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Individual risk attitude and narrow framing of risks: implications for the equity premium puzzle*

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1 Introduction

It is well-known that standard macroeconomic (rational) asset pricing models fail to account for several financial market puzzles. Mehra and Prescott (1985), in a framework without frictions and market imperfections, find an inconsistency between standard asset pricing theory and empirical evidence about financial markets: by using plausible preference parameters (values of the risk aversion parameter γ not greater than 5), standard asset pricing theory (Consumption-based CAPM - CCAPM) is not able to explain the large equity premium (excess equity return) that emerges from major industrialized economies' financial data. On the other hand, if in the CCAPM we implement empirical data for the consumption growth rate and consumption volatility, and we use plausible values for the time preference and risk aversion parameters, we find that the risk-free rate is too high with respect to its empirical counterpart (Weil, 1989).

Mehra and Prescott (2008) argue that "*The puzzle cannot be dismissed lightly, since much of our economic intuition is based on the very class of models that fall short so dramatically when confronted with financial data. It underscores the failure of paradigms central to financial and economic modeling to capture the characteristic that appears to make stocks comparatively so risky. Hence, the viability of using this class of models for any quantitative assessment, say, for instance, to gauge the welfare implications of alternative stabilization policies, is thrown open to question*".

Over the last 25 years, many attempts to resolve the puzzles have been made, by generalizing several key features of the Mehra and Prescott (1985) model: alternative assumptions on preferences, survival bias, incomplete markets and market imperfections, stock market limited participation and problems of temporal

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aggregation (Mehra, 2008). No one of these explanations has proved fully satisfactory.

The critical point is that in standard asset pricing models (with different preference specifications), the time-varying risk associated with equity investments (which drives the equity premium) is related to the covariance (correlation) between equity returns and consumption growth rate. But if consumption changes are poorly correlated with stock returns (and this is what emerges from the data) we have no guarantee that consumption falls as stock returns decline: therefore, there is no increase in risk aversion and in the equity premium demanded for holding stocks in equilibrium.

Hence, what appears to be necessary in standard asset pricing models is a factor able to generate changes in risk aversion independently (totally or partially) of consumption fluctuations. But we know that in standard "rational" models utility functions are defined over consumption levels. Habit formation models introduce the possibility of time-varying risk aversion, but they reach outcomes only partially satisfactory (and in particular in the set-up proposed by Campbell and Cochrane, 1999). So, the basic question to be addressed is whether there is the possibility to have an investor which gets utility from a source different than consumption and, at the same time, an investor which has a time-varying risk aversion, independently of consumption changes.

Recent behavioral finance/economics models try to provide an answer, by departing from the assumption of perfect rationality of economic agents. Starting from relevant experimental evidence (Tversky and Kahneman, 1981), these models introduce the concept of bounded rationality and then derive several interesting implications. A strand of this literature adopts the basic idea that economic agents derive utility not only from consumption but also from fluctuations in their financial (equity) wealth (*prospect theory* - Kahneman and Tversky, 1979). In particular, these fluctuations affect investor's risk aversion, independently of their correlation with consumption dynamics. Moreover, individual preferences are characterized by the so-called *loss aversion*: economic agent is more sensitive to reductions in his wealth than to increases of the same magnitude.

In this paper we study the quantitative implications of a "behavioral" asset pricing model along the lines proposed in a recursive utility framework by Barberis and Huang (2009) (henceforth, BH model/approach). In this approach investors give more importance to possible setbacks in their financial wealth than to the possibility to take profitable investment opportunities: this fact, combined with the *narrow framing* of risks, makes equity investments very risky and therefore investors demand a large equity premium in equilibrium. In order to test the validity of such approach when compared with financial data, we numerically solve a simple dynamic representative-agent asset pricing model with loss aversion/narrow framing preferences.

In doing so, Barberis and Huang (2008, 2009) provide only a few short illustrative applications relative to U.S. data: they do not compare the predictions of their model with other countries' financial market data. We agree to the view that a theoretical model that attempts to match the empirical evidence about phenomena such as the equity premium and risk-free rate cannot be tested only by using the data of a single economy, although very relevant as the U.S. one. Recent empirical works show that every economy and every stock market can be characterized by a peculiar story that heavily affects its empirical performances (Brown, Goetzmann and Ross, 1995; Jorion and Goetzmann, 1999). Therefore it is important to test the model also with financial data of other industrialized economies (with less-developed stock markets), in order to have the opportunity to uncover potential heterogeneous behaviors among the various countries. We do that in this work, by applying the model to U.S., UK, Italy, Germany, France and Japan. Our empirical results show that the BH approach seems to be robust also when it is applied to countries other than U.S.

Moreover, Barberis and Huang (2008, 2009) do not attempt "to infer" exact plausible values for the so-called "narrow framing degree". Instead, as a second step, we reverse the basic problem by solving numerically the same model in order to assess the exact values of the narrow framing degree that rationalize the observed empirical values of the equity premium and risk-free rate. We find that such narrow framing values are generally quite small and usually not greater than 0.24 (with the exception of several outliers greater than 1 for Germany); moreover, they are in line with seminal works on asset pricing models with loss aversion/narrow framing preferences.

Finally, our numerical results show that there is an interesting and conflicting relationship between the risk aversion parameter and the narrow framing degree in affecting the risk-free rate dynamics. In fact, the simultaneous increase of these parameters affects the behavior of the risk-free interest rate in an opposite way. We know that in the CCAPM, for values consistent with a plausible parameterization of individual preferences, the increase in risk aversion implies, as a net effect, the increase in the interest rate (Campbell, 2003): the growth of current consumption "wins" over the precautionary saving motive. On the other hand, the increase in the narrow framing parameter lowers safe returns, because of the increase in stock market risk which implies an increase in the purchase of safe assets and therefore a reduction in interest rates. The crucial point is that the last effect, by adding to the precautionary saving one, is able to overturn the "standard" effect related to the increase in risk aversion, by lowering the risk-free rate: as a final result, the model is able to match the empirical evidence.

The present work can be seen as a further contribution to the study of the role and importance of behavioral preference parameters in explaining the individual financial decision making processes more strictly related to individual psychology and irrationality.

2 Financial wealth fluctuations and narrow framing

2.1 Theoretical basics

In traditional models[1] the economic agent typically adopts the following behavior: he merges the new choices he faces with those already faced, then he

[1] We refer to models based on traditional Von Neumann-Morgenstern utility functions, defined on consumption and/or wealth.

controls if the new "gamble" improve or not the future distribution of wealth and/or consumption. But starting with the seminal work of Tversky and Kahneman (1981), experimental evidence on financial decision-making under risk has shown that individuals often do not behave as in traditional models. In many cases, when people evaluate risk, they often engage in *narrow framing*: that is, they often evaluate risks in isolation, separately from other risks they are already facing. As remarked by Barberis and Huang (2009), "*…… narrow framing means that, when an agent is deciding whether to accept a gamble, he uses a utility function that depends directly on the outcome of the gamble, not just indirectly via the gamble's contribution to his total wealth*".

The classic proof of such behavior is due to the paper of Tversky and Kahneman (1981). Their work points out a clear contradiction: economic agents are faced with two concurrent decisions and they make a sub-optimal choice, opting for a dominated strategy. What happens is that instead of focusing on the *combined* outcome of the two decisions (i.e. on the outcome that determines their final wealth), individuals focus on the outcome of each decision *separately*.

There are different situations where we can find a similar individual behavior. For example, we can think about stock market non-participation and the equity home bias: in both cases profitable portfolio diversification opportunities are rejected.

Kahneman (2003) argues that when an agent evaluates a new gamble, the distribution of the gamble, separately considered, is much more "accessible" than the distribution of his overall wealth once the new gamble has been merged with his other risks. The expression "accessible" refers to the fact that many decisions are based on easily interpretable information; and this consideration is based on the idea that many choices are made intuitively rather than through effortful reasoning.

Consistently with the explanation of Kahneman (2003) and in support of it, we can recall the seminal contribution provided by Simon (1982). Simon remarks that individuals' cognitive resources are limited: this element forces individuals to simplify the space of the choice problem, which appears unmanageable for his excessive complexity.

On these premises, we can naturally think of financial markets as a field where we can apply the theoretical approach we are discussing about. In fact, few sectors of human activity are characterized by a so huge quantity of information as it occurs in stock markets (Slovic, 1972). Such information is highly accessible by everyone because it can be daily reached by means of newspapers, tv-news, internet, etc. But the crucial point here is the correct and optimal processing of information.

As argued in Magi (2009), the coming of new technologies, providing quickly information about world stock market movements, has highly contributed to increase individuals' difficulties in exploiting at the maximum the huge amount of information available to them. In fact, although a large amount of information means more accuracy in evaluating alternative choices, as argued by Simon (1982), an amount of information in excess, given the individual's bounded cognitive resources, makes the decision space unmanageable; and in order to simplify this space, economic agents make narrow framing (and not "overall") evaluations. As a consequence, this behavior implies that individuals make the choice that is *apparently* the best one. The overall evaluation of the problem would lead to a better choice than that effectively made, but the lack of the "optimal" skills in processing information leads to the sub-optimal choice. The overall framing is involuntary declined in favor of the narrow one because of the lack of such optimal skills in processing information. We observe that this framework is in line with recent works on the so-called "rational inattention" (Sims, 2003; Reis, 2005) and it can rationalize a key assumption in Sims (2003): individuals only devote small fractions of their capacity in observing and processing information.

In this work we will use a representative agent model and this fact could have a not negligible impact on aggregation issues. By using a representative agent model, we are implicitly assuming that individual narrow framing preferences hold in the aggregate too. It seems plausible that if all individuals have preferences characterized by the same degree of narrow framing and loss aversion, by aggregating such individuals the same type of preferences should be preserved. Obviously in this way we rule out the possibility of heterogeneity in individual preferences and abilities to process information.

More generally, regarding the relationship between narrow framing and asset prices in a full equilibrium setting, Barberis and Huang (2009) argue that first it is important to think carefully about where, if anywhere, narrow framing is likely to have a significant impact on asset prices. One way to think about this is to consider a model more ambitious than the one that we will actually adopt in the next sub-section, namely a heterogeneous agent model with two groups of investors, where one group consists of standard expected utility (EU) maximizers while investors in the second group are narrow framers. In such a model, the narrow framers are unlikely to have much of an effect on the prices of assets with close substitutes - if they did, that would present an attractive opportunity for the EU investors, who would then trade aggressively against the narrow framers, reducing their impact on prices. On the other hand, narrow framers would have a more significant impact on the prices of assets *without* close substitutes - in this case, it would be much riskier for the EU agents to trade against them.

But Barberis and Huang stress the fact that, at the moment, a tractable way of analyzing a formal heterogeneous agent model of this kind does not exist.[2] Then, a possible approach, as in many other lines of asset pricing research, is to start by studying a homogeneous agent model. In taking this approach, we need to be careful to pick an application where the prediction of the homogeneous agent model is at least qualitatively similar to the prediction of the more realistic heterogeneous agent model. With this goal, in this paper we take the equity premium as our application (as done in Barberis and Huang, 2008, 2009). If narrow framing affects the equity premium in a homogeneous agent model, it is likely to also affect the equity premium in a heterogeneous agent model: since the aggregate stock market does not have a close substitute, it would be too risky for EU maximizers to trade aggressively against the narrow framers. The effect of the narrow framers on the equity premium would therefore remain intact, at least in part.

[2] Gabaix (2008) assumes the possible existence of at least two types of agents in the real world: prospect theory (narrow framing) investors (individuals) and expected utility investors (institutions). Then, he suggests the first steps of a possible analytical framework for incorporating these features into the BH approach. The proposal of Gabaix can be an interesting starting point for future research along this direction.

Even if the implications of a homogeneous agent model for the equity premium are *qualitatively* similar to those of a heterogeneous agent model, we must acknowledge that they may be *quantitatively* different. Even if EU maximizers cannot fully reverse the effect of narrow framers on the pricing of the aggregate stock market, they may partially offset it. As such, Barberis and Huang (2009) conclude that "*the equity premium that we obtain in a homogeneous agent economy should be thought of as an upper bound on the equity premium that we would obtain in a more realistic heterogeneous agent economy*".

2.2 The Barberis-Huang (BH) approach: the basic model

Barberis, Huang and Santos (2001) [BHS henceforth] propose a new approach in order to solve some financial market puzzles. They introduce a new source of utility for the representative agent, besides the usual one (levels of consumption). The basic idea is the following: economic agents derive utility not only from consumption but also directly from fluctuations in the value of their financial wealth, and such fluctuations heavily affect investors' risk aversion, regardless of their correlation with consumption growth. This idea has its origins in the seminal work by Kahneman and Tversky (1979), which introduced the so-called prospect theory, based on prospect-type utility: economic agent derives utility not from consumption/wealth levels but from their changes, evaluated with respect to a reference level. Therefore the utility function is defined on gains and losses and captures the so-called *loss aversion*, i.e. the fact that the agent is more sensitive to reductions in his wealth rather than to increases of the same magnitude.

Obviously, this approach is in contrast with standard asset pricing models,[3] which assume that economic agents only care of their future utility deriving from consumption levels. But in the economic literature there are many contributions that, by means of theoretical arguments and experimental works, show that standard explanations of individual attitudes towards risk are widely questionable and often wrong (see Rabin, 1998, 2002). As stressed by Rabin (2002), "*….Our attitudes towards risk are driven instead primarily by attitudes towards change in wealth levels.*" In the BHS approach, the motivating idea is that after a big loss in the stock market, the investor may experience a sense of *regret* over his decision to invest in stocks. As a consequence, "*he may interpret this loss as a sign that he is a second-rate investor, thus dealing his ego a painful blow; and he may feel humiliation in front of friends and family when word leaks out*" (Barberis, Huang and Santos, 2001).

The BHS model has been modified by the same authors in a subsequent contribution (we refer to it as "BH model"). The analytical novelty introduced by Barberis and Huang (2009) is the use of recursive utility (Epstein-Zin-Weil utility)[4] in its standard formulation,

$$U_t = W\left(C_t, \mu(U_{t+1})\right) \tag{1}$$

[3] For extensive survey articles on CCAPM models see Campbell (2003) and Kocherlakota (1996).
[4] See Epstein and Zin (1989, 1991) and Weil (1989).

where $\mu(U_{t+1})$ is the certainty equivalence of future utility U_{t+1}, given the time t information. The function $W(\bullet)$ is an aggregator which combines future utility U_{t+1} with current consumption C_t in order to generate current utility U_t. Usually, in this kind of literature, the aggregator function assumes the CES (Constant Elasticity of Substitution) form,

$$W(C,x)=\left[(1-\beta)C^{\rho}+\beta x^{\rho}\right]^{\frac{1}{\rho}} \tag{2}$$

with $0<\beta<1$, $0\neq\rho<1$, while for the certainty equivalence we assume a functional form with homogeneity of degree one,

$$\mu(kx)=k\mu(x),\ k>0.$$

By adopting the BH preference formulation, the maximization problem of the representative investor modifies as follows:

$$\text{Max}\quad U_t=W\left(C_t,\mu(U_{t+1})+b_0E_t\left[v(G_{S,t+1})\right]\right) \tag{3}$$

$$\text{s.t.}\qquad W_{t+1}=(W_t-C_t)R_{W,t+1} \tag{4}$$

where
$$W(C,x)=\left[(1-\beta)C^{\rho}+\beta x^{\rho}\right]^{\frac{1}{\rho}} \tag{5}$$

$$\mu(x)=\left[E(x^{\delta})\right]^{\frac{1}{\delta}},\qquad 0\neq\delta<1 \tag{6}$$

$$G_{S,t+1}=\theta_{S,t}(W_t-C_t)(R_{S,t+1}-R_f) \tag{7}$$

$$v(G)=\begin{cases}G & \text{for}\quad G\geq 0\\ \lambda G & \phantom{\text{for}}\quad G<0\end{cases}\qquad \lambda>1 \tag{8}$$

We can have two (or more) financial assets: for example a risky asset (equity) with gross rate of return $R_{S,t+1}=1+r_{S,t+1}$ between t and t+1, and a risk-free asset with safe return R_f; $\theta_{S,t}$ is the fraction of post-consumption wealth invested in the risky asset. $R_{W,t+1}$ measures the return on the overall individual wealth, i.e. on the individual's "market portfolio", between t and t+1. Obviously, the composition of this portfolio depends on the number of assets we take into account. In general, we could also consider n assets.

The first term of the utility function (3) is what we usually find in standard asset pricing models. The novelty is the second term, $v(G_{t+1})$, which represents utility deriving from individual stock wealth changes: in other words, utility deriving from fluctuations in individual's risky financial wealth.[5] In particular, G_{t+1} is the gain or loss obtained by the agent on his equity investments between t and t+1. The utility (disutility) deriving to the investor from this gain (loss) is measured by the function $v(\bullet)$. We note that we have both the narrow framing of risks, introduced by the presence of $v(\bullet)$, and the loss aversion, introduced by the particular (piecewise linear) form of $v(\bullet)$. Figure 1 makes clear the fundamental feature of representative agent's preferences, the loss aversion, i.e. the higher sensitivity to stock wealth setbacks rather than to increases of the same magnitude.

As we can see by (7), the reference level for measuring the gain/loss is given by the initial value of risky financial asset parameterized with the risk-free asset. The idea is that investor will be satisfied if $R_{S,t+1} > R_f$ and unsatisfied vice versa (BHS, 2001). Finally, the certainty equivalence has been specified in a very simple form (equation 6), widely used in the literature.

Utility function with loss aversion

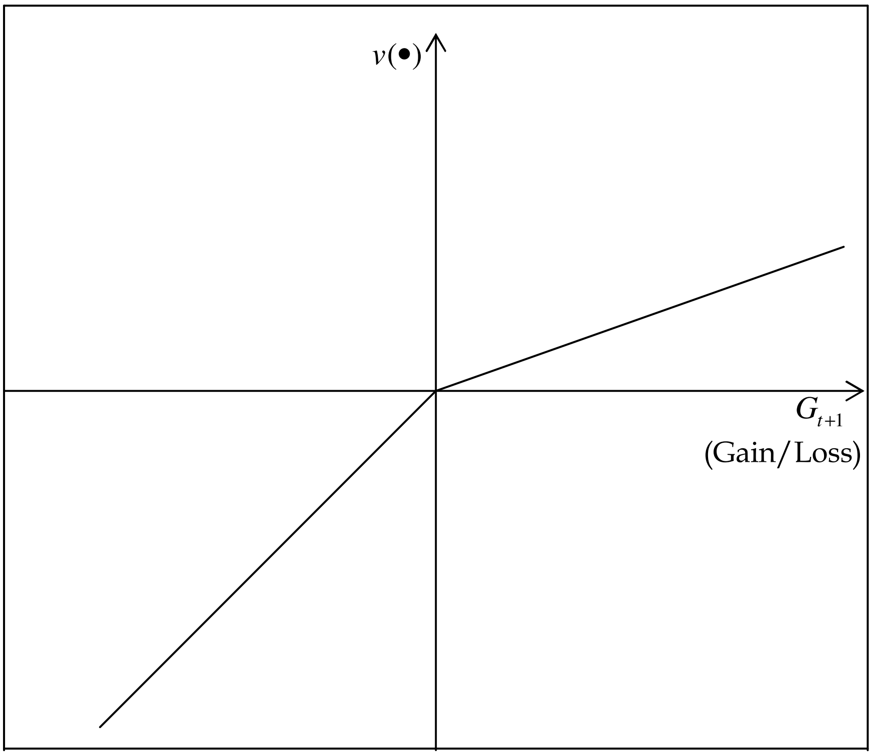

Figure 1

In this context, another crucial issue is the frequency by which the investor evaluates his financial situation, checking his stock market performances. Following the results obtained by Benartzi and Thaler (1995), Barberis, Huang and Santos (2001) consider "the year" as standard evaluation period. The time horizon of equity investments is usually longer, 3-5 years, but it is reasonable assuming that economic

[5] We only consider risky asset fluctuations: the time t+1 risk-free return is known with certainty at time t, and therefore there is no risk in the changes of the safe asset.

agent seriously checks his financial market performances at least once a year. This assumption is confirmed by some elements: we file taxes once a year, receive our most comprehensive mutual fund reports once a year, and institutional investors scrutinize their money managers' performances most carefully on an annual basis.

What about b_0? In the BH model this parameter plays a very important role. It is a non-negative parameter which permits us to control for the importance of utility deriving from financial wealth changes when compared with the utility deriving from consumption. At the same time, b_0 can also be interpreted as the *narrow framing* degree of a risky investment. If we set $b_0 = 0$ we have the standard consumption-based asset pricing model.

3 First-order optimality conditions

We consider a simple representative agent economy in general equilibrium, where preferences are described by equations (3) – (8). We consider the presence of three assets: a risk-free financial asset in zero net supply, and two risky assets, the stock market (S) and a non-financial (N) asset (human capital or housing wealth), both in positive net supply. The overall wealth return $R_{W,t+1}$ of equation (4) is therefore the following:

$$R_{W,t+1} = \left(\theta_{S,t} R_{S,t+1} + \theta_{N,t} R_{N,t+1} + (1-\theta_{S,t} - \theta_{N,t}) R_f\right).$$

In line with Barberis and Huang (2008, 2009) we assume that $\rho = \delta = 1-\gamma$,[6] where γ has the usual meaning. In this way we get

$$W(C,x) = \left[(1-\beta)C^{1-\gamma} + \beta x^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

and $$\mu(x) = \left[E(x^{1-\gamma})\right]^{\frac{1}{1-\gamma}}, \qquad 0 < \gamma \neq 1$$

The agent maximizes the function (3), which becomes as follows:

$$U_t = \left\{(1-\beta)C_t^{1-\gamma} + \beta\left([E_t(U_{t+1}^{1-\gamma})]^{\frac{1}{1-\gamma}} + b_0 E_t v(G_{S,t+1})\right)^{1-\gamma}\right\}^{\frac{1}{1-\gamma}}, \tag{9}$$

subject to the usual constraints.

Now we consider an equilibrium where:

[6] This assumption is standard in the literature. It implies that the separation between risk aversion and intertemporal elasticity of substitution is lost, but in our setting this is not a problem.

1) the risk-free rate is a constant R_f ;

2) consumption and equity returns grow over time according to the following log-returns:

$$\log(\frac{C_{t+1}}{C_t}) = g_C + \sigma_C \varepsilon_{C,t+1}$$

$$\log(R_{S,t+1}) = g_S + \sigma_S \varepsilon_{S,t+1}$$

where

$$\begin{pmatrix} \varepsilon_{C,t} \\ \varepsilon_{S,t} \end{pmatrix} \sim \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \omega_{C,S} \\ \omega_{C,S} & 1 \end{pmatrix} \right]$$

and where g_i and σ_i are mean log returns (mean log growth rate for consumption) and standard deviations (of log stock returns and log consumption growth), $\omega_{C,S}$ is the correlation between log consumption growth and log stock returns and stochastic shocks ε_i are exogenous;

3) the consumption-wealth ratio is a constant α. Hence we have

$$R_{W,t+1} = \frac{W_{t+1}}{W_t - C_t} = \frac{1}{1-\alpha} \frac{C_{t+1}}{C_t}$$

and

$$\log(R_{W,t+1}) = g_W + \sigma_W \varepsilon_{W,t+1}$$

where $g_W = g_C + \log(\frac{1}{1-\alpha})$, $\sigma_W = \sigma_C$ and $\varepsilon_{W,t+1} = \varepsilon_{C,t+1}$.

4) the fraction of total wealth invested in the stock market is constant over time:

$$\theta_{S,t} = \frac{S_t}{S_t + N_t} = \theta_S, \quad \forall t$$

where S_t is total equity wealth and N_t is total non-financial wealth.

Barberis and Huang (2009) show that with these assumptions, the first order conditions of optimality are the following:

$$1 = \left[\beta R_f E_t \left((\frac{C_{t+1}}{C_t})^{-\gamma} \right) \right] \cdot \left[\beta E_t \left((\frac{C_{t+1}}{C_t})^{-\gamma} R_{W,t+1} \right) \right]^{\frac{\gamma}{1-\gamma}} \tag{10}$$

$$0=\frac{E_t\left[(\frac{C_{t+1}}{C_t})^{-\gamma}(R_{S,t+1}-R_f)\right]}{E_t\left[(\frac{C_{t+1}}{C_t})^{-\gamma}\right]}+ \\ +b_0 R_f(\frac{\beta}{1-\beta})^{\frac{1}{1-\gamma}}(\frac{1-\alpha_t}{\alpha_t})^{\frac{-\gamma}{1-\gamma}}E_t\left[v(R_{S,t+1}-R_f)\right] \tag{11}$$

$$0=\frac{E_t\left[(\frac{C_{t+1}}{C_t})^{-\gamma}(R_{W,t+1}-R_f)\right]}{E_t\left[(\frac{C_{t+1}}{C_t})^{-\gamma}\right]}+ \\ +b_0 R_f(\frac{\beta}{1-\beta})^{\frac{1}{1-\gamma}}(\frac{1-\alpha_t}{\alpha_t})^{\frac{-\gamma}{1-\gamma}}\theta_{S,t}E_t\left[v(R_{S,t+1}-R_f)\right] \tag{12}$$

where $\alpha_t \equiv C_t / W_t$ is the consumption-wealth ratio. By exploiting the assumptions 1–4, equations (10) - (12) can be modified further as follows:[7]

$$\alpha=1-\beta^{\frac{1}{\gamma}}R_f^{\frac{1-\gamma}{\gamma}}e^{\frac{1}{2}(1-\gamma)\sigma_C^2} \tag{13}$$

$$0=b_0 R_f(\frac{\beta}{1-\beta})^{\frac{1}{1-\gamma}}(\frac{1-\alpha}{\alpha})^{\frac{-\gamma}{1-\gamma}}[e^{g_S+\frac{1}{2}\sigma_S^2}-R_f+(\lambda-1)\cdot \\ \cdot[e^{g_S+\frac{1}{2}\sigma_S^2}N(\hat{\varepsilon}_S-\sigma_S)-R_f N(\hat{\varepsilon}_S)]]+e^{g_S+\frac{1}{2}\sigma_S^2-\gamma\sigma_S\sigma_C\omega}-R_f \tag{14}$$

$$0=b_0 R_f(\frac{\beta}{1-\beta})^{\frac{1}{1-\gamma}}(\frac{1-\alpha}{\alpha})^{\frac{-\gamma}{1-\gamma}}\theta_S[e^{g_S+\frac{1}{2}\sigma_S^2}-R_f+(\lambda-1)\cdot \\ \cdot[e^{g_S+\frac{1}{2}\sigma_S^2}N(\hat{\varepsilon}_S-\sigma_S)-R_f N(\hat{\varepsilon}_S)]]+\frac{1}{1-\alpha}e^{g_C+\frac{1}{2}\sigma_C^2-\gamma\sigma_C^2}-R_f \tag{15}$$

with $\hat{\varepsilon}_S=\frac{\log(R_f)-g_S}{\sigma_S}$ and where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.[8]

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[7] For details see the Mathematical Appendix A.
[8] See the Mathematical Appendix B.

Now, given some parameter values, and in particular those that describe individual preferences, we can solve the equations (13) – (15) finding the unknowns α , R_f e g_S , in order to determine the riskfree interest rate and the equity premium generated by the model.[9] We note that in our framework the equity premium is given by

$$EP = E_t(r_{S,t+1}) - r_f$$

We know that

$$\log(R_{S,t+1}) \equiv \log(1 + r_{S,t+1}) = g_S + \sigma_S \varepsilon_{S,t+1} \Rightarrow$$

$$\Rightarrow r_{S,t+1} = e^{g_S + \sigma_S \varepsilon_{S,t+1}} - 1,$$

then, taking the expected values at time t on both sides we have

$$E_t(r_{S,t+1}) = e^{g_S + \frac{\sigma_S^2}{2}} - 1$$

In deriving the expression above we exploit the fact that if a random variable x has a standard normal distribution then

$$E(e^{a+bx}) = e^{a + \frac{b^2}{2}}.$$

Once we have found R_f and g_S in the three-equation system (13) - (15), we can find the equity premium according to the following expression:

$$EP = (e^{g_S + \frac{\sigma_S^2}{2}} - 1) - r_f = e^{g_S + \frac{\sigma_S^2}{2}} - R_f . \tag{16}$$

4 Quantitative implications of the model: matching the empirical evidence

In this section we numerically solve (by working on first order optimality conditions) the simple dynamic representative-agent asset pricing model outlined above, with the aim to match the empirical evidence observed in industrialized economies about the equity premium and the riskfree rate. We exploit the results provided by the numerical solution of equations (13) – (15) in order to test the validity of the BH model.

Then, as a second step, in the next section we reverse the problem by solving the same model in order to infer the "exact" levels of the narrow framing degree

[9] The solution of equations (13) – (15) is run with MatLab. Codes are available upon request by the author.

which are coherent with the observed empirical values of the equity premium and risk-free rate.

Barberis and Huang (2008, 2009) provide only a few short illustrative applications relative to U.S. data. Here we will see how the model behaves when it is compared with financial data of other countries; we analyse the cases of U.S., UK, Italy, Germany, France and Japan.

In Table 1 we have the empirical values of the equity premium and riskfree rate to which we will refer for evaluating the ability of the model in matching the data, while Table 2 shows consumption dynamics and stock return parameters that will be implemented in the model. We use data drawn from Campbell (2003), for the time periods reported in Tables 1 and 2.

Concerning preference parameters (β , γ, λ), we implement values that are consistent with the most important results found in the theoretical and empirical literature (see Tversky and Kahneman, 1992; Kocherlakota, 1996; Barberis and Thaler, 2003; Magi, 2007, 2009). This holds also for the numerical exercises we will run in section 5.

For β and γ we use values largely used in the prevailing literature (Kocherlakota, 1996, Campbell, 2003, Barberis and Huang, 2008, 2009), keeping in mind that regarding risk aversion parameter there is much controversy in the literature. Kocherlakota (1996) summarizes the prevailing view by observing that "a vast majority of economists believe that values of γ above ten (or, for that matter, above five) imply highly implausible behavior on the part of individuals." This consideration is a reference to the arguments of Pratt (1964), which claims that risk aversion parameter has to be not greater than 3 (see Ljungqvist and Sargent, 2004, chapter 13). For the intertemporal discount factor β we take in all cases a value equal to 0.98, which corresponds to a *time preference* (intertemporal discount rate) equal to 0.02. However, β has little effect on attitudes to risk and setting it to 0.98 ensures that the risk-free rate is not too high.

Concerning b_0, in the present section we use a range of different values, in order to verify the behavior of the model as the level of narrow framing varies. We remark that at the moment there are only a few research articles where we can find plausible attempts to estimate the narrow framing degree.[10] The same consideration holds for the loss aversion parameter, but in this case the literature is more developed (Tversky and Kahneman, 1992; Benartzi and Thaler, 1995; Berkelaar, Kouwenberg and Post, 2004; Gomes, 2005).

Finally, about the calibration of θ_S, we take values which are close to data drawn from households' wealth of the various countries and however we remark that the results are quite similar over a range of values of θ_S.

A) U.S.

We start with the case of U.S., with data of the period 1891-1998 (drawn from Campbell, 2003). We solve the model by taking a value of θ_S equal to 0.25 (see

[10] We will attempt to provide a contribution in this direction in the next section. However, Guiso (2009) provides a first interesting approach to the *econometric* estimate of the narrow framing degree.

Table 2), but we would obtain quantitatively similar results by using slightly different values, for example 0.20 or 0.30 (as done by Barberis and Huang, 2008, 2009).[11]

First, we observe that the cases with the parameter triples (γ, λ, b_0) = (1.5, 3, 0) and (γ, λ, b_0) = (3, 3, 0) have not been reported because we have the same results obtained with $\lambda = 2.25$: when $b_0 = 0$ the value of λ becomes completely insignificant because it goes out of the model (see equations 14 and 15). From Table 3 we see that the model is able to produce a large equity premium together with a low risk-free rate, in accord with the empirical evidence. In particular, we highlight the possibility to get low interest rates and large equity premiums (by means of the raise in $E_t r_{S,t+1}$ and therefore g_S) by increasing the narrow framing degree b_0. Obviously, with the same level of γ, this behavior will be stronger as much as loss aversion λ will be higher. Our results about U.S. are in line with Barberis and Huang (2008, 2009).

The mechanism in action is that previously discussed in section 2. If investor derives utility also directly from changes in his equity wealth, and, via the parameter λ, is more sensitive to losses than to gains of the same magnitude, he finds the stock market particularly risky: hence, in order to hold equities in equilibrium, he wants to be compensated by a high risk premium. At the same time, the narrow framing made on the stock market and the strong aversion towards financial setbacks (loss aversion) imply an increasing demand for safe risk-free assets and the deferment of consumption to the future: this explains the reduction in risk-free rates. We note that interest rate declines as we introduce narrow framing and loss aversion in the model (i.e. when b_0 is different from zero), but also, by keeping γ and b_0 constant, when we raise the level of loss aversion λ. This fact also holds for the increase in the equity premium.

Moreover, what is evident is that the effects on interest rate and equity premium are remarkable in particular at a first step of the introduction of *behavioral* (narrow framing) preferences, i.e. when we shift from a narrow framing degree equal to zero (standard asset pricing model) to a level of 0.05: for higher levels of narrow framing the effect is still present but it seems to appear with less intensity. This point will be more evident and clear in the next section.

When the narrow framing degree is zero we collapse to the standard model and the only preference parameter that matters (together with β) is consumption risk aversion γ. In this case equations (13) – (15) become the following

$$\alpha = 1 - \beta^{\frac{1}{\gamma}} R_f^{\frac{1-\gamma}{\gamma}} e^{\frac{1}{2}(1-\gamma)\sigma_C^2} \tag{17}$$

$$0 = e^{g_S + \frac{1}{2}\sigma_S^2 - \gamma\sigma_S\sigma_C\omega} - R_f \tag{18}$$

[11] We note that this observation holds for all cases we will see.

$$0=\frac{1}{1-\alpha}e^{g_C+\frac{1}{2}\sigma_C^2-\gamma\sigma_C^2}-R_f \tag{19}$$

and what said above (we come back to the standard model) should be more clear.

From Table 3 we can see that with $b_0=0$, increasing values of γ imply slightly increasing values of the equity premium, but in particular, consistently with standard theory, they imply a substantial increase in r_f. The increase in risk aversion means that economic agent becomes more reluctant to substitute consumption in different states of the world and, at the same time, the elasticity of intertemporal substitution (EIS) decreases:[12] investor prefers current consumption over future consumption and for increasing current consumption he has to borrow, pushing up interest rates. Keep in mind that an increase in γ also induces a higher demand for risk-free financial assets (precautionary saving), with the consequent increase in their prices and decrease in their interest rates; but at the same time, the effect related to current consumption growth turns out to be prevailing: as a net final effect the risk-free rate goes up.[13]

B) UK

Let's see the case of UK (for the period 1919-1998). As shown in Table 4, the qualitative behavior of the model is similar to that of the U.S. case. The introduction of the narrow framing of risks and of loss aversion imply a reduction in riskfree rate and an increase in the equity premium. In particular, in correspondence of the parameter triple (γ, λ, b_0) = (1.5, 3, 0.05), we are able to match almost perfectly the "benchmark" empirical values (reported in Table 1). Also in this case, as done above, we have omitted to analyse cases as the parameter triple (γ, λ, b_0) = (1.5, 3, 0), because with no narrow framing we find the same outcomes of the case with (γ, λ, b_0) = (1.5, 2.25, 0).

In the third part of Table 4 we have implemented higher values of b_0 than that calibrated in the US case: the idea is to stress the fact that when γ shifts from 1.5 to 3, if the level of loss aversion λ remains at 2.25, even marked increases of the narrow framing degree beyond a given threshold are not enough for reducing significantly r_f. In fact, increasing loss aversion as well becomes necessary: with λ equal to 3 (see the last part of Table 4) the model produces, together with a large equity premium, even a negative safe interest rate, and this with "moderate" levels of narrow framing. This means that, beyond certain levels, the narrow framing degree tends to lose its importance and to reduce its impact on interest rates and equity returns.

[12] We remember that, despite the use of recursive preferences, the negative relation between EIS and risk aversion is still a characteristic of the model, given the starting assumptions ($\rho=\delta=1-\gamma$): the value of EIS decreases as risk aversion increases and vice versa.

[13] Obviously, this relationship holds for values not so high of the risk aversion parameter γ (Campbell, 2003). When the value of γ is very high (for example greater than 15), i.e. absolutely implausible, the precautionary saving motive wins over the current consumption growth one, and the risk-free rate declines.

Finally, we note another interesting behavior of the model: an increase in risk aversion γ, by keeping constant the other preference parameters, is not always followed by an increase in risk-free rate, as standard theory predicts. In particular, when $b_0 \neq 0$ and $\lambda = 3$, from Table 4 we see that in several cases increases in γ imply a reduction in r_f. For example, we have

$$(\gamma,\ \lambda,\ b_0) = (1.5, 3, 0.05) \quad \Rightarrow \quad r_f = 1.17\%$$

and $$(\gamma,\ \lambda,\ b_0) = (3, 3, 0.05) \quad \Rightarrow \quad r_f = -0.34\%$$

or in alternative,

$$(\gamma,\ \lambda,\ b_0) = (1.5, 3, 0.10) \quad \Rightarrow \quad r_f = 0.90\%$$

and $$(\gamma,\ \lambda,\ b_0) = (3, 3, 0.10) \quad \Rightarrow \quad r_f = -0.47\%$$

The crucial point is that the presence of b_0, accompanied by a relative high level of loss aversion, implies a decreasing impact of γ on r_f. In other words, for growing values of b_0 the positive effect of risk aversion on interest rates tends to diminish, up to becoming negative. But if we consider the question carefully, this fact is perfectly consistent with the basic idea of narrow framing preferences, where a critical role is assigned to the framing of risks "in isolation" and to equity losses in affecting investors' financial choices. With regard to the effects on the risk-free rate, the negative effect introduced by the narrow framing of risks turn out to be slightly stronger than the positive effect of risk aversion γ.

C) ITALY

In this case we have outcomes over a limited time period, 1971-1998. As we will see, the Italian case is very similar to the Japanese one. Table 1 shows that for such countries we have empirical values of the equity premium not so large as in other cases. However, they are values not in accordance with standard theory (CCAPM) and therefore contribute to the equity premium puzzle. On these premises, it will be coherent to expect the Italian case to be partially different from the general tendency, which globally confirms the validity of the BH approach, i.e. its ability to match empirical financial data of major industrialized economies. Table 5 shows the results obtained for Italy.

They are not so good as in the U.S. and UK cases. The overall tendency is that expected, but the problem is to get an interest rate close to its empirical value and at the same time an equity premium not so large and close to the empirical evidence. For example, with the parameter triple $(\gamma,\ \lambda,\ b_0)$ = (3, 2.25, 0.05), we are able to get EP = 4.66%, but such a result is not much significant since it is obtained with an interest rate equal to 6.73%. In other words, when the model matches the risk-free rate, it is not able to make the same thing with the equity premium.

On the other hand, the Italian case, for different structural factors, is not so easily comparable to the cases of U.S. and UK, and more generally, to English-speaking countries. Some notable examples of such differences are the historical poor individual attitude toward equity investments, the capital market and portfolio structure, etc.[14] In Italy the equity investment culture is very recent, at the contrary of English-speaking countries, where since many years there exists a higher financial culture and a stronger propensity in using some financial products.[15] Hence, it seems to be plausible that the BH approach, given its distinctive characteristics, is suitable for the Italian reality only approximately.

D) GERMANY

The basic outcomes for Germany are reported in Table 6. The BH model seems to work well: there is the possibility to generate a large equity premium and meanwhile a low risk-free rate, both close to their empirical values. As we can see, the best results are obtained with $\lambda = 3$. Finally, we note that as already seen for US and UK, in some cases (see first section of Table 6) an increase in b_0 beyond certain levels has negligible effects on interest rates and equity premiums.

E) FRANCE

Also in this case we have outcomes consistent with the empirical evidence (see Table 7): an increase in the narrow framing degree implies a reduction of r_f and an increase in the risk premium demanded by investors for holding stocks in equilibrium. Moreover, regarding the effects on interest rates, as already observed for UK, also in this case the increase in b_0 offsets the raise in the risk aversion parameter, with a negative final effect (i.e., the risk-free rate declines).

F) JAPAN

As already mentioned above (see the case of Italy), also for Japan we have results partially at odds with the basic idea of the BH approach. In Table 8 we have reported the most important results which point out that this case presents more problems than the Italian one in satisfying the predictions of the BH approach. As remarked for Italy, also in this case there are some difficulties in generating

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[14] For an analysis of the differences between Italy and U.S. in term of financial market participation and household wealth, see for example Faiella and Neri (2004).

[15] There are many works that emphasize the importance of the diffusion of a certain level of financial culture and literacy, in order to further encourage the development of some countries' stock markets and to increase the possibility of convenient financial portfolio diversification. See for example Haliassos and Hassapis (2001) and Alessie, Lusardi and Van Rooij (2007).

simultaneously a large equity premium and a low risk-free rate. Moreover, this task is complicated by the fact that in the Japanese case the interest rate is lower (one percentage point) than in the Italian case.

The analysis of the results presented above allows to state that, overall, the BH approach seems to be robust also for financial data of countries different than U.S. However, the aspects to be emphasized here are two.

First, the possibility for economic agents to get direct utility not only from consumption but also from fluctuations in their financial (equity) wealth, combined with loss aversion, contributes to improve the performances of frictionless and complete markets models in explaining the actual empirical behavior of the equity risk premium and risk-free rate.

The second point to be highlighted, which is not present in Barberis and Huang (2008, 2009), is the relationship between the risk aversion parameter γ and the narrow framing degree b_0. Our numerical results show that there is a conflicting relationship between the risk aversion parameter and the narrow framing degree. In fact, the simultaneous increase of these parameters affects the behavior of the risk-free interest rate in an opposite way. We know that in the CCAPM, for values consistent with a plausible parameterization of individual preferences, the increase in risk aversion implies, as a net effect, the increase in the interest rate (Campbell, 2003): the growth of current consumption "wins" over the precautionary saving motive. On the other hand, the increase in the narrow framing parameter lowers safe returns, because of the increase in stock market risk which implies an increase in the purchase of safe assets and therefore a reduction in interest rates. The crucial point is that the last effect, by adding to the precautionary saving one, is able to overturn the "standard" effect related to the increase in risk aversion: as a final result, the model is able to match the empirical evidence.

However, the necessity and the importance of investigating the investors' narrow framing degree from different perspectives appears to be evident, in order to further verify the role and weight of narrow framing in economic agents' cognitive decision-making processes.

5 Assessing the exact level of the narrow framing degree

In this section we reverse the problem analysed above, and given the values of some preference parameters, we solve equations (13) and (14) in the unknowns α and b_0. In this case equation (15) is redundant: we have two unknowns, hence two equations are enough. Obviously, we are particularly interested into the narrow framing degree and not into the value of α. The basic idea is to find the exact levels of b_0 that rationalize the observed empirical values of the equity premium and risk-free rate.

From equation (16) we know that

$$EP = (e^{g_S + \frac{\sigma_S^2}{2}} - 1) - r_f = e^{g_S + \frac{\sigma_S^2}{2}} - R_f$$

Given the empirical value of the equity premium (EP) and the riskfree (gross) rate $R_f = 1 + r_f$, we have that

$$g_S = \ln\left(EP + R_f\right) - \frac{\sigma_S^2}{2}$$

With the value of g_S calculated according to the previous expression and the empirical value of R_f, we are able to numerically solve equations (13) - (14) in order "to infer" the "optimal" value of the narrow framing degree b_0.

Tables 10-15 report outcomes derived from quantitative applications relative to U.S., UK, Italy, Germany, France and Japan (see Table 9 for parameter values implemented in solving equations 13-14). We remark the fact that in some cases we have no acceptable solution for b_0 because the solution turns out to be negative, and since b_0 is by definition a non-negative parameter such a solution is not significant.

For the U.S. economy we note that with risk aversion (γ) constant, the level of the narrow framing degree declines as the loss aversion increases (see Table 10). This is consistent with theoretical predictions: when loss aversion raises, the narrow framing degree needed to obtain a given equity premium may be declining, because equity investments are already perceived as particularly risky by means of the increase in λ.

By comparing the results obtained with the same value of the loss aversion parameter but with different levels of risk aversion, we see that the implied narrow framing parameter declines as the level of risk aversion increases. This relationship makes sense according to intuition and basic theory and it is a feature of investors' behavior not only for U.S. but also for UK and Japan (see Table 11 and 15).

In fact, we observe that the same numerical exercise with UK and Japanese data provides very similar outcomes. The first regularity of the U.S. case is confirmed (inverse relationship between b_0 and λ with γ constant), and this holds also for the second regularity: by keeping loss aversion constant, as γ increases the implied narrow framing degree declines (with similar values for U.S. and UK, with smaller values for Japan).

Otherwise, Italy and France share some peculiar results (see Tables 12 and 14). In these two cases, with constant risk aversion b_0 declines as loss aversion raises, as for U.S., UK and Japan, but with λ constant, b_0 slightly increases as risk aversion raises; this last behavior is different from the other three cases and might appear just a bit counterintuitive, but changes in b_0 are so small that it seems to be negligible.

In the Italian case we have very small changes in narrow framing when we consider movements in risk aversion by keeping loss aversion constant. This fact may be interpreted as a signal that Italian investors require almost no change in their narrow framing degree when consumption risk aversion is increasing. About this point, in the French case we have a similar behavior, but with slightly larger changes in the narrow framing parameter.

Finally, the case of Germany seems to be a little controversial. In every section of Table 13, in correspondence of λ equal to 2.75, we have a value of b_0 that can be considered as an "outlier" if compared with the outcomes found for the other countries. However, further sensitivity analyses (not reported in Table 13) show that the strange behavior of the German case is due for the large part to the high level of mean (log) stock return g_S (9.21%, the highest among the six countries considered here). For example, with risk aversion and loss aversion equal, respectively, to 3 and 2.75, by reducing g_S of 1% (8.21%) we obtain $b_0 = 0.14$ (against 1.14); by reducing g_S of 0.5% (8.71%) we obtain $b_0 = 0.31$. Moreover, starting from the second section of Table 13, the behavior of the narrow framing parameter as risk aversion increases, is equal to that of Italian and French investors, but by comparing the first and second section of Table 13 we have instead an opposite behavior, as for U.S., UK and Japan; anyway, movements in b_0 are very small.

Overall, we stress the fact that the values of the narrow framing parameter that rationalize the observed empirical values of the equity premium and risk-free rate are in all cases not greater than 0.24 (with the exception of several outliers greater than 1 for Germany). This empirical regularity is substantially in accordance with the results found in section 4, and, more important, our results concerning b_0 are in line with the values implemented for several simulations in seminal works about asset pricing models with loss aversion/narrow framing preferences: Barberis, Huang and Santos (2001), Barberis and Huang (2001), Barberis, Huang and Thaler (2006), and very recently Barberis and Huang (2008, 2009). In such works the basic values of the narrow framing degree able to match empirical evidence are usually not greater than 1, but more often not greater than 0.50. In the papers quoted above, in many cases, with a value of 0.10, the model is able to match empirical evidence also with regard to other different stock market puzzles (equity home bias, stock market non-participation).[16] The present work contributes further to the investigation of the relevance of behavioral preference parameters, in trying "to capture" the individual preferences more strictly related to individual psychology and irrationality.

6 Concluding remarks

In this paper we study the quantitative implications of a "behavioral" asset pricing model along the lines proposed in a recursive utility framework by Barberis and Huang (2009) (BH approach). In particular, in order to test the validity of such approach when compared with financial data and stock market puzzles, we numerically solve a dynamic representative-agent asset pricing model with loss aversion/narrow framing preferences.

Barberis and Huang (2008, 2009) do not compare the predictions of their model with financial market data different than U.S. ones. Moreover, they do not attempt "to infer" plausible parameter values for the narrow framing degree. We apply the model to U.S., UK, Italy, Germany, France and Japan. Our empirical results show that the BH approach seems to be robust also when it is applied to countries other than U.S. We find interesting heterogeneous behaviors between the various countries.

As a second step, we reverse the basic problem by solving numerically the same model in order to infer the exact values of the narrow framing degree that

[16] About the equity home bias puzzle see also Magi (2009).

rationalize the observed empirical values of the equity premium and risk-free rate. We find that such narrow framing parameter values are generally quite small and however usually not greater than 0.24, with the exception of several outliers in the German case.

Finally, our numerical results show that there is an interesting and conflicting relationship between the risk aversion parameter and the narrow framing degree in affecting the risk-free rate dynamics. In fact, the simultaneous increase of these parameters affects the behavior of the risk-free interest rate in an opposite way. We know that in the C-CAPM, for values consistent with a plausible parameterization of individual preferences, the increase in risk aversion implies, as a net effect, the increase in the interest rate (Campbell, 2003): the growth of current consumption "wins" over the precautionary saving motive. On the other hand, the increase in the narrow framing parameter lowers safe returns, because of the increase in stock market risk which implies an increase in the purchase of safe assets and therefore a reduction in interest rates. The crucial point is that the last effect, by adding to the precautionary saving one, is able to overturn the "standard" effect related to the increase in risk aversion: as a final result, the model produces a low risk-free rate, matching therefore the empirical evidence.

TABLES

Table 1 (data drawn from Campbell, 2003)

	US	UK	ITA	GER	FRA	JAP
Period	1891-1998	1919-1998	1971-1998	1978-1997	1973-1998	1970-1999
EP	6.72%	8.67%	4.68%	8.67%	8.30%	5.10%
r_f	2.0%	1.25%	2.37%	3.22%	2.71%	1.38%

Empirical values for the riskfree rate and equity premium
(annual real values; arithmetic averages)

Table 2 (data drawn from Campbell, 2003)

	US	UK	ITA	GER	FRA	JAP
Period	1891-1998	1919-1998	1971-1998	1978-1997	1973-1998	1970-1999
g_C	1.80%	1.55%	2.20%	1.68%	1.23%	3.20%
σ_C	3.22%	2.88%	1.70%	2.43%	2.91%	2.55%
σ_S	18.60%	22.17%	27.0%	20.10%	23.42%	21.90%
$\omega_{C,S}$	0.45	0.42	-0.033	-0.151	-0.117	0.40
θ_S	0.25	0.25	0.15	0.15	0.15	0.20

Model's parameters, annual real values (arithmetic averages)

Table 3 – U.S. (1891-1998)

β	γ	λ	b_0	r_f	g_S	EP
0.98	1.5	2.25	0	4.71%	3.28%	0.43%
0.98	1.5	2.25	0.05	3.20%	5.65%	4.45%
0.98	1.5	2.25	0.10	2.87%	6.15%	5.33%
0.98	1.5	2.25	0.15	2.75%	6.34%	5.65%
0.98	1.5	3	0.05	2.32%	7.0%	6.80%
0.98	1.5	3	0.10	1.98%	7.51%	7.70%
0.98	1.5	3	0.15	1.86%	7.68%	8.0%
0.98	3	2.25	0	7.20%	6.0%	0.84%
0.98	3	2.25	0.05	4.75%	6.77%	4.12%
0.98	3	2.25	0.10	3.96%	7.0%	5.16%
0.98	3	2.25	0.15	3.66%	7.10%	5.57%
0.98	3	3	0.05	2.87%	7.30%	6.58%
0.98	3	3	0.10	2.0%	7.53%	7.70%
0.98	3	3	0.15	1.74%	7.60%	8.15%

Our elaboration

Table 4 – UK (1919-1998)

β	γ	λ	b_0	r_f	g_S	EP
0.98	1.5	2.25	0	4.34%	2.20%	0.42%
0.98	1.5	2.25	0.05	2.32%	5.36%	5.81%
0.98	1.5	2.25	0.10	2.0%	5.85%	6.66%
0.98	1.5	2.25	0.15	1.89%	6.02%	6.95%
0.98	1.5	3	0.05	1.17%	7.10%	8.85%
0.98	1.5	3	0.10	0.90%	7.51%	9.58%
0.98	1.5	3	0.15	0.82%	7.63%	9.80%
0.98	1.5	3	0.50	0.71%	7.80%	10.10%
0.98	3	2.25	0	6.50%	4.64%	0.85%
0.98	3	2.25	0.05	2.90%	5.72%	5.62%
0.98	3	2.25	0.10	2.10%	5.94%	6.66%
0.98	3	2.25	0.50	1.55%	6.10%	7.38%
0.98	3	2.25	2	1.47%	6.11%	7.47%
0.98	3	3	0.05	-0.34%	6.58%	9.80%
0.98	3	3	0.10	-0.47%	6.61%	9.96%

Our elaboration

Table 5 - Italy (1971-1998)

β	γ	λ	b_0	r_f	g_S	EP
0.98	1.5	2.25	0	5.43%	1.62%	-0.024%
0.98	1.5	2.25	0.05	4.15%	5.71%	5.65%
0.98	1.5	2.25	0.10	3.78%	6.84%	7.27%
0.98	1.5	2.25	0.50	3.41%	7.98%	8.92%
0.98	1.5	3	0.05	3.42%	7.94%	8.86%
0.98	1.5	3	0.10	3%	9.23%	10.74%
0.98	1.5	3	0.50	2.63%	10.33%	12.36%
0.98	1.5	3	2	2.56%	10.54%	12.68%
0.98	3	2.25	0	8.86%	4.80%	-0.048%
0.98	3	2.25	0.05	6.73%	7.14%	4.66%
0.98	3	2.25	0.10	5.85%	8.07%	6.58%
0.98	3	3	0.05	5.38%	8.56%	7.60%
0.98	3	3	0.10	4.19%	9.78%	10.17%
0.98	3	3	2	3%	10.96%	12.72%

Our elaboration

Table 6 – Germany (1978-1997)

β	γ	λ	b_0	r_f	g_S	EP
0.98	1.5	2.25	0	4.58%	2.34%	-0.12%
0.98	1.5	2.25	0.05	3.54%	5.70%	4.48%
0.98	1.5	2.25	0.10	3.30%	6.44%	5.53%
0.98	1.5	2.25	0.50	3.06%	7.20%	6.60%
0.98	1.5	2.25	2	3.0%	7.35%	6.82%
0.98	1.5	3	0.05	3.02%	7.32%	6.74%
0.98	1.5	3	0.10	2.74%	8.18%	7.97%
0.98	1.5	3	0.50	2.50%	8.94%	9%
0.98	3	2.25	0	7%	4.55%	-0.21%
0.98	3	2.25	0.05	5.18%	6.62%	3.84%
0.98	3	2.25	0.10	4.60%	7.27%	5.13%
0.98	3	2.25	0.50	3.94%	7.96%	6.55%
0.98	3	3	0.05	4.14%	7.75%	6.10%
0.98	3	3	0.10	3.40%	8.53%	7.70%
0.98	3	3	0.20	3%	8.94%	8.56%
0.98	3	3	0.50	2.78%	9.18%	9.05%

Our elaboration

Table 7 – France (1973-1998)

β	γ	λ	b_0	r_f	g_S	EP
0.98	1.5	2.25	0	3.84%	0.91%	-0.12%
0.98	1.5	2.25	0.05	2.48%	5.31%	5.90%
0.98	1.5	2.25	0.10	2.26%	6.0%	6.87%
0.98	1.5	3	0.01	2.95%	3.80%	3.80%
0.98	1.5	3	0.03	2.10%	6.51%	7.59%
0.98	1.5	3	0.05	1.81%	7.40%	8.86%
0.98	1.5	3	0.10	1.58%	8.10%	9.86%
0.98	3	2.25	0	5.47%	2.35%	-0.25%
0.98	3	2.25	0.05	2.79%	5.37%	5.66%
0.98	3	2.25	0.10	2.24%	5.96%	6.85%
0.98	3	2.25	0.50	1.82%	6.41%	7.76%
0.98	3	2.25	2	1.75%	6.48%	7.90%
0.98	3	3	0.01	4%	4%	2.97%
0.98	3	3	0.03	1.81%	6.41%	7.77%
0.98	3	3	0.05	1.09%	7.18%	9.33%
0.98	3	3	0.10	0.70%	7.59%	10.18%

Our elaboration

Table 8 – Japan (1970-1999)

β	γ	λ	b_0	r_f	g_S	EP
0.98	1.5	2.25	0	6.98%	4.68%	0.35%
0.98	1.5	2.25	0.05	5.96%	6.88%	3.76%
0.98	1.5	2.25	0.10	5.53%	7.80%	5.20%
0.98	1.5	2.25	0.50	4.96%	8.97%	7.10%
0.98	1.5	3	0.05	5.33%	8.20%	5.84%
0.98	1.5	3	0.10	4.75%	9.41%	7.78%
0.98	1.5	3	0.50	4.13%	10.68%	9.83%
0.98	1.5	3	2	4%	10.94%	10.26%
0.98	3	2.25	0	12%	9.60%	0.75%
0.98	3	2.25	0.05	10.44%	10.51%	3.34%
0.98	3	2.25	0.10	9.61%	11.0%	4.72%
0.98	3	3	0.05	9.45%	11.10%	5.0%
0.98	3	3	0.10	8.20%	11.77%	7.0%
0.98	3	3	0.50	6.48%	12.70%	9.82%

Our elaboration

Table 9 (Data drawn from Campbell, 2003; our elaboration for g_S)

	USA	UK	ITA	GER	FRA	JAP
Period	1891-1998	1919-1998	1971-1998	1978-1997	1973-1998	1970-1999
EP	6.72%	8.67%	4.68%	8.67%	8.30%	5.10%
r_f	2.0%	1.25%	2.37%	3.22%	2.71%	1.38%
g_S	6.63%	7.0%	3.17%	9.21%	7.70%	3.88%
σ_C	3.22%	2.88%	1.70%	2.43%	2.91%	2.55%
σ_S	18.6%	22.17%	27%	20.10%	23.42%	21.90%
$\omega_{C,S}$	0.45	0.42	-0.033	-0.151	-0.117	0.40

Model's parameters, annual real values (arithmetic averages)

Table 10 – U.S. (1891-1998)

β	γ	λ	r_f	EP	b_0
0.98	1.5	2.25	2.0%	6.72%	---
0.98	1.5	2.5	2.0%	6.72%	0.20
0.98	1.5	2.75	2.0%	6.72%	0.069
0.98	1.5	3	2.0%	6.72%	0.041
0.98	3	2.25	2.0%	6.72%	---
0.98	3	2.5	2.0%	6.72%	0.19
0.98	3	2.75	2.0%	6.72%	0.066
0.98	3	3	2.0%	6.72%	0.040
0.98	4	2.25	2.0%	6.72%	---
0.98	4	2.5	2.0%	6.72%	0.19
0.98	4	2.75	2.0%	6.72%	0.064
0.98	4	3	2.0%	6.72%	0.039
0.98	5	2.25	2.0%	6.72%	---
0.98	5	2.5	2.0%	6.72%	0.18
0.98	5	2.75	2.0%	6.72%	0.063
0.98	5	3	2.0%	6.72%	0.038

Our elaboration

Table 11 – UK (1919-1998)

β	γ	λ	r_f	EP	b_0
0.98	1.5	2.25	1.25%	8.67%	---
0.98	1.5	2.5	1.25%	8.67%	---
0.98	1.5	2.75	1.25%	8.67%	0.10
0.98	1.5	3	1.25%	8.67%	0.044
0.98	3	2.25	1.25%	8.67%	---
0.98	3	2.5	1.25%	8.67%	---
0.98	3	2.75	1.25%	8.67%	0.094
0.98	3	3	1.25%	8.67%	0.042
0.98	4	2.25	1.25%	8.67%	---
0.98	4	2.5	1.25%	8.67%	---
0.98	4	2.75	1.25%	8.67%	0.092
0.98	4	3	1.25%	8.67%	0.041
0.98	5	2.25	1.25%	8.67%	---
0.98	5	2.5	1.25%	8.67%	---
0.98	5	2.75	1.25%	8.67%	0.091
0.98	5	3	1.25%	8.67%	0.041

Our elaboration

Table 12 – Italy (1971-1998)

β	γ	λ	r_f	EP	b_0
0.98	1.5	2.25	2.37%	4.68%	0.0167
0.98	1.5	2.5	2.37%	4.68%	0.0125
0.98	1.5	2.75	2.37%	4.68%	0.0100
0.98	1.5	3	2.37%	4.68%	0.0083
0.98	3	2.25	2.37%	4.68%	0.0169
0.98	3	2.5	2.37%	4.68%	0.0126
0.98	3	2.75	2.37%	4.68%	0.0100
0.98	3	3	2.37%	4.68%	0.0083
0.98	4	2.25	2.37%	4.68%	0.0170
0.98	4	2.5	2.37%	4.68%	0.0127
0.98	4	2.75	2.37%	4.68%	0.0101
0.98	4	3	2.37%	4.68%	0.0084
0.98	5	2.25	2.37%	4.68%	0.0172
0.98	5	2.5	2.37%	4.68%	0.0128
0.98	5	2.75	2.37%	4.68%	0.0102
0.98	5	3	2.37%	4.68%	0.0085

Our elaboration

Table 13 – Germany (1978-1997)

β	γ	λ	r_f	EP	b_0
0.98	1.5	2.25	3.22%	8.67%	---
0.98	1.5	2.5	3.22%	8.67%	---
0.98	1.5	2.75	3.22%	8.67%	1.15
0.98	1.5	3	3.22%	8.67%	0.239
0.98	3	2.25	3.22%	8.67%	---
0.98	3	2.5	3.22%	8.67%	---
0.98	3	2.75	3.22%	8.67%	1.14
0.98	3	3	3.22%	8.67%	0.237
0.98	4	2.25	3.22%	8.67%	---
0.98	4	2.5	3.22%	8.67%	---
0.98	4	2.75	3.22%	8.67%	1.15
0.98	4	3	3.22%	8.67%	0.239
0.98	5	2.25	3.22%	8.67%	---
0.98	5	2.5	3.22%	8.67%	---
0.98	5	2.75	3.22%	8.67%	1.17
0.98	5	3	3.22%	8.67%	0.242

Our elaboration

Table 14 – France (1973-1998)

β	γ	λ	r_f	EP	b_0
0.98	1.5	2.25	2.71%	8.30%	---
0.98	1.5	2.5	2.71%	8.30%	0.190
0.98	1.5	2.75	2.71%	8.30%	0.083
0.98	1.5	3	2.71%	8.30%	0.053
0.98	3	2.25	2.71%	8.30%	---
0.98	3	2.5	2.71%	8.30%	0.196
0.98	3	2.75	2.71%	8.30%	0.085
0.98	3	3	2.71%	8.30%	0.055
0.98	4	2.25	2.71%	8.30%	---
0.98	4	2.5	2.71%	8.30%	0.199
0.98	4	2.75	2.71%	8.30%	0.087
0.98	4	3	2.71%	8.30%	0.056
0.98	5	2.25	2.71%	8.30%	---
0.98	5	2.5	2.71%	8.30%	0.205
0.98	5	2.75	2.71%	8.30%	0.089
0.98	5	3	2.71%	8.30%	0.057

Our elaboration

Table 15 – Japan (1970-1999)

β	γ	λ	r_f	EP	b_0
0.98	1.5	2.25	1.38%	5.10%	0.021
0.98	1.5	2.5	1.38%	5.10%	0.014
0.98	1.5	2.75	1.38%	5.10%	0.0104
0.98	1.5	3	1.38%	5.10%	0.0083
0.98	3	2.25	1.38%	5.10%	0.0196
0.98	3	2.5	1.38%	5.10%	0.013
0.98	3	2.75	1.38%	5.10%	0.0097
0.98	3	3	1.38%	5.10%	0.0078
0.98	4	2.25	1.38%	5.10%	0.0189
0.98	4	2.5	1.38%	5.10%	0.0125
0.98	4	2.75	1.38%	5.10%	0.0094
0.98	4	3	1.38%	5.10%	0.0075
0.98	5	2.25	1.38%	5.10%	0.0181
0.98	5	2.5	1.38%	5.10%	0.012
0.98	5	2.75	1.38%	5.10%	0.0090
0.98	5	3	1.38%	5.10%	0.0072

Our elaboration

MATHEMATICAL APPENDIX

A) Derivation of equations (13) - (15).

Now we see how we obtain equations (13) – (15) in the text. Barberis and Huang (2009) show that from the general maximization problem with n assets, if we suppose that a) asset 1 is the risk-free asset, b) the gross risk-free return is the reference point for evaluating stock wealth fluctuations and c) $\theta_{i,t} > 0 \ \ \forall i > 1$, we get the two following first order optimality conditions (FOC):[17]

$$1 = \left[\beta R_f E_t \left((\frac{C_{t+1}}{C_t})^{-\gamma} \right) \right] \cdot \left[\beta E_t \left((\frac{C_{t+1}}{C_t})^{-\gamma} R_{W,t+1} \right) \right]^{\frac{\gamma}{1-\gamma}} \qquad \text{(A1)}$$

$$0 = \frac{E_t\left[(\frac{C_{t+1}}{C_t})^{-\gamma} (R_{i,t+1} - R_{f,t}) \right]}{E_t\left[(\frac{C_{t+1}}{C_t})^{-\gamma} \right]} + \qquad i = 2, \ldots\ldots, n \qquad \text{(A2)}$$
$$+ b_0 I\{i > m\} R_{f,t} (\frac{\beta}{1-\beta})^{\frac{1}{1-\gamma}} (\frac{1-\alpha_t}{\alpha_t})^{\frac{-\gamma}{1-\gamma}} E_t\left[v(R_{i,t+1} - R_{f,t}) \right]$$

These expressions are obtained by using dynamic programming techniques. We note that equation (A1) is equation (10) in the text. In equation (A2) we have an indicator function, $I\{i > m\}$, which means that we have 1 for values $i > m$ and 0 for values $i \le m$. Given the goal of investigating the equity premium and risk-free rate puzzles, as suggested by Barberis and Huang (2009), we do not fix a predetermined value for r_f, a third condition is particularly useful: we obtain it by doing the "weighted" sum of equations contained in (A2), with the i-th equation weighted with $\theta_{i,t}$. We get what follows:

[17] We are also assuming that superscripts of the aggregator and certainty equivalence functions are equal to $1-\gamma$.

$$0=\frac{E_t\left[(\frac{C_{t+1}}{C_t})^{-\gamma}(R_{W,t+1}-R_{f,t})\right]}{E_t\left[(\frac{C_{t+1}}{C_t})^{-\gamma}\right]}+$$
$$+b_0R_{f,t}(\frac{\beta}{1-\beta})^{\frac{1}{1-\gamma}}(\frac{1-\alpha_t}{\alpha_t})^{\frac{-\gamma}{1-\gamma}}\sum_{i=m+1}^{n}E_t\left[\theta_{i,t}v(R_{i,t+1}-R_{f,t})\right] \tag{A3}$$

Now, by considering that the model presented in the text has three assets and only one of them is narrowly framed (equity), equations (A2) and (A3) become as equations (11) and (12). We have $n=3$, $m=2$ and $3=S$. Then, by exploiting assumptions 1 – 4 in the text, from equations (10) – (12) we get equations (13) – (15). Let's start by equation (10). By substituting into this last expression the consumption growth rate, the total wealth return, and by using the other assumptions, we have:

$$1=\left[\beta R_f E_t\left(e^{-\gamma g_C-\gamma\sigma_C\varepsilon_{C,t+1}}\right)\right]\cdot\left[\beta E_t\left(e^{-\gamma g_C-\gamma\sigma_C\varepsilon_{C,t+1}}\cdot e^{g_W+\sigma_W\varepsilon_{W,t+1}}\right)\right]^{\frac{\gamma}{1-\gamma}}$$

where $g_W=g_C+\log(\frac{1}{1-\alpha})$, $\sigma_W=\sigma_C$ and $\varepsilon_{W,t+1}=\varepsilon_{C,t+1}$. Solving expected values, after some steps we have

$$1=\left[\beta R_f e^{-\gamma g_C+\frac{\gamma^2\sigma_C^2}{2}}\right]\cdot\left[\beta E_t\left(e^{(1-\gamma)g_C+(1-\gamma)\sigma_C\varepsilon_{C,t+1}}\cdot\frac{1}{1-\alpha}\right)\right]^{\frac{\gamma}{1-\gamma}} \Rightarrow$$

$$\Rightarrow\quad 1=\left[\beta R_f e^{-\gamma g_C+\frac{\gamma^2\sigma_C^2}{2}}\right]\beta^{\frac{\gamma}{1-\gamma}}\left(\frac{1}{1-\alpha}\right)^{\frac{\gamma}{1-\gamma}}\left(e^{(1-\gamma)g_C+\frac{(1-\gamma)^2\sigma_C^2}{2}}\right)^{\frac{\gamma}{1-\gamma}} \Rightarrow$$

$$\Rightarrow\quad 1=\beta^{\frac{1}{1-\gamma}}R_f\left(\frac{1}{1-\alpha}\right)^{\frac{\gamma}{1-\gamma}}e^{\frac{\gamma\sigma_C^2}{2}} \Rightarrow$$

$$\Rightarrow\quad 1-\alpha=\beta^{\frac{1}{\gamma}}R_f^{-\frac{\gamma-1}{\gamma}}e^{-\frac{1}{2}(\gamma-1)\sigma_C^2} \tag{A4}$$

In solving expected values, as already pointed out in the text, we exploit the fact that if a random variable x is distributed as a standard normal distribution, then

$$E(e^{a+bx})=e^{a+\frac{b^2}{2}}.$$

Hence, from (A4), after some simple steps, we get equation (13).

Now we move to equation (11). By means of some algebraic passages is very easy to show that the first term on the right-hand side is equal to

$$e^{g_S+\frac{\sigma_S^2}{2}-\gamma\omega\sigma_S\sigma_C}-R_f \tag{A5}$$

Instead, we have to pay more attention to the second term on the right-hand side, i.e. the term with the expected value. For this term we have:

$$E_t v(R_{S,t+1}-R_f)=E_t v(e^{g_S+\sigma_S\varepsilon_{S,t+1}}-R_f)=$$

$$E_t[I_{R_{S,t+1}\geq R_f}(e^{g_S+\sigma_S\varepsilon_{S,t+1}}-R_f)+I_{R_{S,t+1}<R_f}\lambda(e^{g_S+\sigma_S\varepsilon_{S,t+1}}-R_f)]=$$

$$E_t[I_{g_S+\sigma_S\varepsilon_{S,t+1}\geq\log(R_f)}(e^{g_S+\sigma_S\varepsilon_{S,t+1}}-R_f)]+E_t[I_{g_S+\sigma_S\varepsilon_{S,t+1}<\log(R_f)}\lambda(e^{g_S+\sigma_S\varepsilon_{S,t+1}}-R_f)]=$$

$$e^{g_S}E_t\left(I_{\varepsilon_{S,t+1}\geq\hat{\varepsilon}_S}e^{\sigma_S\varepsilon_{S,t+1}}\right)-R_fE_t\left(I_{\varepsilon_{S,t+1}\geq\hat{\varepsilon}_S}\right)+\lambda e^{g_S}E_t\left(I_{\varepsilon_{S,t+1}<\hat{\varepsilon}_S}e^{\sigma_S\varepsilon_{S,t+1}}\right)-\lambda R_fE_t\left(I_{\varepsilon_{S,t+1}<\hat{\varepsilon}_S}\right)=$$

where we are using an indicator function and

$$\hat{\varepsilon}_S=\frac{\log(R_f)-g_S}{\sigma_S}.$$

Now we have to solve the four terms of the last expression. In doing so, solving the expected values with indicator functions is obviously necessary, and we do that by taking into account that, in general,

$$E(I_{\varepsilon<\hat{\varepsilon}})=N(\hat{\varepsilon})$$

$$E(I_{\varepsilon<\hat{\varepsilon}}e^{a\varepsilon})=e^{\frac{a^2}{2}}N(\hat{\varepsilon}-a)$$

In our specific case we have:

$$=e^{g_S+\frac{\sigma_S^2}{2}}N(\sigma_S-\hat{\varepsilon}_S)-R_fN(-\hat{\varepsilon}_S)+\lambda e^{g_S+\frac{\sigma_S^2}{2}}N(\hat{\varepsilon}_S-\sigma_S)-\lambda R_fN(\hat{\varepsilon}_S)=$$

By using the fact that $N(x)=1-N(-x)$ and $N(-x)=1-N(x)$, we can write

$$=e^{g_S+\frac{\sigma_S^2}{2}}-R_f+e^{g_S+\frac{\sigma_S^2}{2}}[\lambda N(\hat{\varepsilon}_S-\sigma_S)-N(\hat{\varepsilon}_S-\sigma_S)]-R_f[\lambda N(\hat{\varepsilon}_S)-N(\hat{\varepsilon}_S)]=$$

$$=e^{g_S+\frac{\sigma_S^2}{2}}-R_f+(\lambda-1)[e^{g_S+\frac{\sigma_S^2}{2}}N(\hat{\varepsilon}_S-\sigma_S)-R_fN(\hat{\varepsilon}_S)]. \tag{A6}$$

By combining equations (A5) and (A6) we get equation (14). For obtaining equation (15) starting from equation (12), we note that we have to simply calculate the expression for the first term of the right-hand side, since the second term contains the expected value of function $v(\bullet)$ we have calculated above. The first term of the right-hand side is equal to

$$\frac{1}{1-\alpha}e^{g_C+\frac{\sigma_C^2}{2}-\gamma\sigma_C^2}-R_f\,. \tag{A7}$$

Finally, by combining (A6) and (A7) we get equation (15).

B) Cumulative distribution and error functions.

Let's see some things about the expression $N(\cdot)$ that we find in the three equations (13) – (15) in the text. It is the cumulative distribution function (cdf) of the standard normal distribution, and in our case it indicates the expected value of an *indicator function*, $v(\cdot)$. In equations (13) – (15) we have that

$$E(I_{\varepsilon<\hat{\varepsilon}})=N(\hat{\varepsilon})$$

where we have formalized what said above by using an indicator function.

The cdf of the standard normal distribution is, by definition,

$$P(X\leq x)=N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt=\int_{-\infty}^{x}f(t)dt \tag{B1}$$

where $f(t)=\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ is the standard normal density function.

In solving the 3-equations model of the text, $N(\cdot)$ is implemented in MatLab in function of the so-called *erf* (*error function*), because the MatLab recognizes such a kind of function and use it for solving numerically the relative equations. We can get the *erf* by multiplying times 2 the integral of the density function of a gaussian distribution with mean 0 and variance ½. We have:

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_{0}^{x}e^{-t^2}dt \tag{B2}$$

Such a function has some properties that we will exploit next.

What we are looking for is a way for writing down $N(x)$ as a function of $erf(x)$. We will find the solution by inverting a notorious relationship of mathematical statistics, which gives us the possibility to write down the *erf* as a function of $N(x)$. The relationship is the following one:

$$erf(x) = 2N(x\sqrt{2}) - 1 \tag{B3}$$

where
$$N(x\sqrt{2}) = \int_{-\infty}^{x\sqrt{2}} f(t)dt \tag{B4}$$

Before inverting equation (B3), we verify such relationship by substitution. To this aim we substitute (B4) into the right-hand side of (B3), obtaining

$$2\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x\sqrt{2}} e^{-\frac{t^2}{2}} dt\right] - 1 \tag{B5}$$

Now we operate a change of variable within the integral. Let be

$$z = \frac{t}{\sqrt{2}} \;\Rightarrow\; z^2 = \frac{t^2}{2},$$

which implies $t = z\sqrt{2}$, $dt = \sqrt{2}dz$ and, for the change of the upper-bound of the integral, $t \le x\sqrt{2} \Rightarrow z\sqrt{2} \le x\sqrt{2} \Rightarrow z \le x$. After some substitutions and simplifications into (B5), we have

$$2\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2} \sqrt{2}dz\right] - 1 = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-z^2} dz - 1 \tag{B6}$$

Now, we exploit that

$$erf(+\infty) = \frac{2}{\sqrt{\pi}} \int_{0}^{+\infty} e^{-z^2} dz = 1 \tag{B7}$$

And we note also that
$$\int_{0}^{+\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{2}.$$

By substituting (B7) into (B6) we get

$$\frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-z^2} dz - \frac{2}{\sqrt{\pi}} \int_{0}^{+\infty} e^{-z^2} dz = \frac{2}{\sqrt{\pi}}\left[\int_{-\infty}^{x} e^{-z^2} dz - \int_{0}^{+\infty} e^{-z^2} dz\right] =$$

$$=\frac{2}{\sqrt{\pi}}\left[\int_{-\infty}^{0} e^{-z^2}dz+\int_{0}^{x} e^{-z^2}dz-\int_{0}^{+\infty} e^{-z^2}dz\right]=$$

$$=\frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-z^2}dz=erf(x)$$

The last expression is obtained exploiting the fact that e^{-z^2} is a symmetrical function divided in two parts by the vertical axis and hence the following relationship holds:

$$\int_{0}^{+\infty} e^{-z^2}dz=\int_{-\infty}^{0} e^{-z^2}dz=\frac{\sqrt{\pi}}{2}. \tag{B8}$$

Once proved equation (B3), we invert it in order to write down $N(x)$ as a function of the *error function*. At the end, we will prove that

$$erf(x)=2N(x\sqrt{2})-1\Rightarrow N(x)=\frac{1}{2}\left[erf\left(\frac{x}{\sqrt{2}}\right)+1\right]$$

We have:

$$erf(x)=2N(x\sqrt{2})-1\Rightarrow N(x\sqrt{2})=\frac{1}{2}erf(x)+\frac{1}{2}\Rightarrow$$

$$\Rightarrow\int_{-\infty}^{x\sqrt{2}} f(t)dt=\frac{1}{2}erf(x)+\frac{1}{2} \tag{B9}$$

We can rewrite the last part of the right-hand side, ½, in a particular way, by using the following relationship (which holds because we are working with the standard normal distribution):

$$\int_{-\infty}^{0} f(t)dt=\int_{-\infty}^{x} f(t)dt-\int_{0}^{x} f(t)dt=\frac{1}{2} \tag{B10}$$

By substituting (B10) into (B9) we have:

$$\int_{-\infty}^{x\sqrt{2}} f(t)dt=\frac{1}{2}erf(x)+\int_{-\infty}^{x} f(t)dt-\int_{0}^{x} f(t)dt\Rightarrow$$

$$\Rightarrow\int_{-\infty}^{x\sqrt{2}} f(t)dt=\frac{1}{2}erf(x)+N(x)-\int_{0}^{x} f(t)dt\Rightarrow$$

$$\Rightarrow N(x)=\int_{-\infty}^{x\sqrt{2}} f(t)dt+\int_{0}^{x} f(t)dt-\frac{1}{2}erf(x)\Rightarrow$$

$$\Rightarrow N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x\sqrt{2}} e^{-\frac{t^2}{2}}dt+\frac{1}{\sqrt{2\pi}}\int_{0}^{x} e^{-\frac{t^2}{2}}dt-\frac{1}{2}erf(x) \qquad \text{(B11)}$$

Now we operate another change of variable, both in the first and second integral of the right-hand side of (B11). We have

$$z=\frac{t}{\sqrt{2}} \;\Rightarrow\; z^2=\frac{t^2}{2}$$

$t=z\sqrt{2}$, $dt=\sqrt{2}dz$ and, for the change of the upper-bound of the first integral, $t\le x\sqrt{2}\Rightarrow z\sqrt{2}\le x\sqrt{2}\Rightarrow z\le x$. For the upper-bound of the second integral we have $t\le x\Rightarrow z\sqrt{2}\le x\Rightarrow z\le\frac{x}{\sqrt{2}}$.[18] By doing the relative substitutions into (B11) we get

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2}\sqrt{2}dz+\frac{1}{\sqrt{2\pi}}\int_{0}^{x/\sqrt{2}} e^{-z^2}\sqrt{2}dz-\frac{1}{2}\left(\frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-z^2}dz\right)\Rightarrow$$

$$\Rightarrow N(x)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-z^2}dz+\frac{1}{\sqrt{\pi}}\int_{0}^{x/\sqrt{2}} e^{-z^2}dz-\frac{1}{\sqrt{\pi}}\int_{0}^{x} e^{-z^2}dz$$

By multiplying both sides times 2 and collecting terms we have:

$$2N(x)=\frac{2}{\sqrt{\pi}}\left[\int_{-\infty}^{x} e^{-z^2}dz+\int_{0}^{x/\sqrt{2}} e^{-z^2}dz-\int_{0}^{x} e^{-z^2}dz\right]\Rightarrow$$

$$\Rightarrow 2N(x)=\frac{2}{\sqrt{\pi}}\left[\left(\int_{-\infty}^{x} e^{-z^2}dz-\int_{0}^{x} e^{-z^2}dz\right)+\int_{0}^{x/\sqrt{2}} e^{-z^2}dz\right]\Rightarrow$$

$$\Rightarrow 2N(x)=\frac{2}{\sqrt{\pi}}\left[\int_{-\infty}^{0} e^{-z^2}dz+\int_{0}^{x/\sqrt{2}} e^{-z^2}dz\right] \qquad \text{(B12)}$$

Using equation (B8) we can rewrite (B12) as follows:

[18] The lower-bound remains zero.

$$2N(x) = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-z^2} dz + \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{2}} e^{-z^2} dz$$

but we can also rewrite it in the following way:

$$2N(x) = erf(+\infty) + erf\left(\frac{x}{\sqrt{2}}\right) \Rightarrow$$

$$\Rightarrow 2N(x) = 1 + erf\left(\frac{x}{\sqrt{2}}\right)$$

Finally, we have what we are looking for:

$$N(x) = \frac{1}{2}\left[erf\left(\frac{x}{\sqrt{2}}\right) + 1\right] \qquad \text{(B13)}$$

Equation (B13) is the expression used in solving (with MatLab) equations (13) – (15) in the three unknowns α, R_f and g_S and equations (13) and (14) in the unknowns α and b_0.

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