VALUING R&D INVESTMENTS WITH A JUMP-DIFFUSION PROCESS

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Abstract

Traditional tools based on DCF methods fail to capture the value of R&D projects because of their dependence on future events that are uncertain at the time of the initial decision. We consider a continuous-time framework where information arrives both continuously and discontinuously. This is modelled by a jump-diffusion process. This assumption better decribes the evolution of asset value due to the risky nature of many real investments. The main contribution of this paper is to derive a closed-form solution for the multicompound option to value sequential investment opportunities when the underlying asset may reasonably undergo the possibility of jumps in value.

key words: multicompound options; sequential investments; jump-diffusion process.

JEL Classification: G 12; G 13; G 30; C 69

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1 Introduction

As several researchers have noted R&D ventures are essentially real growth options. The value of these early projects derives not so much from their expected cash flows as from the follow-on opportunities they may create. Although traditional tools fail to capture the value of these investments, because of their dependence on future events that are uncertain at the time of the initial decision, firms engage the pilot to get started a multi-stage process that may eventually reach a commercial phase of launching the new product. Take the example of developing a new drug. Investing in R&D in the pharmaceutical industry, begins with research that leads with some probability to a new compound; such a project continues with testing and concludes with the construction of a production facility and the marketing of the product. Because many early investments can be seen as chains of interrelated projects, the earlier of which is prerequisite for those to follow, they can be evaluated as multicompound options which involve sequential decisions to exercise the options to invest only when the R&D outcomes are successful. Compound options have been extensively used in corporate finance to evaluate investment opportunities. For example, Geske (1979) suggested that when a company has common stock and coupon bonds outstanding, the firm's stock can be viewed as a call option on a call option. Carr (1988) analyzed sequential compound options, which involve options to acquire subsequent options to exchange an underlying risky asset for another risky asset. Gukhal (2003) derives analytical valuation formulas for compound options when the underlying asset follows a jump-diffusion process. Agliardi and Agliardi (2006) study multicompound options in the case of time-dependent volatility and interest rate. This assumption seems more suitable due to the sequential nature of many early projects. Multicompound options are merely N-fold options of options. Basically the procedure consists of solving N-nested Black-Scholes partial differential equations: at the first step the underlying option is priced according to the Black-Scholes method; then, compound options are priced as options on the securities whose values have already been found in the earlier steps. Roll (1977), Whaley (1981), Geske and Johnson (1894) and Selby and Hodges (1987) also study compound options.

The objective of this paper is to study the multicompound options approach to value sequential investment opportunities when the underlying asset follows a jump-diffusion process. Many authors have suggested that incorporating jumps in option valuation models may explain some of the large empirical biases exhibited by the Black-Scholes model. This is true because the assumption of jump-diffusion process better decribe the evolution of asset value due to the risky nature of many early investments. For istance, many new business ventures are subject to several, qualitatively different sources of risk. There is the uncertainty associated with the market factors outside the control of the firm, such as demand for the product and production costs. There is the exogenous risk associated with the actions of a competitor. Finally, there is the technical uncertainty which is idiosyncratic to the firm. Traditional option methodology assumes a continuous cash-flow generation process which is inadequate when these types of risk jointly determine the value of a new venture. However, in many cases, closed-form solutions for valuing options with jump-diffusion process are not available. The main contribution of this paper is to derive a pricing formula for multicompound options when the jump distribution is log-normal; in doing so, we integrate work on multicompound options[1] by Agliardi and Agliardi [3] with that on compound options[2] by Gukhal [13].

The paper is organized as follows. Section 2 reviews the literature on real options and its application to the valuation of R&D ventures and start-up companies. This is followed by a description of the economic model in Section 3. Section 4 derives a closed-form solution for multicompound options in which the equation for the underlying process is replaced by a more general mixed diffusion-jump process. An extension to pricing sequential expansion options is presented in section 5. Section 6 concludes the paper.

2 Literature review

A number of existing research contribution has previously analyzed various aspects of optimal sequential investment behaviour for firm facing multi-stage projects. Staging investment involves firms either with some degree of flexibility in proceeding with investment or when there is a maximum rate at which outlays or construction can proceed, that is, it takes time-to-build. The real option literature have studied the R&D process as a contingent claim on the value of the underlying cash flows on completion of the R&D project. Majd and Pindyck (1987) develop a continuous investment model with time-to-build.

[1] In this paper the multicompound option c_N is expressed in the form: $c_N(S,t) = SN_N(h_N(t) + \sqrt{\int_t^{T_N} \sigma^2(\tau)d\tau}..., h_1(t) + \sqrt{\int_t^{T_1} \sigma^2(\tau)d\tau}; \Xi_N^{(N)}(t)) - \sum_{j=1}^{N} X_j e^{-\int_t^{T_j} r(\tau)d\tau} N_{N+1-j}(h_N(t), ..., h_j(t); \Xi_{N+1-j}^{(N)}(t))$, where $h_k(t) = \left(\ln \frac{S}{S_k^*} + \int_t^{T_k} (r(\tau) - \frac{\sigma^2(\tau)}{2})d\tau\right) / \left(\int_t^{T_k} \sigma^2(\tau)d\tau\right)^{\frac{1}{2}}$ and $\Xi_k^{(N)}(t)$ denotes a k-dimension correlation matrix with typical element $\rho_{ij}(t) = \left(\int_t^{T_j} \sigma^2(\tau)d\tau / \int_t^{T_i} \sigma^2(\tau)d\tau\right)^{\frac{1}{2}}$ for $1 \leq i \leq j \leq k, t \leq T_k$.

[2] If Y has a log-normal distribution the value of the compound call option is given by: $\sum_{n_1=0}^{\infty} \frac{e^{-\lambda T_1}(\lambda T_1)^{n_1}}{n_1!} K_1 e^{-rT_1} N[a_2] + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{e^{-\lambda T_1}(\lambda T_1)^{n_1}}{n_1!} \frac{e^{-\lambda \upsilon}(\lambda \upsilon)^{n_2}}{n_2!} \left\{S_0 N_2[a_1, b_1, \rho_{1T}] - Ke^{-rT} N_2[a_2, b_2, \rho_{1T}]\right\}$, where $a_1 = \frac{\ln(S_0/S_1^*) + (\mu_{JD1} + \sigma_{JD1}^2/2)T_1}{\sigma_{JD1}\sqrt{T_1}}$, $a_2 = a_1 - \sigma_{JD1}\sqrt{T_1}$, $b_1 = \frac{\ln(S_0/K) + (\mu_{JD1} + \sigma_{JD}^2/2)T}{\sigma_{JD}\sqrt{T}}$, $b_2 = b_1 - \sigma_{JD}\sqrt{T}$ and $\rho_{1T} = \frac{cov(x_{T_1}, x_T)}{\sqrt{var(x_{T_1})var(x_T)}}$.

They solve an investment problem in which the project requires a fixed total investment to complete, with a maximum instantaneous rate of investment. Pindyck (1993) also takes into account market and technical uncertainty. Myers and Howe (1997) present a life cycle model of investments in pharmaceutical R&D programs; the problem is solved using Monte Carlo simulation. Childs and Triantis (1999) develop and numerically implement a model of dynamic R&D investment that highlights the interactions across projects. Schwartz and Moon (2000) have also studied R&D investment projects in the pharmaceutical industry using a real options framework. In this articles, they numerically solve a continuous-time model to value R&D projects allowing for three types of uncertainty: technical uncertainty associated with the success of the R&D process itself, an exogenous chance for obsolescence and uncertainty about the value of the project on completion of the R&D stages. Schwartz (2003) develops and implements a simulation approach to value patents-protected R&D projects based on the real option approach. It takes into account uncertainty in the cost to completion of the project, uncertainty in the cash flows to be generated from the project, and the possibility of catastrophic events that could put an end to the effort before it is completed. Errais and Sadowsky (2005) introduce a general discrete time dynamic framework to value pilot investments that reduce idiosyncratic uncertainty with respect to the final cost of a project. In this model, the pilot phase requires N stages of investment for completion that they value as a compound perpetual Bermudan option. Although the preceding articles suggested the use of more suitable technique when we attempt to value intangible project that are linked to the future opportunities they create, such investments are hard to value, even with the real options approach. The main reason for this is that there are multiple sources of uncertainty in R&D investment projects and that they interact in complicated way. In practice, the bulk of the literature on the R&D valuation have dealt with the development of numerical simulation methods based on optimal stopping time problems. Berk, Green and Naik (2004) develop a dynamic model of multi-stage investment project that captures many features of R&D ventures and start-up companies. Their model assumes different sources of risk and allow to study their interaction in determining the value and risk premium of the venture. Closed-form solutions for important cases are obtained. More recently, a number of articles consider strategic interaction features in R&D. Miltersen and Schwartz (2004) develop a model to analyze patent-protected R&D investment projects when there is multiple sources of uncertainty in R&D stages and imperfect competition in the development and marketing of the resulting product. Grenadier (2002) adds a time-to-build features in a model of option exercise games.

Our study differs from those mentioned above in several crucial respects. We provide a model which relies on simple mathematatics to price options with jump-diffusion process. We emphasize that sequential investments opportunities, as for example R&D projects, can be valued in a continuous-time framework based on the Black-Scholes model.

3 Model and assumptions

Let us consider the investment decision by a venture capital fund that is evaluating a single R&D project. We assume that the commercial phase of the project cannot be launched before a pilot phase consisting on N-stages of investment is completed. The risk free rate in our setting will be denoted by r. Let I be the amount of investment required for completion of any R&D stage. When the R&D is successfully completed, the project will generate a stream of stochastic cash flows, which we model as a mixed diffusion-jump process:

$$dV_t = (\alpha - \lambda k) V_t dt + \sigma V_t dz_t + (Y - 1) V_t dq_t, \tag{1}$$

where α is the instantaneous expected return on the underlying asset; σ is the instantaneous standard deviation of the return, conditional on no arrivals of important new information[3]; dz is the standard Brownian motion; dq is the independent Poisson process with rate λt; $(Y - 1)$ is the proportional change in the asset value due to a jump and $k \equiv E[Y - 1]$; dq and dz are assumed to be independent.

The total uncertainty in the underlying project is posited to be the composition of two type of risk: the systematic risk and the technical risk. The former is generally related to economic fundamentals that causes marginal changes in the asset value. This is associated with demand for the product and production costs and is modelled by a standard geometric Brownian motion. The technical risk which represents the discontinuous arrival of new information has more than a marginal effect on the asset value. This component is modelled by a jump process reflecting the non-marginal impact of information. Usually, such information is specific to the firm: for example, a new drug may be rendered unnecessary by a superior treatment option, the entry by a new competitor who take out a patent for a drug that is targeted to cure the same disease, the possibility of political and technical unpredictable information that will cause V to jump. Assume that the logarithmic jump amplitude, $\ln(Y)$, is normally distributed with mean (μ_J) and variance (σ_J^2); then, the version of Ito's lemma for a diffusion-jump stochastic process is:

$$dx_t = \left(\alpha - \lambda k - \frac{1}{2}\sigma^2\right) dt + \sigma dz_t + \ln(Y) dq_t.$$

As in Merton (1976) we assume that technical uncertainty is completely diversifiable, that is, the firm will not demand any additional return over the risk free rate for being exposed to this source of risk. This fact will allow us to specify

[3] Further, it is possible to include both a time-varying variance and time-varying interest rate; see Agliardi and Agliardi (2003) and Amin (1993) for a discussion of this point.

a unique equivalent risk-neutral measure by setting the market price of risk of q to zero. Although, it may be too strong an assumption for industries where firms may place an important premium on idiosyncratic risk, this assumption seems unlikely to change results significantly (see Errais and Sadowsky, (2005) for further details). In the particular case when the expected change in the asset price is zero, given that the Poisson event occurs (*i.e.*, $k = 0$) [4], by following standard arguments in the financial mathematics literature we can construct the risk-neutral pricing measure under which we will work for the remaining of the paper[5]. The process for the underlying asset value under $\mathbb{Q}$ is given by:

$$dV_t = rV_t dt + \sigma V_t d\tilde{z}_t + (Y - 1) V_t dq_t. \tag{2}$$

In the real options setting, investment opportunities may be viewed as options; thus, the pricing formula for multicompound option can be applied to evaluate the N-stages pilot we described earlier. In more specific terms, let $F(V, t)$ denote the value of a European call option with exercise price I_1 and expiration date T_1. Let us now define inductively a sequence of call options, with value F_k, on the call option whose value is F_{k-1}, with exercise price I_k and expiration date T_k, $k = 1, .., N$, where we assume $T_1 \geq T_2 \geq ... \geq T_N$. Because all the calls are function of the value of the firm V and the time t, the following partial differential equation holds for F_k:

$$\frac{\partial F_k}{\partial t} = rF_k - rV\frac{\partial F_k}{\partial V} - \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F_k}{\partial V^2} - \lambda E\left\{F_k(VY, t) - F_k(V, t)\right\},$$

$t \leq T_k$, $k = 1, .., N$, $T_1 \geq T_2 \geq .. \geq T_N$. The boundary condition is:

$$F_k\left(F_{k-1}(V, T_k), T_k\right) = \max\left(F_{k-1}(V, T_k) - I_k, 0\right),$$

where $F_{k-1}(V, T_k)$ stands for the price of the underlying compound option. Naturally, if $k = 1$ the well-known pricing formula for simple option is obtained:

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda T_1}(\lambda T_1)^n}{n!}\left(VN_1(a_1) - I_1 e^{-rT_1} N_1(b_1)\right),$$

with:

$$a_1 = \frac{\ln\left(\frac{V}{I_1}\right) + \left(r + \frac{\nu^2}{2}\right)T_1}{\nu\sqrt{T_1}}, \qquad b_1 = a_1 - \nu\sqrt{T_1},$$

where $t = 0$ and $\nu^2 = \sigma^2 + \frac{n\sigma_J^2}{T_1}$, conditional on the number of jumps n.

[4] See Merton (1976, pp. 135-136) for a discussion of this point.

[5] We refer the reader to Musiela and Rutkowsky (1998) for additional details.

4 Derivation of the valuation formula

We want to determine the value of the investment opportunity $F_k(V,t)$ in each stage T_k, $k=1,..,N$, of the pilot conditioning on the discontinuous arrival of new information. To simplify notation, we assume that t equals zero. Let V_k^* denote the value of V such that $F_{k-1}(V,T_k)-I_k=0$ if $k>1$ and $V_1^*=I_1$. Moreover, let us set $s_k=\sum_{i=k}^{N} n_i$ the total number of jumps in the interval $[0,T_k]$, $k=1,..,N$, $T_1\geq T_2\geq..\geq T_N$.

Let us define now:

$$b_k=\frac{\ln\left(\frac{V}{V_k^*}\right)+\left(r-\frac{\nu^2}{2}\right)T_k}{\nu\sqrt{T_k}}, \tag{3}$$

and:

$$a_k=b_k+\nu\sqrt{T_k}, \tag{4}$$

where $\nu^2T_k=\sigma^2T_k+s_k\sigma_J^2$, $k=1,..,N$. Moreover, let:

$$\rho_{ij}=\sqrt{\frac{T_j}{T_i}}, \quad \text{for } 1\leq i<j\leq N, \tag{5}$$

the correlation between the logarithmic returns x_{T_j} and x_{T_i} conditioning on the number of jumps s_j and $S_i=s_i-s_j$. For any k, $1\leq k\leq N$, let $\Xi_k^{(N)}$ denote a k-dimension symmetric correlation matrix with typical element $\rho_{ij}=\rho_{N-k+i,N-k+j}$. Let $N_k(b_k,...,b_1;\Xi_k)$ denote the k-dimension multinormal cumulative distribution function, with upper limits of integration $b_1,...,b_k$ and correlation matrix Ξ_k. Finally, let $\sum_{n_k=0}^{\infty}\frac{e^{-\lambda}\lambda^{n_k}}{n_k!}..\sum_{n_1=0}^{\infty}\frac{e^{-\lambda}\lambda^{n_1}}{n_1!}$ denote the joint probability function of k independent Poisson processes with rate λ. We mean the discontinuous arrivals of new information are assumed to be independent of each other[6]. Our aim is to derive a valuation formula for the N-fold multicompound option. Let V_N^* denote the value of V such that $F_{N-1}(V,T_N)-I_N=0$. Then, for V greater than V_N^* the N^{th}- compound call option will be exercised, while for values less than V_N^* it will remain unexercised.

The value of the multicompound option is the expected present value of the resulting cash flows on the completed project :

$$E_0^{\mathbb{Q}}\left[e^{-rT_1}(V-I_1)1_{\varepsilon_1}..1_{\varepsilon_N}\right]+\sum_{j=2}^{N}E_0^{\mathbb{Q}}\left[e^{-rT_j}(-I_j)1_{\varepsilon_j}..1_{\varepsilon_N}\right], \tag{6}$$

[6] See Kocherlakota and Kocherlakota (1992) for a more detailed development of the bivariate Poisson distribution.

where $\varepsilon_k = \{V_k \geq V_k^*\}$, $k = 1, .., N$. The first term in (6) can be written in the form:

$$E_0^{\mathbb{Q}} \left\{ e^{-rT_N} E_{T_N}^{\mathbb{Q}} \left[.. \left\{ e^{-r\tau_1} \left(V - I_1 \right) 1_{\varepsilon_1} \right\} .. \right] 1_{\varepsilon_N} \right\}, \tag{7}$$

$\tau_k = T_k - T_{k+1}$. To examine option pricing when the asset price dynamics include the possibility of non-local changes, we condition the expectation to the number of jumps between any points in time:

$$E_0^{\mathbb{Q}} \left\{ \sum_{n_N=0}^{\infty} \left[.. \left\{ \sum_{n_1=0}^{\infty} \left[e^{-r\tau_1} \left(V - I_1 \right) 1_{\varepsilon_1} \mid n_1 \right] prob\left(n_1 \right) \right\} ..1_{\varepsilon_N} \mid n_N \right] prob\left(n_N \right) \right\}. \tag{8}$$

The evaluation of the expectation requires the calculation of the joint probability function of N independent Poisson processes with rate λt :

$$\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} \left(\lambda T_N \right)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda \tau_1} \left(\lambda \tau_1 \right)^{n_1}}{n_1!} \times$$

$$E_0^{\mathbb{Q}} \left\{ e^{-rT_N} E_{T_N}^{\mathbb{Q}} \left[.. \left\{ e^{-r\tau_1} \left(V - I_1 \right) 1_{\varepsilon_1} \right\} .. \right] 1_{\varepsilon_N} \mid n_1, .., n_N \right\}. \tag{9}$$

To evaluate the first expectation we will work with the logarithmic return x_{T_k}, rather than V. Conditioning on the number of jumps, s_k, $\ln x_{T_k} \sim N\left[\eta, \nu^2 T_k\right]$ where $\eta T_k = \left(r - \frac{\nu^2}{2} \right) T_k$ and $\nu^2 T_k = \sigma^2 T_k + s_k \sigma_J^2$. The price of the multicompound option at time 0 equals:

$$\sum_{n_n=0}^{\infty} \frac{e^{-\lambda T_N} \left(\lambda T_N \right)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda \tau_1} \left(\lambda \tau_1 \right)^{n_1}}{n_1!} \times$$

$$e^{-rT_N} \left\{ \int_{-\infty}^{b_N} N'(y) \left(g\left(y \right) N_{N-1} \left(\hat{a}_{N-1}, .., \hat{a}_1; \hat{\Xi}_{N-1}^{(N-1)} \right) dy + \right) \right.$$

$$\left. - \int_{-\infty}^{b_N} N'(y) \left(I_1 e^{-r(T_1 - T_N)} N_{N-1} \left(\hat{b}_{N-1}, .., \hat{b}_1; \hat{\Xi}_{N-1}^{(N-1)} \right) \right) dy \right\}, \tag{10}$$

where $\hat{a}_k = a_k \left(g\left(y \right), V_k^*, T_k, T_N \right)$, $\hat{b}_k = b_k \left(g\left(y \right), V_k^*, T_k, T_N \right)$ for $k = 1, .., N-1$, and the entries of $\hat{\Xi}_k^{(N-1)}$ are $\beta_{N-1-k+i, N-1-k+j}$, where we define $\beta_{ij} = \frac{-\rho_{ij}}{\sqrt{1-\rho_{ij}^2}}$, for $i < j$. Note that the critical values V_k^* above which the k^{th}-multicompound option will be exercised, are determined recursively and their existence and uniqueness are guaranteed in view of the expression of F_{k-1} (see Remark 3).

The function $g : \mathbb{R} \to \mathbb{R}$ is given by the formula:

$$g(y) = V_0 \exp\left[\left(r - \frac{\nu^2}{2}\right) T_N + \nu\sqrt{T_N} \cdot y\right], \tag{11}$$

where y has a standard Gaussian probability law under $\mathbb{Q}$. Straighforward calculations yield:

$$\hat{a}_k = \frac{\ln\left(\frac{V_0}{V_k^*}\right) + \left(r + \frac{\nu^2}{2}\right)(T_k - T_N) + \left(r - \frac{\nu^2}{2}\right) T_N}{\nu\sqrt{T_k - T_N}} + y\sqrt{\frac{T_N}{T_k - T_N}}, \tag{12}$$

$$\hat{b}_k = \frac{\ln\left(\frac{V}{V_k^*}\right) + \left(r - \frac{\nu^2}{2}\right) T_k}{\nu\sqrt{T_k - T_N}} + y\sqrt{\frac{T_N}{T_k - T_N}}, \text{ for } k = 1, .., N-1. \tag{13}$$

The second integral in (10) can be expressed in terms of the N-dimension multinormal cumulative distribution function by applying the following

Lemma 1 *Let $1 \leq k < N$, and let $\hat{\Xi}_k^{(N-1)}$ be the matrix obtained from $\Xi_k^{(N-1)}$ replacing any element β_{ij} with $\frac{-\rho_{ij}}{\sqrt{1-\rho_{ij}^2}}$, by setting*

$$\alpha_k = \frac{\ln\left(\frac{V}{V_k^*}\right) + \left(r - \frac{\nu^2}{2}\right) T_k}{\nu\sqrt{T_k - T_N}},$$

where α and β are real numbers, the following identity holds:

$$\int_{-\infty}^{b_N} N'(y)\, N_k\left(\alpha_{N-1} + y\beta_{N-1,N}, .., \alpha_{N-k,N} + y\beta_{N-k,N}; \hat{\Xi}_k^{(N-1)}\right) dy =$$

$$N_{k+1}(b_N, ..., b_{N-k}; \Xi_{k+1}^{(N)}).$$

Proof. *by induction after solving the following equation $\frac{b_k}{\sqrt{1-\rho_{k,N}^2}} = \alpha_k$ and $\frac{-\rho_{k,N}}{\sqrt{1-\rho_{k,N}^2}} = \beta_{k,N}$, $k = 1, .., N-1$, for b_k and $\rho_{k,N}$.* ■

Finally, we succeed in writing the first integral in (10) in terms of the cumulative function of the multivariate normal distribution using Lemma 1, after making the following substitution $x = y - \sigma\sqrt{T_N}$; thus we get:

$$\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N}\left(\lambda T_N\right)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda\tau_1}\left(\lambda\tau_1\right)^{n_1}}{n_1!} \times$$
$$\left[V_0 N_N\left(a_N, .., a_1; \Xi_N^{(N)}\right) - I_1 e^{-rT_1} N_N\left(b_N, .., b_1; \Xi_N^{(N)}\right)\right].$$

The second expectation in (6) can be evaluate to give:

$$-\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N}\left(\lambda T_N\right)^{n_N}}{n_N!} .. \sum_{n_j=0}^{\infty} \frac{e^{-\lambda\tau_j}\left(\lambda\tau_j\right)^{n_j}}{n_j!} \times$$
$$\left\{\sum_{j=2}^{N} I_j e^{-rT_j} N_{N+1-j}\left(b_N, .., b_j; \Xi_{N+1-j}^{(N)}\right)\right\}, \quad j = 2, .., N.$$

Hence, we have the following result for the value of a multicompound call option:

Proposition 2 *The value of the multicompound call option F_N with maturity T_N and strike price I_N written on a compound call option F_{N-1} with maturity T_{N-1} and strike price I_{N-1} is given by:*

$$\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N}\left(\lambda T_N\right)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda\tau_1}\left(\lambda\tau_1\right)^{n_1}}{n_1!} \left[V_0 N_N\left(a_N, .., a_1; \Xi_N^{(N)}\right)\right] +$$
$$-\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N}\left(\lambda T_N\right)^{n_N}}{n_N!} .. \sum_{n_j=0}^{\infty} \frac{e^{-\lambda\tau_j}\left(\lambda\tau_j\right)^{n_j}}{n_j!} \times$$
$$\left[\sum_{j=1}^{N} I_j e^{-rT_j} N_{N+1-j}\left(b_N, .., b_j; \Xi_{N+1-j}^{(N)}\right)\right], \quad j = 1, .., N;$$

where the a_is, the b_is and the ρ_{ij}s are as defined previously.

Remark 3 *It can be proved that $\partial_V F_k = N_k(b_k, ..., b_1; \Xi_k^{(k)})$. Thus uniqueness of V_k^* is guaranteed for every k, $1 \leq k \leq N$.*

In the particular case when $\lambda = 0$, the formula reduces to [3]. This proposition is the main result of the paper and forms the basis for the valuation of sequential investment opportunities, as for example R&D ventures, including the possibility of jumps in the underlying asset value.

5 An extension

In Carr [7] sequential exchange opportunities are valued using the techniques of modern option-pricing theory. The vehicle for analysis is the concept of compound exchange option. Accordingly, the real option literature has suggested that sequential expansion opportunities can be viewed as compound exchange options. Trigeorgis (1996) highlighted that many new business ventures, as R&D and start-up projects, can be seen as the base-scale projects plus an option to make additional investments. For example, the opportunities for a firm to continuously expand its technology represents a critical component of the sofware providing industry's investment decisions. The firms' ability to later expand capacity is clearly more valuable for more volatile business with higher returns on project, such as computer software or biotechnology, than it is for traditional business, as real estate or automobile production. Neverthless, the value of these early investments is generally subject to considerable uncertainty, because of their dependence on future events that are uncertain at the time the base-scale takes place. Market factors outside the control of the firm change continuously and have considerable effect on the value of these investment opportunities. Moreover, when the new software product comes together with technological innovations, there is also considerable uncertainty with respect to the actions of a competitor or changes in environment before or soon after technological improvements. For example, a software product may fail because of technological advances in hardware.

In this section we attempt to evaluate sequential technology adoptions as in Carr (1988). As before, we could relax the assumption of a pure diffusion process for the underlying asset value, to illustrate the case where new technology competitors arrive randomly according to an exogenous Poisson distribution. A pricing-formula for multicompound exchange option with jump-diffusion process is obtained.

5.1 The mathematical problem and solution

Since this problem and its solution are extensions of the multicompound call option formula, i will use the same notation and assumptions as much as possible. We consider the valuation of a European sequential exchange option $F_k(V_1, V_2, t)$ which can be exercised at T_k, where $T_1 \geq T_2 \geq ... \geq T_N$. Assume that the prices of both assets follow the same stochastic differential equation (1). Let φ_{12} denote the correlation coefficient between the Wiener processes dz_1 and dz_2; dq_i and dz_i are assumed to be independent as well dq_i and dq_j, $i, j = 1, 2$. As suggested by Margrabe (1978), the valuation problem can be reduced to that of a one-asset option if we treat V_1 as numeraire. Accordingly, we define a new random variable $V = \frac{V_2}{V_1}$, which is again lognormal. The option sells for $F_k(V_1, V_2, t)/V_1 = W_k(V, t)$. The risk-free rate in this market is zero. The functional governing the multicompound exchange option's value $W_k\left(W_{k-1}\left(V,T_k\right),T_k\right)$ is known at expiration to be $\max\left(W_{k-1}\left(V,T_k\right)-q_k,0\right)$ where q_k is the exchange ratio. This problem is analogous to that of section 4 if we treat q_k as the exercise price of the option. Our aim is to derive a valuation formula for the N-fold multicompound exchange option, that is for $W_N(V,t)$, $0\le t\le T_N$. Let V_N^* denote the value of V such that $W_{N-1}\left(V,T_N\right)-q_N=0$ and $V_1^*=q_1$. To simplify notation we will assume again $t=0$. Let us define now:

$$b'_k=\frac{\ln\left(\frac{V}{V_k^*}\right)-\frac{\nu^2}{2}T_k}{\nu\sqrt{T_k}}, \tag{14}$$

and:

$$a'_k=b'_k+\nu\sqrt{T_k}, \tag{15}$$

where $\nu^2=\nu_1^2-2\varphi_{12}\nu_1\nu_2+\nu_2^2$; Finally, we set ρ_{ij} as in (5).

The current value of the multicompound exchange option W_N follows by:

$$E_0^{\mathbb{Q}}\left[\left(V-q_1\right)1_{\varepsilon_1}..1_{\varepsilon_N}\right]+\sum_{j=2}^{N}E_0^{\mathbb{Q}}\left[\left(-q_j\right)1_{\varepsilon_j}..1_{\varepsilon_N}\right]. \tag{16}$$

The derivation of the pricing formula is standard. We assume that the random variable Y has the same log-normal distribution as we described before. In this case the logarithmic return x_{T_k} will have a normal distribution with mean equals $\eta T_k=\left(r-\frac{\nu^2}{2}\right)T_k$ and variance equals $\nu^2T_k=\sigma^2T_k+s_k\sigma_J^2$. The evaluation of the first expectation in (16) requires the calculation of the joint probability function of N independent Poisson processes with rate λt. Solving as in $(7)-(9)$, we obtain:

$$\begin{aligned}&\sum_{n_n=0}^{\infty}\frac{e^{-\lambda T_N}\left(\lambda T_N\right)^{n_N}}{n_N!}..\sum_{n_1=0}^{\infty}\frac{e^{-\lambda\tau_1}\left(\lambda\tau_1\right)^{n_1}}{n_1!}\times\\&\left\{\int_{-\infty}^{b_N}N'(y)\left(\tilde{g}\left(y\right)N_{N-1}\left(\tilde{a}_{N-1},..,\tilde{a}_1;\hat{\Xi}_{N-1}^{(N-1)}\right)\right)dy+\right.\\&\left.-\int_{-\infty}^{b_N}N'(y)\left(q_1N_{N-1}\left(\tilde{b}_{N-1},..,\tilde{b}_1;\hat{\Xi}_{N-1}^{(N-1)}\right)\right)dy\right\},\end{aligned} \tag{17}$$

where $\tilde{g}\left(y\right)$ equals (11), $\tilde{a}_k$ equals (12) and $\tilde{b}_k$ equals (13) if $r=0$. The calculation of the second integral in (16) is straighforward. Finally, in light of Lemma 1, we obtain the following:

Proposition 4 *The value of the sequential exchange option F_N with maturity T_N and strike price q_N written on a exchange option F_{N-1} with maturity T_{N-1} and strike price q_{N-1} is given by:*

$$\sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} \left(\lambda T_N\right)^{n_N}}{n_N!} .. \sum_{n_1=0}^{\infty} \frac{e^{-\lambda \tau_1} \left(\lambda \tau_1\right)^{n_1}}{n_1!} \left[V_{02} N_N \left(\acute{a}_N, .., \acute{a}_1; \Xi_N^{(N)} \right) \right] +$$

$$- \sum_{n_N=0}^{\infty} \frac{e^{-\lambda T_N} \left(\lambda T_N\right)^{n_N}}{n_N!} .. \sum_{n_j=0}^{\infty} \frac{e^{-\lambda \tau_j} \left(\lambda \tau_j\right)^{n_j}}{n_j!} \times$$

$$\left[V_{01} \sum_{j=1}^{N} q_j N_{N+1-j} \left(\acute{b}_N, .., \acute{b}_j; \Xi_{N+1-j}^{(N)} \right) \right], \quad j = 1, .., N;$$

where the $\acute{a}_i s$, the $\acute{b}_i s$ and the $\rho_{ij} s$ are as defined previously.

Of course, when $\lambda = 0$, the formula reduces [7].

6 Summary

We have proposed a multicompound option approach to value sequential investment opportunity where the underlying asset is subject to two types of uncertainty: market and technical uncertainty. The former is generally related to economic fundamentals and always driving the value of a project, while technical uncertainty is idiosyncratic to the firm and associated with the success of the venture itself. These features are modelled by assuming that the underlying asset follows a jump-diffusion process. Finally, by assuming that technical uncertainty is completely diversifiable and that the jump distribution is lognormal, close-form solutions for simple multicompound and for multicompound exchange options are obtained.

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