Process and Product Innovation: a Differential Game Approach to Product Life Cycle[1]

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Abstract

We investigate the timing of adoption of product and process innovation by using a differential game in which firms may invest in both activities. We consider horizontal product innovation that reduces product substitutability, and process innovation that reduces marginal cost. First, we demonstrate that the incentive for cost-reducing investment is relatively higher than the incentive to increase product differentiation. Second, depending on initial conditions, (i) firms activate both types of investment from the very outset to the steady state; (ii) firms initially invest only in one R&D activity and then reach the steady state either carrying out only such activity or carrying out both; (iii) firms do not invest at all in either type of innovation.

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1 Introduction

Casual observation tells that firms carry out both product and process innovations, these being driven either by technological complementarities or by the life cycle of technology. However, these two factors have been investigated separately. One stream of research has focussed upon complementarities within R&D portfolios in monopoly (Athey and Schmutzler, 1995; Lambertini and Orsini, 2000; Lambertini, 2003, 2004; Lin, 2004) or oligopoly (Rosenkranz, 1995; Bonanno and Haworth, 1998; Lin and Saggi, 2002). The bottom line of this stream of research is that R&D efforts in each direction boosts the firms' incentive to carry out analogous effort in the other direction.

On the other hand, the technology life cycle describes the dynamics of product and process innovation. The common view is that product innovation necessarily precedes process innovation. Many reasons have been adduced to explain such belief. Abernathy and Utterback (1975 and 1982) propose a model where firms initially tend to direct most of their R&D resources to product innovation, because the market potential is large and products have to find a proper "dominant design" (Porter, 1983) to match consumers' requirements. As the proliferation of brand new products crowds the market, R&D profitability fades away and firms shift from product to process innovation. Klepper (1996; see also Filson, 2002) highlights the role of firms' innovative capabilities and size in conditioning R&D spending. In particular, he suggests that the returns to product innovation are highest at the very beginning because they depend on the acquisition of new consumers, while the returns to process innovation are very attractive at a later stage because they are proportional to the level of output produced by the firm. This gives a rationale for the fact that firms tend to pursue first product and then process innovation.

Adner and Levinthal (2001) move some critiques to the traditional approach to technology life cycle, that they call a "supply-side" view of technological change. The main limitation they emphasize is the exclusive attention to the internal capabilities of the firm.

They propose instead a "demand side" approach where technology changes are driven by the interaction between technology development and consumers' heterogeneous demands. Unlike what the traditional view maintains, they claim that the first phase is not always characterized by the adoption of product innovation. This will depend on the initial performance of the product on the market. The second phase is still characterized by the predominance of process innovation, but it does not necessarily imply a sharp decrease in the product innovation component. On the contrary, it considers the possibility that there exists a third stage, not yet treated formally in the literature, where innovative activity is a balance of product and process R&D in order to keep the price relatively stable and to increase at the same time the relative performance of the product. Adner and Levinthal's distinction between "new to the world" and "new to the market" technology is crucial to understand which activity firms implement first. The former corresponds to a product whose initial technology is "crude". The product is substantially new for customers and an investment in product innovation is appealing due to the potential market that could be created or expanded. In this situation the conventional life cycle holds, with an initial predominance of product innovation and a subsequent shift to process innovation. The latter refers to a product whose technology is very advanced; the possibilities for improving the functionality of the product are then limited and firms could only invest to reach a larger share of consumers. Initially they find it profitable to implement process innovation to lower the cost of production but after a certain point they switch to product innovation to further expand the market.

Others have stressed the difference between innovations and improvements, i.e., between technological breakthroughs and engineering refinements (see Doraszelski, 2004). This allows one to understand that firms may not wait for future breakthroughs, while there exists an incentive for them to delay the adoption of a new technology until it is sufficiently developed.

This paper bridges the two aformentioned lines or research and investigates the technology life cycle by using a differential game in which firms invest either in process innovation or in product innovation, or in both. In our framework product innovation is conceived in a "horizontal" way as a reduction of product substitutability. Both investment activities generate a positive spillover for the rival.

The setup we adopt in the remainder of the paper allows one to analyse both the innovation protfolio decisions and the life cycle characterising innovative activities. In the light of the conclusions drawn by Adner and Levinthal (2001), and by virtue of the dynamic nature of our approach, we shall privilege the latter aspect rather than the former. In line of principle, there exist four different perspectives: (i) firms activate both types of investment activities from the outset up to the steady state; (ii) firms initially invest only in product differentiation and then reach the steady state either carrying out only product innovation or carrying out both; (iii) firms initially invest only to reduce the cost of production and then arrive at the steady state either carrying out only process innovation or carrying out both; (iv) firms do not invest at all in either type of innovation, and they are driven to the steady state by the depreciation rate. First, we demonstrate that the incentive for cost-reducing investment is relatively higher than the incentive to increase product differentiation. This is due to the fact that, in the present setup, product differentiation is a public good, in that any investment in this direction by either firm fully spills over to the rival via the demand function. Second, we show that the implementation of both innovative activities depends on the initial degree of product substitutability compared with the initial marginal cost of production. In particular, this is surely the case if initial conditions tell that products are very close (or even perfect) substitutes and marginal cost is very close to consumers' reservation price.

The multiplicity of equilibrium paths generated by our model is confirmed by casual observation. The standard case where product innovation precedes process innovation is, e.g., that of CD players or digital cameras, where the new product has been introduced into the market at high prices, and subsequently the abatement of production costs has allowed firms to supply comparable varieties at lower prices. Conversely, the opposite happens when firms restyling existing products once they have already abated production as much as possible. Examples of this is the case can be found in the car industry, for instance this applies for utility cars like Fiat Panda and Lancia Y10, which have undergone a typical process of horizontal differentiation (i.e., restyling) at an advanced stage of their life cycle. Finally, the third type of equilibrium with both innovations taking place at the same time is observed in the PC industry (see Adner and Levinthal, 2001; Adner, 2004).

The outline of the paper is as follows. Section 2 illustrates the basics of the model. Section 3 analyses the cases where firms perform only one type of activity. Section 4 considers the case in which firms invest both in process and in product innovation and provides the main results of the paper concerning the time of adoption. Section 5 concludes the paper.

2 The Model

We build a differential duopoly game over continuous time $t \in [0, \infty)$, where at any instant firms choose the quantity level and the investment level in either product or process innovation, or both. Process innovation is formalized as a reduction in the unit cost of production, while product innovation affects product substitutability as perceived by consumers.

Firms produce differentiated goods. As in Spence (1976) and Singh and Vives (1984), the representative consumer's utility function is a function of the consumption of the two differentiated goods and the numeraire good m, and is given by:

$$u(q_1(t), q_2(t), m) = a(q_1(t) + q_2(t)) - (q_1(t)^2 + q_2(t)^2)/2 - s(t)q_1(t)q_2(t) + m, \tag{1}$$

where $s(t) \in [0, 1]$. Constrained utility maximization for any given price pair $\{p_i(t), p_j(t)\}$

gives rise to the following demand system:

$$p_i(t) = a - q_i(t) - s(t)q_j(t) \quad \forall i \neq j,\ i,\ j = \{1,2\} \tag{2}$$

where the parameter $s(t)$ represents the degree of substitutability between the two products. If $s(t) = 1$ products are completely homogeneous. At the opposite, for $s(t) = 0$, products are independent and each firm acts as a monopolist. At any time t, the output level $q_i(t)$ is produced at a constant marginal cost. Accordingly, the istantaneous cost function is $C_i(c_i, q_i, t) = c_i(t)\, q_i(t)$.

We assume that product differentiation can be affected by firms' R&D investment in product innovation. The dynamics associated to product innovation is described by the kinematic equation:

$$\frac{ds(t)}{dt} \equiv \dot{s} = s(t)\left[-x_i(t) - x_j(t) + \delta\right] \tag{3}$$

where $x_i(t)$ represents the amount of effort made by firm i at time t in order to increase product differentiation through a reduction of $s(t)$.[1] The parameter $\delta \in [0,1]$ indicates the depreciation rate due to ageing of technology, which is common to both firms and constant over time. A characteristic associated with horizontal product differentiation is that is gives rise to a complete spillover effect between the R&D activity undertaken by the firms. Equation (3) can be rewritten as follows:

$$\frac{\dot{s}}{s(t)} = -x_i(t) - x_j(t) + \delta \tag{4}$$

so to highlight that the rate of change of product substitutability over time is linear in the instantaneous investment efforts.

Moreover, firms invest in process innovation; as a consequence, the marginal cost borne by firm i evolves over time as described by the following kinematic equation:

$$\frac{dc_i(t)}{dt} \equiv \dot{c}_i = c_i(t)\left[-k_i(t) - \theta k_j(t) + \eta\right] \tag{5}$$

[1] The idea that s depends upon firms' investment decisions has been investigated in static models by Harrington (1995), Lambertini and Rossini (1998) and Lambertini, Poddar and Sasaki (1998). Recent contributions apply this idea to differential games, e.g. Cellini and Lambertini (2002 and 2004).

where $k_i(t)$ indicates the effort made by firm i to reduce the cost of production. The parameter $\theta \in [0,1]$ measures the positive technological spillover that firm i receives from the process innovative activity of the rival, while $\eta \in [0,1]$ is the depreciation rate, assumed to be common to both firms and constant over time. It is easy to notice that the rate of change of firm i's marginal cost over time is linear in the instantaneous investment efforts, given that:

$$\frac{\dot{c}_i}{c(t)} = -k_i(t) - \theta k_j(t) + \eta. \tag{6}$$

Notice that (5) is indeed a dynamic version of the linear R&D technology employed by d'Aspremont and Jacquemin (1988).

The instantaneous cost of investing in product innovation and in process innovation is respectively given by $\gamma\,[x_i(t)]^2$ and $\beta\,[k_i(t)]^2$. Both types of investment are financed through internal funds. To firm i, hence, investing in both types of R&D implies decreasing returns to innovative activity. The parameters γ and β are inverse measures of product and process innovation R&D efficiency, respectively.

Instantaneous profits are given by:

$$\pi_i(t) = [a - c_i(t) - q_i(t) - s(t)q_j(t)]\,q_i(t) - \gamma\,[x_i(t)]^2 - \beta\,[k_i(t)]^2 \tag{7}$$

We assume that firm i aims at maximizing the discounted profit flow:

$$\Pi_i(t) = \int_0^\infty \pi_i(t)\,e^{-\rho t}dt \tag{8}$$

w.r.t. controls $x_i(t)$, $k_i(t)$ and the market variable $q_i(t)$, under the constraint given by the state dynamics (3) and (5). The discount rate $\rho > 0$ is assumed to be constant and common to both firms.

The corresponding current value Hamiltonian function is:

$$\mathcal{H}_i(t) = e^{-\rho t}\left[\pi_i(t) + \lambda_i(t)\dot{s} + \lambda_{ii}(t)\dot{c}_i + \lambda_{ij}(t)\dot{c}_j\right] \tag{9}$$

where $\lambda_i(t) = \mu_i(t)e^{\rho t}$, $\lambda_{ii}(t) = \mu_{ii}(t)e^{\rho t}$ and $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$, μ_i, $\mu_{ii}(t)$ and $\mu_{ij}(t)$ being the co-state variable associated to $s(t), c_i(t)$ and $c_j(t)$, respectively.

3 The two separated activities

In order to shed light on the relation between product and process innovation, we start by considering the two activites separately. In the next section we will consider the case where firms implement both types of innovative activity.

3.1 Product Innovation

Firms invest only in product innovation aiming at reducing the degree of product substitutability. At this time process innovation is not taken into account, hence $k_i(t) = 0$ and the marginal cost c does not depend on firms' innovative activity. Without loss of generality, we can assume that the marginal cost is constant and common for both firms, hence $c_i = c$. Instantaneous profits are given by:

$$\pi_i(t) = [a - c - q_i(t) - s(t)q_j(t)]\, q_i(t) - \gamma\, [x_i(t)]^2 \tag{10}$$

Firm maximize the discounted profit flow $\Pi_i(t) = \int_0^\infty \pi_i(t)\, e^{-\rho t} dt$ w.r.t. controls $x_i(t)$ and $q_i(t)$, under the only constraint given by (3). The corresponding Hamiltonian function writes:

$$\begin{aligned} \mathcal{H}_i(t) \;=\; & e^{-\rho t} \left\{ [a - c]\, q_i(t) - q_i(t)^2 - \; s(t)q_i(t)q_j(t) - \gamma\, [x_i(t)]^2 + \right. \\ & \left. + \lambda_i(t)s(t)\, [-x_i(t) - x_j(t) + \delta] \right\}. \end{aligned} \tag{11}$$

The initial condition is $s(0) = s_0 \in (0, 1]$. Firms play simultaneously. Firm i's first order conditions (FOCs) on controls are:[2]

$$\frac{\partial \mathcal{H}_i(t)}{\partial q_i(t)} = a - c - 2q_i(t) - s(t)q_j(t) = 0 \Rightarrow q_i^*(t) = \frac{a - c - s(t)q_j(t)}{2} \tag{12}$$

$$\frac{\partial \mathcal{H}_i(t)}{\partial x_i(t)} = -2\gamma x_i(t) - \lambda_i(t)s(t) = 0 \Rightarrow x_i^*(t) = -\frac{s(t)\lambda_i(t)}{2\gamma} \tag{13}$$

[2] For the sake of brevity, in the remainder we omit the indication of exponential discounting.

Note that both (12) and (13) contain the state variable $s(t)$, which is common for both firms. As a consequence, the open-loop solution and the closed-loop memoryless solution do not coincide. The solution concept we adopt is the closed-loop Nash equilibrium. We then take into account the feedback between player i's strategy and player j's state variable. This will lead to an equilibrium characterized by subgame perfection.

We specify firm i's co-state equation containing the feedback effects:

$$-\frac{\partial \mathcal{H}_i(t)}{\partial s(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial q_j(t)} \frac{\partial q_j^*(t)}{\partial s(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial x_j(t)} \frac{\partial x_j^*(t)}{\partial s(t)} = \dot{\lambda}_i - \rho \lambda_i(t) \Rightarrow$$

$$\dot{\lambda}_i = q_i(t) q_j(t) - \frac{s(t) \left[q_i(t)\right]^2}{2} - \frac{\lambda_i(t) \lambda_j(t) s(t)}{2\gamma} + \lambda_i(t) \left[-\delta + x_i(t) + x_j(t) + \rho\right] \tag{14}$$

along with the transversality condition:

$$\lim_{t \to \infty} \mu_i(t)\, s(t) = 0. \tag{15}$$

We introduce the following:

Assumption $q_i(t) = q_j(t) = q(t)$ and $x_i(t) = x_j(t) = x(t)$.

This is the usual symmetry assumption involving no loss of generality as long as one adopts the Nash equilibrium as the solution concept.

Then, from (12), we derive the equilibrium per firm output

$$q(t) = \frac{a - c}{2 + s(t)}, \tag{16}$$

which coincides with the standard outcome of Cournot models with product differentiation (see Singh and Vives, 1984; Cellini and Lambertini, 1998). Hence, given the implied symmetry condition $\lambda_i(t) = \lambda_j(t) = \lambda(t)$, from (13) we know that:

$$-\lambda(t) = \frac{2\gamma x(t)}{s(t)} \tag{17}$$

By symmetry, and using (17), the co-state equation (14) simplifies as follows:

$$\dot{\lambda} = \frac{\left[2 - s(t)\right] \left[q(t)\right]^2 - 4\gamma x(t) \left[3x(t) + \rho - \delta\right]}{2s(t)} \tag{18}$$

From (17) we obtain $x(t)$, which can be differentiated w.r.t. t:

$$\dot{x} = \frac{1}{2\gamma}\left[-\dot{\lambda}s\left(t\right) - \lambda(t)\dot{s}\right] \tag{19}$$

where, following the symmetry assumptions,

$$\dot{s} = s(t)\left[-2x(t) + \delta\right] \tag{20}$$

Then, plugging (18), (17) and (20) into (19) and rearranging, one obtains:

$$\dot{x} = x\left(\rho + x\right) - \frac{s(t)\left(a-c\right)^2\left[2 - s(t)\right]}{4\gamma\left[2 + s(t)\right]^2}. \tag{21}$$

The expression in (21) is valid for all $s(t) \in (0,1]$; if $s(t) = 0$, optimal per period investment is $x(t) = 0$.

We are now in a position to asssess the overall dynamics properties of the case under study, being it fully characterized by (21) and (20). Steady state solutions are obtained by solving the system $\dot{s} = 0$ and $\dot{x} = 0$. Given that explicit solutions are cumbersome, we start by imposing $\dot{x} = 0$ to determine an equilibrium relation between $x(t)$ and $s(t)$:

$$x(s) = \frac{-\gamma\rho\left[2 + s(t)\right] \pm \sqrt{\gamma\left\{\gamma\rho^2\left[2 + s(t)\right]^2 + \left(a-c\right)^2\left[2 - s(t)\right]s(t)\right\}}}{2\gamma\left[2 + s(t)\right]}. \tag{22}$$

The smaller root corresponds to a locus where x has always a negative value, and can therefore be disregarded, being economically meaningless. Therefore, the optimal investment is:

$$x^{Pd}(s) = \frac{-\gamma\rho\left[2 + s(t)\right] + \sqrt{\gamma\left\{\gamma\rho^2\left[2 + s(t)\right]^2 + \left(a-c\right)^2\left[2 - s(t)\right]s(t)\right\}}}{2\gamma\left[2 + s(t)\right]}. \tag{23}$$

where the superscript Pd stands for *product innovation.* Hence, by substituting (23) in (3) and solving for s, we get three solutions:

$$s = 0; \quad s = \frac{(a-c)\left[(a-c) \pm \Psi\right] - 2\delta\gamma\left(\delta + 2\rho\right)}{\left(a-c\right)^2 + \delta\gamma\left(\delta + 2\rho\right)} \tag{24}$$

where $\Psi \equiv \sqrt{\left(a-c\right)^2 - 8\delta\gamma\left(\delta + 2\rho\right)}$. It is immediate to prove that:

Lemma 1 *The solutions* $s = \dfrac{(a-c)\left[(a-c) \pm \Psi\right] - 2\delta\gamma(\delta+2\rho)}{(a-c)^2 + \delta\gamma(\delta+2\rho)}$ *are real if and only if* $(a-c)^2 - 8\delta\gamma(\delta+2\rho) > 0$, *or* $\gamma \leq \dfrac{(a-c)^2}{8\delta(\delta+2\rho)} = \widetilde{\gamma}$. *This also ensures that* $s \in [0,1]$.

The statement of the above lemma amounts to requiring that firms need to face a sufficiently low cost of production for the investment in product innovation to start. We will come back to this point while analyzing the possibility for firms to undertake simultaneously process and product R&D activities.

Considering the stability of the system, the following can be shown to hold:

Proposition 1 *Provided that* $\gamma \leq \widetilde{\gamma}$, *the steady state point*

$$\begin{aligned} s^{Pd} &= \frac{(a-c)\left[(a-c) - \Psi\right] - 2\delta\gamma(\delta+2\rho)}{(a-c)^2 + \delta\gamma(\delta+2\rho)}, \\ x^{Pd} &= \frac{\delta}{2} \end{aligned}$$

is the unique saddle point equilibrium of the game when firms invest only in product innovation.

Proof. The Jacobian matrix of the dynamic system formed by (3) and (21) is (under symmetry):

$$J^{Pd} = \begin{bmatrix} \dfrac{\partial \dot{s}}{\partial s} = \delta - 2x & \dfrac{\partial \dot{s}}{\partial x} = -2s \\ \dfrac{\partial \dot{x}}{\partial s} = \dfrac{(a-c)^2(3s-2)}{2\gamma(2+s)^3} & \dfrac{\partial \dot{x}}{\partial x} = \rho + 2x \end{bmatrix} \tag{25}$$

whose trace and determinant are:

$$\begin{aligned} T\left(J^{Pd}\right) &= \rho + \delta; \\ \Delta\left(J^{Pd}\right) &= (\delta - 2x)(\rho + 2x) + \frac{(a-c)^2(3s-2)s}{\gamma(2+s)^3}. \end{aligned} \tag{26}$$

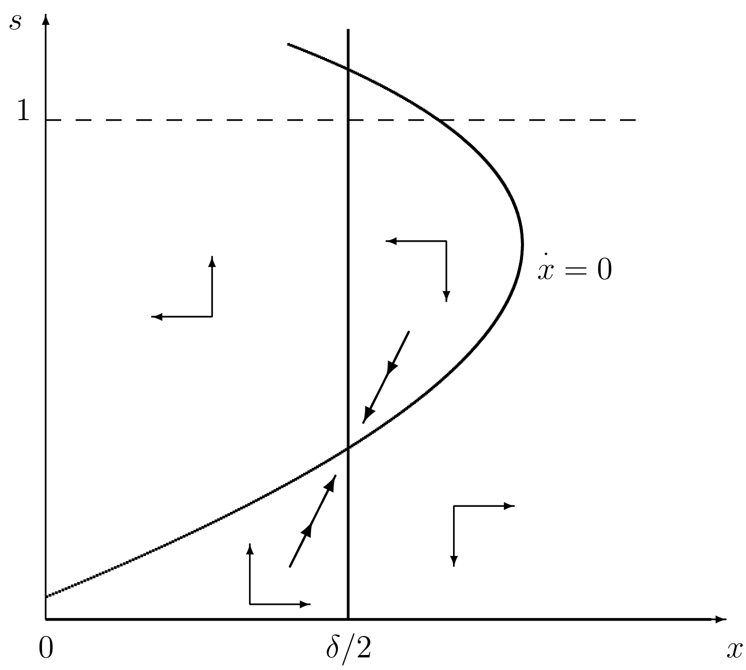
Evaluating $\Delta\left(J^{Pd}\right)$ in $\left(s^{Pd}, x^{Pd}\right)$, we have:

$$\Delta\left(J^{Pd}\right) \propto -16\delta\gamma(\delta+2\rho)\left[(a-c)^2 - 8\delta\gamma(\delta+2\rho)\right]\left[(a-c)^2 + \delta\gamma(\delta+2\rho)\right] \tag{27}$$

where the expression on the r.h.s. is surely negative on the basis of the condition warranting the reality of s^{Pd}. Therefore, $\Delta\left(J^{Pd}\right) < 0$ in $\left(s^{Pd}, x^{Pd}\right)$, which qualifies as the unique saddle point equilibrium. ■

An intuitive illustration of saddle point stability can be obtained by investigating the dynamics of the system in the positive quadrant of the space $\{s, x\}$, which is described in Figure 1. The locus $\dot{s} = 0$ is given by the $s = 0$ and $x = \delta/2$, while the locus $\dot{x} = 0$ draws a curve over the admissable values of s.[3] The dynamics of s and x are respectively summarized by vertical and horizontal arrows.

Figure 1 : The phase diagram in the product innovation case



From the phase diagram, it is clear that this saddlepoint equilibrium can be approached only along the north-est arm of the saddle path. It is also immediate to demonstrate that

[3]The curve $\dot{x} = 0$ may or may not cross twice the vertical line $x = \delta/2$ within the admissable values of $s \in [0, 1]$. In Figure 1 we depict the situation where they cross only once.

$s^{Pd} \in [0,1]$.

Comparative statics exercises can be carried out on $x^{Pd}(s)$ to show that:

$$\frac{\partial x^{Pd}(s)}{\partial s} \propto 2 - 3s(t) \; ; \; \frac{\partial x^{Pd}(s)}{\partial c} < 0. \tag{28}$$

The first partial derivative implies that the incentive to implement product innovation is non-monotone in the degree of product differentiation. In particular, $\partial x^{Pd}(s)/\partial s > 0$ if and only if $s \in [0, 2/3)$, i.e., firms have a strong incentive to differentiate their products varieties if they are close substitutes. As soon as product differentiation is large enough, R&D efforts start decreasing. The second derivative in (28) shows that the optimal investment is everywhere decreasing in the level of marginal cost, for any given $s(t)$ before the steady state is reached. The obvious reason is that a decrease in c, all else equal, entails a higher gross profits and therefore a larger source of internal funds to be invested in R&D activities.

Now examine the effect of a variation in c on the steady state level of substitutability (see Proposition 1), to verify easily that $\partial s^{Pd}/\partial c > 0$ always. The explanations is that, the higher is c, the lower is the amount of internal funds available ti finance R&D at any time during the game; as a result, the steady state degree of substitutability is positively related to marginal production cost. The same applies along the optimal investment path, as it can be verified by inverting $x^{Pd}(s)$ to obtain $s^{Pd}(x)$, and then checking that $\partial s^{Pd}(x)/\partial c > 0$.

3.2 Process Innovation

Consider the case where firms are allowed to undertake only process innovation R&D, hence $x_i(t) = 0$ and the parameter s measuring product substitutability does not change over time. Instantaneous profits are given by:

$$\pi_i(t) = [a - c_i(t) - q_i(t) - sq_j(t)]\, q_i(t) - \beta\, [k_i(t)]^2 \tag{29}$$

Firms maximize the discounted profit flow w.r.t. controls $k_i(t)$ and $q_i(t)$, under the constraints (5). The corresponding Hamiltonian function writes:

$$\begin{aligned}\mathcal{H}_i(t) &= e^{-\rho t}\left\{[a-c(t)]\,q_i(t)-q_i(t)^2- \;sq_i(t)q_j(t)-\beta\,[k_i(t)]^2+\right. \\ &\left.+\lambda_{ii}(t)c_i(t)\,[-k_i(t)-\theta k_j(t)+\eta]+\lambda_{ij}(t)c_j(t)\,[-k_j(t)-\theta k_i(t)+\eta]\right\}.\end{aligned} \tag{30}$$

The initial condition is $c\,(0)=c_0\in(0,a]$. Firm i's first order conditions (FOCs) on controls are:[4]

$$\frac{\partial\mathcal{H}_i(t)}{\partial q_i(t)}=a-c_i(t)-2q_i(t)-sq_j(t)=0\Rightarrow q_i^*(t)=\frac{a-c_i(t)-sq_j(t)}{2} \tag{31}$$

$$\frac{\partial\mathcal{H}_i(t)}{\partial k_i(t)}=-2\beta k_i(t)-\lambda_{ii}(t)c_i(t)-\lambda_{ij}(t)\theta c_j(t)=0\Rightarrow k_i^*(t)=-\frac{[c_i(t)\lambda_{ii}(t)+\theta\lambda_{ij}(t)c_j(t)]}{2\beta} \tag{32}$$

Note that (31) does not contain the state variable (5) of the rival, which is however present in (32). At a first sight there seem then to be a feedback between the R&D decisions, at least for any positive spillover effect.[5] However, it is possible to prove that:

Lemma 2 *The open-loop Nash equilibrium of the game with firms investing only in process innovation is subgame (or Markov) perfect.*

Proof. We show that the present game is a *perfect game* in the sense of Leitmann and Schmitendorf (1978) and Feichtinger (1983).[6] We basically need to prove that no feedback effect are actually present, even for positive spillover levels.

Let us write the co-state equations:

$$-\frac{\partial\mathcal{H}_i(t)}{\partial c_i(t)}-\frac{\partial\mathcal{H}_i(t)}{\partial k_j(t)}\frac{\partial k_j^*(t)}{\partial c_i(t)}=\dot{\lambda}_{ii}-\rho\lambda_{ii}(t)\Leftrightarrow \tag{33}$$

[4] As before, we omit the indication of the exponential discounting.

[5] If $\theta=0$ the two investment plans are completely independent and therefore it evident that no feedback effect operates.

[6] We remind that a differential game is *perfect* whenever the closed-loop equilibrium collapses into the open-loop one, the latter being thus strongly time consistent, i.e. subgame perfect. For further details, see Mehlmann (1988, Ch. 4) and Dockner et al. (2000, Ch. 7).

$$\dot{\lambda}_{ii} = q_i(t) + \lambda_{ii}(t)\left[k_i(t) + \theta k_j(t) + \rho - \eta\right] - \frac{\theta}{2\beta}\lambda_{ji}(t)\left[\lambda_{ij}(t)c_j(t) + \theta\lambda_{ii}(t)c_i(t)\right]$$

$$-\frac{\partial\mathcal{H}_i(t)}{\partial c_j(t)} - \frac{\partial\mathcal{H}_i(t)}{\partial k_i(t)}\frac{\partial k_i^*(t)}{\partial c_j(t)} = \dot{\lambda}_{ij} - \rho\lambda_{ij}(t) \Leftrightarrow \tag{34}$$

$$\dot{\lambda}_{ij} = \lambda_{ij}(t)\left\{\left[k_j(t) + \theta k_i(t) + \rho - \eta\right] - \frac{\theta}{2\beta}\left[2\beta k_i(t) + \lambda_{ii}(t)c_i(t) + \theta\lambda_{ij}(t)c_j(t)\right]\right\}$$

where

$$\frac{\partial\mathcal{H}_i(t)}{\partial k_j(t)}\frac{\partial k_j^*(t)}{\partial c_i(t)}$$

captures the feedback effects, and partial derivatives $\partial k_j^*(t)/\partial c_i(t)$ are calculated using the optimal values of investment as from (32), i.e.

$$k_j^*(t) = -\frac{c_j(t)\lambda_{jj}(t) + \lambda_{ji}(t)\theta c_i(t)}{2\beta} \tag{35}$$

These conditions must be evaluated along with the initial conditions $\{c_i(0)\} = \{c_{0,i}\}$ and the transversality conditions

$$\lim_{t\to\infty}\mu_{ij}\left(t\right)c_j\left(t\right) = 0,\ \ i,j = 1,2. \tag{36}$$

From (34) one can note that $\dot{\lambda}_{ij} = 0$ in $\lambda_{ij}(t) = 0$. We can rewrite the expression for the optimal investment of firm i as follows:

$$k_i^*(t) = -\frac{\lambda_{ii}(t)c_i(t)}{2\beta} \tag{37}$$

which entails that $\partial k_i^*(t)/\partial c_j(t) = 0$. This holds for both firms due to the underlying symmetry of the model. Feedback (cross-) effects are nil along the equilibrium path and hence the open-loop is a degenerate closed-loop one, and it is strongly time consistent, or equivalently, subgame perfect.[7] ■

[7] For a dynamic R&D game where knowledge accumulation has a strategic effect during the game itself, see Doraszelski (2003).

On the basis of Lemma 2, we can proceed with the characterisation of the open-loop solution. The first order conditions on control as well as the transversality conditions are the same as above, while the co-state equations simplify as follows:

$$-\frac{\partial \mathcal{H}_i(t)}{\partial c_i(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial k_j(t)} \frac{\partial k_j^*(t)}{\partial c_i(t)} = \dot{\lambda}_{ii} - \rho \lambda_{ii}(t) \Leftrightarrow \dot{\lambda}_{ii} = q_i(t) + \lambda_{ii}(t) \left[k_i(t) + \theta k_j(t) + \rho - \eta \right] \quad (38)$$

$$-\frac{\partial \mathcal{H}_i(t)}{\partial c_j(t)} - \frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} \frac{\partial k_i^*(t)}{\partial c_j(t)} = \dot{\lambda}_{ij} - \rho \lambda_{ij}(t) \Leftrightarrow \dot{\lambda}_{ij} = \lambda_{ij}(t) \left\{ \left[k_j(t) + \theta k_i(t) + \rho - \eta \right] \right\} \quad (39)$$

Moreover, we can differentiate w.r.t. time the expression for the optimal R&D effort that appears in (32) to get the dynamic equation of $k_i(t)$:

$$\dot{k}_i = -\frac{1}{2\beta} \left\{ c_i(t) \dot{\lambda}_{ii} + \lambda_{ii}(t) \dot{c}_i + \theta \left[c_j(t) \dot{\lambda}_{ij} + \lambda_{ij}(t) \dot{c}_j \right] \right\}. \quad (40)$$

with $\dot{\lambda}_{ii}$ and $\dot{\lambda}_{ij}$ that obtain from (38) and (39) respectively. Moreover, from (32) we get an explicit expression for $\lambda_{ii}(t)$, i.e.

$$\lambda_{ii}(t) = -\frac{2\beta k_i(t) + \theta \lambda_{ij}(t) c_j(t)}{c_i(t)}$$

which can used to further simplify (32). As to the second co-state variable, its dynamic equation (39) must be trated autonomously and, by imposing stationarity, i.e. $\dot{\lambda}_{ij} = 0$, we obtain $\lambda_{ij}(t) = 0$. This yields:

$$\dot{k}_i = \rho k_i(t) - \frac{c_i(t) q_i(t)}{2\beta} \quad (41)$$

As before, we introduce the symmetry assumption:

Assumption $q_i(t) = q_j(t) = q(t)$ and $k_i(t) = k_j(t) = k(t)$.

We further assume $c_i(t) = c_j(t) = c(t)$ and from (31) obtain the equilibrium per firm output $q = \frac{a - c(t)}{2 + s}$. It is now immediate to rewrite (41) as

$$\dot{k} = \rho k(t) - \frac{c(t) \left[a - c(t) \right]}{2\beta (2 + s)}. \quad (42)$$

Steady state solutions are obtained by solving the system $\dot{k} = 0$ and $\dot{c} = 0$. We start by imposing the stationarity condition $\dot{k} = 0$ to determine an equilibrium relation between $k(t)$ and $c(t)$:

$$k^{Ps}(c) = \frac{c(t)\,[a - c(t)]}{2\beta\rho\,(2+s)} \tag{43}$$

where the superscript Ps stands for *process innovation*. Hence, by substituting (43) in (5) and solve for c, we obtain three solutions, namely:

$$c = 0;\; c = \frac{1}{2}\left\{a \pm \frac{\sqrt{(1+\theta)\,[a^2(1+\theta) - 8\beta\eta\rho\,(2+s)]}}{1+\theta}\right\} \tag{44}$$

Before proceeding, it is immediate to note that:

Lemma 3 *The solutions* $c = \frac{1}{2}\left\{a \pm \frac{\sqrt{(1+\theta)\,[a^2(1+\theta) - 8\beta\eta\rho\,(2+s)]}}{1+\theta}\right\}$ *are real if and only if* $a^2(1+\theta) - 8\beta\eta\rho\,(2+s) > 0$, *or* $s < \dfrac{a^2(1+\theta)}{8\beta\eta\rho} - 2 = \widetilde{s}$.

As a consequence, firms then invest in process innovation only when goods are sufficiently differentiated.

We now check the stability conditions of the system. The following can be shown to hold:

Proposition 2 *Provided that* $s \leq \widetilde{s}$, *the steady state point*

$$\begin{aligned} c^{Ps} &= \frac{1}{2}\left\{a - \frac{\sqrt{(1+\theta)\,[a^2(1+\theta) - 8\beta\eta\rho\,(2+s)]}}{1+\theta}\right\}, \\ k^{Ps} &= \frac{\eta}{1+\theta} \end{aligned}$$

is the unique saddle point equilibrium of the game when firms invest only in process innovation.

Proof. Under symmetry, the Jacobian matrix of the dynamic system formed by (5) and (42) is:

$$J^{Ps}=\left[\begin{array}{ll}\dfrac{\partial\dot{c}}{\partial c}=\eta-k(1+\theta) & \dfrac{\partial\dot{c}}{\partial k}=-c(1+\theta)\\ \dfrac{\partial\dot{k}}{\partial c}=-\dfrac{a-2c}{2\beta(2+s)} & \dfrac{\partial\dot{k}}{\partial k}=\rho\end{array}\right] \tag{45}$$

whose trace and determinant are:

$$\begin{aligned}T\left(J^{Ps}\right) &= \rho+\eta-k(1+\theta); \\ \Delta\left(J^{Ps}\right) &= [\eta-k(1+\theta)]\rho-\frac{c(a-2c)(1+\theta)}{2\beta(2+s)}.\end{aligned} \tag{46}$$

Evaluating $\Delta\left(J^{Ps}\right)$ in $\left(c^{Ps},k^{Ps}\right)$, we verify that:

$$\Delta\left(J^{Ps}\right)\propto-8\beta\eta\rho(2+s)(1+\theta) \tag{47}$$

where the expression on the r.h.s. is negative. Accordingly, the determinant is negative in correspondence of the steady state point $\left(c^{Ps},k^{Ps}\right)$, which qualifies as the unique saddle point equilibrium of the game. ■

As in the product innovation case, we may resort to the phase diagram to illustrate saddle point stability. The locus $\dot{c}=0$ is given by the $c=0$ and $k=\eta/(1+\theta)$, while the locus $\dot{k}=0$ draws a curve in the space $\{c,k\}$ as described in Figure 2.

As in the previous setting, we now proceed to some comparative statics exercises on $k^{Ps}(c)$ and c^{Ps}, to obtain the following:

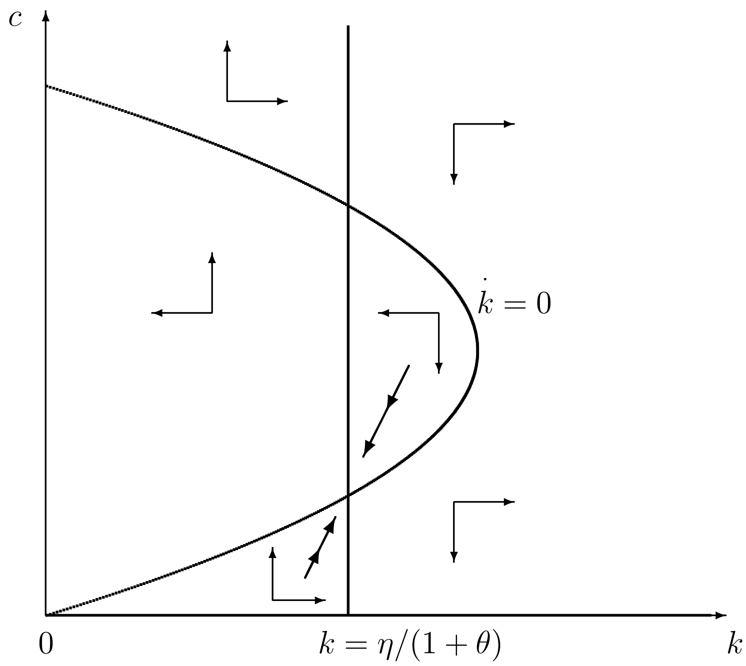
$$\frac{\partial k^{Ps}(c)}{\partial c}\propto a-2c(t)\ ;\ \frac{\partial k^{Ps}(c)}{\partial s}<0;\ \frac{\partial c^{Ps}}{\partial s}>0. \tag{48}$$

The first derivative in (48) shows that the optimal R&D effort for process innovation is non-monotone in the level of the marginal cost, with R&D efforts initially increasing as $c(t)$ departs from c_0 (provided $c_0\in(a/2,a)$), and then decreasing as soon as $c(t)$ goes below $a/2$.[8] The remaining two properties in (48) are intuitive, since (i) the higher

[8]Note that c^{Ps} is surely lower than $a/2$.

is the substitutability between products, the lower are gross profits at any instant; as a consequence, internal funds for process R&D shrink; and (ii), steady state marginal cost is positively related to substitutability, which replicates the propery we have already highlighted in the previous subsection concerning product innovation. As for (i), also in this case a higher level of substitutability decreases the available funds for financing R&D for process innovation.

Figure 2 : The phase diagram in the process innovation case



4 Process and Product Innovation

We now analyse the most interesting case, where firms implement simultaneously process and product innovation. Instantaneous profits are given by (7) and firms maximize the discounted profit flow w.r.t. controls $x_i(t)$, $k_i(t)$ and $q_i(t)$, under the constraints represented by (3) and (5). The corresponding Hamiltonian function writes:

$$\begin{aligned}\mathcal{H}_i(t) &= e^{-\rho t}\left\{[a-c(t)]\,q_i(t)-q_i(t)^2-\ s(t)q_i(t)q_j(t)-\gamma\,[x_i(t)]^2-\beta\,[k_i(t)]^2+\right. \\ &\quad +\lambda_i(t)s(t)\,[\delta-x_i(t)-x_i(t)]+\lambda_{ii}(t)c_i(t)\,[\eta-k_i(t)-\theta k_j(t)]+ \\ &\quad \left.+\lambda_{ij}(t)c_j(t)\,[\eta-k_j(t)-\theta k_i(t)]\right\}\end{aligned} \tag{49}$$

We assume $\eta=\delta$, i.e. the depreciation rates associated to process innovation technology and to product innovation technology are exactly the same.

Firms play simultaneously. Firm i's first order conditions (FOCs) on controls are equivalent to (12), (13) and (32). The co-state equations for the closed-loop Nash equilibrium also coincide with those derived when treating the two types of innovation separately, i.e., (14), (38) and (39). As a consequence, the dynamic equations of controls coincide with (19) and (42). Imposing stationarity on states and controls, we find the following equilibrium expressions:

$$\begin{aligned}s^*(c) &= \frac{(a-c)^2-2\delta\gamma\,(\delta+2\rho)-(a-c)\,\Psi}{(a-c)^2+\delta\gamma\,(\delta+2\rho)} \\ c^*(s) &= \frac{1}{2}\left\{a-\frac{\sqrt{(1+\theta)\,[a^2\,(1+\theta)-8\beta\eta\rho\,(2+s)]}}{1+\theta}\right\}\end{aligned}$$

where $\Psi\equiv\sqrt{(a-c)^2-8\delta\gamma\,(\delta+2\rho)}$. Provided that $s^*(c)$ and $c^*(s)$ belong to $\mathbb{R}$, then also $s^*(c)\in[0,1]$ and $c^*(s)\in[0,a]$. However, note that

$$s^*(c) \in \mathbb{R} \text{ iff } \rho<\frac{(a-c)^2-8\delta^2\gamma}{16\delta\gamma} \tag{50}$$

$$c^*(s) \in \mathbb{R} \text{ iff } \rho<\frac{a^2\,(1+\beta)}{8\beta\delta\gamma\,(2+s)} \tag{51}$$

Now, it is apparent that there surely exists a range of admissible values for the discount rate satisfying (51), while it is possible that $(a-c)^2<8\delta^2\gamma$. If so, then (50) is impossible, which entails that there is no real solution for the product innovation problem. In such a case, firms only proceed to activate R&D for process innovation, while product differentiation remains at the initial condition s_0. This leads to the following Lemma:

Lemma 4 *Firms' incentives towards R&D for process and product innovation can be charaterised in therms of their time preferences:*

- *if* $\frac{a^2(1+\beta)}{8\beta\delta\gamma(2+s)} > \frac{(a-c)^2 - 8\delta^2\gamma}{16\delta\gamma} > 0$, *then (i)* $\forall\rho \in \left[0, \frac{(a-c)^2 - 8\delta^2\gamma}{16\delta\gamma}\right)$ *firms activate both process and product innovation; (ii)* $\forall\rho \in \left(\frac{(a-c)^2 - 8\delta^2\gamma}{16\delta\gamma}, \frac{a^2(1+\beta)}{8\beta\delta\gamma(2+s)}\right)$, *firms invest in process innovation only;*
- *if* $\frac{(a-c)^2 - 8\delta^2\gamma}{16\delta\gamma} > \frac{a^2(1+\beta)}{8\beta\delta\gamma(2+s)} > 0$, *then (i)* $\forall\rho \in \left[0, \frac{a^2(1+\beta)}{8\beta\delta\gamma(2+s)}\right)$ *firms activate both process and product innovation; (ii)* $\forall\rho \in \left(\frac{a^2(1+\beta)}{8\beta\delta\gamma(2+s)}, \frac{(a-c)^2 - 8\delta^2\gamma}{16\delta\gamma}\right)$, *firms invest in product innovation only;*
- *if* $\frac{a^2(1+\beta)}{8\beta\delta\gamma(2+s)} > 0 > \frac{(a-c)^2 - 8\delta^2\gamma}{16\delta\gamma}$, *then firms can only activate process R&D* $\forall\rho \in \left[0, \frac{a^2(1+\beta)}{8\beta\delta\gamma(2+s)}\right)$.

Lemma 4 states that, in general, the incentive for cost-reducing investment is relatively higher then the incentive to increase product differention. The explanation of this result can be found in the fact that, while investing in product differentiation entails a full spillover via the demand function, any effort aimed at decreasing marginal cost involves only a partial spillover measured by $\theta \in [0,1]$.

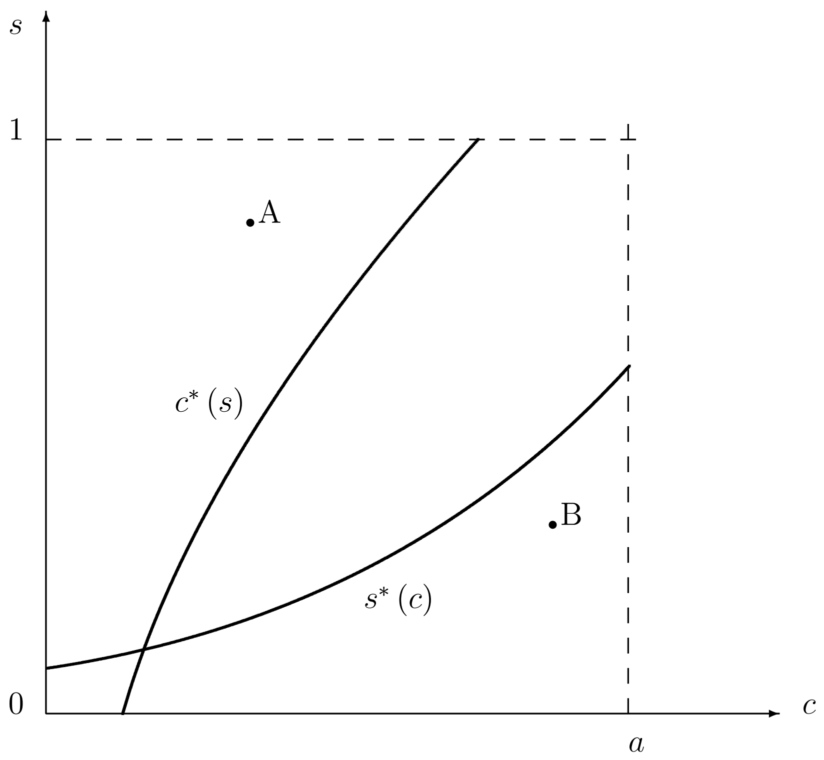
Now we focus on the situation where real solutions exist on both sides, to evaluate the bearings of initial conditions on the optimal investment portfolio of firms. That is, provided both types of innovations are viable, shall we expect to observe either one or the other, or both in equilibrium?

To address this question, we may carry out a partition of the state space $\{c, s\}$ where we draw the equilibrium solutions $(c^*(s), s^*(c))$, as in Figure 1. Both (50) and (51) are concave in the other state variable, and for admissible values of parameters $\{a, \beta, \gamma, \delta, \theta, \rho\}$ cross only once within the rectangular region defined by $c^*(s) \in [0, a]$ and $s^*(c) \in [0, 1]$.

This identifies four regions in the relvant space of states. Since $s^*(c)$ solves $\dot{s} = 0$, for all $s > s^*(c)$ we know that $\dot{s} < 0$. This implies that in such a region, firms invest in product innovation. Likewise, $c^*(s)$ solves $\dot{c} = 0$; accordingly, for all $c > c^*(s)$, we have $\dot{c} < 0$. In this region, firms ivest in cost-reducing R&D activities. These two regions overlap in the cone to the north-east of the equilibrium point E. Therefore, if initial condition (c_0, s_0) identify a point belonging to the cone, then firms activate both types of investment from the very outset to the steady state. Otherwise, if, say, initial conditions identify point A to the north-west of both equilibrium loci, then firms initially invest only in product differentiation. Depending upon whether optimal trajectories cross the cone or not, firms may reach the steady state either (i) by carrying out only product innovation, allowing the marginal cost to adjust via the exogenous depreciation rate, or (ii) by carrying out both. The same argument goes through, *mutatis mutandis*, if initial conditions identify a point like B. In the residual region to the south-west of the equilibrium point, firms do not invest at all in either type of innovation, and they are driven to the steady state by the depreciation rate.

This discussion ultimately entails that the present model provides a theoretical framework corresponding to the main message conveyed by the empirical research in this field, revealing that firms first operate product innovation and later on implement process innovation. However, this theoretical setup also allows for other equilibrium solutions where process innovation comes first, or both types of activities coexist over the life cycle. This is consistent with Adner and Levinthal (2001), whose 'demand side' approach does not necessarily imply a well defined hierarchy between the two types of innovation.

Figure 3 : Process and Product Innovation



Now we come to the evaluation of the stability properties of the complete model with both types of innovation. Given that, in this case, the relevant Jacobian matrix is 4×4, one cannot draw the phase diagram. The Jacobian matrix is:

$$J^* = \begin{bmatrix} \frac{\partial \dot{s}}{\partial s} = \delta - 2x & \frac{\partial \dot{s}}{\partial x} - 2s & \frac{\partial \dot{s}}{\partial c} = 0 & \frac{\partial \dot{s}}{\partial k} = 0 \\ \frac{\partial \dot{x}}{\partial s} = \frac{(a-c)^2(3s-2)}{2\gamma(2+s)^3} & \frac{\partial \dot{x}}{\partial x} = \rho + 2x & \frac{\partial \dot{x}}{\partial c} = \frac{(a-c)(2-s)s}{2\gamma(2+s)^2} & \frac{\partial \dot{x}}{\partial k} = 0 \\ \frac{\partial \dot{c}}{\partial s} = 0 & \frac{\partial \dot{c}}{\partial x} = 0 & \frac{\partial \dot{c}}{\partial c} = \eta - k(1+\theta) & \frac{\partial \dot{c}}{\partial k} = -c(1+\theta) \\ \frac{\partial \dot{k}}{\partial s} = \frac{c(a-c)}{2\beta(2+s)^2} & \frac{\partial \dot{k}}{\partial x} = 0 & \frac{\partial \dot{k}}{\partial c} = -\frac{a-2c}{2\beta(2+s)} & \frac{\partial \dot{k}}{\partial k} = \rho \end{bmatrix} \tag{52}$$

whose characteristic equations yields four eigenvalues. Unfortunately, assessing the sign of such eigenvalues analytically is not feasible as their expressions are cumbersome. Moreover, we cannot obtain the explicit solutions for c^* and s^* as a function of parameters only, as the system

$$\begin{aligned} c - c^*(s) &= 0 \\ s - s^*(c) &= 0 \end{aligned} \tag{53}$$

is made up by equations whose degree is higher than four.

However, we may resort to numerical calculations, which can be performed as follows. We use the solutions:

$$x^* = \frac{\delta}{2}\,;\; k^* = \frac{\delta}{1+\theta} \tag{54}$$

and set the numerical values of parameters:

$$a = 1; \beta = \gamma = \frac{1}{2}; \theta = \frac{1}{5}; \delta = \rho = \frac{1}{20}. \tag{55}$$

Then, we may (i) solve the system (53) numerically, and (ii) compute the eigenvalues $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ of J^*. Given (55), we have:

$$c^* \simeq 0.0042\,;\; s^* \simeq 0.0076 \tag{56}$$

$$\begin{aligned} \zeta_1 &\simeq 0.1288\,;\; \zeta_2 \simeq 0.0808 \\ \zeta_3 &\simeq -0.0308\,;\; \zeta_4 \simeq -0.0288. \end{aligned} \tag{57}$$

Using instead:

$$a = 8; \beta = \gamma = \frac{1}{2}; \theta = \frac{1}{2}; \delta = \frac{1}{20}; \rho = \frac{1}{40}, \tag{58}$$

we obtain:

$$c^* \simeq 0.002\,;\; s^* \simeq 0.0001 \tag{59}$$

$$\begin{aligned} \zeta_1 &\simeq 0.0999\,;\; \zeta_2 \simeq 0.0499 \\ \zeta_3 &\simeq -0.0249\,;\; \zeta_4 \simeq -0.0249. \end{aligned} \tag{60}$$

In general, repeating the same exercise for admissible parameter values, one can verify that the outcome is regularly $\zeta_1, \zeta_2 > 0$ while $\zeta_3, \zeta_4 < 0$. Therefore, the equilibrium is a saddle point.

5 Concluding Remarks

In this paper we have investigated the product/technology life cycle by using a differential game in which firms may invest in product and process innovation. We have considered horizontal product innovation that reduces product substitutability, and process innovation reducing marginal production cost. The solution concept we have adopted is the closed-loop Nash equilibrium.

The main question we have addressed, in the light of the existing debate based upon empirical evidence, is about the relative timing of adoption of product and process innovations. We have shown that firms may activate both activities simultaneously, or one at a time, depending upon the initial conditions of the state variables, i.e., initial levels of product homogeneity *vis à vis* the marginal cost of production. In particular, four situations may arise: (i) firms activate both types of investment from the very outset to the steady state; (ii) firms initially invest only in product differentiation and then reach the steady state either carrying out only product innovation or carrying out both; (iii) firms initially invest only to reduce the cost of production and then arrive at the steady state either carrying out only process innovation or carrying out both; (iv) firms do not invest at all in either type of innovation, and they are driven to the steady state by the depreciation rate. The taxonomy of equilibria generated by the theoretical setup we have analysed is in line with the conclusions drawn by Adner and Levinthal (2001), pointing out that product/technology life cycle may not necessarily follow the first-product-then-process pattern outlined by conventional wisdom.

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