k\sigma S_c \beta^{-1}) \end{bmatrix} \quad (156)$$

Trace and determinant of submatrix A_{11} are, respectively:

$$\text{tr}(A_{11}) = 1 + \frac{1}{\beta} - \frac{\sigma k S_c}{\beta} \quad (157)$$

$$\det(A_{11}) = \frac{1 + k\sigma S_c \phi_\pi}{\beta} \quad (158)$$

It is immediate to verify that condition (115) is automatically verified if $\phi_\pi > 0$. In the same way, condition (117) is verified if $\phi_\pi > 1$, while by setting $\phi_\pi > 0$ it is sufficient to verify condition (118). This ensures that submatrix A_{11} has one root inside and one outside the unit circle. To get another root inside the unit circle we need the following

condition to be satisfied:

$$\frac{|1 - \psi|}{\beta} < 1 \quad (159)$$

which delivers the result stated in the text.

When both conditions $\phi_\pi > 1$ and (159) are satisfied, two roots are inside and one is outside the unit circle. If, on the other hand, $\phi_\pi < 1$, then two roots will already be inside the unit circle. Therefore, to get one root inside the unit circle, we require the following additional condition on fiscal policy:

$$\frac{|1 - \psi|}{\beta} > 1 \quad (160)$$

which implies:

$$\psi < 1 - \beta \quad \psi > 1 + \beta \quad (161)$$

This proves the result for the current absolute inflation targeting rule (75).
Q.E.D.

C.4.2 Expected inflation rule (76)

Given the monetary rule under (76), matrix Γ and B in the system (74) is given by:

$$\Gamma = \begin{bmatrix} \beta^{-1} & -k\beta^{-1} & 0 \\ -\sigma S_c \beta^{-1} (1 - \phi_\pi) & 1 - k\sigma S_c \beta^{-1} (1 - \phi_\pi) & 0 \\ \alpha_{31} & -k\alpha_{31} + \alpha_y & \beta^{-1} (1 - \psi) \end{bmatrix} \quad (162)$$

where:

$$\begin{aligned} \alpha_{\pi 3} &= bR_1\eta_b - 1 - \frac{2b\pi\phi_\pi}{\beta} + \frac{\phi_\pi}{\beta} \left[1 - \frac{k}{\beta} \sigma S_c (\phi_\pi - 1) \right] + \frac{R_1(1 - \psi)}{\beta} \left(b + \frac{1}{R_1} \right) + \\ &\quad + \frac{(1 - \psi)}{\beta} \phi_\pi + \frac{b(1 - \psi) R_1 \eta_b}{\beta} \phi_\pi - \frac{bR_1^2}{\beta\pi} (1 - \psi) \eta_b \beta \eta_{cg} \\ \alpha_{y2} &= \frac{k\phi_\pi}{\beta} \left[1 + \frac{1}{\beta} - \frac{\sigma S_c k (\phi_\pi - 1)}{\beta} \right] + \frac{k}{\beta} \left(bR_1\eta_b - 1 - \frac{2b\pi\phi_\pi}{\beta} \right) \\ \alpha_{31} &= -\frac{\sigma S_c (1 - \phi_\pi) \alpha_{y2} + \alpha_{\pi 3}}{\beta} \end{aligned}$$

Again, we observe that the structure of matrix Γ in (162) is upper-left triangular. We can then concentrate on the submatrix A_{11} given by:

$$A_{11} = \begin{bmatrix} \beta^{-1} & -k\beta^{-1} \\ -\sigma S_c \beta^{-1} (1 - \phi_\pi) & 1 - k\sigma S_c \beta^{-1} (1 - \phi_\pi) \end{bmatrix} \quad (163)$$

To get determinacy for the full system we require that two eigenvalues of the system to be outside the unitary circle and one inside, since public debt b_{2t} is a predetermined variable. Trace and determinant of submatrix (163) are given, respectively, by:

$$\text{tr}(A_{11}) = 1 + \frac{1}{\beta} - \frac{\sigma k S_c (1 - \phi_\pi)}{\beta} \quad (164)$$

$$\det(A_{11}) = \frac{1}{\beta} \quad (165)$$

From the Schur-Cohn criterion, it is immediate to check that condition (115) is fully satisfied. Condition (117) implies:

$$\sigma k S_c (\phi_\pi - 1) + 2(1 + \beta) > 0 \quad (166)$$

which is satisfied if and only if $\phi_\pi > 1$. From (118), we get:

$$\sigma k S_c (1 - \phi_\pi) > 0 \quad (167)$$

which can be rewritten as:

$$-\sigma k S_c (\phi_\pi - 1) < 0 \quad (168)$$

which is satisfied if and only if $\phi_\pi > 1$, as well.

The remaining part of the proof follows exactly the same steps described for the current inflation targeting rule outlined earlier. *Q.E.D.*

C.5 Proof of proposition 5

Rule (77) requires a different setting for the analysis, given the time-indexing of the system. Therefore, let us define the vector $Z_t = [R_{1t}, \pi_t, y_t, b_{2t}]'$. Matrix Γ in this case is:

$$\Gamma = \begin{bmatrix} 0 & \phi_\pi & 0 & 0 \\ 0 & \frac{1}{\beta} & -\frac{k}{\beta} & 0 \\ \sigma S_c & -\frac{\sigma S_c}{\beta} & \left(1 + \frac{k\sigma S_c}{\beta}\right) & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \frac{1-\psi}{\beta} \end{bmatrix} \quad (169)$$

where:

$$\alpha_{\pi 4} = \frac{bR_1\eta_b}{\pi\beta} - \frac{1}{\beta} + \phi_\pi + \frac{(1-\psi)}{\beta} \left(b + \frac{1}{R_1}\right) - \frac{bR_1^2(1-\psi)}{\pi} \eta_b \eta_{cg}$$

$$\alpha_{y 3} = \frac{k}{\beta} \left[\frac{(1-\psi)}{\beta} \left(b + \frac{1}{R_1}\right) - 1 + \frac{b\eta_b R_1}{\pi} \right]$$

$$\alpha_{R 1} = \frac{\pi b \eta_b}{\beta} + \frac{b R_1}{\beta}$$

$$\alpha_R = \frac{(1-\psi)}{\beta} \left(1 + \frac{b R_1 \eta_b}{\pi}\right)$$

$$\begin{aligned}
\alpha_{41} &= \alpha_{y3}\sigma S_c - \alpha_R \\
\alpha_{42} &= \phi_\pi \alpha_{R1} - \frac{1}{\beta} (\sigma S_c \alpha_{y3} + \alpha_{\pi 4}) \\
\alpha_{43} &= \alpha_{y3} + \frac{k}{\beta} (\sigma S_c \alpha_{y3} + \alpha_{\pi 4})
\end{aligned}$$

To get determinacy, we need two roots of the matrix (169) to be outside the unit circle and two inside. Given the upper block-triangular structure of the matrix Γ in (169), we can concentrate on the submatrix Γ_{11} , given by:

$$\Gamma_{11} = \begin{bmatrix} 0 & \phi_\pi & 0 \\ 0 & \frac{1}{\beta} & -\frac{k}{\beta} \\ \sigma S_c & -\frac{\sigma S_c}{\beta} & \left(1 + \frac{k\sigma S_c}{\beta}\right) \end{bmatrix} \quad (170)$$

We apply the Schur-Cohn criterion in order to detect the presence of two roots inside and one outside the unit circle for submatrix Γ_{11} in (170). In this case, $A_0 = \beta^{-1}$, $A_1 = \beta^{-1}$, $A_2 = -(1 + \beta^{-1} + \beta^{-1}\sigma k S_c)$. It is then immediate to check that by applying conditions (120)-(121) for Case 1 we get a contradiction. Therefore, from condition (122) of Case 2, we get (after simplifying):

$$\sigma S_c k (\phi_\pi - 1) > 0 \quad (171)$$

which is satisfied if and only if $\phi_\pi > 1$. On the other hand, from condition (123) we find:

$$-2(1 + \beta) + \sigma S_c k (\phi_\pi - 1) < 0 \quad (172)$$

which is certainly satisfied if and only if:

$$\phi_\pi < 1 + \frac{2(1 + \beta)}{\sigma S_c k} \quad (173)$$

Finally, in order to verify condition (124), after substituting out the definitions for A_0 , A_1 , A_2 given previously and simplifying, we need to check if the following inequality is satisfied:

$$\frac{\phi_\pi^2 k^2 \sigma^2 S_c^2}{\beta} + \phi_\pi k \sigma S_c \left(1 + \frac{1}{\beta} + \frac{\sigma S_c k}{\beta}\right) + (1 - \beta) > 0 \quad (174)$$

which is certainly satisfied since $\beta < 1$. These conditions ensures we have two roots inside and one outside the unit circle. To get another root inside the unit circle, condition (159) needs to be satisfied. As we have seen before, this implies condition (79), stated in the text.

On the other hand, if condition $\phi_\pi > 1$ is not satisfied, then the same reasoning applied in the proof of Proposition 2 can be repeated here without any further change.

Q.E.D.

C.6 Proof of proposition 6

With rule (86) matrix Γ of the system (74) can be written as:

$$\Gamma = \begin{bmatrix} \frac{1}{\beta} & -\frac{k}{\beta} & 0 \\ \sigma S_c \left(\phi_\pi - \frac{1}{\beta} \right) & \frac{k\sigma S_c}{\beta} + 1 + \sigma S_c \phi_y & 0 \\ \gamma_{31} & \gamma_{32} & \frac{(1-\psi)}{\beta} \end{bmatrix} \quad (175)$$

where:

$$\gamma_{y1} = \phi_y \left(1 + \frac{k\sigma S_c}{\beta} + \sigma S_c \phi_y \right) - \frac{b\pi\phi_y}{\beta} (1 + \eta_b) + \frac{k}{\beta} (1 - bR_1\eta_b) - \frac{k\phi_\pi}{\beta}$$

$$\gamma_{y2} = \frac{(1-\psi)}{\beta} (1 + bR_1\eta_b) \phi_y$$

$$\begin{aligned} \gamma_{\pi1} = & \frac{b\eta_b}{\beta} (R_1 - \pi\phi_\pi) + \frac{\phi_\pi}{\beta} (1 - b\pi) + \phi_y \sigma S_c \left(\phi_\pi - \frac{\sigma S_c}{\beta} \right) \frac{1}{\beta} + \\ & + (1-\psi) R_1 \left(\frac{1}{\beta} \left(b + \frac{1}{R_1} \right) - \frac{b}{\pi} \eta_b R_1 \eta_{cg} \right) \end{aligned}$$

$$\gamma_{\pi2} = \frac{(1-\psi)}{\beta} (1 + bR_1\eta_b) \phi_\pi$$

$$\gamma_{31} = \frac{1}{\beta} (\sigma S_c \gamma_{y1} - \gamma_{\pi1}) - \gamma_{y1} \sigma S_c \phi_\pi - \gamma_{\pi2}$$

$$\gamma_{32} = -\frac{k}{\beta} (\sigma S_c \gamma_{y1} - \gamma_{\pi1}) - \gamma_{y1} (1 + \sigma S_c \phi_y) - \gamma_{y2}$$

Even in this case, to get determinacy we need two roots of matrix (175) outside and one inside the unit circle. Given the upper-left triangular structure of the matrix Γ in (175), we can concentrate on the 2×2 submatrix G_{11} , here given by:

$$G_{11} = \begin{bmatrix} \frac{1}{\beta} & -\frac{k}{\beta} \\ \sigma S_c \left(\phi_\pi - \frac{1}{\beta} \right) & \left(\frac{k\sigma S_c}{\beta} + 1 + \sigma S_c \phi_y \right) \end{bmatrix} \quad (176)$$

Therefore, trace and determinant of matrix (176) are, respectively, given by:

$$\text{tr}(G_{11}) = 1 + \frac{1}{\beta} + \frac{\sigma k S_c}{\beta} + \sigma k S_c \phi_y \quad (177)$$

$$\det(G_{11}) = \frac{1 + \sigma S_c (\phi_y - k\phi_\pi)}{\beta} \quad (178)$$

From Schur-Cohn criterion, condition (115) can be split in two parts: i) $\det > 1$ is immediately satisfied, given the assumption $\phi_\pi, \phi_y > 0$; ii) $\det > -1$, identifies the following bound:

$$k\phi_\pi + \phi_y > \frac{1-\beta}{\sigma S_c \beta} \quad (179)$$

Condition (117) is immediately satisfied, given that ϕ_π, ϕ_y are assumed to be positive. On the other hand, condition (118) directly implies:

$$k(\phi_\pi - 1) + \phi_y(1 - \beta) > 0 \quad (180)$$

With conditions (179) and (180) we have that one root of submatrix G_{11} in (176) will be inside and another root will be outside the unit circle. To get determinacy for the full system subsumed by matrix (175) we need another root to be inside the unit circle. This is obtained by considering condition (159): this implies condition (89), stated in the text. However, if conditions (179)-(180) are violated, then the same reasoning applied in the proof of Proposition 2 applies here.

Q.E.D.

C.7 Proof of proposition 7

With rule (92) matrix Γ of system (74) is now given by:

$$\Gamma = \begin{bmatrix} \frac{1}{\beta} & -\frac{k}{\beta} & 0 \\ \frac{\sigma S_c(1-\phi_\pi)}{\beta} & \frac{k\sigma S_c}{\beta} + 1 + \sigma S_c\phi_y & 0 \\ \lambda_{31} & \lambda_{32} & \frac{(1-\psi)}{\beta} \end{bmatrix} \quad (181)$$

where:

$$\begin{aligned} \lambda_\pi &= \frac{\phi_\pi}{\beta^2} [1 + k\sigma S_c(1 - \phi_\pi)] + \frac{bR_1(1 - \psi)}{\beta\pi} \eta_b\phi_\pi - \frac{\phi_y\sigma S_c(1 - \phi_\pi)}{\beta} - \frac{bR_1^2(1 - \psi)\eta_b\eta_{cg}}{\pi} + \\ &+ \frac{b\eta_b R_1}{\beta\pi} - \frac{b\pi\eta_b}{\beta^2}\phi_\pi - \frac{bR_1}{\beta^2}\phi_\pi - \frac{1}{\beta} + \frac{(1 - \psi)}{\beta^2} \left(b + \frac{1}{R_1} \right) + \frac{(1 - \psi)}{\beta^2} \\ \lambda_{y1} &= \frac{\pi b\eta_b\phi_y}{\beta} + \frac{bR_1\phi_y}{\beta} + \frac{(1 - \psi)}{\beta} (\phi_y + k\phi_\pi) - \frac{k\pi b\eta_b\phi_\pi}{\beta^2} + \frac{k\pi b\eta_b\phi_\pi}{\beta\pi} - \frac{kbR_1\phi_\pi}{\beta^2} - \\ &+ \frac{k(1 - \pi)}{\beta} \left(b + \frac{1}{R_1} \right) \frac{k\phi_\pi}{\beta^2} + \left(\frac{k\phi_\pi}{\beta} - \phi_y \right) \left[1 + \sigma S_c\phi_y + \frac{k\sigma S_c(1 - \phi_\pi)}{\beta} \right] \\ \lambda_y &= \frac{bR_1(1 - \psi)\eta_b\phi_y}{\beta\pi} \\ \lambda_{31} &= -\frac{\sigma S_c(1 - \phi_\pi)\lambda_{y1} + \lambda_\pi}{\beta} \end{aligned}$$

To get determinacy, we need two roots of matrix (181) to be outside and one inside the unit circle. Once again, given the upper-left triangular structure of (181) we can concentrate on the 2×2 submatrix H_{11} , here given by:

$$H_{11} = \begin{bmatrix} \frac{1}{\beta} & -\frac{k}{\beta} \\ \frac{\sigma S_c(1-\phi_\pi)}{\beta} & \left(\frac{k\sigma S_c}{\beta} + 1 + \sigma S_c\phi_y \right) \end{bmatrix} \quad (182)$$

Trace and determinant of H_{11} in (182) are, respectively, given by:

$$\text{tr}(H_{11}) = 1 + \frac{1}{\beta} + \frac{\sigma k S_c (1 - \phi_\pi)}{\beta} + \sigma S_c \phi_y \quad (183)$$

$$\det(H_{11}) = \frac{1 + \sigma S_c \phi_y}{\beta} \quad (184)$$

Condition (115) is immediately satisfied, given that $\phi_\pi, \phi_y > 0$. Condition (117) implies the following upper bound:

$$\phi_\pi = 1 + \frac{2(1 + \beta)}{k\sigma S_c} + \phi_y \frac{(1 + \beta)}{k} \quad (185)$$

On the other hand, condition (118) implies:

$$k(\phi_\pi - 1) + \phi_y(1 - \beta) > 0 \quad (186)$$

which is certainly satisfied if $\phi_\pi > 1$. With conditions (185) and (186), one root of submatrix H_{11} in (186) will be inside and another root outside the unit circle. To get determinacy for the full system we need the additional conditions on fiscal policy, which will capture the position of the third root. Implementing condition (159) will imply (94), stated in the text.

When one of (185) or (186), or both, are violated, then the same reasoning applied in the proof of Propositions 2-4 applies here, originating bounds (95).

Q.E.D.

C.8 Proof of proposition 8

With rule (42) matrix Γ of system (XX) is now given by:

$$\Delta = \begin{bmatrix} \delta_{11} & \delta_{12} & -\delta_{13} & 0 \\ 0 & \frac{1}{\beta} & -\frac{k}{\beta} & 0 \\ \sigma S_c & \frac{\sigma S_c}{\beta} & \left(1 + \frac{k\sigma S_c}{\beta}\right) & 0 \\ -\delta_{41} & \delta_{42} & \delta_{43} & \frac{(1-\psi)}{\beta} \end{bmatrix} \quad (187)$$

where vector Z_t is given by as $Z_t = [R_{1t}, \pi_t, y_t, b_{2t}]'$. where:

$$\begin{aligned}
\delta_{11} &= \rho + \phi_y \sigma S_c \\
\delta_{12} &= \frac{\phi_\pi - \phi_y \sigma S_c}{\beta} \\
\delta_{13} &= \frac{k}{\beta} (\phi_\pi - \phi_y \sigma S_c) + \phi_y \\
\delta_{41} &= \left[\phi_y \sigma S_c + \rho - \frac{b}{\beta} (\pi \eta_b + R_1) \right] (\rho + \phi_y \sigma S_c) + \\
&\quad + \delta_y \sigma S_c + \frac{(1-\psi)}{\beta} \left(1 + \frac{b R_1}{\pi} \eta_b \right) \\
\delta_{42} &= \frac{1}{\beta} [\sigma S_c \delta_y - (\phi_\pi - \phi_y \sigma S_c) \delta_{\rho 1} - \delta_{\pi 1}] \\
\delta_y &= \phi_y - \frac{k}{\beta} \left[\frac{b \eta_b R_1}{\pi} - 1 + \frac{(1-\psi)}{\beta} \left(b + \frac{1}{R_1} \right) - \phi_y \sigma S_c \right] \\
\delta_{\rho 1} &= \rho + \phi_y \sigma S_c - \frac{b}{\beta} (\pi \eta_b + R_1) \\
\delta_{\pi 1} &= \frac{1}{\beta} \left[\frac{b \eta_b R_1}{\pi} - 1 + \frac{(1-\psi)}{\beta} \left(b + \frac{1}{R_1} \right) - \phi_y \sigma S_c \right] - \frac{b R_1^2 (1-\psi) \eta_b \eta_{cg}}{\pi}
\end{aligned}$$

Matrix (187) is upper-left triangular. In this case, to get determinacy we need two roots inside and two outside the unit circle, since both R_{1t} and b_{2t} are predetermined. We can start by focusing on the 3×3 submatrix D_{11} given by:

$$D_{11} = \begin{bmatrix} \delta_{11} & \delta_{12} & -\delta_{13} \\ 0 & \frac{1}{\beta} & -\frac{k}{\beta} \\ \sigma S_c & \frac{\sigma S_c}{\beta} & \left(1 + \frac{k \sigma S_c}{\beta} \right) \end{bmatrix} \quad (188)$$

By applying apply now the Schur-Cohn criterion for 3×3 matrix, from (119), we have:

$$A_0 = -\frac{\rho}{\beta} \quad (189)$$

$$A_1 = \frac{1}{\beta} + \rho + \frac{\rho}{\beta} + \rho \frac{k \sigma S_c}{\beta} + \frac{\phi_y \sigma S_c}{\beta} + \phi_\pi \frac{k \sigma S_c}{\beta} \quad (190)$$

$$A_2 = \rho + \phi_y \sigma S_c + \frac{1}{\beta} + 1 + \frac{k \sigma S_c}{\beta} \quad (191)$$

Let us start with Case I. From condition (120), we get: $k(\phi_\pi + \rho - 1) + (1 - \beta)\phi_y < 0$. Moreover, from condition ((121)), we get a contradiction, given our assumptions on parameters' sign. Consider now Case II. Condition (122) implies:

$$k(\phi_\pi + \rho - 1) + (1 - \beta)\phi_y > 0 \quad (192)$$

which is satisfied if $\phi_\pi > 1$ condition (123) is immediately satisfied. By applying condition (124), we get:

$$\begin{aligned} \frac{\rho^2}{\beta^2} - \frac{\rho}{\beta} \left(\rho + \phi_y \sigma S_c + \frac{1}{\beta} + 1 + \frac{k\sigma S_c}{\beta} \right) + \frac{\rho}{\beta} + \rho + \\ + \frac{1}{\beta} + \frac{k\sigma S_c}{\beta} (\phi_\pi + \rho) + \phi_y \frac{\sigma S_c}{\beta} - 1 > 0 \end{aligned} \quad (193)$$

After adding and subtracting $\frac{k\sigma S_c}{\beta}$ to (193) and rearrange, we get:

$$\begin{aligned} \phi_y \frac{\sigma S_c}{\beta} (1 - \rho) + \frac{k\sigma S_c}{\beta} (\phi_\pi + \rho - 1) + \frac{k\sigma S_c}{\beta} \frac{(\beta - \rho)}{\beta} + \\ + \left[\frac{\rho^2}{\beta^2} - \frac{\rho^2}{\beta} + \rho - 1 + \frac{1}{\beta} - \frac{\rho}{\beta^2} \right] > 0 \end{aligned} \quad (194)$$

Adding and subtracting $\frac{\rho}{\beta}$ to the term in square bracket of (194) and rearrange, we get that the inequality in (194) can be satisfied if and only if:

$$\left(1 - \frac{\rho}{\beta} \right) (1 - \rho) \frac{(1 - \beta)}{\beta} > 0 \quad (195)$$

which is satisfied if and only if $\rho < \beta$, as stated in the text, since $\beta < 1$, by assumption. When these conditions are satisfied, one root will be inside and two outside the unit circle. To get determinacy for the system captured by matrix (187) we need another root inside the unit circle, which is obtained by setting:

$$\frac{|1 - \psi|}{\beta} < 1 \quad (196)$$

which, after taking advantage of the absolute value properties, delivers the result stated in (98).

When condition (96) or (97), or both, are not satisfied we require the following condition on fiscal policy, such that:

$$\frac{|1 - \psi|}{\beta} > 1 \quad (197)$$

which implies condition (99) stated in the text.

Q.E.D.

C.9 Proof of proposition 9

As in previous case, we can concentrate our attention on the 2×2 submatrix given by:

$$\Gamma_{11} = \begin{bmatrix} \frac{(1 + \eta_{RS} \phi_\pi)}{\beta} & -\frac{(k - \eta_{RS} \phi_y)}{\beta} \\ \varphi_1 & \varphi_2 \end{bmatrix} \quad (198)$$

The determinant is given by:

$$\det(\Gamma_{11}) = \frac{1 - \alpha_R \sigma S_c \phi_y + \phi_\pi \eta_{RS} (1 - k \alpha_R S_c)}{\beta (1 - \alpha_{R1} \sigma S_c \phi_y)} \quad (199)$$

The trace is:

$$\text{tr}(\Gamma_{11}) = \frac{1 + \eta_{RS} \phi_\pi}{\beta} + \frac{\sigma S_c (k - \eta_{RS} \phi_y) (1 - \alpha_R \phi_\pi) + \beta - \beta \alpha_R \sigma S_c \phi_y}{\beta (1 - \sigma S_c \alpha_{R1} \phi_y)} \quad (200)$$

Condition $\det > 1$ from (115) implies :

$$1 - \alpha_R \sigma S_c \phi_y + \phi_\pi \eta_{RS} (1 - k \alpha_R S_c) > \beta (1 - \alpha_{R1} \sigma S_c \phi_y) \quad (201)$$

which, after rearrangement becomes:

$$\phi_\pi > \frac{\phi_y \sigma S_c (\alpha_R - \beta \alpha_{R1}) - (1 - \beta)}{\eta_{RS} (1 - \alpha_{R1} k S_c)} \equiv \bar{\phi}_{\pi 1} \quad (202)$$

On the other hand, condition $\det > -1$, is satisfied if and only if:

$$\phi_\pi > \frac{\sigma S_c \phi_y (\alpha_R + \beta \alpha_{R1}) - (1 + \beta)}{\eta_{RS} (1 - k S_c \alpha_R)} \equiv \bar{\phi}_{\pi 2} \quad (203)$$

Bounds $\bar{\phi}_{\pi 1}$ and $\bar{\phi}_{\pi 2}$ previously defined are both upper bounds. To establish which of the two bounds in (202)-(203) are binding, let us verify if $\bar{\phi}_{\pi 2} > \bar{\phi}_{\pi 1}$. This condition is verified if and only if:

$$\sigma S_c \phi_y (\alpha_R + \beta \alpha_{R1}) - (1 + \beta) > \phi_y \sigma S_c (\alpha_R - \beta \alpha_{R1}) - (1 - \beta) \quad (204)$$

which is verified if:

$$\phi_y > \frac{1}{\sigma S_c \alpha_{R1}} \quad (205)$$

Therefore, if condition (205) is verified, bound $\bar{\phi}_{\pi 2}$ given in (203) applies.

Moreover condition (118) is satisfied if and only if:

$$\begin{aligned} \sigma S_c k + 2(1 + \beta) - \beta \sigma S_c \phi_y (\alpha_R + \alpha_{R1}) - \sigma S_c (\alpha_R + \alpha_{R1}) \phi_y - \sigma S_c \phi_y \eta_{RS} > \\ \phi_\pi (\sigma S_c \alpha_{R1} k + \eta_{RS} k \alpha_R S_c - 2\eta_{RS} + \sigma S_c \phi_y \eta_{RS}) \end{aligned} \quad (206)$$

which, after rearrangement, becomes:

$$\phi_\pi < \frac{\sigma S_c k + 2(1 + \beta) - \sigma S_c \phi_y (\alpha_R + \alpha_{R1}) (1 + \beta) - \sigma S_c \phi_y}{\sigma S_c \alpha_{R1} k + \eta_{RS} k \alpha_R S_c - 2\eta_{RS} + \sigma S_c \phi_y \eta_{RS}} \quad (207)$$

As ancillary result, it is not difficult to prove that $(\alpha_{R1} - \alpha_R) = 1$. From condition (117), we get:

$$\phi_\pi > \frac{(1 - \beta) \sigma S_c \phi_y + \eta_{RS} \sigma S_c \phi_y - \sigma S_c k}{\eta_{RS} \sigma S_c \phi_y + k \alpha_R (\sigma S_c - \eta_{RS})} \equiv \bar{\phi}_{\pi 3} \quad (208)$$

Therefore, by mixing up the above conditions, we get that determinacy obtains if and only if:

$$\operatorname{argmax} \{ \bar{\phi}_{\pi 2}, \bar{\phi}_{\pi 3} \} < \phi_{\pi} < \frac{k - \eta_{RS} \phi_y}{k \alpha_R (1 + \eta_{RS}) + \eta_{RS} \phi_y} \quad (209)$$

$$\frac{1}{\sigma S_c \alpha_{R1}} < \phi_y \quad (210)$$

Conditions (209)-(210) identify the presence of one root inside and another outside the unit circle for submatrix (198). To check determinacy for the whole system, we need an additional conditions on the fiscal policy side in order to have an additional roots inside the unit circle. This is obtained by setting:

$$\frac{|1 - \psi|}{\beta} < 1 \quad (211)$$

which is equivalent to state:

$$1 - \beta < \psi < 1 + \beta \quad (212)$$

Alternatively, if one or all of (209)-(210) do not hold, conditions (211) should be replaced by:

$$\frac{|1 - \psi|}{\beta} > 1 \quad (213)$$

Or, equivalently:

$$1 - \beta < \psi; \quad \psi > 1 + \beta \quad (214)$$

Q.E.D.

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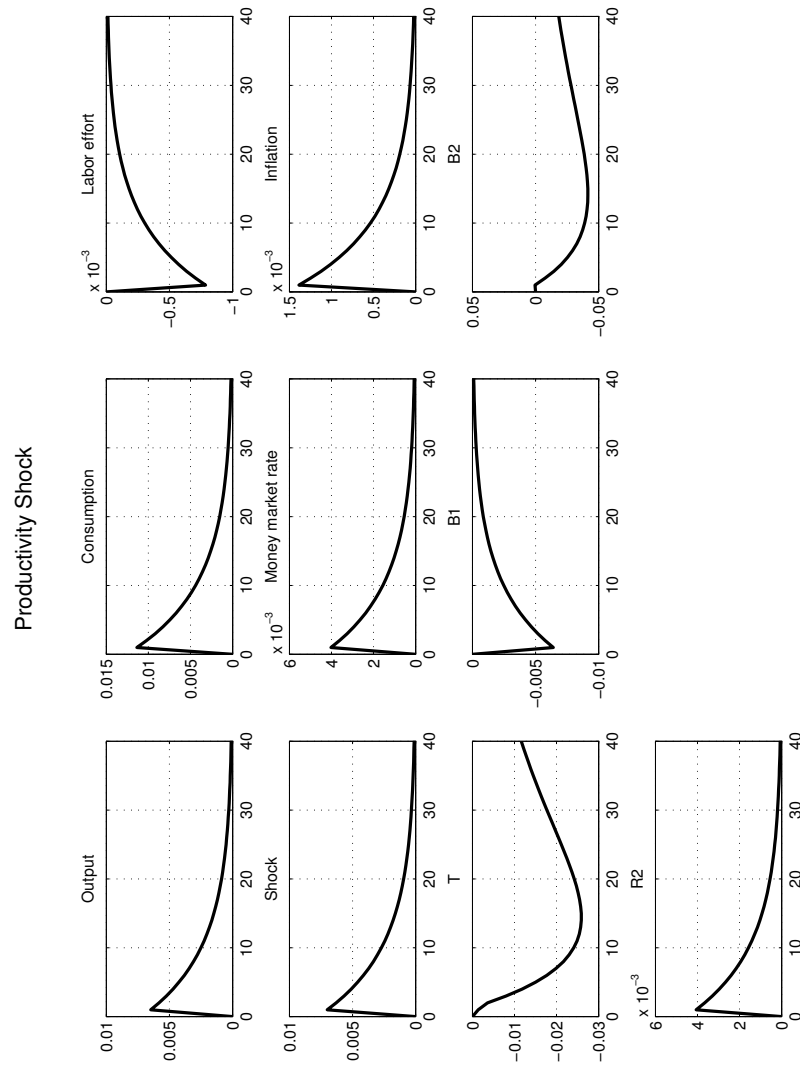
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Table 1: Parameter Values

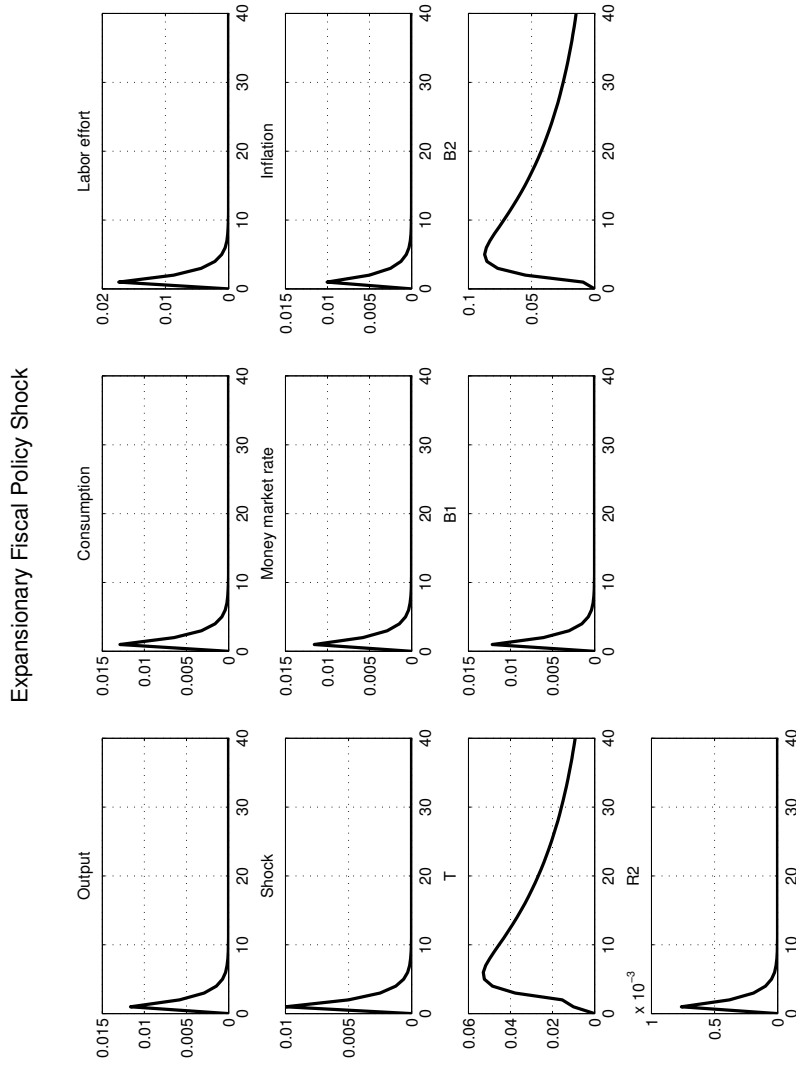
Parameter	Value
β	0.99
σ	0.5
η	1
α	0.67
δ	2/3
θ	6
ψ	0.05
ρ_A	0.9
ρ_G	0.5
σ_A	0.007
σ_G	0.01

Figure 1: Impulse responses: technology shock



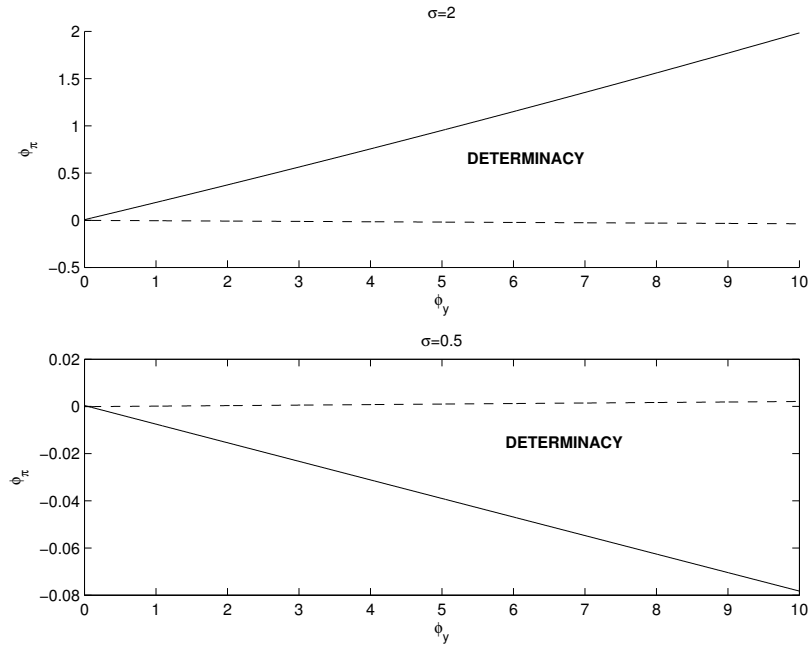
Response of each variable of the system after a one percentage standard deviation technology shock

Figure 2: Impulse responses: fiscal policy shock



Response of each variable of the system after a one percentage standard deviation fiscal policy shock

Figure 3: Determinacy regions for model with weakly separable utility



Both panels represent the simulations relative to bounds established by condition (111) by varying parameter ϕ_y . The top panel is obtained by setting $\sigma = 2$, while bottom panel is obtained with $\sigma = 0.5$.



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