A Simple Approach to CAPM, Option Pricing and Asset Valuation

Riccardo Cesari (*)
Università di Bologna, Dip. Matemates, viale Filopanti, 5
40126 Bologna, Italy
E-mail: rcesari@economia.unibo.it

Carlo D'Adda
Università di Bologna, Dip. Scienze economiche, strada Maggiore, 45
40125 Bologna, Italy
E-mail: dadda@spbo.unibo.it

Abstract

In this paper we propose a simple, intuitive approach to asset valuation in terms of marginal contributions to the characteristics (moments) of the market portfolio. Considering only the first two moments, mean and variance, the valuation equation is shown to correspond to Sharpe's CAPM. A risk-neutral pricing formula is easily derived, showing the equivalence between CAPM and the Black and Scholes' model. Extensions to higher moments like skewness and kurtosis are straightforward, providing a generalized valuation equation. Finally, the generalized equation is derived in a different, more rigorous way, as a result of a classical intertemporal general equilibrium model.

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(*) Corresponding author

1. Motivation

The Capital asset pricing model and the option pricing theory are two of the best known and most important developments in the subject of Finance. The first model is provided by William Sharpe (1964) even if Tobin (1958), Treynor (1965), Lintner (1965) and Mossin (1966) reached similar results during the same period and all of them are indebted to the Markowitz (1952, 1959) portfolio model.

The Option pricing theory, on the other hand, derives from the seminal paper of Black and Scholes (1973), in which an arbitrage argument is developed to solve the old problem of pricing option contracts[1] in a completely new way.

Nowadays, the two models have become the cornerstones of any financial *curriculum studiorum*.

In particular, a student of Economics learns about CAPM during his second year courses and the golden formula he finds in his handbook[2] is:

$$\overline{R}_j = R_F + \beta_{jM}(\overline{R}_M - R_{RF}) \qquad (1)$$

where $\overline{R}_j \equiv E_t(R_j)$ is the expected (at time t) rate of total return of stock j, R_{RF} is the risk-free rate, R_M is the market rate of return and β_{jM} is the beta coefficient, measuring the risk of the stock and defined by the covariance between R_j and R_M divided by the variance of R_M.

If our student is clever enough, he will understand that CAPM, as a capital asset *pricing* model, is an equilibrium model to price financial assets of any kind, even if standard implementation is usually limited to common stocks. In fact, if S(t) is the price at time t of asset j and M(t) is the price (index) of the market, using the definition of rate of return between current time t and a future date T (excluding dividends for simplicity):

$$R_j = \frac{S(T) - S(t)}{S(t)} \qquad (2)$$

and, substituting into Equation 1, he can obtain the CAPM formula in price terms:

$$S(t) = \frac{E_t(S(T))}{1 + R_{RF}} - \frac{E_t(M(T) - M(t)(1 + R_{RF}))}{(1 + R_{RF})Var_t(M(T))} Cov_t(S(T), M(T)) \qquad (3)$$

The interpretation of the price formula is straightforward: the current price of the asset is the future price expected today $E_t(S(T))$ and discounted at the risk-free rate minus a risk adjustment that depends on the covariance between the asset and the market.

In a more compact form:

$$S(t) = P_{RF} E_t(S(T)) + P_M Cov_t(S(T), M(T)) \tag{4}$$

where P_{RF} is the current price of a zero coupon bond, giving one unit of money at time T, and P_M is the (negative) price of one unit of risk (i.e. covariance).

The following year, during his Finance classes, our student learns about option pricing using a completely different set-up and obtaining a completely different result for the price of derivative assets.
In the simplest case of a European call option, which gives the right to buy a specified asset (underlying) at a given date T, paying a given amount K (strike price), the celebrated Black and Scholes (1973) model provides the price of the call:

$$C(t) = S(t)\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$
where (5)
$$d_1 = \frac{\ln(S(t)/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

r is the constant, continuously compounded risk-free rate and $\Phi(x)$ is the probability of a number less than or equal to x according to the standard normal distribution function.
Even if he recognises that continuous and discrete compounding are equivalent, in the sense that $e^{-r(T-t)}$ is the same as $\frac{1}{1+R_{RF}}$, the two approaches will still appear to be quite different.
Can they be compatible?
Presented in different contexts, by different teachers, in different academic years, the two models seem to belong to different sections of Finance and the link between them, if it exits, appears completely lacking[3].
Moreover, CAPM is very general, concerning stocks, bonds and derivative assets, including puts and calls, but option pricing is less special than it appears if you bear in mind that common stocks are call options written on the assets of the firm and corporate bonds are equivalent to default-free bonds plus a short position in puts. Are they therefore two competing models[4] of asset pricing?

In the following sections we shall show that the two models are special cases of a more general valuation equation. In particular, we shall present a simple and intuitive pricing model, which includes CAPM (section 2), risk-neutral pricing and therefore the option approach (section 3), a direct route to generalizations with explicit expressions for prices and testable restrictions on risk premia (section 4) and a derivation of the same pricing function through a more rigorous intertemporal general equilibrium model (section 5). A final section concludes the paper.

2. Intuition

In (micro)economics, goods are priced at the margin (utility of a marginal quantity). We could, therefore, try to use this principle in finance, to price financial assets.

In order to do this, we need a simple, basic assumption according to which a financial good (or asset) is just a bundle of characteristics. Just as consumer goods are physical objects with physical characteristics (see Lancaster (1966)), financial assets are random variables (random processes) with moments as characteristics: mean, variance, skewness etc.

Each asset is priced at the margin, in terms of its marginal contributions to the measures of the characteristics of the global market portfolio, the price of the asset being the sum of price, P_i, times marginal quantity, Mch_i, of each characteristics:

$$S(t) = P_1(t)Mch_1(t) + P_2(t)Mch_2(t) + \dots\dots\dots \quad (6)$$

If this is the intuition, let us give the simplest conceivable example.

Example 1. Suppose that the expected future (time T) value (the mean or first moment) is the only relevant characteristic.

The issue of g units of asset S in a perfectly competitive market has a differential effect on the characteristic of the market portfolio:

$$\Delta ch_1(t) = E_t(M(T)+gS(T))-E_t(M(T)) = gE_t(S(T))$$

and the marginal characteristic is just the limit of the ratio $\dfrac{\Delta ch_1(t)}{g}$ as the quantity g goes to zero:

$$Mch_1(t) \equiv \lim_{g \downarrow 0} \frac{\Delta ch_1(t)}{g}$$

Therefore the price of the asset is the marginal effect in the characteristic multiplied by the price of the characteristic:

$$S(t) = P_1(t)Mch_1(t) = P_1(t)E_t(S(T)) \quad (7)$$

What about $P_1(t)$? By definition, it is the competitive price of one unit of

the characteristic (the mean).
If a default-free zero-coupon bond exists maturing at T and paying at that time one unit of money, its price $P_{RF}(t)=e^{-r(T-t)}$ must satify Equation 7:

$$P_{RF}(t)=P_1(t)E_t(1)=P_1(t)= e^{-r(T-t)}$$

so that the price of the first characteristic can be identified with the discount function, and Equation 7 becomes:

$$S(t) = e^{-r(T-t)}E_t(S(T)) \tag{8}$$

If the mean is the only relevant characteristic, prices reflect the risk-neutral valuation principle: the price of an asset is the expected future value discounted at the risk-free rate.

Example 2. Suppose, as a second step, that mean and variance (the first two moments) are the relevant characteristics. The issue of g units of asset S implies the following differential effects:

$$\begin{aligned} \Delta ch_1(t) &= E_t(M(T)+gS(T))-E_t(M(T)) = gE_t(S(T)) \\ \Delta ch_2(t) &= Var_t(M(T)+gS(T))-Var_t(M(T)) \\ &= g^2\, Var_t(S(T))+2gCov_t(S(T),M(T)) \end{aligned} \tag{9}$$

Therefore, taking the limits:

$$Mch_i(t) \equiv \lim_{g\downarrow 0} \frac{\Delta ch_i(t)}{g} \qquad i = 1,2$$

$$\begin{aligned} S(t) &= P_1(t)Mch_1(t) + P_2(t)Mch_2(t) \\ &= P_1(t)E_t(S(T)) + P_2(t)2Cov_t(S(T), M(T)) \end{aligned} \tag{10}$$

Once again, we have to identify $P_1(t)$ and $P_2(t)$.
The zero-coupon bond and the market portfolio can be used to invert the price formula, obtaining P_1 and P_2 :

$$P_{RF}(t) = P_1(t)E_t(1)+2P_2(t)Cov_t(1,M(T)) = P_1(t)$$

$$M(t) = P_{RF}(t)E_t(M(T))+2P_2(t)Var_t(M(T))$$

so that $P_2(t)$ is obtained in terms of observable variables:

$$P_2(t) = \frac{M(t) - P_{RF}(t)E_t(M(T))}{2Var_t(M(T))}$$

Substituting, we have Sharpe's CAPM of Equation 3:

$$S(t) = P_{RF}(t)E_t(S(T)) + \frac{M(t) - P_{RF}(t)E_t(M(T))}{Var_t(M(T))} Cov_t(S(T), M(T))$$

Note that the asset price is made by a risk-neutral component plus a risk-adjustment component.

3. Equivalent pricing functions

Let us write Equation 10 in an equivalent form, collecting P_{RF} and the expectation operator:

$$S(t) = e^{-r(T-t)} E_t\left[S(T) + 2\frac{P_2(t)}{P_{RF}(t)} M(T)(S(T) - E_t(S(T)))\right] \qquad (11)$$

In this way, the asset price appears as the discounted (natural) expectation of a risk-adjusted argument, the expression in square brackets.
Now define an expectation operator $\hat{E}_t$ such that:

$$\hat{E}_t(S(T)) \equiv E_t\left[S(T) + 2\frac{P_2(t)}{P_{RF}(t)} M(T)(S(T) - E_t(S(T)))\right]$$

Clearly, the risk-adjustment in $\hat{E}_t$ has been made through the probability distribution, not the argument.
We can, therefore, write Equation 11 as:

$$S(t) \equiv e^{-r(T-t)} \hat{E}_t(S(T)) \qquad (12)$$

and, comparing it with Equation 8 of Example 1, it should be no surprise that $\hat{E}_t$ is called the risk-neutral expectation operator.
But how can we obtain $\hat{E}_t$ from an operational point of view?
Let μ be the compounded average rate of return of asset S. We have, by definition:

$$S(t) \equiv E_t(e^{-\mu(T-t)} S(T)) \equiv \hat{E}_t(e^{-r(T-t)} S(T))$$

so that the risk-neutral expectation $\hat{E}_t$ is the natural expectation E_t with the average rate μ substituted by the risk-free rate r. Given that μ in the theory of probability is called drift coefficient, the previous result is an application of Girsanov's theorem of drift change (Duffie, 1992 p. 237).

Example 3. Consider the case of a European call option, giving the right, at maturity T, to the payoff C(T)=max(0, S(T)-K).
According to the two-moment pricing of Example 2, we have the call price:

$$C(t) = P_{RF}(t)E_t(C(T)) + 2P_2(t)Cov_t(C(T), M(T)) = P_{RF}(t)\hat{E}_t(C(T))$$

The first expression is the *natural pricing* function, which requires the calculation of natural expectations, the second one is the *risk-neutral pricing* function, requiring to substitute the average rate μ with r.
For example, if S(T) and M(T) are jointly normally distributed[5] we obtain the natural pricing formula:

$$C(t) = P_{RF}(t)[\frac{\sigma_S}{\sqrt{2\pi}} \exp(-\frac{(K - S(t)e^{\mu\tau})^2}{2\sigma_S^2}) + (S(t)e^{\mu\tau} - K)\Phi(\frac{S(t)e^{\mu\tau} - K}{\sigma_S})]$$

$$+ P_2(t)2Cov_t(S(T), M(T))\Phi(\frac{S(t)e^{\mu\tau} - K}{\sigma_S})$$

$$\sigma_S \equiv \frac{\sigma^2}{2\mu}(e^{2\mu\tau} - 1) \qquad \tau \equiv T - t$$

and the equivalent risk-neutral pricing formula:

$$C(t) = e^{-r\tau}\frac{\hat{\sigma}_S}{\sqrt{2\pi}} \exp(-\frac{(K - S(t)e^{r\tau})^2}{2\hat{\sigma}_S^2}) + (S(t) - Ke^{-r\tau})\Phi(\frac{S(t)e^{r\tau} - K}{\hat{\sigma}_S})$$

with

$$\hat{\sigma}_S \equiv \sqrt{\frac{\sigma^2(\exp(2r\tau) - 1)}{2r}} \qquad \tau \equiv T - t$$

If S(T) and M(T) are jointly lognormally distributed[6] we obtain the natural two-moment pricing formula:

$$C(t) = P_1 E_t(C(T)) + P_2(t)\, 2Cov_t(C(T),M(T))$$

where:

$$E_t(C(T)) = \exp(\mu_S + \frac{\sigma_S^2}{2})\Phi(\frac{\mu_S + \sigma_S^2 - \log K}{\sigma_S}) - K\Phi(\frac{\mu_S - \log K}{\sigma_S})$$

$$\begin{aligned}
&\mathrm{Cov}_t(C(T), M(T)) = \\
&\qquad \exp(\mu_S + \frac{\sigma_S^2}{2} + \mu_M + \frac{\sigma_M^2}{2} + \rho_{SM}\sigma_S\sigma_M)\Phi(\frac{\mu_S + \sigma_S^2 + \rho_{SM}\sigma_S\sigma_M - \log K}{\sigma_S}) \\
&\qquad - \exp(\mu_S + \frac{\sigma_S^2}{2} + \mu_M + \frac{\sigma_M^2}{2})\Phi(\frac{\mu_S + \sigma_S^2 - \log K}{\sigma_S}) \\
&\qquad - K\exp(\mu_M + \frac{\sigma_M^2}{2})\left[\Phi(\frac{\mu_S + \rho_{SM}\sigma_S\sigma_M - \log K}{\sigma_S}) - \Phi(\frac{\mu_S - \log K}{\sigma_S})\right]
\end{aligned}$$

$$\mu_S \equiv \log(S(t)) + (\mu - \sigma^2/2)\tau \qquad \sigma_S^2 \equiv \sigma^2\tau \qquad \tau \equiv T - t$$

and the risk-neutral formula is given by the Black and Scholes model in Equation 5.

4. Extensions

Lognormality and other non normal distributions suggest that not only mean and variance but also higher moments like skewness and kurtosis should be included in the pricing function[7].

Along the line of section 2 we calculate the differential effect on the market portfolio of g units of asset S in terms of third (skewness) and fourth (kurtosis) central moments, and let g go to zero in order to get a marginal effect.

The differences in the market moments before and after the issue of g units of asset S are, respectively:

$$\Delta ch_3(t) = E[[(M(T)+gS(T))-E(M(T)+gS(T))]^3] - E[(M(T)-E(M(T)))^3] =$$

$$\begin{aligned}
&E[g^3(S(T)-E(S(T)))^3]+ \\
&3E[g^2(S(T)-E(S(T)))^2(M(T)-E(M(T)))]+ \\
&3E[g(S(T)-E(S(T)))(M(T)-E(M(T)))^2]
\end{aligned} \qquad (13)$$

$$\Delta ch_4(t) = E[[(M(T)+gS(T))-E(M(T)+gS(T))]^4] - E[(M(T)-E(M(T)))^4] =$$

$$\begin{aligned}
&E[g^4(S(T)-E(S(T)))^4]+ \\
&4E[g^3(S(T)-E(S(T)))^3(M(T)-E(M(T)))]+
\end{aligned} \qquad (14)$$

$$6E[g^2(S(T)-E(S(T)))^2(M(T)-E(M(T)))^2]+$$
$$4E[g(S(T)-E(S(T)))(M(T)-E(M(T)))^3]$$

so that, at the margin, we obtain the following equation:

$$S(t) = P_1(t)E_t(S(T)) + P_2(t)2Cov_t(S(T),M(T))+ P_3(t)3Cosk_t(S(T),M(T)) + P_4(t)4Coku_t(S(T),M(T)) \quad (15)$$

where:

$$Cosk_t(S(T),M(T)) \equiv E[(S(T)-E(S(T)))(M(T)-E(M(T)))^2]$$

can be defined as co-skewness between the asset and the market and

$$Coku_t(S(T),M(T)) \equiv E[(S(T)-E(S(T)))(M(T)-E(M(T)))^3]$$

can be defined analogously as co-kurtosis[8].

In this case, four different (observable) assets are required to substitute the (unknown) prices of the characteristics, P_1,P_2,P_3,P_4, but if a risk free zero-coupon bond exists its price is always $P_{RF}=P_1$.

Note that, in general, higher moments can be expressed in terms of covariances:

$$Cosk_t(S(T),M(T)) = Cov_t(S(T),M^2(T))-2E_t(M(T))Cov_t(S(T),M(T))$$

$$Coku_t(S(T),M(T)) = Cov_t(S(T),M^3(T))-3E_t(M(T))Cov_t(S(T),M^2(T))+ 3E_t^2(M(T))Cov_t(S(T),M(T)).$$

so that, substituting into the price function, we obtain:

$$\begin{aligned} S(t) = {} & P_{RF}(t)E_t(S(T)) + \tilde{P}_2(t)Cov_t(S(T),M(T)) \\ & + \tilde{P}_3(t)Cov_t(S(T),M^2(T)) + \tilde{P}_4(t)Cov_t(S(T),M^3(T)) \end{aligned} \quad (16)$$

As in the standard two-moment CAPM, it is possible to translate the valuation equation in return terms. From Equation 16 we obtain:

$$\begin{aligned} E_t(R_j) = {} & R_{RF} + \pi_2(t)Cov_t(R_j,R_M) \\ & + \pi_3(t)Cov_t(R_j,R_M^2) + \pi_4(t)Cov_t(R_j,R_M^3) \end{aligned} \quad (17)$$

which is clearly an extension of the Sharpe's model: the covariances measure the risk factors and the π's represent explicit forms of the market prices of risks.

In general, with respect to the classical, linear CAPM, a nonlinear relation holds between asset returns and the market portfolio, induced by higher moment preferences. Empirical analysis is required to assess the relevant risk factors beyond the linear relation with market returns[9].

5. General equilibrium approach

Let us consider the classical intertemporal consumption-investment model (e.g. Merton, 1982) of a representative agent with additive, concave utility in consumption, $U(C_t, t)$, N financial assets with prices $S_i(t)$ and total returns $R_i(t)$ and wealth W_t at the beginning of period t, before the choice of the optimal consumption C_t and portfolio allocations $x_i(t)$ of residual wealth, with $\sum_{i=1}^{N} x_i(t) = 1$.

Following the Bellman approach to stochastic dynamic programming[10] we have the constrained problem in terms of utility value function J:

$$\begin{cases} J(W_t, t) = \max_t (U(C_t, t) + E_t(J(W_{t+1}, t+1))) \\ W_{t+1} = (W_t - C_t)(1 + \sum_{i=1}^{N} x_i(t) R_i(t+1)) \\ \sum_{i=1}^{N} x_i(t) = 1 \end{cases}$$

giving, by derivation, the envelope condition, $J_W(W_t,t)=U_C(C_t,t)$ and the stochastic Euler equation:

$$E_t\left[\frac{J_W(W_{t+1}, t+1)}{J_W(W_t, t)}(1 + R_i(t+1))\right] = 1$$

or

$$S_i(t) = E_t\left[\frac{J_W(W_{t+1}, t+1)}{J_W(W_t, t)}(S_i(t+1) + D_i(t+1))\right] \tag{18}$$

In the case of a one-period default-free zero coupon bond we have:

$$P_{RF}(t) = E_t\left(\frac{J_W(W_{t+1}, t+1)}{J_W(W_t, t)}\right) \tag{19}$$

so that, ignoring dividends and using the property that

E(XY)=E(X)E(Y)+Cov(X,Y), the valuation Equation (18) becomes:

$$S_i(t) = P_{RF}(t)E_t(S_i(t+1)) + Cov_t\left(S_i(t+1), \frac{J_W(W_{t+1}, t+1)}{J_W(W_t, t)}\right) \qquad (20)$$

Noting that, from a Taylor expansion, the marginal utility can be written as:

$$\begin{aligned} J_W(W_{t+1}, t+1) &= J_W(W_t, t+1) + J_{WW}(W_t, t+1)(W_{t+1} - W_t) \\ &+ \frac{1}{2} J_{WWW}(W_t, t+1)(W_{t+1} - W_t)^2 + \ldots\ldots \end{aligned}$$

the price equation (20) becomes:

$$\begin{aligned} S_i(t) = P_{RF}(t)E_t(S_i(t+1)) + \tilde{P}_2(t)Cov_t\left(S_i(t+1), W(t+1)\right) + \\ \tilde{P}_3(t)Cov_t\left(S_i(t+1), W^2(t+1)\right) + \ldots\ldots\ldots \end{aligned} \qquad (21)$$

where W is the aggregate wealth and the global market portfolio.

We have therefore obtained the same valuation function in Equation 16 of section 4 following an intertemporal general equilibrum approach.

6. Conclusions

In recent years, the proliferation of financial assets of many types has been enormous. This paper tries to explore whether the apparent multiplicity of rights and obligations may be tackled through one simple valuation approach, in which any asset is evaluated through its marginal contribution to the relevant characteristics (moments) of the market portfolio. For example, in a Gaussian world, asset prices are obtained through the marginal contributions to two basic characteristics, expected value and variance. The equivalent risk-neutral pricing function is easily obtained, so that the valuation formula agrees both with the CAPM and the Black and Scholes' no-arbitrage pricing of options. Extensions to three or more characteristics are simply obtained by considering the marginal contributions of each asset to first, second and higher-order market moments, providing a generalized valuation equation. Finally, the generalized equation is derived in a different, more rigorous way, as a result of a classical intertemporal general equilibrium model of optimal consumption and investment decisions.

References

Black F. and Scholes M. 1973. "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, vol. 81, no. 3 (May/June): 637-654

Cootner, P. H. (ed.) 1964. *The Random Character of Stock Market Prices*, Cambridge, Ma: Mit Press

Cox, J.C. and M. Rubinstein. 1985. *Option Markets*. Englewood Cliffs, NJ: Prentice Hall.

Dittmar R.F. 2002. "Nonlinear Pricing Kernels, Kurtosis Preference, and Evidence from the Cross Section of Equity Returns". *Journal of Finance*, vol. 57, no. 1 (February): 369-403

Duffie D. 1992, *Dynamic Asset Pricing Theory*, Princeton, NJ: Princeton University Press

Fang H. and T. Lai. 1997. "Co-Kurtosis and Capital Asset Pricing". *Financial Review*, vol. 32, no. 2 (May): 293-307

Kendall, M. and A. Stuart. 1977. *The Advanced Theory of Statistics. Volume 1. Distribution Theory*, 4th edition. London: Charles Griffin & Co.

Lancaster, K.J. 1966. "A New Approach to Consumer Theory" *Journal of Political Economy*, vol. 74, April: 132-157

Leland, H. E. 1999. "Beyond Mean-Variance: Performance Measurement in a Nonsymmetrical World" *Financial Analysts Journal*, vol. 55, no. 1: 27-36

Lintner, J. 1965. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets.", *Review of Economics and Statistics*, vol. 47, 13-37

Markowitz, H. 1952. "Portfolio Selection." *Journal of Finance*, vol. 7, 77-91

Markowitz, H. 1959. *Portfolio Selection: Efficient Diversification of Investments*, New Haven: Yale University Press

Merton, R.C. 1982. "On the Microeconimic Theory of Investment under Uncertainty", in Arrow, K.J. and Intrilligator, M.D. 1982. *Handbook of Mathematical Economics*, Vol.II, Amsterdam: North-Holland, ch. 13

Mossin, J. 1966. "Equilibrium in a Capital Asset Market." *Econometrica*, vol. 34: 768-783

Rendleman, R.J. Jr. 1999. “Option Investing from a Risk-Return Perspective.” *Journal of Portfolio Management*, vol. 25, May: 109-121

Rubinstein , M. 1976. “The Valuation of Uncertain Income Streams and the Pricing of Options.” *The Bell Journal of Economics and Management Science*, vol. 7: 407-425

Sharpe, W. F. 1964. “Capital Asset Prices: a Theory of Market Equilibrium under Conditions of Risk.” *Journal of Finance*, vol. 19: 425-442

Sharpe, W.F., G.J. Alexander and J.V. Bailey. 1995. *Investments*. 5th edition. Englewood Cliffs, NJ: Prentice Hall.

Tobin, J. 1958. “Liquidity Preference as Behavior towards Risk.*” Review of Economic Studies*, vol. 25: 65-86

Treynor, J. 1965. “How to Rate Management of Investment Funds.*” Harvard Business Review*, vol. 43: 63-75

VV.AA. 1992*. From Black-Scholes to Black Holes. New Frontiers in Options.* London: Risk Magazine Ltd

Notes

[1] Early models can be found in Cootner (ed.) (1964). More recent developments are collected in VV.AA. (1992).

[2] For example Sharpe, Alexander and Bailey (1995), chapter 10.

[3] For example, in the handbook of Sharpe, Alexander and Bailey (1995), options are analysed ten chapters and 400 pages later than CAPM.

[4] During the last thirty years only a few authors have addressed the question of the relation between CAPM and option pricing. Black and Scholes (1973), in their famous paper, derived the link using the abstract approach of stochastic calculus and a continuous-time version of CAPM. In discrete time, Rubinstein (1976) analysed the relation under the special assumption of lognormality and Cox and Rubinstein (1985, p.185) and Rendleman (1999) under the binomial model of price dynamics.

[5] In particular assume that the asset price $S(t)$ is a diffusion process with drift $\mu S(t)$ and diffusion coefficient σ so that the conditional distribution of $S(T)$ given $S(t)$ is normal with mean $S(t)\exp(\mu(T-t))$ and variance $\sigma^2/2(\exp(2\mu(T-t))-1)/\mu$.

[6] In particular assume that the asset price $S(t)$ is a diffusion process with drift $\mu S(t)$ and diffusion coefficient $\sigma S(t)$ so that the conditional distribution of $\log(S(T))$ given $S(t)$ is normal with mean $\log(S(t))+(\mu-\sigma^2/2)(T-t)$ and variance $\sigma^2(T-t)$.

[7] Skewness is a moment used to measure the asymmetry of the probability distribution around the mean. A symmetrical distribution has a skewness equal to zero but it must be reckoned that some special distributions exist having third (and all odd-order moments) equal to zero but that are not symmetrical. See Kendall and Stuart (1977), p.87. On the other hand, kurtosis measures the fatness of the tails of the distrubution, i.e. the probability of extreme events. For normal distributions, kurtosis is 3 times the square of the variance.

[8] In the case of normality, the first product-moment (coskewness) is zero and the second (cokurtosis) is $3Cov(X,M)Var(M)$.

[9] For recent empirical tests of a four-moment CAPM see Fang and Lai (1997) and Dittmar (2002). Skewness from lognormal returns is considered in Leland (1999).

[10] According to the Bellman principle, the optimal consumption-investment path over the agent's time horizon must be such that at any point in time it must be optimal for the remaining period.