On the Dynamic Consistency of Optimal Monetary Policy[1]

Roberto Cellini§ and Luca Lambertini#
§ Dipartimento di Economia e Metodi Quantitativi
Università di Catania
Corso Italia 5, 95129 Catania, Italy
fax 39-095-370574, e-mail cellini@mbox.unict.it
Dipartimento di Scienze Economiche
Università di Bologna
Strada Maggiore 45, 40125 Bologna, Italy
fax 39-051-2092664, e-mail lamberti@spbo.unibo.it

February 7, 2003

[1] We thank Paolo Manasse, Paolo Onofri and Riccardo Rovelli for useful comments and discussion. The usual disclaimer applies.

Abstract

The literature on the time inconsistency of optimal monetary policy puts forward the idea that a central bank may strategically exploit the first mover advantage against the private sector, manipulating expectations so as to achieve a higher level of employment and output. We argue that this view is largely ill-founded. We show that the dynamic version of the basic model used in this literature is an optimal control model yielding a time consistent and stable solution to the central banker's problem, where prices are stable and the output reaches the full employment level in steady state. Then we extend it to include a strategic private sector, which transforms the initial setup into a differential game. We prove that such a game has a strongly time consistent open-loop Nash equilibrium, as well as a time consistent Stackelberg open-loop equilibrium with the bank leading, where, however, the bank cannot gain as compared to the simultaneous game. With the private sector leading, inflation may arise in equilibrium if output is below the full employment level.

Keywords: monetary policy, time consistency, consumption, investment, differential games, optimal control

JEL Classification: E21, E52, E61

1 Introduction

A widely debated issue in the literature on macroeconomic policy is that concerning the time (in)consistency of optimal monetary policy, or *rules vs discretion*. This discussion dates back to Kydland and Prescott (1977). Broadly speaking, the idea behind their analysis and the subsequent literature is that a central bank may strategically exploit an advantage in a Stackelberg game against the private sector (consumers and producers), manoeuvring the inflation rate (through an appropriate choice of the money supply or the nominal interest rate) in order to achieve a higher level of employment and ultimately increase aggregate output. To this aim, the bank finds it convenient to announce that she will manage monetary policy so as to stabilise prices, and then - provided the private sector takes this announcement at face value and adjust inflationary expectations accordingly - she finds it optimal to produce a positive inflation rate (the so-called 'surprise') driving the economy to the desired levels of employment and output.

This view on the objectives and behaviour of a central bank originated a debate on how to build a credibility for a central bank, and the desirability of commitment devices limiting her discretion. Eventually, this materialised into the need of mandatory tasks for central bankers, supposed to confine themselves to stabilise prices without trying to affect the demand side of economic systems.[1] Under uncertainty, or incomplete information, announcements and the credibility of monetary policy may rely upon reputational effects generated by a repeated game.[2]

Here we argue that the established wisdom on the time inconsistency of optimal monetary policy is largely ill-founded. In support of this claim, we propose a properly dynamic version of the basic model initially investigated by Kydland and Prescott (1977) and Barro and Gordon (1983a), showing that it translates into a single-agent (i.e., non-strategic) optimal control model yielding a time consistent and stable solution to the central banker's problem, where prices are stable and the output reaches the full employment level in steady state.

Then, we extend the model to allow for an explicitly strategic behaviour

[1] See Turnovsky and Brock (1980), Barro and Gordon (1983a,b), Lucas and Stokey (1983), Persson, Persson and Svensson (1987). See also Alesina and Tabellini (1987) and, for an assessment of this discussion, see Blinder (1997, 2000) and Svensson (1999).

[2] See Backus and Driffill (1985), Barro (1986), Cukierman and Liviatan (1991) and Schultz (1996).

on the part of the private sector, which transforms the initial setup into a differential game. We prove that such a game has a strongly time consistent (i.e., subgame perfect) open-loop Nash equilibrium, as well as a time consistent Stackelberg open-loop equilibrium with the bank leading, where, however, the bank cannot gain as compared to the simultaneous game. In both cases, the optimal inflation rate is nil if the private sector has perfect foresight. The time consistency property also holds if the private sector leads. In this case, under perfect foresight, the monetary policy produces a positive inflation rate if the aggregate output is lower than the full employment output (and conversely).

The remainder of the paper is structured as follows. Section 2 contains a summary of the static model of Kydland and Prescott (1977) and Barro and Gordon (1983a), and a discussion of the related issues. Section 3 examines an optimal control model of the central bank's monetary policy. A differential game between the central bank and the private sector is proposed in section 4, where strongly time consistent open-loop Nash and Stackelberg equilibria are investigated. Concluding remarks are in section 5.

2 The original model

First, we briefly summarise the setup originally proposed by Kydland and Prescott (1977) and then extended by Barro and Gordon (1983a,b) and many others.[3] The central bank manoeuvres the inflation rate π_t to minimise the following quadratic loss function:

$$L_t = \pi_t^2 + (y_t - \overline{y})^2 \tag{1}$$

under the constraint given by the Phillips curve of the economy:

$$y_t = y_{t-1} + \beta \left[\pi_t - E(\pi_t)\right] \tag{2}$$

where $E(\pi_t)$ defines the expected inflation rate.[4] Substituting the Phillips curve (2) in L_t and taking the first order condition w.r.t. π_t, we obtain:

$$\frac{\partial L_t}{\partial \pi_t} = 2\left[\pi_t + y_{t-1} - \overline{y} + \beta\left(\pi_t - E(\pi_t)\right)\right] = 0 \tag{3}$$

[3] See Persson and Tabellini (1990, 1999, 2000) for exhaustive overviews of the related literature.

[4] Behind the above formulation of the Phillips curve, one can model a downward-sloping labour demand function: $\ell_t = N - [w_t - \pi_t]$ where ℓ_t is employment and w_t is the nominal wage rate, and the union sets the nominal wage so as to maximise the expected wage bill.

i.e.:

$$\pi_t^* = \frac{\beta E(\pi_t) - (y_{t-1} - \overline{y})}{1+\beta}. \tag{4}$$

Now, as in Kydland and Prescott (1977), suppose (i) the bank, at the outset, announced that she would manage monetary policy so as to stabilise prices, that is, to attain $\pi_t = 0$; and (ii) private agents took this announce at face value. If so, then $E(\pi_t) = 0$, which entails an inflationary surprise on the basis of (4). According to Kydland and Prescott, this ultimately implies that the optimal monetary policy is time inconsistent, as the (*ex post*) optimal policy does not coincide with the (*ex ante*) announcement. Alternatively, as in Barro and Gordon (1983a), impose on (3) that expectations be rational, i.e., $E(\pi_t) = \pi_t$. This yields $\pi_t^* = -(y_{t-1} - \overline{y}) > 0$ for all $y_{t-1} < \overline{y}$, that is, there exists a (positive) inflationary bias if the aggregate output in the previous period fell short of the full employment level.

One of the usual interpretations of the above story is the following: the central bank plays a dynamic game against the private agents, the strategy space being defined by the inflation rate (for the bank) and expectations on the same rate (for the private agents). The pair $\{\pi_t^*, E(\pi_t) = \pi_t^*\}$ is a Nash equilibrium of this game, given that (a) the monetary policy is *ex post* optimal, given expectations $E(\pi_t)$, and (b) private agents make the "best choice" as concerns expectations (i.e., expectations are correct, $E(\pi_t) = \pi_t$) given the monetary policy (cf. Persson and Tabellini, 1999, p. 1410, *inter alia*). However, the Nash equilibrium is not Pareto-efficient. More important, if the bank acts as a leader, a dynamic inconsistency emerges, as shown by previous results by Kydland (1975, 1977) concerning the dynamic inconsistency of open-loop Stackelberg equilibria of differential games.

Our contention is that (i) the above setup is not a game, because the private agents' behaviour is unspecified (i.e., expectations *can not be* strategies); (ii) it is not dynamic, as it is solved statically by substitution of the Phillips curve constraint into the central bank's loss function, and then taking the first order condition on the inflation rate, which (iii) is not a best reply against private agents. Additionally, (iv) this is not a Stackelberg model (which also follows as a corollary to point (i)) where the bank acts as a leader. Even if one accepts the idea that expectations be considered as strategies (so that the private sector would set $E(\pi_t) = \pi_t$ in order to eliminate the inflationary surprise), the discussion carried out in the literature, focussing on the possibility for the central bank to adopt a commitment transforming her into the leader is, in our opinion, not well posed. The game where first the private sector rationally fixes expectations and then the central bank chooses monetary policy must be kept apart from the alternative game where the central bank decides the monetary policy and then the public sets expectations.[5] In other terms, the solution concept must be common knowledge at the outset and cannot be modified afterwards. If so, then there is no issue of time inconsistency of the optimal monetary policy.

Again concerning point (i), one should note that, if the private sector is assumed to have rational expectations, then this piece of information cannot be used after taking the first order condition of the central bank. Rather, it must be inserted explicitly into the Phillips curve and the loss function at the outset. This, of course, entails that the loss function (1) reduces to:

$$L_t = \pi_t^2 + (y_{t-1} - \overline{y})^2 \tag{5}$$

whose first derivative w.r.t. π_t is simply:

$$\frac{\partial L_t}{\partial \pi_t} = 2\pi_t = 0 \tag{6}$$

which implies $\pi_t^* = 0$.

The above argument entails that, *if* consumers are endowed with rational expectations *and if* this is part of the central bank's problem from the outset, then the only equilibrium outcome for monetary policy is a full price stabilisation, i.e., no inflation at all. In this case, *ex ante*, the central bank cannot announce anything but $\pi_t^* = 0$ and the resulting monetary policy is credible.

The source of the discussion carried out in Kydland and Prescott (1977) and Barro and Gordon (1983a), *inter alia*, is that the nature of expectations is initially considered as parametric and then becomes rational *only after* having taken the first order condition for the minimisation of the loss function (1). Our contention is that expectations can be taken as given if and only if the central bank does not know *a priori* whether expectations are rational or not. If so, then the best she can do is setting $E(\pi_t) = \alpha\pi_t$, with $\alpha > 0$. This yields the following reformulation of the model:

$$y_t = y_{t-1} + \beta(1-\alpha)\pi_t \tag{7}$$

$$L_t = \pi_t^2 + [y_{t-1} + \beta(1-\alpha)\pi_t - \overline{y}]^2 \tag{8}$$

[5] As well as, of course, from the game where the central bank and the private sector act simultaneously.

and the related first order condition:

$$\frac{\partial L_t}{\partial \pi_t} = 2\left[\pi_t + \left(\beta\left(1-\alpha\right)\pi_t + y_{t-1} - \overline{y}\right)\beta\left(1-\alpha\right)\right] = 0 \tag{9}$$

entailing:

$$\pi^{ss} = -\frac{\beta\left(1-\alpha\right)\left[y_{t-1} - \overline{y}\right]}{1 + \beta^2\left(1-\alpha\right)^2} \tag{10}$$

with

$$\pi^* = 0 \text{ in } \begin{cases} \alpha = 1 \\ y_{t-1} = \overline{y} \\ \text{both.} \end{cases} \tag{11}$$

On these basis, the purpose of this paper consists in investigating the dynamic version of this problem, first as an optimal control problem for a single agent (the central bank) and then as a differential game between the central bank and the private economy, after having properly defined the objective function of the latter.

3 The optimal control problem

This is the optimal control problem based on the static version of the time-(in)consistency model (Kydland and Prescott, 1977; Barro and Gordon, 1983a). The instantaneous loss function of the central bank is:

$$L(t) = \left[\pi(t)\right]^2 + \left[y(t) - \overline{y}\right]^2 \tag{12}$$

where it appears that the desirable inflation target is zero, and $\overline{y}$ is the full employment output. Current output is the state variable evolving according to the following expectation-augmented Phillips curve:

$$\dot{y} = \beta\left[\pi(t) - E(\pi)\right] \tag{13}$$

which obtains by rewriting the usual discrete time formulation:

$$y(t) = y(t-1) + \beta\left[\pi(t) - E(\pi)\right] \tag{14}$$

as

$$y(t) - y(t-1) = \beta\left[\pi(t) - E(\pi)\right] \tag{15}$$

and then taking the limit of this expression for $\Delta t \to 0$.

We generically assume that $E(\pi) \equiv \alpha\pi(t)$, with $\alpha > 0$, so that:

$$\dot{y} = \beta(1-\alpha)\pi(t) \tag{16}$$

where $\alpha = 1$ denotes the case where private agents have perfect foresight.

The objective function of the bank is:

$$\min_{\pi} \int_0^\infty e^{-\rho t} L(t) dt \tag{17}$$

where ρ denotes the discounting factor (assumed to be constant over time). Hence, the current value Hamiltonian of the bank rewrites as:

$$\mathcal{H}(t) = e^{-\rho t}\left\{[\pi(t)]^2 + [y(t) - \overline{y}]^2 + \lambda(t)\beta(1-\alpha)\pi(t)\right\}. \tag{18}$$

where the inflation rate is the control variable and the production is a state variable. The first-order condition and the adjoint equation of this dynamic problem are:

$$\frac{\partial \mathcal{H}(t)}{\partial \pi(t)} = 2\pi(t) + \lambda(t)\beta(1-\alpha) = 0 \tag{19}$$

$$\Rightarrow \lambda(t) = -\frac{2\pi(t)}{\beta(1-\alpha)} \text{ and } \dot{\pi} = -\dot{\lambda}\frac{\beta(1-\alpha)}{2} \tag{20}$$

$$-\frac{\partial \mathcal{H}(t)}{\partial y(t)} = \dot{\lambda} - \rho\lambda(t) \Rightarrow \dot{\lambda} = \rho\lambda(t) - 2[y(t) - \overline{y}]. \tag{21}$$

This allows to write the kinematic equation of the control variable:

$$\dot{\pi} = \beta(1-\alpha)\left[\frac{\rho\pi(t)}{\beta(1-\alpha)} + y(t) - \overline{y}\right] \tag{22}$$

It is immediate to show that this solution gives rise to a steady state. Indeed, $\dot{\pi} = 0$ for:

$$\pi^{ss} = -\frac{\beta(1-\alpha)[y(t) - \overline{y}]}{\rho} \tag{23}$$

Now notice that

$$\pi^{ss} = 0 \text{ in } \begin{cases} \alpha = 1 \\ y = \overline{y} \\ \text{both.} \end{cases} \tag{24}$$

The above steady state solution is qualitatively equivalent to the solution of the static problem outlined in section 2 (see eq. (4)). This of course drastically differs from the static solution (obtained by substitution), where $E(\pi) = 0$ entails an optimal inflationary surprise on the part of the central banker (see above, eq. (11)).

Finally, solving the dynamic system made up by the Phillips curve and (22), we can easily verify that it yields the following steady state (given expectations):

$$y^{ss} = \overline{y}\,;\ \pi^{ss} = 0. \tag{25}$$

It is worth stressing that this solution obtains independently of the size of α, i.e., for any kind of expectations on the behaviour of prices.

Alternatively, a solution to $\left\{\dot{y} = 0\,,\ \dot{\pi} = 0\right\}$ is also $\{\alpha = 1\,;\ \pi^{ss} = 0\}$, for any y, which means that the aggregate income level is indeterminate. This is an obvious consequence of the fact that, if $\alpha = 1$, $\dot{y} = 0$ irrespective of the levels of the aggregate output and the inflation rate. Therefore, the bank can only set $\pi = 0$ to minimise the inflation component of the loss function (12), and the monetary policy has no influence at all on the performance of aggregate output. To further stress this point, note that imposing $\alpha = 1$ at the outset (that is, assuming that private agents have perfect foresight) amounts to eliminating the dynamic constraint (13) from the central bank's problem, which *de facto* becomes a static one.

It is worth emphasizing that, since the private agents *do not* exhibit any strategic behaviour (i.e., this is a single-agent optimal control problem), the solution outlined above is strictly time consistent. Moreover, it is also stable in the saddle point sense. To verify this, write the dynamic system in matrix form:

$$\begin{bmatrix} \dot{y} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} 0 & \beta\,(1-\alpha) \\ \beta\,(1-\alpha) & \rho \end{bmatrix} \begin{bmatrix} y \\ \pi \end{bmatrix} + \begin{bmatrix} 0 \\ -\beta\,(1-\alpha)\,\overline{y} \end{bmatrix}$$

and examine the Jacobian matrix of the problem:

$$J = \begin{bmatrix} 0 & \beta\,(1-\alpha) \\ \beta\,(1-\alpha) & \rho \end{bmatrix}$$

whose trace and determinant are:

$$Tr\,(J) = \rho > 0\,;\ \Delta\,(J) = -\beta^2\,(1-\alpha)^2 < 0 \text{ for all } \alpha \in (0,1)\,. \tag{26}$$

Therefore we may state:

Proposition 1 *The steady state* $\{y^{ss} = \overline{y}\,,\, \pi^{ss} = 0\}$ *is a stable and time consistent solution for all* $\alpha \in (0,1)$. *The solution* $\{\alpha = 1\,;\, \pi^{ss} = 0\}$, *although it is generated by a time-consistent monetary policy, is unstable in that the aggregate output diverges from the full employment output* $\overline{y}$.

Summing up, the basic model contains no issue such as a choice between adopting simple policy rules or a discretional behaviour.

4 The differential game

Now introduce an explicit objective function for the private agents. Suppose they want to maximise the discounted flow of consumption, net of the cost associated with their investments in output-increasing activities. Define the aggregate (instantaneous) amount of such investments as $I(t)$. At any t, private agents choose $I(t)$ so as to maximise:

$$c(t) = y(t) - bI(t) - d\left[I(t)\right]^2,\; b, d > 0, \tag{27}$$

i.e., investment involves a convex cost. The dynamic behaviour of output modifies as follows:

$$\dot{y} = \beta\pi(t)\left(1-\alpha\right) + I(t) - \delta y(t), \tag{28}$$

where $\delta \in [0,1]$ is a constant depreciation rate. The loss function of the bank writes the same as before:

$$L(t) = \left[\pi(t)\right]^2 + \left[y(t) - \overline{y}\right]^2;$$

therefore, the Hamiltonians are:

$$\mathcal{H}_B(t) = e^{-\rho t}\left\{\left[\pi(t)\right]^2 + \left[y(t)-\overline{y}\right]^2 + \lambda(t)\left[\beta\left(1-\alpha\right)\pi(t) + I(t) - \delta y(t)\right]\right\} \tag{29}$$

for the central bank, and

$$\mathcal{H}_P(t) = e^{-\rho t}\left\{y(t) - bI(t) - d\left[I(t)\right]^2 + \mu(t)\left[\beta\left(1-\alpha\right)\pi(t) + I(t) - \delta y(t)\right]\right\} \tag{30}$$

for the (aggregate) private agents.

4.1 The Nash game

Assume that the central bank and the private agents move simultaneously at each instant in time, and consider the open-loop solution concept. First order conditions (FOCs) are (initial and transversality conditions, as well as the indication of time are omitted for brevity):

$$\begin{aligned}\frac{\partial \mathcal{H}_B}{\partial \pi} &= 2\pi + \beta(1-\alpha)\lambda = 0 \\ \frac{\partial \mathcal{H}_P}{\partial I} &= \mu - b - 2dI = 0\end{aligned} \tag{31}$$

A first remark is now in order. Both first order conditions on controls (31) are independent of the state variable y. Hence, the open-loop Nash equilibrium is strongly time consistent (or Markov-perfect). Therefore, the optimal monetary policy *is* strongly time consistent as well (and subgame perfect).[6]

The adjoint equation involving co-state variables are:

$$\begin{aligned}\dot{\lambda} &= \lambda(\rho+\delta) - 2(y-\overline{y}) \\ \dot{\mu} &= \mu(\rho+\delta) - 1\end{aligned} \tag{32}$$

Now, from (31), we obtain:

$$\lambda = -\frac{2\pi}{\beta(1-\alpha)} \; ; \; \dot{\pi} = -\frac{\beta(1-\alpha)}{2}\dot{\lambda} \tag{33}$$

$$\mu = 2dI + b \; ; \; \dot{I} = \frac{\dot{\mu}}{2d} \tag{34}$$

Accordingly, we can write the kinematic equations of controls as follows:

$$\begin{aligned}\dot{\pi} &= \pi(\rho+\delta) + \beta(y-\overline{y})(1-\alpha) \\ \dot{I} &= \frac{(2dI+b)(\rho+\delta)-1}{2d}\end{aligned} \tag{35}$$

[6] Of course this is not true in general, when a game is solved through an open-loop Nash equilibrium. For a comprehensive exposition of the classes of open-loop differential games generating strongly time consistent Nash equilibria, see Dockner *et al.* (2000, ch. 7). However, it is always true that the open-loop Nash equilibrium is at least weakly time consistent (that is, it is never time inconsistent).

and solving the system $\left\{\dot{\pi} = 0,\ \dot{I} = 0\right\}$ we obtain the unique steady state solution to the open-loop problem:

$$\pi^{ss} = -\frac{\beta\left(y - \overline{y}\right)\left(1 - \alpha\right)}{\rho + \delta}\ ;\ I^{ss} = \frac{1 - b\left(\rho + \delta\right)}{2d\left(\rho + \delta\right)}\ . \tag{36}$$

Note that π^{ss} is smaller than the optimal inflation rate in the single-agent optimal control problem investigated in the previous section, due to the presence of the depreciation rate δ.

Now consider the issue of stability. Since the expression of $\dot{I}$ in (35) does not depend on (y, π), we can treat the central bank and the private sector separately. The Jacobian matrix of the system $\left\{\dot{y} = 0,\ \dot{\pi} = 0\right\}$ is J_B, while that of the system $\left\{\dot{y} = 0,\ \dot{I} = 0\right\}$ is J_P :

$$J_B = \begin{bmatrix} -\delta & \beta\left(1 - \alpha\right) \\ \beta\left(1 - \alpha\right) & \rho + \delta \end{bmatrix}$$

$$J_P = \begin{bmatrix} -\delta & 1 \\ 0 & \rho + \delta \end{bmatrix}$$

whose trace and determinants are:

$$\begin{aligned} Tr\left(J_B\right) &= \rho > 0\,;\ \Delta\left(J_B\right) = -\delta\left(\rho + \delta\right) - \beta^2\left(1 - \alpha\right)^2 < 0\ ; \\ Tr\left(J_P\right) &= \rho > 0\,;\ \Delta\left(J_P\right) = -\delta\left(\rho + \delta\right) < 0\ . \end{aligned} \tag{37}$$

Therefore, the following holds:

Proposition 2 *The open-loop Nash equilibrium $\{\pi^{ss}, I^{ss}\}$ is strongly time consistent and stable in the saddle point sense, in the whole admissible range of parameters.*

Since the presence of strategic agents in the private sector ultimately entails that, in general, the economy will not reach the full employment output in steady state, this model seems to support the views of, e.g., Alan Blinder (2000) on the scope for economic policy. In a game with a strategic private sector, the central bank has only one instrument and two tasks, and there appears to be some room for fiscal policy to cope with the demand side. However, as in the optimal control model presented in the previous section, the central bank is not facing a choice between rules and discretional behaviour. Strictly speaking, there is no discretional behaviour that could be adopted here, since the model produces a strongly time consistent open-loop equilibrium.

4.2 Extension: the Stackelberg game

Now turn to the sequential play setting. Observing (31), or their transformations (33-34), we see that the co-state variables of each player are independent of the control variable of the rival. Therefore, both Stackelberg games (with the central bank leading and private agents following, and the opposite) yield time consistent open-loop solutions, since the Stackelberg games are *uncontrollable* for the leader (see Xie, 1997; Dockner *et al.*, 2000, ch. 5).[7] To begin with, we shall focus on the game where the central bank plays the leader's role against the private sector.

In the light of the long-standing discussion concerning the possibility that the central bank takes an advantage on the private sector by strategically exploiting a first mover advantage, we can examine the Stackelberg differential game with the central bank as the leader. In such a case, her Hamiltonian looks as follows:

$$\begin{aligned}\mathcal{H}_B(t) &= e^{-\rho t}\left\{[\pi(t)]^2 + [y(t)-\overline{y}]^2 + \right. \\ &\left. \lambda(t)\left[\beta(1-\alpha)\pi(t) + I(t) - \delta y(t)\right] + \theta(t)\left[\mu(\rho+\delta)-1\right]\right\}\end{aligned} \tag{38}$$

where, from (34) we obtain $I(t) = \mu(t)/(2d)$; $\dot{\mu} = \mu(\rho+\delta) - 1$ comes from (32); and $\theta(t)$ is an additional co-state variable applied to the dynamic constraint represented by the co-state equation of the follower. Therefore, (38) can be reformulated as:

$$\mathcal{H}_B(t) = e^{-\rho t}\left\{[\pi(t)]^2 + [y(t)-\overline{y}]^2 + \lambda(t)\left[\beta(1-\alpha)\pi(t) + \frac{\mu(t)}{2d} - \delta y(t)\right] + \right. \\ \left. \theta(t)\left[\mu(\rho+\delta)-1\right]\right\} \tag{39}$$

[7] Calvo (1978) shows that a time inconsistency issue arises in a dynamic game between the central bank and the private sector, where the Hamiltonian functions are not linear-quadratic in control and state variables and consumer preferences also depend upon liquidity, i.e., a nominal wealth effect. In such a case, monetary policy may strategically affect the consumption path, therefore becoming time inconsistent. See also Chang (1998).

Taking the FOCs, we obtain (as above, we omit the indication of time and exponential discounting):

$$\frac{\partial \mathcal{H}_B}{\partial \pi} = 2\pi + \beta(1-\alpha)\lambda = 0 \tag{40}$$

$$-\frac{\partial \mathcal{H}_B}{\partial y} = \dot{\lambda} - \rho\lambda \Rightarrow \tag{41}$$

$$\dot{\lambda} = \lambda(\rho+\delta) - 2(y - \overline{y})$$

$$-\frac{\partial \mathcal{H}_B}{\partial \mu} = \dot{\theta} - \rho\theta \Rightarrow \tag{42}$$

$$\dot{\theta} = -\delta\theta - \frac{\lambda}{2d}$$

where it is immediate to verify that (40-41) are independent of (42) and, in fact, coincide with the FOCs of the Nash game (see eqs. (31) and (32) above), which directly entails the following:

Proposition 3 *The open-loop Stackelberg equilibrium with the central bank leading, which is time consistent, is observationally equivalent to the open-loop Nash equilibrium.*

That is, there is no first mover advantage to be exploited by the central bank, in this setting. In general, this will hold as long as real and nominal effects do not interact, which is typically the case in every linear-quadratic formulation of a monetary policy problem, with no need of resorting to closed-loop or feedback equilibria.

Now examine the Stackelberg game with the private sector leading. the relevant Hamiltonian function of the leader is:

$$\mathcal{H}_P(t) = e^{-\rho t}\left\{y(t) - bI(t) - d\left[I(t)\right]^2 + \right. \tag{43}$$

$$\left. \mu(t)\left[-\frac{\lambda(t)}{2}\beta^2(1-\alpha)^2 + I(t) - \delta y(t)\right] + \eta(t)\left[\lambda(t)(\rho+\delta) - 2(y(t)-\overline{y})^2\right]\right\},$$

where $\eta(t)$ is the additional co-state pertaining to the dynamics of the central bank's co-state variable $\lambda(t)$, as defined by (32), and $\lambda(t)\beta^2(1-\alpha)^2/2 = \pi(t)$ obtains from (31).

The FOCs are:

$$\begin{aligned}
\frac{\partial \mathcal{H}_P}{\partial I} &= \mu - b - 2dI = 0 && (44)\\
-\frac{\partial \mathcal{H}_P}{\partial y} &= \dot{\mu} - \rho\mu \Rightarrow && (45)\\
\dot{\mu} &= \mu(\rho+\delta) + 2\eta - 1 &&\\
-\frac{\partial \mathcal{H}_P}{\partial \lambda} &= \dot{\eta} - \rho\eta \Rightarrow && (46)\\
\dot{\eta} &= \frac{\mu}{2}\beta^2(1-\alpha)^2 - \delta\eta &&
\end{aligned}$$

First, note that (44) yields the same dynamics of private investment as in (34). Then, imposing $\dot{\eta} = 0$, we obtain $\eta = \mu\beta^2(1-\alpha)^2/(2\delta)$. Consequently, using (45), we can write:

$$\dot{I} = \frac{1}{2d\delta}\left\{(b+2dI)\left[\delta(\rho+\delta) + \beta^2(1-\alpha)^2\right] - \delta\right\} = 0 \tag{47}$$

which can be solved together with $\dot{y} = 0$, to yield:

$$I^{ss} = \frac{\delta - b\left[\delta(\rho+\delta) + \beta^2(1-\alpha)^2\right]}{2d\left[\delta(\rho+\delta) + \beta^2(1-\alpha)^2\right]} \tag{48}$$

which is the steady state investment effort of the private sector, and

$$y = \frac{\delta - \left[\delta(\rho+\delta) + \beta^2(1-\alpha)^2\right]\left[b - 2d\beta(1-\alpha)\pi\right]}{2b\delta\left[\delta(\rho+\delta) + \beta^2(1-\alpha)^2\right]} \tag{49}$$

which still depends upon the inflation rate π. Using (33), we find the equilibrium inflation rate:

$$\pi^{ss} = -\frac{\beta(1+\alpha)\left\{\delta - \left[\delta(\rho+\delta) + \beta^2(1-\alpha)^2\right](b + 2d\delta\overline{y})\right\}}{2d\left[\beta^2(1-\alpha^2) + \delta(\rho+\delta)\right]\left[\delta(\rho+\delta) + \beta^2(1-\alpha)^2\right]} \tag{50}$$

that can be plugged into (49) to obtain the expression defining the steady state aggregate output:

$$y^{ss} = \frac{\delta(\rho+\delta) - b(\rho+\delta)\left[\delta(\rho+\delta) + \beta^2(1-\alpha)^2\right] + \Gamma}{2d\left[\beta^2(1-\alpha^2) + \delta(\rho+\delta)\right]\left[\delta(\rho+\delta) + \beta^2(1-\alpha)^2\right]} \tag{51}$$

where

$$\Gamma \equiv 2d\beta^2 \left(1-\alpha^2\right) \left[\beta^2 \left(1-\alpha^2\right) + \delta \left(\rho+\delta\right)\right] \overline{y}. \tag{52}$$

If the private sector has perfect foresight, i.e., $\alpha = 1$, we can simplify the steady state expressions of aggregate output and inflation as follows:

$$y^{ss} = \frac{1 - b\left(\rho+\delta\right)}{2d\delta\left(\rho+\delta\right)}\,;\ \pi^{ss} = \frac{\beta\delta\left[\left(\rho+\delta\right)\left(b+2d\delta\overline{y}\right)-1\right]}{d\delta^2\left(\rho+\delta\right)^2} \tag{53}$$

with

$$\pi^{ss} > \text{ for all } \overline{y} > \frac{1 - b\left(\rho+\delta\right)}{2d\delta\left(\rho+\delta\right)} = y^{ss}, \tag{54}$$

and conversely. therefore, we may state our final result:

Proposition 4 *The open-loop Stackelberg equilibrium with the private sector leading is time consistent. If the private sector has perfect foresight, the optimal monetary policy yields a positive (negative) inflation rate whenever the aggregate output is below (above) the full employment level.*

We omit for brevity the analysis of the stability properties of both Stackelberg steady states, which are both saddle points.

5 Concluding remarks

We have re-examined the issue of time (in)consistency of optimal monetary policy, analysing an optimal control model where the central bank acts as a single agent, and a differential game where the central bank interacts with the private sector, the objective of the latter being the maximisation of the discounted consumption flow. We have shown that, in the settings considered here, optimal monetary policy is indeed time consistent, although it does not necessarily lead to the full stabilisation of prices. This holds irrespective of whether the game is played simultaneously or sequentially, since both Nash and Stackelberg ope-loop equilibria are strongly time consistent (that is, subgame perfect), due to the linear-quadratic form of the objective functions.

References

[1] Alesina, A. and G. Tabellini (1987), "Rules and Discretion with Noncoordinated Monetary and Fiscal Policies", *Economic Inquiry*, **25**, 619-30.

[2] Backus, D. and J. Driffill (1985), "Inflation and Reputation", *American Economic Review*, **75**, 530-38.

[3] Barro, R.J. (1986), "Reputation in a Model of Monetary Policy with Imperfect Information", *Journal of Monetary Economics*, **17**, 1-20.

[4] Barro, R.J. and D.B. Gordon (1983a), "A Positive Theory of Monetary Policy in a Natural Rate Model", *Journal of Political Economy*, **91**, 589-610.

[5] Barro, R.J. and D.B. Gordon (1983b), "Rules, Discretion and Reputation in a Model of Monetary Policy", *Journal of Monetary Economics*, **12**, 101-21.

[6] Blinder, A.S. (1997), "Distinguished Lecture on Economics in Government: What Central Bankers Could Learn from Academics - And Vice Versa", *Journal of Economic Perspectives*, **11**, 3-19.

[7] Blinder, A.S. (2000), "Central-Bank Credibility: Why Do We Care? How Do We Build It?", *American Economic Review*, **90**, 1421-31.

[8] Calvo, G. (1978), "On the Time Consistency of Optimal Policy in a Monetary Economy", *Econometrica*, **46**, 1411-28.

[9] Chang, R. (1998), "Credible Monetary Policy in an Infinite Horizon Model: Recursive Approaches", *Journal of Economic Theory*, **81**, 431-61.

[10] Cukierman, A. and N. Liviatan (1991), "Optimal Accomodation by Strong Policymakers under Incomplete Information", *Journal of Monetary Economics*, **27**, 99-127.

[11] Dockner, E.J, S. Jørgensen, N. Van Long and G. Sorger (2000), *Differential Games in Economics and Management Science*, Cambridge, Cambridge University Press.

[12] Kydland, F. (1975), "Noncooperative and Dominant Player Solutions in Discrete Dynamic Games", *International Economic Review*, **16**, 321-35.

[13] Kydland, F. (1977), "Equilibrium Solutions in Dynamic Dominant-Player Models", *Journal of Economic Theory*, **15**, 307-24.

[14] Kydland, F. and E. Prescott (1977), "Rules rather than Discretion: The Inconsistency of Optimal Plans", *Journal of Political Economy*, **85**, 473-92.

[15] Lucas, R.E. Jr. and N.L. Stokey (1983), "Fiscal and Monetary Policy in an Economy without Capital", *Journal of Monetary Economics*, **12**, 55-93.

[16] Persson, T. and G. Tabellini (1990), *Macroeconomic Policy, Credibility and Politics*, Chur, Harwood Academic Publishers.

[17] Persson, T. and G. Tabellini (1999), "Political Economics and Macroeconomic Policy", in Taylor, J.B. and M. Woodford (eds), *Handbook of Macroeconomics*, Vol. 1c, Amsterdam, North-Holland, 1397-1482.

[18] Persson, T. and G. Tabellini (2000), *Political Economics. Explaining Economic Policy*, Cambridge, MA, MIT Press.

[19] Persson, M., T. Persson and L.E.O. Svensson (1987), "Time Consistency of Fiscal and Monetary Policy", *Econometrica*, **55**, 1419-31.

[20] Schultz, C. (1996), "Announcements and Credibility of Monetary Policy", *Oxford Economic Papers*, **48**, 673-80.

[21] Svensson L.E.O. (1999), "Inflation Targeting as a Monetary Policy Rule", *Journal of Monetary Economics*, **43** 607-54.

[22] Turnovsky, S.J. and W.A. Brock (1980), "Time Consistency and Optimal Government Policies in Perfect Foresight Equilibrium", *Journal of Public Economics*, **13**, 183-212.

[23] Xie, D. (1997), "On Time Consistency: A Technical Issue in Stackelberg Differential Games", *Journal of Economic Theory*, **76**, 412-30.