A Stochastic Optimal Control Model of Pollution Abatement

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Abstract

We model a dynamic monopoly with environmental externalities, investigating the adoption of a tax levied on the firm's instantaneous contribution to the accumulation of pollution. The latter process is subject to a shock, which is i.i.d. across instants. We prove the existence of an optimal tax rate such that the monopoly replicates the same steady state welfare level as under social planning. Yet, the corresponding output level, R&D investment for environmental friendly technologies and surplus distribution necessarily differ from the socially optimal ones.

Keywords: Optimal taxation; Environmental externalities; Stochastic shock, Social planning

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1 Introduction

A lively debate is currently taking place on the need to preserve the environment from the negative consequences of pollution generated by industrial activities.[1] A crucial aspect is the lack of incentives on the part of profit-seeking firms to carry out R&D projects to generate new environmental-friendly technologies.

To this effect, policy makers may adopt several forms of regulation and taxation/subsidization policies to induce firms to internalise externalities and invest accordingly. The standard approach to this problem consists in introducing a Pigouvian tax or subsidy rule whereby firms pay or receive an amount of money proportional to the aggregate current stock of pollutants generated by the industry as a whole. We propose an alternative policy design, where the tax is levied on the marginal contribution to the accumulation process followed by pollution. Accordingly, this policy is close in spirit to the adoption of a quality standard, such as the use of filters capturing CO_2, in order to decrease the amount of pollutants emitted by a car per mile.

We evaluate this perspective in a simple optimal control model where the market is monopolistic. To add a realistic feature to our framework, we allow for the presence of a stochatic shock affecting the accumulation of pollution, in such a way that the resulting optimal tax rate depends on the expected value of the shock. Our main result is that there exists a tax policy (i) inducing the firm to invest in R&D for a greener technology and (ii) yielding the same steady state social welfare as under social planning. However, the two allocations characterising, respectively, the regulated monopoly and the first best differ under all remaining respects, i.e., price, output, R&D investment and surplus distribution.

The remainder of the paper is structured as follows. The model is laid out in section 2. Section 3 contains the analysis of the regulated monopoly. the first best allocation is described in section 4, while section 5 investigates the optimal design of taxation. Concluding remarks are in section 6.

2 The setup

Consider a monopolistic single-product firm facing the instantaneous demand function $p(t) = a - q(t)$, where $a > 0$ is the reservation price and $q(t) \in$

[1] For an exhaustive account, see Stern (2007) and Jørgensen *et al.* (2009).

$[0, a-c]$ is the output level. The production cost is linear in $q(t)$ with unit cost $c \in (0, a)$. The production process involves a negative environmental externality $S(t)$, that accumulates according to the dynamics

$$\dot{S}(t) = b(t)q(t) - \frac{\delta S(t)}{\theta(t)}. \tag{2.1}$$

This evolutionary structure features a depreciation rate $\delta > 0$, which is also affected by a stochastic shock in its slope, in the form of a random variable $\theta(t)$, i.i.d. over time, with mean $E(\theta) = 1$ and variance $Var(\theta) = \sigma_\theta^2 > 1$. For future reference, we define the mean of the reciprocal as $E(\theta^{-1}) = w > 1$, by Jensen's inequality.[2] The assumption that the dynamics of the stock of pollution is subject to shocks has been introduced to capture the idea, largely discussed in the current debate on global warming and the anthropic responsibility in its evolution, that our knowledge of this matter is still incomplete and subject to natural factors beyond human control. In particular, our way of modelling (2.1) refers to uncertainty affecting measures of the rate at which the atmosphere can absorb and eliminate CO_2-equivalent emissions, especially if one takes into account deforestation.[3]

To create an incentive for the monopolist to invest in R&D so as to make its productive technology more environmental-friendly, the government imposes an instantaneous Pigouvian taxation. Usually, the Pigouvian tax is levied on the total amount of the externality (see Karp and Livernois, 1994; Benchekroun and Long, 1998; 2002, *inter alia*). Here, we propose an alternative policy design, whereby the firm is subject to an instantaneous tax equal to $\tau b(t)$, i.e., what is being taxed is indeed the rate $b(t)$ at which a unit of final product contributes to the increase in the stock of pollution. The coefficient $b(t)$ is thus a further state variable whose dynamic equation is a linear one:

$$\dot{b}(t) = -k(t) + \eta b(t), \tag{2.2}$$

with $\eta > 0$, and decreasing in $k(t) \geq 0$, which is the instantaneous R&D effort carried out by the firm. A plausible economic interpretation of $b(t)$ is to see it as the environmental obsolescence rate of technology, measuring the

[2]For examples of analogous approaches in literature on industrial organization, see Klemperer and Meyer (1986) or Lambertini (2006).

[3]According to the Department of Economics and Social Affairs of UNO, "ongoing deforestation accounts for about 8% of the world's annual carbon emissions" (DESA, 2009, p. 86).

growth rate of the external damage involved by the use of technologies that become increasingly more polluting as time goes by.

The R&D technology used by the firm involves an instantaneous cost measured by $\Gamma(t) = zk^2(t)$, where z is a positive constant. The problem for the monopolistic firm consists in maximizing w.r.t controls $k(t)$ and $q(t)$ the expected value of the following payoff functional:

$$J \equiv \int_0^\infty e^{-\rho t}((p(t) - c)q(t) - \Gamma(t) - \tau b(t))dt, \tag{2.3}$$

subject to:

$$\begin{cases} \dot{b}(t) = -k(t) + \eta b(t) \\ \dot{S}(t) = b(t)q(t) - \dfrac{\delta S(t)}{\theta(t)} \\ b(0) = b_0 > 0 \\ S(0) = S_0 > 0. \end{cases} \tag{2.4}$$

This is a modified (monopolistic) version of a dynamic oligopoly game with environmental effects examined in Dragone *et al.* (2009). The main differences consist in (i) the presence of a shock affecting the accumulation of the environmental externality; (ii) the functional form of the dynamics of $b(t)$, that here is such that the model takes a linear-quadratic form; (iii) the tax is levied on the monopolist's contribution to pollution and not on the overall stock of pollution itself.

3 The monopoly optimum

The current value Hamiltonian function reads as:

$$H(\cdot) = (a - c - q(t))q(t) - zk^2(t) - \tau b(t) + \tag{3.1}$$
$$+\lambda(t)(-k(t) + \eta b(t)) + \mu(t)\left(b(t)q(t) - \frac{\delta S(t)}{\theta(t)}\right),$$

where $\lambda(t)$ is the costate variable associated to the state $b(t)$ and $\mu(t)$ is the one associated to the other state $S(t)$. Because of the aleatory effect, the monopolist is supposed to maximize the expected value of the Hamiltonian, $E(H)$. From now on, we will drop the time argument for brevity.

What follows is the list of the necessary conditions for the maximization of $E(H)$, adjoint equations and transversality conditions (an application of

Pontryagin's Maximum Principle in a stochastic framework can be found in Lambertini, 2005):[4]

$$\frac{\partial E(H)}{\partial k} = -2zk - \lambda = 0, \tag{3.2}$$

$$\frac{\partial E(H)}{\partial q} = a - c - 2q + \mu b = 0, \tag{3.3}$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial E(H)}{\partial b} = (\rho - \eta)\lambda - \mu q + \tau, \tag{3.4}$$

$$\dot{\mu} = \rho\mu - \frac{\partial E(H)}{\partial S} = (\rho + \delta w)\,\mu, \tag{3.5}$$

$$\lim_{t \longrightarrow \infty} e^{-\rho t}\lambda(t) = 0, \qquad \lim_{t \longrightarrow \infty} e^{-\rho t}\mu(t) = 0. \tag{3.6}$$

Note that in (3.5) we make use of the expected value $E(\theta^{-1}) = w$. By differentiating (3.2) and (3.3) w.r.t. time, (2.4), (3.4) and (3.5) amount to the following state-control dynamical system:

$$\begin{cases} \dot{b} = -k + \eta b \\ \dot{S} = bq - \delta w S \\ \dot{k} = (\rho - \eta)k - \dfrac{(a-c-2q)q}{2zb} - \dfrac{\tau}{2z} \\ \dot{q} = \dfrac{1}{2}\left(\dfrac{-a+c+2q}{b}\right)[-k + (\eta + \rho + \delta w)b] \end{cases} . \tag{3.7}$$

Proposition 3.1. *The model admits a unique steady state $P^* = (b^*, S^*, k^*, q^*)$, whose coordinates are, respectively,*

$$b^* = \frac{\tau}{2\eta(\rho - \eta)z}, \quad S^* = \frac{\tau(a-c)}{4\eta(\rho - \eta)z\delta w}, \quad k^* = \frac{\tau}{2(\rho - \eta)z}, \quad q^* = \frac{a-c}{2}.$$

Proof. Solving (3.7) yields a unique stationary point P^*, whose coordinates are all strictly positive if $\rho - \eta > 0$. □

[4] We omit the explicit exposition of second order conditions for a maximum as they are satisfied by construction.

Clearly, if $\rho \in [0, \eta)$, then τ must be negative in order for the vector (b^*, S^*, k^*, q^*) to be economically meaningful. In this range, the fact that discounting is lower than the environmental obsolescence rate entails that the only feasible policy takes the form of a subsidy. Conversely, for all $\rho > \eta$, τ must be positive, i.e., the regulator has to tax the firm to induce the entrepreneur to carry out a positive amount of R&D.

Note that, while k^* is a function of τ, q^* is not. This immediately implies that, by adopting this policy, the regulator is providing the firm with an incentive to carry out R&D while leaving unaffected the choice of the optimal monopoly output (and therefore the corresponding price level).

As a clear consequence of the dynamics of the model, the only coordinate affected by uncertainty is S^*, i.e., the steady state level of the pollution stock is a function of w. The Jacobian matrix of (3.7) evaluated at P^* is:

$$J(P^*) = \begin{pmatrix} \eta & 0 & -1 & 0 \\ \dfrac{a-c}{2} & -\delta w & 0 & \dfrac{\tau}{2\eta(\rho-\eta)z} \\ 0 & 0 & \rho-\eta & \dfrac{\eta(a-c)(\rho-\eta)}{\tau} \\ 0 & 0 & 0 & \rho+\delta w \end{pmatrix}.$$

Proposition 3.2. *P^* is a saddle point for the system (3.7).*

Proof. $J(P^*)$ has the negative eigenvalue $\lambda_1 = -\delta w$, and the positive eigenvalues $\lambda_2 = \rho + \delta w$, $\lambda_3 = \eta$ and that is sufficient to deduce that P^* represents a saddle point equilibrium for (3.7). □

Note that, if $\rho > \eta$, the stable subspace $E(P^*)$ is spanned by the vector $(0, 1, 0, 0)$, that is, on the S-axis the time trajectory of the stock of pollution asymptotically heads towards the level S^*.

Since the model features a single agent, there obviously exists a unique feedback stationary strategy coinciding with the open-loop solution. In particular, the related optimal value function $V(b, S)$ satisfying the Hamilton-Jacobi-Bellman equation is linear-quadratic in b and linear in S.

3.1 Welfare and profit assessment

Let π^*, CS^* and SW^* be the profit, the consumer surplus and the social welfare functions evaluated at the steady state P^*. We have that:

$$\pi^* = (a - c - q^*)q^* - z(k^*)^2 - \tau b^* = \frac{(a-c)^2}{4} - \frac{\tau^2(3\eta - 2\rho)}{4\eta(\rho-\eta)^2 z}, \tag{3.8}$$

independent of w.

Proposition 3.3. *1. If $\rho > \frac{3\eta}{2}$, then $\pi^* > 0$ for every τ.*

2. If $0 < \rho < \frac{3\eta}{2}$, then:

$$\pi^* > 0 \ \forall\ \tau \in \left(\min\left\{\mp(a-c)(\rho-\eta)\sqrt{\frac{\eta z}{3\eta - 2\rho}}\right\},\right.$$
$$\left.\max\left\{\mp(a-c)(\rho-\eta)\sqrt{\frac{\eta z}{3\eta - 2\rho}}\right\}\right).$$

Proof. Trivially, the expression (3.8) is strictly positive irrespective of the value of τ if $\rho > \frac{3\eta}{2}$, whereas if $\rho < \frac{3\eta}{2}$, the positivity is ensured if τ belongs to the interval $(-\tau_1,\ \tau_1)$, where $\tau_1 = \max \pm(a-c)(\rho-\eta)\sqrt{\frac{\eta z}{3\eta - 2\rho}}$. □

The consumer surplus at equilibrium reads:

$$CS^* = \frac{(q^*)^2}{2} + \tau b^* = \frac{(a-c)^2}{8} + \frac{\tau^2}{2\eta(\rho-\eta)z}.$$

Finally, the social welfare at equilibrium follows:

$$SW^* = CS^* - S^* + \pi^* = \frac{3(a-c)^2}{8} - \frac{\tau^2}{4(\rho-\eta)^2 z} - \frac{\tau(a-c)}{4\eta(\rho-\eta)z\delta w}.$$

Proposition 3.4. *$SW^* > 0$ for every τ belonging to the interval:*

$$\left(\min\left\{-\frac{(a-c)(\rho-\eta)}{2}\left(\frac{1}{\eta\delta w} \mp z\sqrt{\frac{1}{\eta^2 z^2 \delta^2 w^2} + \frac{6}{z}}\right)\right\},\right.$$
$$\left.\max\left\{-\frac{(a-c)(\rho-\eta)}{2}\left(\frac{1}{\eta\delta w} \mp z\sqrt{\frac{1}{\eta^2 z^2 \delta^2 w^2} + \frac{6}{z}}\right)\right\}\right).$$

Proof. It suffices to solve the inequality $SW^* > 0$ with respect to τ. □

4 The first best

Now we briefly expose the first best solution that would be attained if the firm were run by a benevolent planner maximising the discounted flow of social welfare w.r.t. q and k. The planner's Hamiltonian is:

$$\begin{gathered}H_P(\cdot) = (a-c-q)q - \frac{q^2}{2} - S - zk^2 + \\ +\lambda(-k+\eta b) + \mu\left(bq - \frac{\delta S}{\theta}\right).\end{gathered} \tag{4.1}$$

Taking the necessary conditions on the expected value of $H_P(\cdot)$ and following the same procedure as in the previous section, we obtain the following steady state coordinates (the subscript P stands for *planner*):

$$\begin{aligned}b_P &= \frac{(a-c)(\rho+\delta w)}{1+2\eta z(\rho+\delta w)^2(\rho-\eta)} \\ S_P &= \frac{2(a-c)^2\eta(\rho-\eta)(\rho+\delta w)^3 z}{\delta w\left[1+2\eta z(\rho+\delta w)^2(\rho-\eta)\right]^2} \\ k_P &= \frac{(a-c)(\rho+\delta w)\eta}{1+2\eta z(\rho+\delta w)^2(\rho-\eta)} \\ q_P &= \frac{2(a-c)\eta(\rho-\eta)(\rho+\delta w)^2 z}{1+2\eta z(\rho+\delta w)^2(\rho-\eta)}\end{aligned} \tag{4.2}$$

Note that all of these coordinates are affected by the shock. The associated profits and consumer surplus are:

$$\pi_P = \frac{(a-c)^2\eta(2\rho-3\eta)(\rho+\delta w)^2 z}{\left[1+2\eta z(\rho+\delta w)^2(\rho-\eta)\right]^2} \tag{4.3}$$

$$CS_P = \frac{1}{2}\left[\frac{2(a-c)\eta(\rho-\eta)(\rho+\delta w)^2 z}{1+2\eta z(\rho+\delta w)^2(\rho-\eta)}\right]^2 \tag{4.4}$$

Hence, the resulting social welfare level is $SW_P = \pi_P + CS_P - S_P$. Before proceeding any further, it is worth stressing the following straightforward result:

Proposition 4.1. *If* $\rho > \dfrac{3\eta}{2}$, *then* $\pi_P > 0$.

We are now in a position to address the following question, i.e., whether the policy maker regulating the behaviour of a profit-maximising monopolist can design an optimal tax rate τ so as to replicate the same welfare performance associated to the first best allocation we have just sketched. This must be done under the non-negativity constraint concerning the firm's profits, as established in Proposition 3.3.1. In doing so, we shall confine our attention to the parameter range $\rho > 3\eta/2$, in order for the palnning equilibrium to be sustainable under our partial equilibrium appoach, i.e., in absence of any other industrial sector that could be taxed to raise the money neccesary for the survival of the public monopoly for all $\rho \in [0, 3\eta/2)$.

5 Designing the optimal taxation

The policy maker's problem consists in solving

$$\Delta SW = SW_P - SW^* = 0 \tag{5.1}$$

w.r.t. τ, with

$$\Delta SW = \frac{1}{8}\left[\frac{2\tau^2}{(\rho-\eta)^2 z} + \frac{2(a-c)\tau}{\delta\eta(\rho-\eta)wz} - 3(a-c)^2 + \Psi\right] \tag{5.2}$$

where

$$\Psi \equiv \frac{8\eta(a-c)^2(\rho+\delta w)^2 z\left[2\delta\eta(\rho-\eta)^2 w(\rho+\delta w)^2 z - 2\rho(\rho-\eta) - \delta\eta w\right]}{\delta\left[1+2\eta z(\rho+\delta w)^2(\rho-\eta)\right]w} \tag{5.3}$$

Equation (5.1) has two real roots in τ, $\tau_- < 0 < \tau_+$.[5] On this basis, we can state our final result:

Proposition 5.1. *For all $\rho > \frac{3\eta}{2}$, there exist a tax (τ_+) allowing the policy maker to replicate at the monopoly equilibrium the social welfare performance associated with the first best.*

The negative solution must be discarded in view of Proposition 4.1. As a last remark, again recollecting Proposition 3.1, it is worth pointing out that such a policy can only reproduce the aggregate surplus created by this

[5]We omit the lengthy expressions of the two roots for the sake of brevity.

industry, while the output and the R&D effort will necessarily differ across regimes. To see this, it's sufficient to compare q^* against q_P: while the former is constant (and coincides with the standard output that we usually observe in a monopoly equilibrium with the same demand and cost functions), the latter clearly accounts for the stochastic process affecting the accumulation of pollution. Additionally, one may observe that $k^* = k_P$ obtains in correspondence of a value of τ that does not solve (5.1).

6 Concluding remarks

In a dynamic monopoly model with environmental externalities, we have investigated the possibility of using a tax taylored on the firm's instantaneous contribution to the accumulation of pollution, which is subject to a shock, the latter being i.i.d. across instants. There exists an optimal tax rate such that the industry exactly replicates the same steady state welfare performance as in the first best. However, the corresponding output level, R&D investment for green technologies and surplus distribution necessarily differ from those characterising social planning.

An interesting extension of the foregoing analysis is the design of the same kind of policy in an oligopoly game where each single firm might refrain from investing in environmental friendly technologies due to the usual free riding incentive usually associated with strategic interplay. This is left for future research.

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