Capital Accumulation and Horizontal Mergers in Differential Oligopoly Games[1]

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Abstract

The existing (static) literature stresses the relevance of capital inputs in determining whether any given merger is (i) profitable and (ii) socially efficient, or not. We take a differential game approach to the same issue, proposing two different models based, respectively, on the capital accumulation dynamics introduced by Ramsey and Solow, respectively. We show that the change in the steady state size of productive plants induced by a merger may play a decisive role in determining whether such a merger is profitable, or socially efficient. However, unlike the static contributions in the same vein, we show that the parameter sets where, respectively, firms find it convenient to merge, and the merger is welfare-increasing, do not intersect at all, irrespectively of the capital accumulation dynamics being considered. This entails that a regulator concerned with the welfare performance of an industry should prevent firms from carrying out any horizontal merger.

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1 Introduction

The seminal paper by Salant *et al.* (1983) has introduced a famous puzzle about the profitability of horizontal mergers in a Cournot oligopoly with a technology characterised by constant returns to scale, common to all firms. According to their analysis, we should not worry about horizontal mergers unless it involves almost all the firms in the industry.[1] However, their model drastically underestimates the incentive towards merging, as shown by Perry and Porter (1985) by introducing fixed costs into the picture. They assume that the aggregate amount of capital is fixed at the industry level, and the associated fixed costs are distributed across firms in proportion to their individual holdings of the capital factor. By doing so, Perry and Porter allow the model to account for the intuitive fact that a merger gives rise to a new firm that is bigger than its parts (previously independent firms), and the new productive unit may produce a larger output at any given marginal costs, as compared to the rivals (as well as the previously independent firms that participated to the merger). This perspective leads Perry and Porter to find that, contrary to the conclusions reached by Salant *et al.* (1983), the price increase generated by the merger can often suffice to compensate for the decrease in the output of the merged firm, and therefore may make such a merger a profitable one. The increase in productive efficiency due to the synergies associated with the merger may be sufficiently large to more than compensate for the decrease in consumer surplus. If so, then the merger is also socially convenient.

[1]The incentive to merge investigated by Salant *et al.* (1983) can be re-examined in terms of the profitability of an output contraction by a subset of firms in the market, relative to the aggregate output of firms outside such a subset. This is done by Gaudet and Salant (1991, 1992), broadly confirming the conclusions of Salant *et al.* (1983).

Farrell and Shapiro (1990) interpret mergers as concentration-increasing transfers of industry-specific capital among firms. These transfers affect industry structure, therefore inducing changes in the Cournot equilibrium of the oligopoly market under consideration. Farrell and Shapiro characterise the conditions under which such transfers raise the market price, and examine the social and private incentives to merge, accounting for the fact that mergers alter the distribution of outputs across firms as well as the aggregate output level. They find that small firms typically have insufficient incentives to merge, while large firms have excessive incentives to do so. They also show that a merger is socially more attractive the more concentrated is production among the non-participant firms.[2]

The main stream of literature dealing with horizontal merger assumes Cournot behaviour to model market interaction. Deneckere and Davidson (1985) consider Bertrand behaviour, and show that any merger is profitable in such a case, since non-merging firms will match any price increase induced by the merger, as long as prices are strategic complements (i.e., products are demand substitutes). Hence, it appears that the nature of market competition strongly affects the profitability of mergers.[3]

Notwithstanding the fact that mergers heavily affect the evolution of an industry over time, mergers have usually been investigated in static settings. To the best of our knowledge, there exists two relevant exceptions. Relying on

[2]Similarly, McAfee and Williams (1992) show that (i) horizontal mergers are more likely to be welfare enhancing the more concentrated is the ownership of the nonmerging firms; and (ii) mergers that create a new largest firm, or increase the size of the largest firm, reduce welfare.

[3]On this point, see also Kwoka (1989). Other extensions account for the case of a non-linear market demand curve. See Cheung (1992), Fauli-Oller (1997) and Hennessy (2000).

Ericson and Pakes (1995), Gowrisankaran (1999) relates merger decisions to entry and investment decisions in a dynamic (stocastic) model, showing that mergers allow firms to better deal with shocks. In a more recent paper, Dockner and Gaunersdorfer (2001) assess the profitability of horizontal mergers in a dynamic Cournot game with sticky prices, first introduced by Simaan and Takayama (1978) and then further investigated by Fershtman and Kamien (1987), *inter alia*. Dockner and Gaunersdorfer show that any merger is profitable in such a setting, since the anticompetitive effects due to the merger are strong enough to benefit both the merging and the non-merging firms alike. From the social standpoint, mergers are always detrimental to welfare.

In general, the existing literature stresses the relevance of capital (or capacity) in determining whether any given merger is (i) profitable and (ii) socially efficient, or not. Hence, merger analysis represents a natural field of application of dynamic games with capital accumulation to build up productive capacity over time. We take this approach in this paper, where we propose two different models based, respectively, on the capital accumulation dynamics introduced by Ramsey (1928) and Solow (1956), respectively. We show that, indeed, the change in the relative (steady state) size of productive plants induced by a merger may play a decisive role in determining whether such a merger is profitable, or socially efficient. However, unlike the existing (static) contributions in the same vein, we show that the parameter sets where, respectively, firms find it convenient to merge, and the merger is welfare-increasing, do not intersect at all, irrespectively of the capital accumulation dynamics being considered. This points to a clearcut conclusion, namely, that a regulator concerned with the welfare performance of an industry should prevent firms from carrying out any horizontal merger.

The remainder of the paper is organised as follows. Section 2 contains the layout of the trwo models. The Ramsey model is described in section 3, while section 4 investigates the Solow model. Consluding remarks are in section 5.

2 The setup

We consider two well known dynamic settings. In both models, the market exists over $t \in [0\,,\,\infty)$, and is served by N firms producing a homogeneous good. Let $q_i(t)$ define the quantity sold by firm i at time t. The marginal production cost is constant and equal to c for all firms. Firms compete *à la* Cournot, the demand function at time t being:

$$p(t) = A - Q(t)\,,\ Q(t) \equiv \sum_{i=1}^{N} q_i(t)\ . \tag{1}$$

In order to produce, firms must accumulate capacity or physical capital $k_i(t)$ over time. The two models we consider in the present paper are characterised by two different kinematic equations for capital accumulation.

A] The Ramsey (1928) setting, with the following capital accumulation dynamics:

$$\frac{\partial k_i(t)}{\partial t} = f(k_i(t)) - q_i(t) - \delta k_i(t)\,, \tag{2}$$

where $f(k_i(t)) = y_i(t)$ denotes the output produced by firm i at time t. Also in this setting, we assume $f' \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f'' \equiv \partial^2 f(k_i(t))/\partial k_i(t)^2 < 0$. In this case, capital accumulates as a result of intertemporal relocation of unsold output $y_i(t) - q_i(t)$.[4] This can be interpreted in two ways. The first consists in viewing this setup as

[4]In the Ramsey model, firms operate at full capacity in steady state, where any investment is just meant to make up for depreciation.

a corn-corn model, where unsold output is reintroduced in the production process. The second consists in thinking of a two-sector economy where there exists an industry producing the capital input which can be traded against the final good at a fixed price equal to one. In this model, the control variable is $q_i(t)$, and the state variable is $k_i(t)$.

B] The Nerlove-Arrow (1962) or Solow (1956) setting, with the relevant dynamic equation being:

$$\frac{\partial k_i(t)}{\partial t} = I_i(t) - \delta k_i(t) \,, \tag{3}$$

where $I_i(t)$ is the investment carried out by firm i at time t, and $\delta > 0$ is the constant depreciation rate. The instantaneous cost of investment is $C_i\left[I_i\left(t\right)\right] = b\left[I_i\left(t\right)\right]^2$, with $b > 0$. We also assume that firms operate with a non-increasing returns technology $q_i(t) = f(k_i(t))$, with $f' \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f'' \equiv \partial^2 f(k_i(t))/\partial k_i(t)^2 \leq 0$. The demand function rewrites as:[5]

$$p(t) = A - \sum_{i=1}^{N} f(k_i(t)) \,. \tag{4}$$

Here, the control variable is the instantaneous investment $I_i(t)$, while the state variable is obviously $k_i(t)$. For the sake of tractability, in the remainder of the paper we will deal with model [B] under the assumption that $f\left(k_i\left(t\right)\right) = k_i(t)$.

In both models, the kinematic equation of player i's state variable is unaffected by the state and control variables of rivals. That is, strategic interaction among firms takes place through instantaneous profits only. We

[5] Again, notice that the assumption $q_i(t) = f(k_i(t))$ entails that firms always operate at full capacity.

follow this route in order to keep our models in line with the original formulations of dynamics (2-3). However, the analysis could be easily extended to account for the interaction between state and control variables of all players in the state dynamics without significantly changing our conclusions. A sufficient condition for all the ensuing results to continue to hold is that the kinematic equations of state variables be additively separable in state and control variables (see, e.g., Mehlmann, 1988, ch. 4).

2.1 The Ramsey model

Under the dynamic constraint (2), the Hamiltonian of firm i is:

$$\begin{aligned}\mathcal{H}_i &= \Big\{ e^{-\rho t} \left[A - q_i(t) - Q_{-i}(t) - c\right] q_i(t) + \lambda_{ii}(t) \left[f(k_i(t)) - q_i(t) - \delta k_i(t)\right] + \\ &\quad + \sum_{j \neq i} \lambda_{ij}(t) \left[f(k_j(t)) - q_j(t) - \delta k_j(t)\right] \Big\} , \end{aligned} \tag{5}$$

where $Q_{-i}(t) = \sum_{j \neq i} q_j(t)$.

First of all, we would like to briefly highlight a technical feature of the game, which is discussed in detail by Cellini and Lambertini (2001). The first order condition concerning the control variable is:

$$\frac{\partial \mathcal{H}_i(t)}{\partial q_i(t)} = A - 2q_i(t) - Q_{-i}(t) - c - \lambda_{ii}(t) = 0 \; . \tag{6}$$

Now examine at the co-state equation of firm i calculated for the state variable of firm i herself, for the closed-loop solution of the game:

$$\begin{aligned} -\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} - \sum_{j \neq i} \frac{\partial \mathcal{H}_i(t)}{\partial q_j(t)} \frac{\partial q_j^*(t)}{\partial k_i(t)} &= \frac{\partial \lambda_{ii}(t)}{\partial t} - \rho \lambda_{ii}(t) \Rightarrow \\ \frac{\partial \lambda_{ii}(t)}{\partial t} &= \lambda_{ii}(t) \left[\rho + \delta - f(k_i(t))\right] \end{aligned} \tag{7}$$

with $\partial q_j^*(t)/\partial k_i(t) = 0$, as it emerges from the best reply function obtained from the analogous to (6):

$$q_j^*(t) = \frac{A - c - Q_{-j}(t) - \lambda_{jj}(t)}{2} \; ; \tag{8}$$

Moreover, (8) also suffices to establish that the adjoint co-state equation:

$$-\frac{\partial \mathcal{H}_i(t)}{\partial k_j(t)} - \sum_{h \neq j} \frac{\partial \mathcal{H}_i(t)}{\partial q_h(t)} \frac{\partial q_h^*(t)}{\partial k_j(t)} = \frac{\partial \lambda_{ij}(t)}{\partial t} - \rho \lambda_{ij}(t) \tag{9}$$

is redundant since $\mu_{ij}(t) = \lambda_{ij}(t)e^{-\rho t}$ does not appear in firm i's first order condition (6) on the control variable. This amounts to saying that the Ramsey game belong to the so-called class of 'linear state games' (see Mehlmann, 1988, ch. 4; and Dockner *et al.*, 2000, section 7.3), since the first order condition on the control of firm i does not contain the state variable pertaining to the same firm or any of her rivals. Therefore, the open-loop solution produces a strictly time-consistent (or, equivalently, subgame perfect or Markov perfect) equilibrium, admitting the solution $\lambda_{ij}(t) = 0$ for all $j \neq i$ at any time t.

The best reply function (8) can be differentiated w.r.t. time to get

$$\frac{dq_i(t)}{dt} = \frac{-\sum_{j \neq i}(dq_j(t)/dt) - d\lambda_i(t)/dt}{2}. \tag{10}$$

Thanks to (7), the expression in (10) simplifies as follows:

$$\frac{dq_i(t)}{dt} = \frac{1}{2}\left[\left(f'(k_i(t)) - \rho - \delta\right)\lambda_i(t) - \sum_{j \neq i} \frac{dq_j(t)}{dt}\right]. \tag{11}$$

In order to simplify calculations and to obtain an analytical solution, we adopt the following assumption, based on firms' *ex ante* symmetry:

$$\sum_{j \neq i} q_j(t) = (N-1)q_i(t) \tag{12}$$

so that

$$\sum_{j \neq i} \frac{dq_j(t)}{dt} = \frac{(N-1)dq_i(t)}{dt}.$$

Thanks to symmetry, in the remainder we drop the index of the firm. As a further simplification, we also drop the indication of time. Using (12), together with (8) and (7), we rewrite (11) as follows:

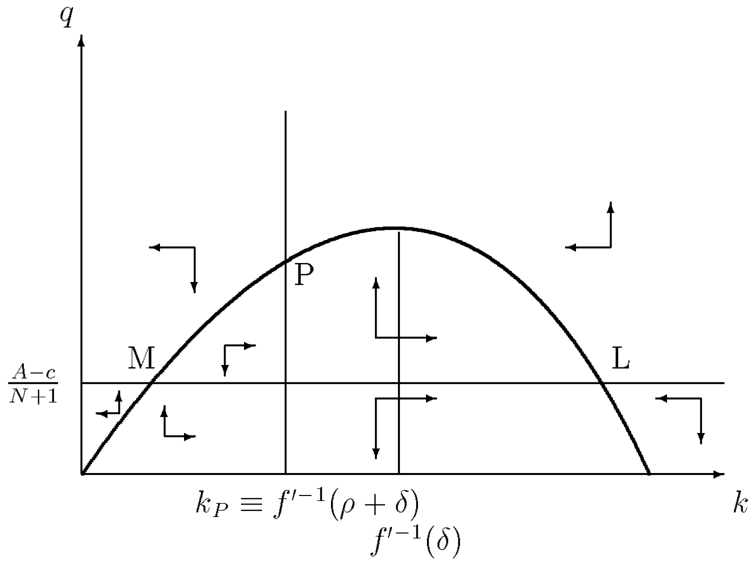
$$\frac{dq}{dt} = (f'(k) - \rho - \delta) \cdot \frac{[A - c - (N+1)q]}{N+1}, \tag{13}$$

yielding the steady state solutions:

$$q^{ss} = \frac{A-c}{N+1} \; ; \; f'(k) = \rho + \delta \, . \tag{14}$$

We are now able to draw a phase diagram in the space $\{k, q\}$, in order to characterise the steady state equilibrium. The locus $\dot{q} \equiv dq/dt = 0$ is given by $q^{ss} = (A-c) / (N+1)$ and $f'(k) = \rho + \delta$ in figure 1. Notice that the horizontal locus $q^{ss} = (A-c) / (N+1)$ denotes the usual equilibrium solution we are well accustomed with from the existing literature dealing with static market games. The two loci partition the space $\{k, q\}$ into four regions, where the dynamics of q is determined by (13), as summarised by the vertical arrows. The locus $\dot{k} \equiv dk/dt = 0$ as well as the dynamics of k, depicted by horizontal arrows, derive from (2). Steady states, denoted by M, L along the horizontal arm, and P along the vertical one, are identified by intersections between loci.

Figure 1: Cournot competition



Note that figure 1 illustrates one of five possible configurations, due to the fact that the position of the vertical line $f'(k) = \rho + \delta$ is independent of demand parameters, while the horizontal locus $q^{ss} = (A - c) / (N + 1)$ shifts upwards (downwards) as A (c and N) increases. Therefore, we obtain one of the following five regimes:

[1] There exist three steady state points, with $k_M < k_P < k_L$ (this is the situation depicted in figure 1).

[2] There exist two steady state points, with $k_M = k_P < k_L$.

[3] There exist three steady state points, with $k_P < k_M < k_L$.

[4] There exist two steady state points, with $k_P < k_M = k_L$.

[5] There exists a unique steady state point, corresponding to P.

An intuitive explanation for the above taxonomy can be provided, in the following terms. The vertical locus $f'(k) = \rho + \delta$ identifies a constraint on optimal capital embodying firms' intertemporal preferences, i.e., their common discount rate. Accordingly, maximum output level in steady state would be that corresponding to (i) $\rho = 0$, and (ii) a capacity such that $f'(k) = \delta$. Yet, a positive discounting (that is, impatience) induces producers to install a smaller steady state capacity, much the same as it happens in the well known Ramsey model. For this reason, we define this level of k as the *optimal capital constraint*, and label it as $\widehat{k}$. When the reservation price A is very large (or c and N are low), points M and L either do not exist (regime [5]) or fall to the right of P (regimes [2], [3], and [4]). Under these circumstances, the capital constraint is operative and firms choose the capital accumulation corresponding to P. As we will see below, this is fully consistent with the dynamic properties of the steady state points.

Notice that, since both steady state points located along the horizontal locus entail the same levels of sales. As a consequence, point L is surely inefficient in that it requires a higher amount of capital. Point M, as already mentioned above, corresponds to the optimal quantity emerging from the static version of the game. It is hardly the case of emphasising that this solution encompasses both monopoly (either when $N = 1$ or when $D = 0$) and perfect competition (as, in the limit, $N \to \infty$). In point M, $d\pi_i(t)/dq_i(t) = 0$, that is, the marginal instantaneous profit is nil.

The stability analysis of the system $\left\{\dot{k}, \dot{q}\right\}$, based upon the trace Tr and determinant Δ of its Jacobian matrix, leads to the following. The trace is $Tr = \rho > 0$, while the determinant Δ varies according to the point where it is evaluated:[6]

[6] For further details, see Cellini and Lambertini (1998).

Regime [1] In M, $\Delta < 0$, hence this is a saddle point. In P, $\Delta > 0$, so that P is an unstable focus. In L, $\Delta(\Xi) < 0$, and this is again a saddle point, with the horizontal line as the stable arm.

Regime [2] In this regime, M Coincides with P, so that we have only two steady states which are both are saddle points. In $M = P$, the saddle path approaches the saddle point from the left only, while in L the stable arm is again the horizontal line.

Regime [3] Here, P is a saddle; M is an unstable focus; L is a saddle point, as in regimes [1] and [2].

Regime [4] Here, points M and L coincide. P remains a saddle, while $M = L$ is a saddle whose converging arm proceeds from the right along the horizontal line.

Regime [5] Here, there exists a unique steady state point, P, which is also a saddle point.

We can sum up the above discussion as follows. The unique efficient and non-unstable steady state point is P if $k_P < k_M$, while it is M if the opposite inequality holds. Such a point is always a saddle. Individual equilibrium output is $q^{ss} = (A - c) / (N + 1)$ if the equilibrium is identified by point M, or the level corresponding to the optimal capital constraint $\widehat{k}$ if the equilibrium is identified by point P. The reason is that, if the capacity at which marginal instantaneous profit is nil is larger than the optimal capital constraint, the latter becomes binding. Otherwise, the capital constraint is irrelevant, and firms' decisions in each period are solely driven by the unconstrained maximisation of single-period profits. It is apparent that, in the present setting, firms always operate at full capacity. When the optimal output is q^{ss}, per-firm instantaneous profits in steady state are

$$\pi^{ss} = \frac{(A-c)^2}{(N+1)^2} \tag{15}$$

while they are $\pi^{ss} = f\left(\widehat{k}\right)\left\{A - c - Nf\left(\widehat{k}\right)\right\}$ if optimal output is $\widehat{k}$.

Now consider the fact that the solution driven by demand and marginal cost parameters, giving rise to profits (15), exactly replicates the standard Cournot solution of the static game with the same demand and cost conditions. Therefore, m firms, with $m \in (1, N]$, will find it profitable to horizontally merge iff:

$$\frac{(A-c)^2}{m(N-m+2)^2} > \frac{(A-c)^2}{(N+1)^2}, \tag{16}$$

which amounts to saying that the following inequality must be satisfied:

$$(N+1)^2 > m(N-m+2)^2 \tag{17}$$

that obviously coincides with the condition found in the static model by Salant *et al.* (1983).

This, without further investigation, suffices to state the following:

Proposition 1 *In the Ramsey model, the profit incentives towards horizontal mergers are the same as in the static Cournot model with constant marginal cost, as long as optimal output is determined by demand and marginal cost parameters only, both before and after the merger.*

Now examine the case where the Ramsey equilibrium prevails, before as well as after the merger. In such a case, the merger is profitable iff:

$$\frac{1}{m}\left[A - c - (N-m+1)f\left(\widehat{k}\right)\right]f\left(\widehat{k}\right) > \left[A - c - Nf\left(\widehat{k}\right)\right]f\left(\widehat{k}\right), \tag{18}$$

that is, iff:

$$(1-m)\left[A-c-(N+1)f\left(\widehat{k}\right)\right]>0\,. \tag{19}$$

Notice that the Ramsey equilibrium price is $p^{ss}=A-c-Nf\left(\widehat{k}\right)\geq c$ before the merger, and $p^{ss}=A-c-(N-m+1)f\left(\widehat{k}\right)\geq c$ after the merger, with $N+1>N>N-m+1$ for all $m>1$. Hence, in line of principle, it is admissible that $A-c<(N+1)f\left(\widehat{k}\right)$. Therefore, we can state:

Proposition 2 *Suppose the Ramsey equilibrium prevails both before and after the merger. In such a case,*

- *if* $A-c>(N+1)f\left(\widehat{k}\right)$, *then*

  $$(1-m)\left[A-c-(N+1)f\left(\widehat{k}\right)\right]<0$$

  always, and no merger is profitable, irrespective of the value of m;

- *if* $A-c<(N+1)f\left(\widehat{k}\right)$, *then*

  $$(1-m)\left[A-c-(N+1)f\left(\widehat{k}\right)\right]>0 \text{ for all } m>1\,,$$

  so that any merger is strictly profitable.

There remains a third case, namely, that where the pre-merger equilibrium entails an optimal output equal to $q^{ss}=(A-c)/(N+1)$, while the post-merger equilibrium entails an output equal to $f\left(\widehat{k}\right)$. This situation may arise if the decrease in the number of firms shifts up the horizontal arm in figure 1 sufficiently to drive the market equilibrium (point M) to the right of the Ramsey equilibrium (point P). In such a case, the relevant expression measuring the profitability of the merger involving m firms is:

$$\frac{1}{m}\left[A-c-(N-m+1)f\left(\widehat{k}\right)\right]f\left(\widehat{k}\right)-\frac{(A-c)^2}{(N+1)^2} \tag{20}$$

or:

$$\frac{\left[A-c-(N+1)f\left(\widehat{k}\right)\right]\left[(N+1)^2 f\left(\widehat{k}\right)-m\left(A-c+(N+1)f\left(\widehat{k}\right)\right)\right]}{m(N+1)^2}, \tag{21}$$

which may take either sign, depending upon the shape of $f\left(\widehat{k}\right)$ and the size of $\widehat{k}$.

It is worth noting that it is *not possible* for the efficient stable steady state[7] to be determined by the Ramsey condition before the merger, and by the demand and cost parameters after the possible merger. Indeed, the merger unambiguously leads to an upwards shift of the horizontal arm in figure 1, so that, if the saddle point lies on the vertical arm before the merger, it must lie on the vertical arm also after the merger.

In the remainder of the section, we take a social welfare analysis perspective. In particular, we compare the steady state social welfare of the two cases where the horizontal merger has occurred or not. Two points are worth stressing. First, we confine our analysis to the steady state point. Second, we keep referring to the two cases as to the pre- (or before-) merger and post- (or after-) merger, even if we do not adopt a dynamic perspective where merger occurs over time, but - rather - a comparative statistcs approach. We consider the sum of the consumer surplus and the individual profits of the present firms as the appropriate measure of the social welfare level: $SW = CS + \sum \pi$. Three cases must be distinguished.

a) If the individual firm's optimal output is determined by demand and cost parameters alone, both before and after the merger (i.e., the optimal steady state point lies along the horizontal arm of figure 1), the merger is detrimental from a social welfare perspective. Indeed, the aggregate

[7] Here, of course, "stable" has to be interpreted in the saddle sense.

output is lower and the market price is higher after the merger, as compared to the steady state associated with the pre-merger setting.

b) If the optimal output is determined by the Ramsey condition, both before and after the merger (i.e., the saddlepoint steady state lies on the vertical arm of figure 1), the merger is detrimental from a social welfare perspective. Also in this case, the aggregate output is lower and the market price is higher after the merger, as compared to the steady state associated with the pre-merger setting.

c) If the optimal output is determined by demand and cost parameters before the merger, while it is determined by the Ramsey condition after the merger, the social welfare is larger after the merger iff

$$\frac{N(A-c)}{(1+N)(1+N-m)} < f\left(\widehat{k}\right) < \frac{(2+N)(A-c)}{(1+N)(1+N-m)}. \qquad (22)$$

This derives from the simple comparison between the pre-merger social welfare, which is $SW^{pre} = N(2+N)(A-c)^2/[2(1+N)^2]$, and the post-merger social welfare, that is $SW^{post} = f\left(\widehat{k}\right)(1+N-m)[f\left(\widehat{k}\right)(-1-N+m)+2(A-c)]$.

By comparing the conclusion about the individual profitability of the merger, as it derives from the analysis of individual profits, and the social convenience of the merger, we can state the following:

Proposition 3 *Under the two cases where either (i) the individually optimal output is determined by demand and cost parameters alone, both before and after the merger or (ii) the optimal output is determined by the Ramsey condition, both before and after the merger, then the merger can be individually profitable, according to appropriate conditions on parameter, but it is socially* *detrimental. If the optimal output is determined by demand and cost parameters before the merger, while it is determined by the Ramsey condition after the merger, then the merger can be individually profitable or not, depending on the parameter configuration; likewise, it can be socially convenient or not, according to the parameter configuration. However, the parameter region where the merger is individually profitable does not intersect with the parameter region where the merger improves social welfare.*

While the proof of the first part of the Proposition is trivial, the proof of the second part requires cumbersom though easy calculation. When the pre-merger setting is associated with the steady state dictated by demand and cost parameters, while the post-merger setting is associated whith the Ramsey condition, it is easily checked that the individual profit is larger after the merger, as compared to the pre-merger situation, iff $f\left(\widehat{k}\right) \in (f_1, f_2)$, while social welfare is larger after the merger, iff $f\left(\widehat{k}\right) \in (f_3, f_4)$,where:

$$\begin{aligned} f_1 &= (A-c)/(1+N) \\ f_2 &= m(A-c)/[(1+N)(N+1-m)] \\ f_3 &= N(A-c)/[(1+N)(N+1-m)] \\ f_4 &= (2+N)(A-c)/[(1+N)(N+1-m)] \end{aligned} \tag{23}$$

Note that $\max(f_1, f_2) < \min(f_3, f_4)$, so that the values of $f\left(\widehat{k}\right)$for which the merger is individually profitable for the firms, are always stricly smaller than the values of $f\left(\widehat{k}\right)$ under which the merger improves the social welfare.

To account for the possibility of price competition, one can extend the analysis to the case of differentiated products (see Cellini and Lambertini, 1998). In such a case, since there exists a steady state replicating the Nash equilibrium of the static game, it needs no proof to conclude that the results obtained by Deneckere and Davidson (1985) holds unmodified in the

Bertrand-Ramsey model as well, i.e., any merger is strictly profitable under Bertrand competition with differentiated products.

2.2 The Nerlove-Arrow-Solow model

When capital accumulates according to (3), the relevant Hamiltonian for firm i is:

$$\begin{aligned}\mathcal{H}_i &= e^{-\rho t}\left\{\left[A - k_i(t) - \sum_{j\neq i} k_j(t) - c\right] k_i(t) - b\left[I_i(t)\right]^2 + \right. \\ &\left. +\lambda_{ii}(t)\left[I_i(t) - \delta k_i(t)\right] + \sum_{j\neq i}\lambda_{ij}(t)\left[I_j(t) - \delta k_j(t)\right]\right\} . \end{aligned} \tag{24}$$

Necessary conditions for the closed-loop memoryless equilibrium are:

$$\begin{aligned}&(i)\ \frac{\partial \mathcal{H}_i(t)}{\partial I_i(t)} = 0 \Rightarrow -2bI_i(t) + \lambda_{ii}(t) = 0 \Rightarrow I_i^*(t) = \lambda_{ii}(t)/(2b) \\ &(ii)\ -\frac{\partial \mathcal{H}_i(t)}{\partial k_i(t)} - \sum\nolimits_{j\neq i}\frac{\partial \mathcal{H}_i(t)}{\partial I_j(t)}\frac{\partial I_j^*(t)}{\partial k_i(t)} = \frac{\partial \lambda_{ii}(t)}{\partial t} - \rho\lambda_{ii}(t) \Rightarrow \\ &\Rightarrow \frac{\partial \lambda_{ii}(t)}{\partial t} = (\rho+\delta)\,\lambda_{ii}(t) + 2k_i(t) + \sum\nolimits_{j\neq i} k_j(t) - (A-c) \\ &(ii')\ -\frac{\partial \mathcal{H}_i(t)}{\partial k_j(t)} - \sum\nolimits_{h\neq j}\frac{\partial \mathcal{H}_i(t)}{\partial I_h(t)}\frac{\partial I_h^*(t)}{\partial k_j(t)} = \frac{\partial \lambda_{ij}(t)}{\partial t} - \rho\lambda_{ij}(t)\ ,\end{aligned} \tag{25}$$

with the transversality conditions:

$$\lim_{t\to\infty} \mu_{ij}(t)\cdot k_i(t) = 0 \text{ for all } i,j\ . \tag{26}$$

Now observe that, on the basis of (25-i), we have:

$$\frac{\partial I_j^*(t)}{\partial k_i(t)} = 0 \text{ for all } i,j\ . \tag{27}$$

Moreover, condition (25-ii'), which yields $\partial\lambda_{ij}(t)/\partial t$, is redundant in that $\lambda_{ij}(t)$ does not appear in the first order conditions (25-i) and (25-ii). Therefore, the closed-loop solution degenerates into an open-loop solution.[8]

Differentiating (25-i) w.r.t. time we obtain:

$$\frac{\partial I_i(t)}{\partial t} = \frac{1}{2b} \cdot \frac{\partial \lambda_{ii}(t)}{\partial t} \ . \tag{28}$$

Then, replace (25-i) into (25-ii), to get the following expression for the dynamics of the costate variable $\lambda_{ii}(t)$:

$$\frac{\partial \lambda_{ii}(t)}{\partial t} = 2b\,(\rho+\delta)\,I_i(t) + 2k_i(t) + \sum_{j\neq i} k_j(t) - (A-c) \ , \tag{29}$$

which can be plugged into (28), that rewrites as:

$$\frac{\partial I_i(t)}{\partial t} = (\rho+\delta)\,I_i(t) + \frac{1}{2b}\left[2k_i(t) + \sum_{j\neq i} k_j(t) - (A-c)\right] \ , \tag{30}$$

The discussion carried out so far establishes the Nerlove-Arrow-Solow game is a linear state game, whose open-loop solution produces a strictly time-consistent (Markov perfect) equilibrium.[9]

Invoking symmetry across firms and simplifying, we can rewrite (30):

$$\frac{\partial I(t)}{\partial t} = \frac{1}{2b}\left[2b\,(\rho+\delta)\,I(t) - A + c + (N+1)k(t)\right] \ , \tag{31}$$

with the righ hand side being zero at:

$$I(t) = \frac{A - c - (N+1)k(t)}{2b\,(\rho+\delta)} \ , \tag{32}$$

[8]Note that, however, the open-loop solution does not coincide with the feedback solution, where each firm holds a larger capacity and sells more than in the open-loop equilibrium (see Reynolds, 1987).

[9]For further details, and the proof of this result for a generic technology $f\,(k_i\,(t))$ with non-increasing returns, see Cellini and Lambertini (2001).

while $\partial k(t)/\partial t = 0$ at:

$$k^{ss} = \frac{A - c}{N + 1 + 2b\delta\,(\rho + \delta)}\,. \tag{33}$$

Using (3) and (31), it can be verified that the steady state identifies a saddle point (see Cellini and Lambertini, 2001).

The corresponding pre-merger steady state profits are:

$$\pi^{ss} = \frac{(A - c)^2\,[1 + b\delta\,(2\rho + \delta)]}{[N + 1 + 2b\delta\,(\rho + \delta)]^2}\,. \tag{34}$$

Now consider the perspective of a merger involving m firms, $m \in (1, N]$. The post-merger profits are:

$$\pi^{ss} = \frac{(A - c)^2\,[1 + b\delta\,(2\rho + \delta)]}{[N - m + 2 + 2b\delta\,(\rho + \delta)]^2}\,, \tag{35}$$

with the merger being profitable iff:

$$[N + 1 + 2b\delta\,(\rho + \delta)]^2 - m\,[N - m + 2 + 2b\delta\,(\rho + \delta)]^2 > 0\,. \tag{36}$$

The l.h.s. of (36) is equal to zero at:

$$m = N + \frac{3}{2} + 2b\delta\,(\rho + \delta) \pm \frac{\sqrt{4N + 5 + 8b\delta\,(\rho + \delta)}}{2}\,. \tag{37}$$

It is easy to check that $m_- < N < m_+$ for all

$$N > [1 + 2b\delta\,(\rho + \delta)]\,. \tag{38}$$

Provided that the above condition is satisfied, the inequality (36) is met for all $m \in (m_-\,, N]$. Otherwise, if $N > [1 + 2b\delta\,(\rho + \delta)]$, $m_- > N$ and no admissible merger is profitable. Simple comparative statics show the following properties:

$$\frac{\partial m_-}{\partial N} > 0\,;\ \frac{\partial m_-}{\partial b} > 0\,;\ \frac{\partial m_-}{\partial \delta} > 0\,;\ \frac{\partial m_-}{\partial \rho} > 0\,. \tag{39}$$

The foregoing discussion proves:

Proposition 4 *In the Solow-Nerlove-Arrow model, there exists a critical threshold*

$$m^* = N + \frac{3}{2} + 2b\delta\left(\rho+\delta\right) - \frac{\sqrt{4N+5+8b\delta\left(\rho+\delta\right)}}{2}$$

$$\textit{with } m^* < N \textit{ for all } N > \left[1+2b\delta\left(\rho+\delta\right)\right] ,$$

beyond which the horizontal merger is profitable. Such a threshold is monotonically increasing in $\{N, b, \delta, \rho\}$. *If* $N < \left[1+2b\delta\left(\rho+\delta\right)\right]$, *then no profitable merger can take place.*

As far as the social welfare is concerned, let us consider the sum of the consumer surplus and the total profits as the social welfare index. The consumer surplus is $CS = (nk)^2/2$, where n is the appropiate number of firms operating in the market, while the profits are given by equations (34) and (35). It is simple to compute the social welfare in the steady state, under the two alternative settings, corresponding to the cases where the merger occurs (SW^{post}) or does not (SW^{pre}). We are interested in evaluating the sign of the difference $DSW = SW^{post} - SW^{pre}$; if it is positive, the merger improves the social welfare. After simple substitutions and calculations, we obtain:

$$DSW \propto \frac{(1+N-m)(3+N-m+2b\delta(\delta+2\rho)}{[2+N-m+2b\delta(\delta+\rho)]^2} - \frac{N[2+N+2b\delta(\delta+2\rho)]}{[1+r+2b\delta(\delta+\rho)]^2} \tag{40}$$

which is negative for all $1 < m < N$. Consequently, we can state the following:

Proposition 5 *In the Solow-Nerlove-Arrow accumulation model, the merger can be individaully profitable or not, depending on the configuration of parameters, but it is always detrimental from the social welfare standpoint.*

It is worth stressing that the above Proposition is in sharp contrast with the result by Perry and Porter (1985). Moreover, in Cellini and Lambertini

(2002), it is shown that the Solow-Nerlove-Arrow setting yields the same steady state outlined above, when one considers direct demand function, *à la* Bertrand. This amounts to saying that this setting is the dynamic counterpart of Kreps and Scheinkman (1983). Accordingly, here it is not possible to properly distinguish between quantity and price competition, as firms' strategy space is defined over investment efforts only. As a consequence, the results derived by Deneckere and Davidson (1985) cannot hold in the present model.

3 Conclusions

In the foregoing analysis, we have taken a differential game approach to study the individual profitability and the social convenience of horizontal mergers involving oligopolistic firms. We have taken into account different types of capital accumulation: the Ramsey model, where capital accumulation takes place through unsold output (i.e., consumption postponement) and the Solow model, where capital accumulation requires a costly investment. The individual profitability and the social efficiency, evaluated at the steady state points, generally depend on the parameters configuration. However, in neither model a horizontal merger can simultaneously be both privately and socially convenient. This is in sharp contrast with the conclusions reached by the static analysis of the same problem (Perry and Porter, 1985; Farrell and Shapiro, 1990, *inter alia*) and suggests that a public agency in charge of regulating merger behaviour should forbid any such operations.

References

[1] Başar, T., and Olsder, G.J. (1982, 1995^2), *Dynamic Noncooperative Game Theory*, San Diego, Academic Press.

[2] Cellini, R. and L. Lambertini (1998), "A Dynamic Model of Differentiated Oligopoly with Capital Accumulation", *Journal of Economic Theory*, **83**, 145-55.

[3] Cellini, R. and L. Lambertini (2000), "Non-Linear Market Demand and Capital Accumulation in a Differential Oligopoly Game", working paper no. 370, Department of Economics, University of Bologna.

[4] Cellini, R. and L. Lambertini (2001), "Differential Oligopoly Games where the Closed-Loop Memoryless and Open-Loop Equilibria Coincide", working paper no. 402, Department of Economics, University of Bologna.

[5] Cellini, R. and L. Lambertini (2002), "Price vs Quantity in a Dynamic Duopoly Game with Capacity Accumulation", working paper no. 449, Department of Economics, University of Bologna.

[6] Cheung, F.K. (1992), "Two Remarks on the Equilibrium Analysis of Horizontal Mergers", *Economics Letters*, **40**, 119-23.

[7] Deneckere, R. and C. Davidson (1985), "Incentives to Form Coalitions with Bertrand Competition", *RAND Journal of Economics*, **16**, 473-86.

[8] Dockner, E.J. and A. Gaunersdorfer (2001), "On the Profitability of Horizontal Mergers in Industries with Dynamic Competition", *Japan and the World Economy*, **13**, 195-216.

[9] Dockner, E.J, S. Jørgensen, N. Van Long and G. Sorger (2000), *Differential Games in Economics and Management Science*, Cambridge, Cambridge University Press.

[10] Ericson, R. and A. Pakes (1995), "Markov-Perfect Industry Dynamics: A Framework for Empirical Work", *Review of Economic Studies*, **62**, 53-82.

[11] Farrell, J. and C. Shapiro (1990), "Horizontal Mergers: An Equilibrium Analysis", *American Economic Review*, **80**, 107-26.

[12] Fauli-Oller, R. (1997), "On Merger Profitability in a Cournot Setting", *Economics Letters*, **54**, 75-79.

[13] Fershtman, C. and M.I. Kamien (1987), "Dynamic Duopolistic Competition with Sticky Prices", *Econometrica*, **55**, 1151-64.

[14] Gaudet, G. and S.W. Salant (1991), "Increasing the Profits of a Subset of Firms in Oligopoly Models with Strategic Substitutes", *American Economic Review*, **81**, 658-65.

[15] Gaudet, G. and S.W. Salant (1992), "Towards a theory of Horizontal Mergers", in Norman, G. and M. La Manna (eds), *The New Industrial Economics: Recent Developments in Industrial Organization, Oligopoly and Game Theory*, Cheltenham, Edward Elgar.

[16] Gowrisankaran, G. (1999), "A Dynamic Model of Endogenous Horizontal Mergers", *RAND Journal of Economics*, **30**, 56-83.

[17] Hennessy, D.A. (2000), "Cournot Oligopoly Conditions under which Any Horizontal Merger Is Profitable", *Review of Industrial Organization*, **17**, 277-84.

[18] Kreps, D. and J. Scheinkman (1983), "Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes", *Bell Journal of Economics*, **14**, 326-37.

[19] Kwoka, J.E. (1989), "The Private Profitability of Horizontal Mergers with Non-Cournot and Maverick Behavior", *International Journal of Industrial Organization*, **7**, 403-11.

[20] Martin, S. (1993), *Advanced Industrial Economics*, Oxford, Blackwell.

[21] McAfee, R.P. and M.A. Williams (1992), "Horizontal Mergers and Antitrust Policy", *Journal of Industrial Economics*, **40**, 181-87.

[22] Mehlmann, A. (1988), *Applied Differential Games*, New York, Plenum Press.

[23] Perry, M.K. and R.H. Porter (1985), "Oligopoly and the Incentive for Horizontal Merger", *American Economic Review*, **75**, 219-27.

[24] Ramsey, F.P. (1928), "A Mathematical Theory of Saving", *Economic Journal*, **38**, 543-59. Reprinted in Stiglitz, J.E. and H. Uzawa (1969, eds), *Readings in the Modern Theory of Economic Growth*, Cambridge, MA, MIT Press.

[25] Reynolds, S.S. (1987), "Capacity Investment, Preemption and Commitment in an Infinite Horizon Model", *International Economic Review*, **28**, 69-88.

[26] Salant, S.W., S. Switzer and R.J. Reynolds (1983), "Losses from Horizontal Merger: The Effects of an Exogenous Change in Industry Structure on Cournot-Nash Equilibrium", *Quarterly Journal of Economics*, **98**, 185-213.

[27] Simaan, M. and T. Takayama (1978), "Game Theory Applied to Dynamic Duopoly Problems with Production Constraints", *Automatica*, **14**, 161-66.

[28] Solow, R. (1956),"A Contribution to the Theory of Economic Growth", *Quarterly Journal of Economics*, **70**, 65-94.