Persuasive Advertising in Oligopoly: A Linear State Differential Game

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Abstract

We investigate a linear state differential oligopoly game with advertising, under either Cournot or Bertrand competition. We show that a unique saddlepoint equilibrium exists in both cases if the marginal cost of advertising is sufficiently low. Then, we prove that Bertrand competition entails more intense advertising than Cournot competition. This is due to the fact that enhancing reservation prices is more relevant to firms when market competition is tougher. Ultimately, this may entail that Cournot outperforms Bertrand when it comes to social welfare.

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1 Introduction

After the pioneering contributions of [20] and [16], a wide debate on the nature and effects of advertising has developed. The existing literature on dynamic models of advertising can be broadly partitioned into two classes: the first establishes a direct relationship between the rate of change in sales (or market shares) and the advertising efforts of firms (see, e.g., [7], [9] and [15]); the second considers advertisement as an instrument to increase the stock of goodwill or reputation and establishes a link between the advertising efforts of a firm and her market demand ([11], [13], [17] and [4]).[1]

We present a dynamic oligopoly model with differentiated products, where firms compete either *à la* Cournot or *à la* Bertrand in the market phase, and each firm invests in advertising activities aimed at increasing its market size, or equivalently the consumers' reservation price for its product. The advertising activity which we consider in the present dynamic framework is of a persuasive nature, has long-run effects and presents positive external effects that spill over to the rival firms. In modelling this issue, we elaborate upon [3].

As to the market decision, we consider an oligopoly with linear demands for substitute goods.[2]

Our main results can be summarised as follows. First, we show that our model is a linear state game (see [6]), i.e., the state variables enter both the state equations and the objective functions linearly. Therefore, the solution of the differential oligopoly game benefits both from the easy analytical procedure of the open-loop concept and from the strong time consistency

[1] See [6, ch. 11], [8], [10], [12] and [14] for exhaustive surveys.

[2] As is well known, this approach encompasses monopolistic competition and homogenous oligopoly as special subcases (see [1], [19] and [21]).

property of the closed-loop concept. Second, we characterise the conditions ensuring that the dynamic system converges to a unique and economically meaningful steady state. We also prove that, under the same conditions, such a steady state is stable in the saddlepoint sense. The third result concerns the comparison of advertising incentives under price-setting and quantity-setting behaviour, respectively. We find that the parameter region where an internal solution does exist is wider under Cournot competition; moreover, in the subset of the parameter space where the steady state exists irrespective of the nature of market competition, there appears that advertising efforts are more intense under price competition than under quantity competition. This is clearly due the fact that, with substitute goods, Bertrand behaviour entails a harsher competition which drives firms to invest a larger amount of resources to increase reservation prices as compared to what they do under a milder type of competition such as Cournot behaviour. As to the social welfare at the market level, we find - consistently - that Cournot competition may lead to a larger social welfare as compared to Bertrand competition. This happens precisely because the tougher competition characterising price-setting behaviour leads firms to higher investment efforts and hence to lower profits that outweigh the larger consumer surplus.

The remainder of the paper is organised as follows. Section 2 illustrates the basic setup. Section 3 carries out the equilibrium analysis and the comparative assessment of optimal advertising efforts. Section 4 presents the results concerning social welfare. Section 5 contains some concluding remarks.

2 The model

As in [3], we consider an oligopoly game played over continuous time $t \in [0, \infty)$. The set of firms is $\mathbb{P} \equiv \{1, 2, 3, ...N\}$. Each firm i produces one differentiated variety of the same good. Let $p_i(t)$ denote the price of good i, and $q_i(t)$ the quantity of good i at time t. Firm i faces the following instantaneous demand function, as in [18]:

$$q_i(t) = \frac{A_i(t)}{1 + S(N-1)} - \frac{p_i(t)\,[1 + S\,(N-2)]}{(1-S)\,[1 + S(N-1)]} + \frac{S \sum_{j \neq i} p_j(t)}{(1-S)\,[1 + S(N-1)]} \quad (1)$$

Variable A_i describes the market size or the reservation price for good i. Parameter $S \in [0, 1)$ measures the degree of substitutability between any pair of differentiated goods. If $S = 0$, goods are independent and each firm becomes a monopolist. In the limit case where $S \to 1$, goods are perfect substitutes and the model collapses into the homogeneous oligopoly model.[3] Market size may be increased by firms through advertising. The dynamics of firm i's market size is:

$$\frac{dA_i(t)}{dt} \equiv \dot{A}_i(t) = k_i(t) + \gamma \sum_{j \neq i} k_j(t) - \delta A_i(t) \quad (2)$$

where k_h is the effort in advertising made by firm h, $\gamma \in [0, 1]$ is a parameter capturing the external effect of the advertising of a firm on the market size of different firms and $\delta \geq 0$ is a depreciation parameter. Given that investment increases market size or reservation price, this is a form of persuasive advertising.

Advertising entails a quadratic cost $\Gamma_i(k_i(t)) = \alpha(k_i(t))^2/2$ with $\alpha > 0$; production entails a linear costs, $c_i(q_i(t)) = cq_i(t)$ with $c > 0$. Prices and

[3] For differential games where S is a state variable that changes through firms' R&D efforts for product differentiation, see [2] and [5].

advertising efforts are controls, while market sizes are states. Each player chooses the path of her control variables over time, in order to maximize the present value of her profit flow, subject to (i) the motion laws regarding the state variables, and (ii) the initial conditions. Formally, the problem of player i may be written as follows. The objective function is:

$$\max_{p_i(t),k_i(t)} J_i \equiv \int_0^\infty \pi_i(t)e^{-\rho t}dt \tag{3}$$

where the factor $e^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant and common to all players. Instantaneous profits are $\pi_i(t) = [p_i(t) - c]\, q_i(t) - \Gamma_i(k_i(t))$. Function (3) is subject to the set of N dynamic constraints of type (2), and to the set of initial conditions $\mathbf{A}(0) = \mathbf{A}_0$. The Hamiltonian function of firm i is

$$\mathcal{H}_i(t) \equiv e^{-\rho t}\left\{\pi_i(t) + \lambda_{ii}(t) \cdot \dot{A}_i + \sum_{j\neq i} \lambda_{ij}(t) \cdot \dot{A}_j\right\} \tag{4}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$ is the costate variable (evaluated at time t) associated by player i with state variable A_j.

3 Equilibrium analysis

With reference to the labels used by [6, ch. 7], the problem at hand is a linear state differential game whose open-loop solution is strongly time consistent. To prove this fact, it suffices to examine the first order conditions (FOCs) and costate equations taken on the Hamiltonian function of firm i:

$$\begin{aligned}\mathcal{H}_i(t) \equiv\; & e^{-\rho t}\Bigg[(p_i(t)-c)\left(\frac{A_i(t)}{1+S(N-1)}-\frac{p_i(t)\,[1+S\,(N-2)]}{(1-S)\,[1+S(N-1)]}+\right.\\ & \left.+\frac{S\sum_{j\neq i}p_j(t)}{(1-S)\,[1+S(N-1)]}\right)-\frac{\alpha}{2}(k_i(t))^2+\\ & +\lambda_{ii}(t)\cdot\left(k_i(t)+\gamma\sum_{j\neq i}k_j(t)-\delta A_i(t)\right)+\\ & +\sum_{j\neq i}\lambda_{ij}(t)\cdot\left(k_j(t)+\gamma\sum_{h\neq j}k_h(t)-\delta A_j(t)\right)\Bigg] \qquad (5)\end{aligned}$$

The FOCs for the open-loop solution are:[4]

$$\frac{\partial\mathcal{H}_i(t)}{\partial p_i(t)}=\frac{A_i(t)}{1+S(N-1)}-\frac{[2p_i(t)-c]\,[1+S\,(N-2)]-S\sum_{j\neq i}p_j(t)}{(1-S)\,[1+S(N-1)]}=0 \qquad (6)$$

$$\frac{\partial\mathcal{H}_i(t)}{\partial k_i(t)}=-\alpha k_i(t)+\lambda_{ii}(t)+\gamma\sum_{j\neq i}\lambda_{ij}(t)=0 \qquad (7)$$

$$\frac{\partial\lambda_{ii}(t)}{\partial t} = \lambda_{ii}(t)\,(\delta+\rho)-\frac{p_i(t)-c}{1+S(N-1)} \qquad (8)$$

$$\frac{\partial\lambda_{ij}(t)}{\partial t} = \lambda_{ij}(t)\,(\delta+\rho) \qquad (9)$$

Now, from (6) we have that $\partial^2\mathcal{H}_i(t)/\partial p_i(t)\partial A_j(t)=0$ for all $j\neq i$. Likewise, from (7) we have $\partial^2\mathcal{H}_i(t)/\partial k_i(t)\partial A_j(t)=0$, since costate equations (8-9) are indeed independent of the state variables. Accordingly, we may state:

Lemma 1 *The game is a linear state one. Hence, the open-loop solution is strongly time consistent, or subgame perfect.*

We can now proceed to characterise the equilibrium. From (7),

$$k_i(t)=\frac{\lambda_{ii}(t)+\gamma\sum_{j\neq i}\lambda_{ij}(t)}{\alpha} \qquad (10)$$

[4] We omit the indication of exponential discounting for the sake of brevity.

which can be differentiated w.r.t. time to obtain the dynamics of the advertising effort:[5]

$$\dot{k}_i = \frac{\dot{\lambda}_{ii} + \gamma \sum_{j \neq i} \dot{\lambda}_{ij}}{\alpha} \tag{11}$$

The next step consists in noting that (9) is a separable differential equation admitting the solution $\lambda_{ij} = 0$ at all t. Therefore, $\dot{k}_i = \dot{\lambda}_{ii}/\alpha$. Moreover, the expression of λ_{ii} can be obtained from (7), as follows:

$$\lambda_{ii} = \alpha k_i \tag{12}$$

which, together with (8), can be plugged into the advertising dynamics:

$$\dot{k}_i = \frac{1}{\alpha} \left[(\rho + \delta)\alpha k_i - \frac{p_i - c}{1 + S(N-1)} \right]. \tag{13}$$

Having rewritten the dynamics of k_i in terms of controls and parameters only, we can introduce the symmetry condition whereby $p_j = p_i = p$, $k_j = k_i = k$ and $A_j = A_i = A$ for all i and j. By doing so, we may solve (6) to yield the optimal pricing rule:

$$p^* = \frac{A\,(1 - S) + c\,[1 + S\,(N-2)]}{2 + S\,(N-3)}. \tag{14}$$

Note that $p^* = 0$ in the limit case where $S \to 1$, as is well known from the static oligopoly literature (see, e.g., [18]). Using (14), (13) rewrites as follows:

$$\dot{k} = (\rho + \delta)k - \frac{(A - c)\,(1 - S)}{\alpha\,[1 + S(N-1)]\,[2 + S\,(N-3)]} \tag{15}$$

and imposing stationarity, $\dot{k} = 0$ yields:

$$k^* = \frac{(A - c)\,(1 - S)}{\alpha(\rho + \delta)\,[1 + S(N-1)]\,[2 + S\,(N-3)]}. \tag{16}$$

[5]Henceforth we will omit the indication of time for the sake of brevity.

As a last step, we may plug (16) into (2) and impose $\dot{A} = 0$ to obtain the steady state level of the reservations price:

$$A_B = \frac{c(1-S)[1+\gamma(N-1)]}{(1-S)[1+\gamma(N-1)] - \alpha\delta(\rho+\delta)[1+S(N-1)][2+S(N-3)]}. \tag{17}$$

This goes along with the steady state advertising investment:

$$k_B = \frac{c(1-S)\delta}{(1-S)[1+\gamma(N-1)] - \alpha\delta(\rho+\delta)[1+S(N-1)][2+S(N-3)]}. \tag{18}$$

It is easily checked that A_B and k_B are both positive iff $\alpha \in [0, \alpha_B)$, where:

$$\alpha_B = \frac{(1-S)[1+\gamma(N-1)]}{\delta(\rho+\delta)[1+S(N-1)][2+S(N-3)]}. \tag{19}$$

The stability properties of the steady state characterised above can be evaluated on the basis of the trace and determinant of the following Jacobian matrix:

$$J_B \equiv \begin{bmatrix} -\delta & 1+(N-1)\gamma \\ -\dfrac{1-S}{\alpha[1+S(N-1)][2+S(N-3)]} & \rho+\delta \end{bmatrix} \tag{20}$$

The trace is $T(J_B) = \rho$, while the determinant is:

$$\Delta(J_B) = \frac{(1-S)[1+\gamma(N-1)]}{\alpha[1+S(N-1)][2+S(N-3)]} - \delta(\rho+\delta) \tag{21}$$

with $\Delta(J_B) < 0$ for all $\alpha \in [0, \alpha_B)$. Consequently, it appears that the steady state is stable in the saddlepoint sense within the same parameter range where the equilibrium values of states and controls are indeed economically acceptable.

We may summarise the foregoing discussion by stating:

Proposition 2 *Provided $\alpha \in [0, \alpha_B)$, the advertising game with Bertrand competition admits a unique saddlepoint equilibrium (A_B, p_B, k_B).*

From [3], we know that the corresponding Cournot game produces a unique saddlepoint equilibrium at (A_C, q_C, k_C) with

$$k_C = \frac{\delta c}{1 + (N-1)\gamma - \alpha\delta(\rho+\delta)(2 + S(N-1))} > 0 \tag{22}$$

$$A_C = \frac{c\,(1 + (N-1)\gamma)}{1 + (N-1)\gamma - \alpha\delta(\rho+\delta)(2 + S(N-1))} > 0 \tag{23}$$

for all $\alpha \in [0, \alpha_C)$, where

$$\alpha_C = \frac{1 + (N-1)\gamma}{\delta(\rho+\delta)(2 + S(N-1))} \tag{24}$$

and $\alpha_B < \alpha_C$ always. This entails the following result:

Lemma 3 *The parameter region where the saddlepoint equilibrium exists is wider under Cournot competition than under Bertrand competition.*

Keeping in mind that α measures the marginal cost of advertising, the reason for this fact appears to be that Cournot competition is softer than Bertrand. Thus, it is easier for firms to finance advertising campaigns when market behaviour follows the Cournot rule.

Accordingly, the two games can be compared only for $\alpha \in [0, \alpha_B)$. In this range, we may compare k_B against k_C and A_B against A_C. Take first the optimal investment levels:

$$\begin{aligned} k_B - k_C &\propto \alpha\delta(\rho+\delta)\,[1 + S(N-1)]\,[2 + S\,(N-3)] \cdot \Psi + \\ &\quad - (1-S)\,[1 + (N-1)\gamma] \\ \Psi &\equiv \alpha\delta(\rho+\delta)\,[2 + S(N-1)] - 1 - (N-1)\gamma \end{aligned} \tag{25}$$

with the r.h.s. of (25) being equal to zero in correspondence of α_B and α_C. This suffices to prove the following result:

Proposition 4 $k_B > k_C$ *for all* $\alpha \in [0, \alpha_B)$ *, irrespective of the value of* γ *and* S.

This is fully consistent with the *ex ante* intuition which would suggest that the incentive to increase reservation prices should be higher under price-setting behaviour, as this is harsher than quantity-setting behaviour, all else equal. Indeed, the above Proposition tells that such incentive is higher under Bertrand competition than under Cournot competition everywhere, provided that the condition for an internal solution to obtain is satisfied.

4 Social welfare

It is immediate to compare the social welfare at the market level, in steady state, under Bertrand and Cournot behaviour.

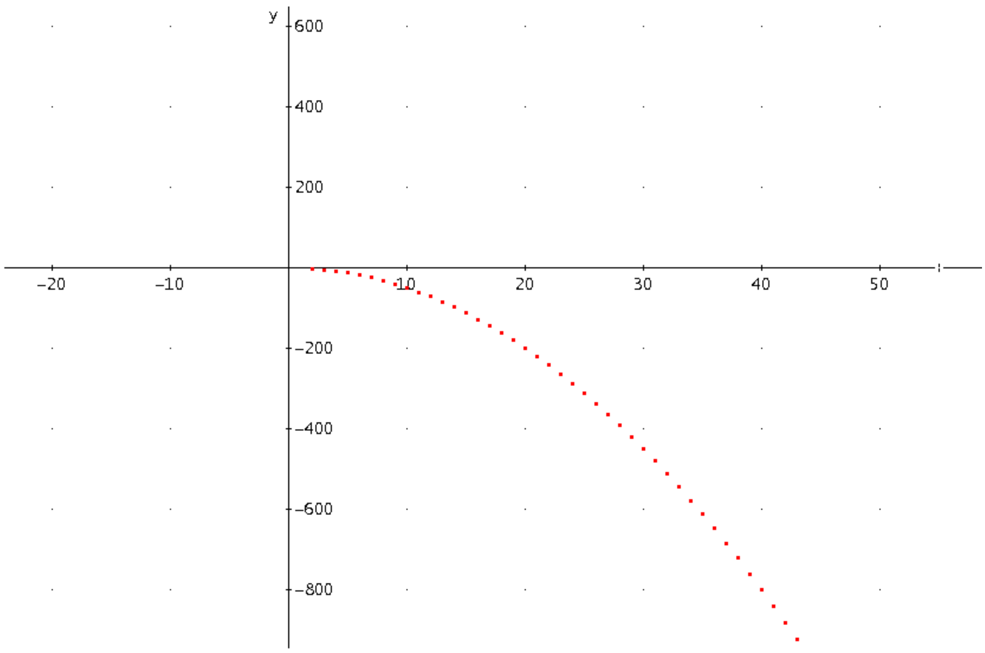
As a social welfare index, we consider the sum of consumer surplus and the profits of firms (net of the cost of advertising investments). Hence, it is

$$SW = N\left[(A-c)q - \left(\frac{1}{2} + S(N-1)\right)q^2 - \frac{\alpha}{2}k^2\right] > 0 \qquad (26)$$

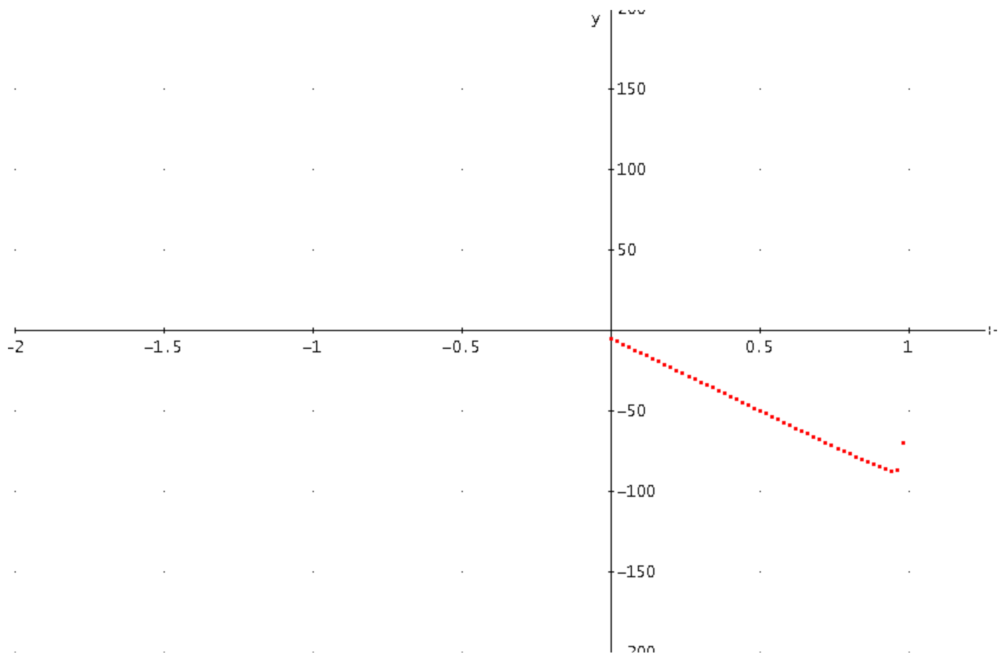
Taking the relevant values of variables from [3], one can find the social welfare in the steady state under Cournot competition (SW^C). Similarly, simple substitutions for the relevant values under price competition allow to obtain the steady state social welfare level under for the Bertrand game (SW^B). The difference between the two indeces is $DSW = SW^B - SW^C$, which is quite a cumbersome expression, depending on the set of parameters $\{\alpha, \gamma, \delta, \rho, c, N, s\}$.[6]

[6] The details of mathematical calculations are omitted for brevity. They are available from the authors upon request.

From an analytical point of view, this difference may take both sings. However, numerical simulations show that, taking severalreasonable parameter regions, the difference is generally negative, indicating that the market social welfare is larger under Cournot behaviour than under Bertrand behaviour. This is due to the fact that, as we know from the foregoing analysis, the harsher competition characterising the Bertrand regime leads firms to larger investment efforts in advertising, and hence to lower profits which outweight the benefits associated with consumer surplus. Figures 1 and 2 show the patterns of DSW under specific parameter configurations. In both cases, we have set $\alpha = 2$, $\gamma = 3/10$, $\delta = 1/100$, $\rho = 2/100$, $c = 1$. Then, in Figure 1 we have set $s = 1/2$ with N varying over the interval of integers [2,50] while in Figure 2, N is set equal to 10 and the difference is plotted against s over the unit interval.



$s = 1/2, N \in [2, 50]$



$s \in [0, 1]\,, N = 10$

In the numerical simulations illustrated above, the social welfare advantage under Cournot is monotonically increasing in the number of existing varieties, while it is non-monotonic in the degree of product substitutability. This pattern, intuitively, may be reversed in correspondence of other specifications of parameter values. Nonetheless, a remarkable feature of the present model is precisely that it allows for non-negligible regions of the parameter space where a reversal of fortune obtains as to the welfare performance of the industry, this being fully consistent with the intuition based upon the steady state equilibrium strategies of firms.

5 Concluding remarks

We have characterised the different incentives to conduct advertising campaigns under either quantity or price competition in a linear state differential game where each firm's advertising effort exerts a positive externality to any other firm in the market. After showing that a unique saddlepoint equilibrium exists in both cases if the marginal cost of advertising is sufficiently low, we have focussed on the parameter range where both equilibria do obtain, proving that Bertrand competition entails more intense advertising than Cournot competition. The intuitive reason is that enhancing the reservation price is more relevant to firms when market competition is tougher. Consistently, social welfare can be larger when firms behave *à la* Cournot, as compared to the Bertrand behaviour.

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