THE OPTIMUM LIFE OF A PATENT WHEN THE TIMING OF INNOVATIONS IS STOCHASTIC

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Abstract. This paper studies the optimal lifetime of a patent in a model where the timing of innovations is uncertain. We assume a Poisson discovery process with a linear hazard function and contractual R & D costs. The invention industry is modelled in three alternative ways: i) monopoly; ii) oligopoly with free entry, iii) perfect competition. Several comparative statics results are derived.

1. Introduction

The patent system plays an important role in remunerating innovative activity and stimulating investment in R & D. But patents entail a social cost because of the inherent monopoly privileges accorded to the patentees. Thus, in deciding the lifetime of a patent, society must balance the gains accruing from faster technological progress against the welfare loss associated with temporary monopoly in the use of the new technology. This paper studies the optimal lifetime of patents in a model where the timing of innovation is stochastic.

The importance of patent policy has been questioned on the grounds of scattered evidence that in many industries other factors, such as secrecy or the gain of a technological lead, are more effective in securing private appropriability of the returns to R & D than patents. But this may be due to the way in which the patent law is actually designed and enforced and not to intrinsic limits of the patent system. In principle, the patent system could be made more effective, though this is not necessarily socially desirable given that patents are costly to society. The relative unimportance of patents is therefore a consequence of policy choices, that could be re-examined and changed (but might also be the correct ones).

The analysis of the optimal patent policy is intrinsically complex, and remains so even if one restricts attention to the problem of the optimum patent lifetime. In fact, it involves a three-stage game: in the first stage, the government fixes the patent grant's duration; in the second stage, firms compete in R & D; in the third stage, firms (possibly distinct from those doing R & D) compete in the product market. The game must be solved backward starting from the equilibrium in the product market; then one must determine the equilibrium in the invention industry, and eventually one can solve the social problem. Moreover, the equilibrium in the product market must be determined in three different scenarios: before the innovation has been completed, when the innovation is protected by the patent, and after the patent expires.

To analyse this complex problem, in a seminal work Nordhaus (1969, 1972) has used the hypothesis of deterministic R & D technology [1]. In Nordhaus' model, investment in R & D instantaneously produces an innovation whose size, measured as the percentage cost reduction that the innovation brings about, is taken to be a function of the investment level.

In addition, Nordhaus made a number of restrictive assumptions: i) innovations are cost reducing and non drastic; ii) there is perfect competition in the product market; iii) there is only one firm doing R & D; iv) there is no uncertainty; v) the patent confers complete protection over the invention; vi) the private and social discount rates coincide. Some of these assumptions have been relaxed in the subsequent literature. The effect of competition in the R & D industry has been analysed by Berkowitz and Kotowitz (1982) and De Brock (1985). These authors compare monopoly and a free entry equilibrium in the invention industry and find that competition in R & D may substantially reduce the optimum patent's life. Rafiquzzaman (1987) has analysed the

[1] Scherer (1972) provides a geometric illustration of the Nordhaus model.

optimal patent term under uncertainty over the size of the cost reduction and has shown that with monopoly in the invention industry uncertainty reduces the otpimal life of the patent. This result is reversed, however, when the invention industry is characterised by competitive conditions (Rafiquzzaman, 1988).

A more recent strand of the literature has considered the case of incomplete patent protection, focussing on the trade-off between patent's length and breadth [2]. In this literature, the socially desired level of R & D is given exogenously, and therefore it is assumed that the innovator must be awarded a given prize (that is, a given level of discounted profits) in order to make that effort. The problem is whether the prize should be awarded through a short and broad patent, or through a more narrow but longer one. Following this approach, results have been derived which do not depend on the structure of the R & D technology.

In particular, Gilbert and Shapiro (1990) have found a sufficient condition for the optimal patent length to be infinite, with patent's breadth adjusted so as to provide the patentee with the appropriate incentive to do R & D. On the basis of this result, they claim that the enphasis on the optimum life of patents has been mispiaced, for the relevant policy dimension is patent breadth, not length. However, Gilbert and Shapiro's result is not general: conditions can be found, which are dual to Gilbert and Shapiro's one, under which optimal patents should confere the widest protection over the invention, and their lifetime should be adjusted to properly incentivate investment in R & D [3]. Moreover, it is likely that the same factors which affect the optimal length of patents also determine the optimal patent breadth when the latter is the relevant policy instrument. Finally, a complete analysis must study also the socially optimal level of the R & D effort and this requires assumptions on the R & D technology.

Though the assumption that the timing of the innovation is deterministic still continues to be used[4], its limits as a model of the innovative activity have been widely recognised. As Dasgupta and Stiglitz (1980a) have put it, it is as though Mother Nature has a patent on all techniques of production with unit cost lower than the current one, and firms have to pay a fee to purchase the right to use one of these techniques, with the fee related to the extent of cost reduction. Actually, the assumption that paying a price a firm can instantaneously and for sure "buy" a certain reduction of its variable production costs seems more appropriate as a description of investment in some fixed production factor than as a model of R & D. Kamien and Schwartz (1974) were the first to criticise Nordhaus' model on this ground. They analyse the effect of varying the patent's length on the equilibrium R & D effort in a model that explicitly incorporates uncertainty on the timing of the innovation. However, they do not analyse the full equilibrium in the R & D industry nor do they address the issue of the optimum patent's length.

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[2] See Gilbert and Shapiro (1990), Klemperer (1990), Gallini (1992).

[3] See Denicolò (1995).

[4] This hypothesis was standard at the time Nordhaus wrote: see also Barzel (1968) and Scherer (1967), who developed the so-called "deterministic auction model".

Starting with Loury (1979) and Lee and Wilde (1980), the standard approach to R & D competition assumes that the date of the innovation is uncertain (see Reinganum 1989 for a survey). Surprisingly, however, in this context the issue of the optimal lifetime of a patent has been left largely unexplored [5].

In this paper we attempt at closing this gap. For sake of simplicity, and also to facilitate a comparison with the previous literature, we have retained some of the early assumptions of Nordhaus. Thus we consider a single cost reducing innovation [6], and assume that the private and social discount rates coincide. Also, we postulate a sharp distinction between the invention industry and the product market. Firms which operate in the invention industry are pure laboratories that produce inventions and innovations. They race for innovating first and the winner patents the innovation which then will be licensed to users operating in the downstream product market.

Besides analysing a stochastic discovery process, we depart from Nordhaus (1969) also in considering alternative assumptions regarding the structure of the invention industry. Specifically, we study monopoly, an oligopolistic equilibrium with a fixed R & D cost and free entry, and the limit case where the fixed cost tends to zero, which loosely speaking corresponds to perfect competition in the invention industry. Moreover, we allow for drastic innovations and imperfect patent protection, so that the patentee's competitors may imitate the innovation or invent around the patent without infringing it. Last but not least, our analysis is compatible with the presence of imperfect competition in the product market, even though special attention is devoted to the case where the product market is perfectly competitive.

In modelling the R & D technology we follow Loury (1979) and Dasgupta and Stiglitz (1980b). They assume a Poisson discovery process with contractual R & D costs. In particular, Dasgupta and Stiglitz (1980b) assume that each R & D project is characterised by an optimal scale, but that each firm can operate many R & D projects simultaneously. Ignoring the integer constraint (as Dasgupta and Stiglitz often do) this is tantamount to assuming a linear (in R & D costs) hazard

[5] The small literature that has addressed this problem is plagued with mistakes. Dasgupta and Stiglitz (1980b, p. 19) claim that the optimal patent life with competition in the invention industry must equate the equilibrium aggregate R & D effort with the socially optimal one. As we shall see in more details in section 8, this claim is false: at the optimal patent life, there is generally under-investment in R & D. La Manna *et al.* (1989) claim that with perfect competition in R & D the optimal patent life is a decreasing function of the interest rate, whereas Proposition 9 below shows that the opposite is true. Martin (1993, pp. 369-372) studies the optimal life of a patent in the context of the model of patent race between Cournot oligopolists of Delbono and Denicolò (1991). However, he assumes that in equilibrium the value of a firm is unaffected by a marginal change in the equilibrium R & D effort. But one cannot invoke the envelope theorem in this context, because there is strategic interaction and therefore the equilibrium R & D effort does not maximise the value of a firm.

[6] However, we allow for drastic innovations. In a sense, a product innovation is equivalent to a drastic cost reducing innovation, for one can imagine that the new product was already known but could have been produced only at prohibitively high costs before the innovation.

function. This hypothesis, though somewhat restrictive, seems justified because there is no strong evidence on the existence of economies of scale in R & D. Moreover, it greatly simplifies the analysis and is therefore retained in this paper [7].

We are able to obtain explicit equations that determine the optimal duration of the patent grant under different hypotheses on the stucture of the invention industry. These equations are non linear but simple enough to allow for definite comparative statics results. In the case of perfect competition in the invention industry (to be defined more precisely later) one can even explicitly solve for the optimal patent life.

Some comparative statics results parallel those obtained in the Nordhaus model or some of its variants. For instance, we show that with free entry, more competition in the invention industry, which is associated with a lower entry cost, entails shorter patent terms. The optimal duration of a patent is longer when it is more costly to do R & D. Under competitive conditions in the product market, a greater elasticity of demand entails a lower deadweight loss of the patent and therefore yields a longer patent length. Other results are new. In fact, the set of exogenous paramenters in our model is different from that of the Nordhaus tradition and it is not always possible to draw analogies. For instance, our model can be used to ask whether large innovations should be protected more or less than small ones, a problem that in Nordhaus-type models would be meaningless because the size of the innovation is endogenous. It turns out that the optimal life of a patent is infinite for sufficiently small innovations [8], and, under perfect competition in the product market, is decreasing in the size of innovations. Also, with perfect competition in the product market, the effect of uncertainty over the size of the cost reduction is to reduce the optimal patent term, independently of the conditions prevailing in the invention industry. This result partially contrasts with that of Rafiquzzaman (1987,1988). Another new result is that when the patent life is set optimally, there is necessarily under-investment in R & D, even if competitive conditions prevail in the invention industry[9]. See the concluding section for a summary of the main results.

The layout of the paper is as follows. In section 2 we describe the equilibrium of the product market before the completion of the innovative process, after the innovation but before the patent expires, and after the patent expires. We focus in particular on the case of perfect competition, illustrating the difference between the cases of drastic and non drastic innovations, and the case of imperfect patent protection. In section 3 the patent race is analysed. We describe three alternative scenarios: research monopoly, oligopoly with free entry in the R & D industry and "perfect

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[7] Given a linear hazard function, the hypothesis that R & D costs are contractual is made for analytical convenience. Indeed, under Loury's specification a linear hazard function yields an interior solution to the firm's profit maximisation problem. If instead one models R & D costs as non contractual like in Lee and Wilde (1980), one has to assume a concave hazard function (at least over the relevant range) to get an interior solution. For a comparison of these two approaches to modelling R & D costs see Reinganum (1989).

[8] Gilbert and Shapiro (1990) have a similar result.

[9] To the best of our knowledge, this question has never been raised in the context of Nordhaus-type models.

competition" in R & D. Section 4 states the social problem in general terms. The three subsequent sections derive the main results for the case of monopoly (sect. 5), oligopoly with free entry (sect. 6) and "perfect competition" in the invention industry (sect. 7), respectively. Section 8 compares the cases of monopoly and free entry, showing that the optimal patent length is shorter under perfect competition than under monopoly, but that may be shorter under monopoly than under free entry if the fixed cost is high enough. We then show in section 9 that, when the length of the patent is set optimally, there is always under-investment in R & D with respect to the social optimum. Section 10 analyses in more details the case of perfect competition in the product market. Section 11 summarises and collects some concluding remarks.

2. The product market

We assume that there is a single innovation in prospect. The innovating firm obtains a patent grant whose lifetime is T. It then licenses the innovation to one or more firms operating in the downstream product market.

In most of our analysis we black-box the product market, which is described by the following three parameters:

→ H, the patentee's profits during the lifetime of the patent;

→ J, the further increase (if any) in social welfare that the innovation brings about during the lifetime of the patent. This increase in social welfare is not captured by the patentee; it may be enjoyed by consumers or by other firms;

→ K, the deadweight loss from the patent. This is the potential increase in social welfare which becomes available to society only when the patent expires; during the lifetime of the patent it is lost due to the monopolistic distortions created by the patent.

H, J and K are flows which we assume to be stationary: in other words, there is no dynamics in the product market apart from that generated by the innovation [10].

[10] In particular, we assume not only that the new technology will not be improved upon, but also that the old technology does not change. In fact, when the innovation is patented, there are incentives to develop the old technology so as to reduce the production cost of those firms that have no access to the new technology, even if the old technology remains an inferior one. This point is elaborated by Beck (1976).

This description of the product market may be compatible with several alternative assumptions on the type of competition and the nature of the innovation. The initial structure of the product market determines the optimal licensing strategy of the innovator (see Kamien (1990) for a survey of the theory of optimal licensing) and the resulting changes in the market equilibrium; thus it determines the values of the parameters H, K and J.

Obviously, H is always positive. It would appear natural to assume that K and J are positive, too. But it is easy to construct non pathological examples where they may be negative.

Consider for instance an asymmetric Cournot duopoly, with one low cost firm and one high cost firm, and with constant marginal costs. As is well known, when the high cost firm reduces slightly its own cost (so that its cost remains the higher one) the equilibrium price decreases, consumers' surplus rises and, quite obviously, the high cost firm makes higher profits. However, the profits of the low cost firm may decrease to such an extent that not only aggregate profits, but also social welfare (i.e. the sum of industry profits and consumers' surplus) decreases. The reason is that the market share of the high cost firm expands and therefore the industry average production cost goes up [11].

Suppose now that the prospective innovation is specific to the high cost firm, and that it is small enough for the above phenomenon to occur. Then the private value of the innovation is positive ($H > 0$), but its social value is negative. This means that $J < 0$; actually $H + J < 0$. Or, suppose that the innovation allows both firms to reduce their initial costs by the same amount. In this case, it is easy to show that the patentee will find it optimal to license the innovation to the low cost firm only. When the patent expires, the new technology becomes available also to the high cost firm which will then be able to reduce its own cost. But, as we have seen, this may actually reduce social welfare, so that $K < 0$ in this case.

Though most of our analysis does not depend on K and J being non negative, to simplify the exposition we shall assume $J, K \geq 0$. It is not difficult to adapt our conclusions to the case where this restriction does not hold.

One case where the above assumption is satisfied is that of perfect competition in the product market. This is the case analysed by Nordhaus (1969) and in most of the subsequent literature. The remainder of this section examines this case, which will be taken up again in section 10 below. Consider a competitive market where initially all firms have access to the same technology and can produce with constant marginal cost c [12]. The innovation reduces production costs to $c - d$. Let $Q(p)$ denote the demand function.

If the innovation is not drastic and there is perfect patent protection, so that the innovation cannot be imitated, the patentee will license the new technology for a fee d per unit of output. Thus, in the post-innovation equilibrium output will stick to the pre-innovation level $Q_0 = Q(c)$ and the patentee's profits will be equal to the cost reduction d times the pre-innovation output. This is the

[11] See Tirole (1988, ch. 10).

[12] Equivalently, one could assume a homogeneous Bertrand oligopoly with constant marginal costs.

area H in figure 1. The deadweight loss created by the patent is given by the triangle K in figure 1. After the patent expires, there is free access to the new technology. Under perfect competition in the product market, the output level rises to the point $Q_1 = Q(c-d)$ where the price equals the new marginal cost and the patentee's profits are driven to zero. In this case, during the lifetime of the patent the patentee reaps the entire social benefit from the innovation ($J = 0$).

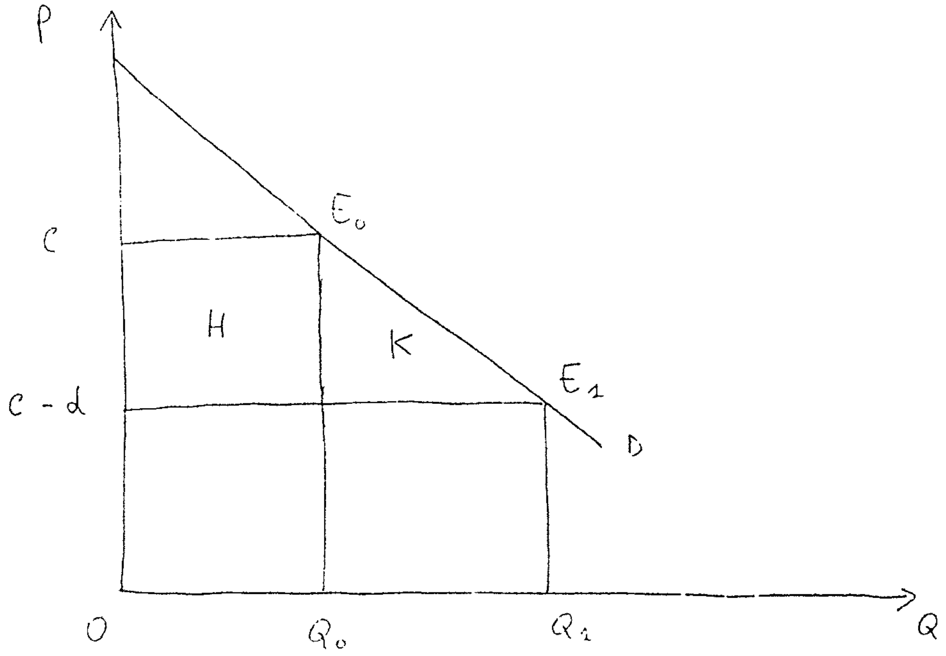

figure 1

But consider now the case of a drastic innovation. An innovation is drastic if the monopoly price associated with the new marginal cost $c-d$ is lower than the old marginal cost c. In this case, the patentee will license the new technology for a fixed fee equal to the area H in figure 2. The main difference with the case of non drastic innovation is that now part of the increase in social welfare which society enjoyes before the patent expires is not captured by the patentee. This corresponds to the area J in figure 2. At time T, the equilibrium price falls to $c-d$ and social welfare is further increased by an amount K.

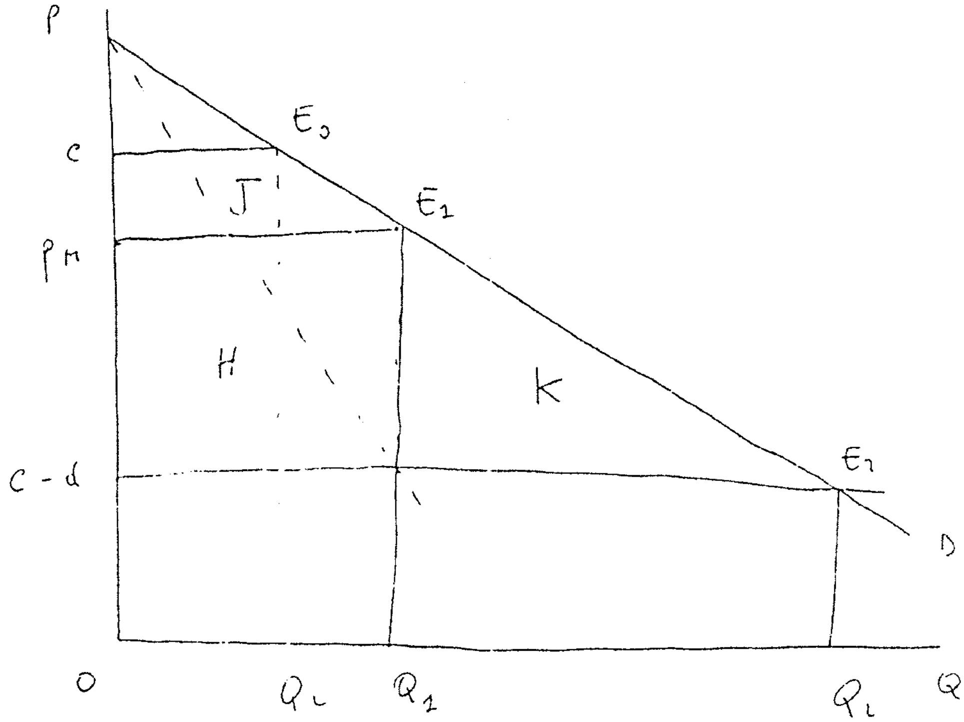

figure 2

A similar picture emerges in the case of imperfect patent protection. The general idea here is that if the patent has a very broad scope, the new production process cannot be imitated and therefore the non-innovating firms will stick to their pre-innovation cost c. But if the patent is more narrowly defined, one can imagine that even the non-innovating firms can develop similar processes without infringing the patent and therefore reduce their costs to a certain extent [13]. We can measure the breadth of the patent by the fraction of the cost reduction that does *not* spill out as freely available technology to the non-innovating firms. Thus, denoting by $(1-\gamma)$ the breadth of the patent, the non innovating firms will have marginal costs equal to $(c-\gamma d)$. See figure 3.

[13] See Nordhaus (1972).

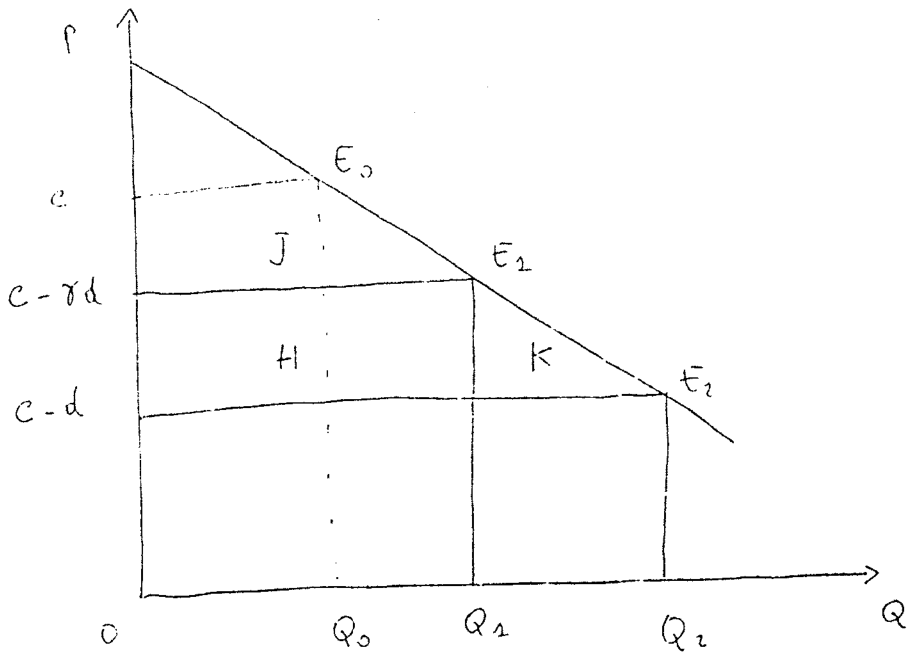

figure 3

Like in the case of a drastic innovation, the main difference with the situation depicted in figure 1 is represented by the area J, which reflects the difference between the social and the private benefit from the innovation during the life of the patent. The only difference between figures 2 and 3 lies in the way the area J is determined.

3. The patent race

In this section we analyse the invention industry. We consider a race for a patentable innovation between n symmetric R & D firms (laboratories). We model the R & D technology as in Loury (1979) and Dasgupta and Stiglitz (1980b). At time $t = 0$, each firm i decides its R & D effort x_i and pays a lump-sum amount αx_i, where α is the constant marginal cost of research effort. The R & D effort determines the expected time of successful completion of the R & D project. Assuming that the date of innovation by firm i is exponentially distributed, and is independent of the date of innovation of other firms, the probability of firm i being successful at or prior to date t is $1 - e^{-x_i t}$.

The payoff function of firm i is the present value of expected profits, net of R & D costs, i.e.:

(1)
$$\Pi_i = \int_0^\infty e^{-\left[\sum_{j=1}^{n} x_j + r\right]t} x_i V \, dt - \alpha x_i = \frac{x_i V}{X_{-i} + x_i + r} - \alpha x_i$$

where r is the interest rate, x_i is i's R & D effort and also i's instantaneous probability of innovating, $X_{-i} = \sum_{j \neq i} x_j$ is the instantaneous probability that one of the $(n-1)$ rivals of firm i innovates, so that $\exp[(-X_{-i} + x_i)t]$ is the probability that no firm has innovated yet by time t, V is the present value of the profits accruing to the winner, and αx_i is the lump-sum cost of R & D.

The prize to the winner V is given by the discounted profits from the patent:

(2)
$$V = \int_0^T H e^{-rt} dt = \frac{z}{r} H$$

where $z = 1 - e^{-rT}$ denotes the fraction of the overall discounted profits H/r that is captured by the patentee.

Each firm chooses its R & D investment to maximise its expected profits (1). The first order condition for a maximum is [14]:

(3)
$$\frac{(X_{-i} + r)V}{(r + X_{-i} + x_i)^2} = \alpha$$

Since all firms are identical, we look for a symmetric Nash equilibrium where $x_i = x$ for all i. Condition (3) then becomes:

(4)
$$\frac{[(n-1)x + r]V}{[(r + nx)^2]} = \alpha$$

We consider two alternative structures of the invention industry. The first one corresponds to $n = 1$, i.e. a research monopoly. This was the hypothesis of Nordhaus. Alternatively, we assume that there is free entry in the invention industry, but that in addition to the variable cost αx_i, firms doing R & D must also pay a fixed cost F. This alternative scenario was first described by Stiglitz (1969) in his discussion of Nordhaus' work and is analysed by Berkowitz and Kotowitz (1982) in the context of a model with a deterministic R & D technology. When F goes to 0, the invention industry tends to become perfectly competitive. As we shall see in more details later (sect. 7), this

[14] It may be easily checked that the second order condition holds.

limit case corresponds to Dasgupta and Stiglitz's (1980b) assumption that, under perfect competition in the invention industry, each firm neglects the impact of a change in its R & D effort on the expected date of innovation.

4. The social problem

The government chooses the value of the patent grant's duration T that maximises social welfare, that is the sum of consumers' surplus and profits, net of R & D costs. We assume that the social rate of discount equals the interest rate r.

Normalising to zero the flow of social welfare before the innovation, the expected discounted social welfare is:

(5)
$$W = \int_0^\infty e^{-(nx+r)t} nxS\,dt - \alpha nx$$

where nx is the instantaneous probability that one firm innovates at time t, conditional on no one having innovated before t, and S is the social value of the innovation:

(6)
$$S = \int_0^T e^{-rt}(H+J)dt + \int_T^\infty e^{-rt}(J+H+K)dt$$

That is, S is the overall discounted social benefit from the new technology, which equals $(H+J)$ before the patent expires and $(H+K+J)$ thereafter. From (5) and (6) one gets:

$$W = nx\left[\frac{\frac{1-e^{-rT}}{r}(H+J)+\frac{e^{-rT}}{r}(H+J+K)}{nx+r}\right] - \alpha nx$$

or

(7)
$$W = \frac{nx}{nx+r}\left[\frac{H+J}{r}+(1-z)\frac{K}{r}\right] - \alpha nx$$

If the innovation process were completed instantaneously, the discounted social benefit would be $(H+J)/r$ plus a fraction $(1-z)$ of K/r. The factor $nx/(nx+r)$ may be interpreted as a further discount factor that accounts for uncertainty over the timing of the innovation. If nx is large, the expected date of innovation is small and therefore this discount factor is close to 1. As nx decreases, society on average must waite longer and longer for the innovation and therefore the discount factor falls to 0.

The policymaker's objective function may also be rewritten as follows:

(8)
$$W = \Pi + \frac{nx\left[\frac{s}{r} + (1-z)\frac{u+\kappa}{r}\right]}{nx + r}$$

where Π denotes the aggregate profits in the invention industry.

When choosing T, the policymaker must take into account that x (and, in the model with free entry, also n) depends on T in a way that is described by the equilibrium condition in the invention industry (4). In the following sections, we consider the cases of monopoly (sect. 5), oligopoly with free entry (sect. 6) and "perfect competition" in the invention industry (sect. 7).

5. Monopoly

In this section we assume that there is only one firm doing R & D, i.e. $n = 1$. The equilibrium R & D effort can be easily found using equation (4). We have $(r + x)^2 = rV/\alpha$, that is

(9)
$$x = \sqrt{\frac{zH}{\alpha}} - r \quad .$$

As noted by Scherer (1972), the patent grant really plays two roles, which he calls the *stimulus effect* and the *Lebensraum effect*. First, the patent grant must be sufficiently long to make room for positive profits to the patentee, for otherwise no one will invest in R & D: this is the Lebensraum effect. Second, the marginal social gain from more investment in R & D induced by the patent must equal the marginal social cost of the protection of the innovation: this is the stimulus effect, which is captured analytically by the first order condition for welfare maximisation.

To put it more prosaically, we must first check that the patent life be sufficiently long to exclude that the equilibrium aggregate R & D effort is zero (i.e., a corner solution) and only then we can proceed to apply the techniques of differential calculus to the society's problem.

Denote by $\overline{T}$ the minimum patent length that induces the monopoly to do R & D. Thus $\overline{T}$ is implicitly defined by the following condition:

(10)
$$1 - e^{-r\overline{T}} = \frac{\alpha}{H} r^2$$

Clearly, for all values of T in the interval $(0, \bar{T})$ the expected social welfare is constant (and equal to 0 under our normalisation). Thus we can restrict our attention to the interval $(\bar{T}, \infty)$

Obviously, the problem would be meaningless if the research monopoly could not be induced to do R & D even if the patent's life were infinite. Thus we assume that $\bar{T}$ is finite, that is [15]:

(11)
$$\sqrt{\frac{H}{\alpha}} > r$$

If this condition is not satisfied, the patent system is ineffective as an instrument of innovation policy and must be replaced with other instruments like R & D subsidies or direct public intervention in the invention industry.

We can now turn to the stimulus effect. Under research monopoly social welfare is:

(12)
$$W = \left[\frac{xz\frac{H}{r}}{x+r} - \alpha x \right] + x\frac{\frac{J}{r} + (1-z)\frac{H+K}{r}}{x+r}$$

where the term inside square brackets represents the profits of the monopolist in the invention industry.

The social problem may now be stated as follows: choose T so as to maximise (12), given that x depends on T (via V) according to (9). But notice that W depends on T only through z. There is a one-to-one correspondence between z and T: given z, T is determined by:

(13)
$$T = -\frac{\log(1-z)}{r}$$

Thus the social problem is to choose z in the interval $(\bar{z}, 1)$, where $\bar{z} = z(\bar{T})$, so as to maximise (12).

[15] If there were a fixed cost to pay to do R & D, condition (11) would have to be strengthened. Let F denote the fixed cost. Then, substituting (9) into (1) one gets the monopoly's profits,

$$\Pi = \left(\sqrt{\frac{zH}{r}} - \sqrt{\alpha r} \right)^2 - F$$

The monopoly would do R & D with $T = \infty$ only if

$$\sqrt{\frac{H}{r}} - \sqrt{F} > \sqrt{\alpha r}$$

Differentiating:

(14)
$$\frac{dW}{dz}=-\frac{x}{x+r}\frac{K}{r}+\frac{r}{(x+r)^2}\left[\frac{J}{r}+(1-z)\frac{(H+K)}{r}\right]\frac{dx}{dz}$$

The first term on the right hand side of (14) is the direct effect of a change in z on social welfare: when z increases, society can appropriate only a smaller fraction of the potential discounted gain K/r. In other words, when T increases society must waite longer and longer to appropriate the welfare triangle K. Clearly, the direct effect of an increase in T on social welfare is always strictly negative provided $K>0$ and $x>0$. It measures the social cost of longer patent terms.

The second term is the indirect effect of z on W through x. Notice that, by the envelope theorem, the derivative of the term inside square brackets in (12) with respect to x is zero. That is, the marginal change in R & D costs associated with a change in x is exactly offset by the marginal change in the privately appropriable fraction z of expected value of the gain H/r from the innovation. Equation (14) shows that the net marginal social gain from a longer patent life is proportional to the sum of three components: J/r and a fraction $(1-z)$ of H/r and K/r. The reason that only a fraction $(1-z)$ of H/r and K/r enters the marginal social benefit is the following: zH/r is offset by the change in R & D costs and zK/r is lost due to monopolistic distortions during the patent's lifetime. The term $r/(x+r)^2$ measures the change of the "discount factor" $x/(x+r)$ due to a change in x.

Factoring the term $1/(x+r)$ out, the derivative of social welfare with respect to z can be rewritten as follows:

$$\frac{dW}{dz}=\frac{1}{x+r}\{-x\frac{K}{r}+\frac{1}{x+r}[J+(1-z)(H+K)]\frac{dx}{dz}\}$$

We shall refer to xK/r as the marginal social cost of increasing the patent life, and to $(dx/dz)[J+(1-z)(H+K)]/(x+r)$ as the corresponding marginal social benefit. The optimum patent lifetime is found equating the marginal social cost and the marginal social benefit.

It is not obvious that the policymaker's problem has a solution, because social welfare might well be always increasing in z (and therefore in T); that is, the marginal social benefit from increasing the patent's duration may always exceed the marginal social cost. With a slight abuse of terminology, if that is the case we shall say that the optimum patent length is infinite.

A simple argument shows that the optimum patent length is indeed finite if $J=0$ and $K>0$. When $T=\infty$ and $J=0$, the social and the private incentive to innovate coincide, so that by the envelope theorem the indirect effect vanishes and only the direct effect is at work. Since the direct effect is always positive if $K>0$ [16], the optimal patent life cannot be infinite.

The next Proposition states more precisely when the optimum patent length is finite.

[16] Notice that by (11) x is always positive if $T=\infty$.

Proposition 1. The optimal life of a patent T^* with monopoly in the invention industry is positive and is associated with a positive R & D effort; it is finite if

(15)
$$J < 2K\left(\frac{1}{r}\sqrt{\frac{H}{\alpha}}-1\right)$$

and is infinite otherwise.

Proof. First of all, notice that:

(16)
$$\frac{dx}{dz}=\frac{1}{2}\sqrt{\frac{H}{z\alpha}}>0$$

Thus, the marginal social benefit is always positive. At $z=\bar{z}$, $x=0$ and therefore the marginal social cost vanishes. This implies that at $z=\bar{z}$, dW/dz is positive, showing that the optimal patent life T^* is greater than $\bar{T}$ (and is therefore positive). In particular, when the patent life is set optimally, the equilibrium R & D effort is positive.

Next we evaluate the first derivative of social welfare (14) at $z=1$ (i.e. $T=\infty$) obtaining:

$$\frac{dW}{dz}=K\left(1-r\sqrt{\frac{\alpha}{H}}\right)+\frac{1}{2}J\sqrt{\frac{\alpha}{H}}$$

The sum of the first two terms on the right hand side is negative because of (11); hence if $J=0$ the optimum T is finite. More generally, the optimum T is finite if:

$$J < 2K\left(\frac{1}{r}\sqrt{\frac{H}{\alpha}}-1\right)$$

When condition (15) is satisfied, the F.O.C. for the social problem reduces to:

(17)
$$-\frac{K}{r}\left(\sqrt{\frac{zH}{\alpha}}-r\right)+\frac{1}{2}\left[\frac{(H+J+K)}{z}-(H+K)\right]=0$$

The first term on the left hand side of (17), which corresponds to the marginal social cost of increasing the patent's duration, is increasing in z, whereas the second term, which corresponds to the marginal social benefit of increasing T, is decreasing in z. Moreover, the marginal social cost vanishes at $z=\bar{z}$ whereas the marginal social benefit is always positive. Hence equation (17) has one real and positive solution, which is the optimum T.

Q.E.D.

The first part of Proposition 1 (and the analogous results of the following section) may be interpreted as a formal statement of Scherer's Lebensraum effect: whenever it is possible, the patent life should be set in such a way that the corresponding equilibrium R & D effort is positive. In other words, the corner solution $x = 0$ is never socially optimal.

Figure 4 represents the two curves corresponding to the marginal social cost and the marginal social benefit of increasing the patent duration, ie.e the two terms of the left hand side of equation (17). The intersection of the two curves is the optimum z, z^*, which corresponds to the optimum patent life T^*. Figure 4 depicts the case where $z^* < 1$ and therefore T^* is finite. If the marginal social benefit curve always lies above the marginal social cost curve over the range $(\bar{z}, 1)$ the optimal patent life is infinite.

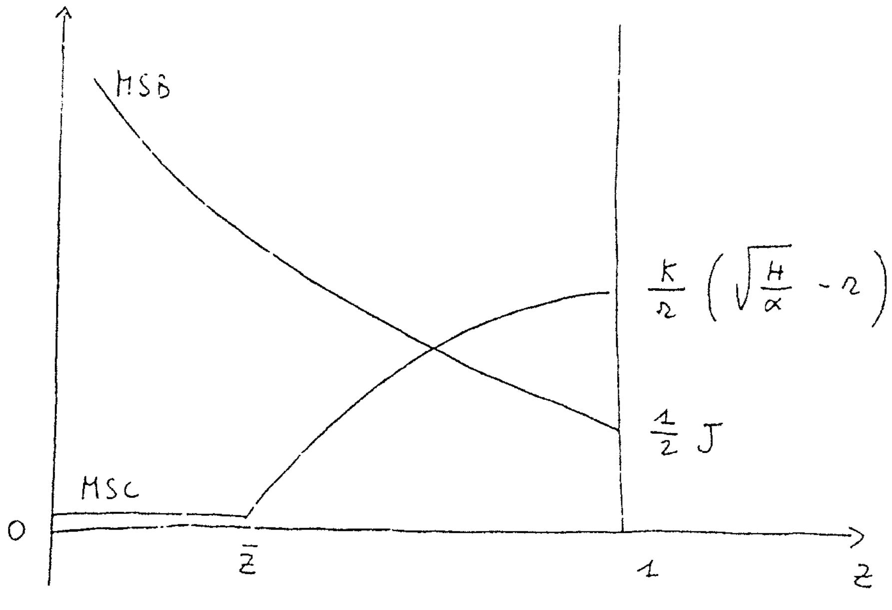

figure 4

Several comparative statics results are summarised in the next Proposition.

Proposition 2. With monopoly in the invention industry, if the optimal lifetime of the patent is finite, then it is:

i) increasing in the cost of R & D α;

ii) decreasing in the deadweight loss associated with the patent K;

iii) increasing in J, the difference between the social and private benefit from the innovation during the life of the patent;

iv) decreasing in the patentee's instantaneous profits H.

Proof. Denote by Θ the left hand side of equation (17) above. Clearly, $\Theta_z < 0$. This implies that for any arbitrary parameter m that influences z^*, the sign of $\partial z^*/\partial m$ equals the sign of Θ_m.

To prove the Proposition it therefore suffices to consider the signs of the following derivatives:

i)

$$\Theta_\alpha = \frac{1}{2}\frac{K}{r}\sqrt{\frac{zH}{\alpha}}\frac{1}{\alpha} > 0$$

ii)

$$\Theta_K = -\frac{1}{r}\sqrt{\frac{zH}{\alpha}} + \frac{1}{2}\left(\frac{1}{z} - 1\right)$$

Evaluated at the stationary point, this reduces to:

$$\Theta_K = \frac{1}{2K}(H + J - zH) > 0$$

iii)

$$\Theta_J = \frac{1}{2z} > 0$$

iv)

$$\Theta_H = -\frac{1}{2}\frac{K}{r}\frac{1}{\sqrt{\frac{z}{H\alpha}}}z^{\frac{3}{2}} + \frac{1}{2}\left(\frac{1}{z} - 1\right)$$

Evaluated at the stationary point, this reduces to:

$$\Theta_H = \frac{1}{2H}(zK + K + J) > 0$$

Q.E.D.

The optimal life of the patent increases with α and J. The reason why it increases with J is obvious. Recall that J is the social benefit from the innovation which is not captured by the patentee and is therefore a positive externality of the R & D activity. The larger J is, the more R & D investment should be stimulated through longer patents. Graphically, an increase in J shifts upward the marginal social benefit curve. Figure 5 illustrates.

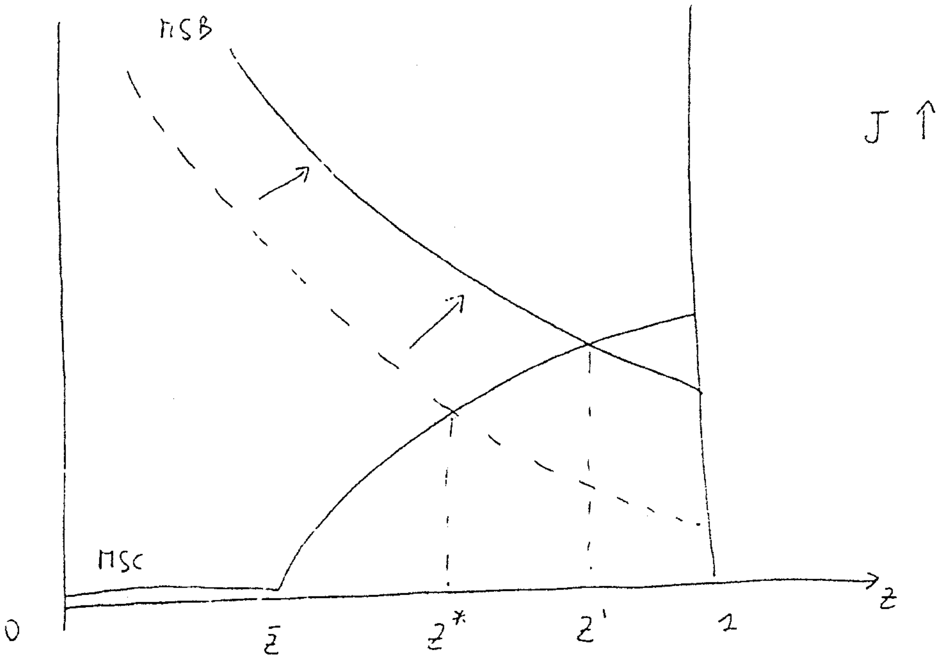

figure 5

The effect of α on T^* is only slightly less intuitive. On the one hand, an increase in α leads to a fall of x and this reduces the marginal social cost of increasing the patent' life. On the other hand, a change in α has two offsetting effects on the marginal social benefit: both dx/dz and $(x+r)$ decrease with α and their ratio is independent of α. Figure 6 illustrates. The marginal social cost curve shifts downward when α increases whereas the marginal social benefit curve is unaffected by a change in α; this leads to an increase in z^*, and hence in the optimal patent's lifetime T^*.

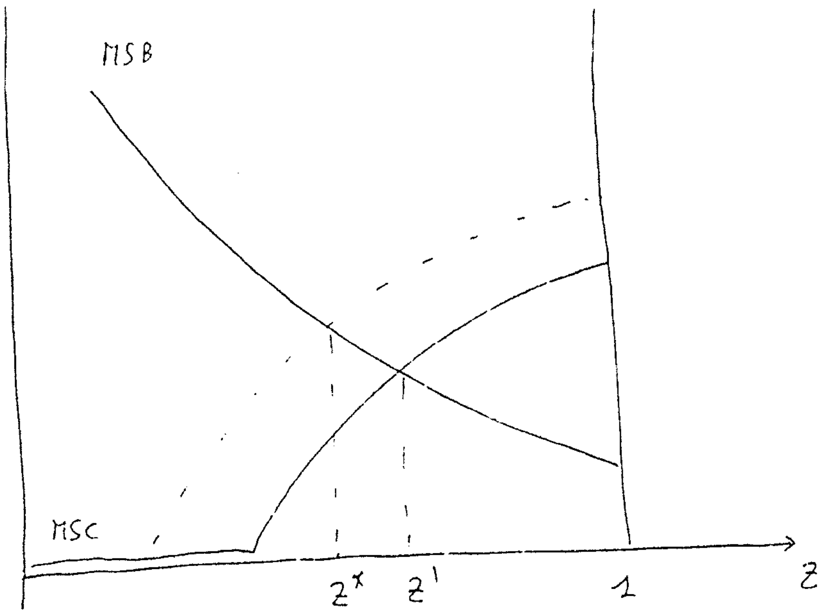

figure 6

The optimal patent life is decreasing in H and K. Since changes in these two variables shift in the same direction both the marginal social cost and the marginal social benefit curves, the graphical analysis is not very useful in this case. However, the intuition behind these results is pretty clear. When H increases (and K and J stay constant), a larger fraction of the social benefit from the innovation is privately appropriable. This reduces the necessity to incentivate R & D through the patent. As far as K itself is concerned, it is obvious that it has an adverse effect on the optimum life of the patent, for K measures the social cost of patent protection.

Suppose indeed this cost vanished (this situation may arise, for instance, if the slope of the demand function at the pre-innovation equilibrium goes to infinity): then, it would clearly be optimal to protect the innovator forever. This follows directly from Proposition 1 if $J > 0$. If instead $J = 0$, the result follows by inspection of figure 4: when $J = K = 0$, both the marginal social cost and the marginal social benefit curves tend to 0 as z tends to 1, and therefore for finite T the latter curve lies above the former one. Analogously, when the term $(\sqrt{H/\alpha}-r)$ tends to 0, the optimal patent life T^* tends to ∞. Though it is obvious, this result is noteworthy and therefore we state it formally in the next Proposition.

Proposition 3. *Under monopoly in the invention industry, the optimal life of the patent tends to infinity if the deadweight loss associated with the patent K and/or the difference* $(\sqrt{H/\alpha}-r)$ *tend to 0.*

A more subtle question is how the optimal life of the patent varies with the discount rate r. When r tends to $\sqrt{H/\alpha}$, by Proposition 3 the optimal patent life tends to infinity; thus for r "high", T^* is an increasing function of r. Moreover, it is easy to show that z is always increasing in r, for the marginal social cost curve shifts downward as r increases. But since z is a function of T and r, this does not imply generally speaking a positive relationship between r and T^*. In fact, we have been able to prove just the following partial result.

Proposition 4. Under monopoly in the invention industry, if $H \leq K$ the optimal life of the patent is increasing with the rate of interest r.

Proof. First of all, notice the following relationship:

(18)
$$\frac{dT}{dr} = \frac{1}{r}\left(\frac{dz}{dr}\frac{1}{1-z} - T\right)$$

The first term inside brackets in (18) is positive whereas the second term is negative.

Next, by implicit differentiation:

(19)
$$\frac{dz}{dr} = \frac{\frac{K}{r^2}\sqrt{H\alpha}\, z^{\frac{3}{2}}}{\frac{3}{2}\frac{K}{r}\sqrt{H\alpha}\, z^{\frac{1}{2}} + \frac{1}{2}(H-K)} > 0$$

If $H = K$, (19) reduces to: $dz/dr = (2z)/(3r)$. Substituting into (18) we get:

$$\frac{dT}{dr} = \frac{1}{r^2}\left(\frac{2}{3}e^{rT} - rT\right) > 0$$

Thus if $H = K$ the optimal length of the patent is an increasing function of the rate of interest. If $H < K$ the term dz/dr is larger than in the case $H = K$ and therefore the same conclusion follows. But if H is sufficiently greater than K the sign of dT/dr may become ambiguous (for r small; when r is large we know that the positive effect must prevail).

Q.E.D.

6. Free entry in the R & D industry

We now turn to the model with free entry. Firms operating in the R & D industry incur a fixed cost F in addition to the variable cost αx_i. The profit of firm i is therefore:

$$\Pi_i = \frac{x_i V}{a + x_i + r} - \alpha x_i - F$$

The presence of the fixed cost does not affect the F.O.C. and the equilibrium condition (4). Now, however, the number of operating firms n is endogenous and is determined by the zero profit condition [17] $(x_i V)/(X_{-i} + x_i + r) - \alpha x_i - F = 0$, which in the symmetric equilibrium becomes:

(20)
$$\frac{xV}{nx + r} - \alpha x - F = 0$$

Equations (4) and (20) determine the equilibrium number of firms n and the individual R & D effort x. Luckily, even if these equations are non linear it is possible to find an explicit solution. Indeed, combining equations (4) and (20) it can be shown that the aggregate R & D effort $X = nx$ is[18]:

[17] As usual, we ignore the integer problem treating n as a continuous variable.

[18] To obtain equation (21) we proceed in the following way. First of all, substitute (4) into (20). This gives:

$$\frac{xV}{nx + r} - \frac{xV}{(nx + r)^2}[r + (n - 1)x] = F$$

which simplifies to:

$$\frac{x^2 V}{(nx + r)^2} = F$$

or

$$(nx + r)^2 = \frac{x^2 V}{F} \qquad (*)$$

Next rewrite (4) as:

$$xV = (nx + r)[V - \alpha(nx + r)]$$

Squaring and using (*) we get:

$$x^2 V^2 = \frac{x^2 V}{F}[V - \alpha(nx + r)]^2$$

or

$$[V - \alpha(r + nx)]^2 = FV$$

whence equation (21) follows.

$$X = \frac{V - \sqrt{FV}}{\alpha} - r$$

Inserting expression (*) of footnote 16 into (4) one also gets:

(21)
$$x = \frac{\sqrt{VF} - F}{\alpha}$$

and then, from (21) and (22),

(22)
$$n = \sqrt{\frac{V}{F}} - \frac{r\alpha}{\sqrt{VF} - F}$$

Clearly, the equilibrium number of firms is a decreasing function of the fixed R & D cost F[19].

We now show that social welfare depends only on the aggregate R & D effort X. To see why, notice that with free entry in the invention industry the aggregate industry profits Π vanish, and therefore equation (8) becomes:

(23)
$$W = \frac{X\left[\frac{J}{r} + (1-z)\frac{H+K}{r}\right]}{X + r}$$

Thus social welfare depends only on the aggregate R & D effort X; it does not depend separately on n and x[20].

Like in the case of monopoly in the invention industry, denote by $\overline{T}$ the minimum patent length that induces a positive equilibrium R & D effort. Notice that the aggregate equilibrium R & D effort X may also be written as:

(24)
$$X = \frac{zH}{r\alpha} - \frac{1}{\alpha}\sqrt{\frac{FzH}{r}} - r$$

[19] It can be checked that if $\sqrt{F} = \sqrt{H/r} - \sqrt{r\alpha}$, so that the profits of the research monopoly vanish (see footnote 15 above), the equilibrium number of firms is 1. Nevertheless, as we shall see in more details in section 8, the behaviour of a monopoly (that is, its reaction to changes in the exogenous parameters) is different from that of an oligopolistic invention industry.

[20] This result is not obvious; notice for instance that aggregate fixed R & D costs nF do depend on n.

Thus $\overline{T}$ is implicitly defined by the following condition:

(25)
$$1-e^{-r\overline{T}}=\frac{r}{H}\left(\sqrt{\frac{F}{4}}+\sqrt{\frac{F}{4}+\alpha r}\right)^2$$

For $T \leq \overline{T}$ a change in the patent grant's duration is completely ineffective. We can therefore restrict our attention to the interval $(\overline{T},\infty)$. Obviously, for the problem to be meaningful we must assume that $\overline{T}$ is finite, that is:

(26)
$$\frac{H}{r}>\left(\sqrt{\frac{F}{4}}+\sqrt{\frac{F}{4}+\alpha r}\right)^2$$

Hereafter we assume that condition (26) holds.

The policymaker chooses T (or, equivalently, z) so as to maximise social welfare, taking into account that X is given by (21).

The first derivative of the objective function is:

$$\frac{dW}{dz}=-\frac{X}{X+r}\frac{(K+H)}{r}+\frac{r}{(X+r)^2}\left[\frac{J}{r}+(1-z)\frac{(H+K)}{r}\right]\frac{dX}{dz}$$

The two terms that appear in this expression may again be interpreted as the marginal social cost and the marginal social benefit from increasing the patent lifetime (or, more precisely, from increasing the fraction z of the total discounted profits H/r that the patentee can appropriate). Like in the case of monopoly, it is convenient to factor out the term $1/(X+r)$, obtaining:

(27)
$$\frac{dW}{dz}=\frac{1}{X+r}\left\{-X\frac{(K+H)}{r}+\frac{J+(1-z)(H+K)}{X+r}\frac{dX}{dz}\right\}$$

and to define the marginal social cost and the marginal social benefit, respectively, as the first and second term inside curley brackets in equation (27).

The marginal social benefit has the same form as under monopoly, whereas the marginal social cost term differs because we now have $(H+K)$ in place of K. Of course, the way in which aggregate R & D effort X is determined is different, and also dX/dz is different from the corresponding term under monopoly. Upon substitution, then, the marginal social benefit will turn out to be different, too.

More precisely, the marginal social cost is:

$$MSC=\frac{(K+H)}{r}\left(\frac{zH}{r\alpha}-\frac{1}{\alpha}\sqrt{\frac{zHF}{r}}-r\right)$$

it is increasing in z and vanishes at $z = \bar{z}$. Using the relationship

(28)
$$\frac{dX}{dz} = \frac{H}{r\alpha} - \frac{1}{2\alpha}\sqrt{\frac{FH}{r}}\frac{1}{\sqrt{z}} > 0$$

the marginal social benefit becomes:

$$MSB = \left[\frac{(H+K+J)}{z} - (H+K)\right]\left(\frac{\sqrt{H} - \frac{1}{2}\sqrt{rF}}{\sqrt{H} - \sqrt{rF}}\right)$$

and is always positive and decreasing in z.

Like in the case of monopoly in the invention industry, we first show that the optimal patent life is always greater than $\bar{T}$; that is, it is always socially optimal to have a positive equilibrium R & D investment (provided, of course, that condition (26) holds, so that a sufficiently long patent's term may induce firms to invest in R & D). Moreover, we determine when the optimal patent life is finite.

Proposition 5. The optimal life of a patent under free-entry is positive and is associated with a positive aggregate R & D effort; it is finite if

(29)
$$J < \frac{(H+K)}{r}\left[\frac{H}{r\alpha} - \frac{1}{\alpha}\sqrt{\frac{HF}{r}} - r\right]\frac{\sqrt{H} - \sqrt{rF}}{\sqrt{H} - \frac{1}{2}\sqrt{rF}}$$

Proof. Let $\bar{z}$ be the value of z implicitly defined by $X(\bar{z}) = 0$, that is $z(\bar{T})$. By (26), $0 < \bar{z} < 1$. Hence, the relevant range of z is $\bar{z} \leq z \leq 1$.

At $z = \bar{z}$ by definition $X(\bar{z}) = 0$ and therefore the marginal social cost vanishes; since the marginal social benefit is always positive, it follows that $T^* > \bar{T} > 0$, implying that when the patent life is set optimally, the aggregate R & D effort X is positive.

At $z = 1$, the marginal social cost is

$$\frac{(K+H)}{r}\left(\frac{H}{r\alpha} - \frac{1}{\alpha}\sqrt{\frac{HF}{r}} - r\right)$$

and the marginal social benefit is

$$J\left(\frac{H - 1/2\sqrt{rFH}}{H - \sqrt{rHF}}\right)$$

If the marginal social benefit at $z = 1$ were greater than the marginal social cost, the first derivative of the objective function would be positive everywhere, so that the optimal patent life would be infinite. The optimum patent life is finite if and only if condition (29) holds.

Q.E.D.

Figure 7 illustrates the marginal social cost and marginal social benefit curves for the case of free entry in the invention industry.

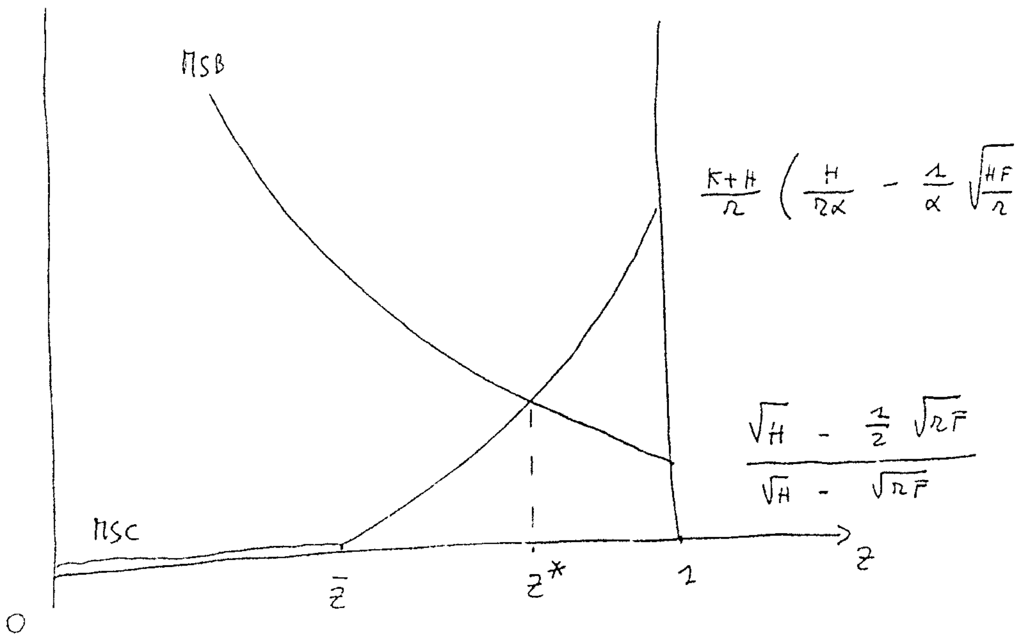

figure 7

Like in the case of monopoly in the invention industry, the optimum patent length is finite if $J = 0$. It is instructive to consider in more details this special case. When $J = 0$ social welfare becomes:

$$W = \frac{H+K}{r}\frac{X(1-z)}{X+r}$$

Thus the social problem reduces to maximising:

$$\Lambda = \frac{X(1-z)}{X+r} \quad (30)$$

This special form of the social welfare function may be explained as follows. When $J = 0$, society in fact does not benefit from the innovation until the patent expires, for firms enter and invest in R & D up to the point where expected profits vanish. Thus social welfare equals $(H + K)/r$ discounted at rate r for a period equal to the expected date of innovation *plus* the length of the patent. The corresponding discount factor is given by (30). The patent's length affects social welfare only through this discount factor.

From the above discussion it is clear that under free entry and $J = 0$ the optimum patent life cannot be infinite[21], because the social benefits from the innovation before T are entirely dissipated by the profit-seeking behaviour of firms and therefore if $T = \infty$ the net social gain from the innovation would vanish. A direct formal proof is as follows: by inspection, $\Lambda(z) \geq 0$. But $\Lambda(\bar{z}) = \Lambda(1) = 0$; this implies that $z = 1$ and $z = \bar{z}$ are actually points of minimum of the objective function, so that $\bar{T} < T^* < \infty$.

Another thing to notice is that when $J = 0$ the objective function Λ does not depend on K neither directly nor through X, and therefore the optimal patent length does not depend on K. This fact is especially noteworthy because in the case of monopoly in the invention industry K is one of the main factors affecting T^*. The next Proposition summarises these conclusions for the case $J = 0$.

Proposition 6. Under free entry in the invention industry and $J = 0$, the optimal patent length is finite and does not depend on the deadweight loss K.

We have seen that when $J = 0$ the optimum patent life is always finite. However, it tends to infinity as:

$$X(z = 1) = \frac{H}{r\alpha} - \frac{1}{\alpha}\sqrt{\frac{HF}{r}} - r$$

tends to zero. This result is the counterpart of Proposition 3 in the case of free entry in the invention industry.

Let us now turn to the general case. The first order condition for a maximum is that the marginal social cost of increasing the patent length equals the marginal social benefit. From (24), (27) and (28) we can write this condition as follows:

(31) $$\left[\frac{(H + K + J)}{z} - (H + K)\right]\left(\frac{\sqrt{H} - \frac{1}{2}\sqrt{rF}}{\sqrt{H} - \sqrt{rF}}\right) - \frac{(K + H)}{r}\left(\frac{zH}{r\alpha} - \frac{1}{\alpha}\sqrt{\frac{zHF}{r}} - r\right) = 0$$

[21] This result is already implicit in Proposition 5.

Let Ψ denote the left hand side of (31). Clearly, $\Psi'_z < 0$. The above discussion also implies that equation (31) has only one root in the interval $(\bar{z}, 1)$, which actually corresponds to the optimum patent life.

We now turn to the comparative statics.

Proposition 7. Under free entry in the invention industry, the optimal patent length is:
i) increasing in the variable cost of R & D α;
ii) increasing in the fixed R & D cost F;
iii) increasing in J, the difference between the social and private benefit from the innovation during the lifetime of the patent grant;
iv) decreasing in the deadweight loss of the patent K, if $J > 0$;
v) decreasing in the patentee's instantaneous profits H.

Proof. Since $\Psi'_z < 0$, it is clear that for any arbitrary parameter m that influences z^*, the sign of $\partial z^* / \partial m$ equals the sign of Ψ'_m.

Points *i*), *ii*) and *iii*) are evident. An increase in α reduces the equilibrium aggregate R & D effort X and therefore shifts downward the marginal social cost curve. The marginal social benefit curve does not move because the effects of α on $(X + r)$ and dX/dz exactly cancel out. An increase in F shifts the marginal social cost curve downward, because it reduces X, and the marginal social benefit curve upward. Finally, an increase in J shifts upward the marginal social benefit curve and does not affect the marginal social cost curve.

More formally, we have:

$$\Psi'_\alpha = \frac{1}{\alpha}\frac{(H+K)}{r}(X + r) > 0$$

$$\Psi'_F = \frac{1}{4}\left[\frac{(J + H + K)}{z} - (H + K)\right]\sqrt{\frac{rH}{F}}\frac{1}{(\sqrt{H} - \sqrt{rF})^2} + \frac{1}{2\alpha}\frac{(H+K)}{r}\sqrt{\frac{Hz}{rF}} > 0$$

$$\Psi'_J = \frac{1}{z}\frac{H - \frac{1}{2}\sqrt{rHF}}{H - \sqrt{rHF}} > 0$$

The effect of K is slightly more complicated, since a change in K shifts both the marginal social cost and the marginal social benefit curves in the same direction. But notice that:

$$\Psi'_K = \frac{1 - z}{z}\frac{X'(z)}{X + r} - \frac{1}{r}X$$

which evaluated at the maximum becomes:

$$\Psi_K = -X'(z)r\frac{J}{H+K} < 0$$

if $J > 0$. As we already know from Proposition 6, if $J = 0$ the optimum patent life is not affected by K.

Last, consider the effect of H. At first sight, H affects in a complicated way both the marginal social cost and the marginal social benefit curves. However, one can distinguish three effects that can be separately signed, and turn out to have the same sign. First, H enters (31) in the form $(H+K)$ whenever K does. Thus this first effect equals the effect of K which we have already determined. Second, when H increases the aggregate R & D effort X increases and therefore the marginal social cost curve shifts upward; the optimal patent life tends to decrease. Third, when H increases the factor $\left(\sqrt{H}-\frac{1}{2}\sqrt{rF}\right)/(\sqrt{H}-\sqrt{rF})$ decreases and therefore the marginal social benefit curve shifts downward. This third effect has therefore the same sign as the first two ones.

Formally, we have:

$$\Psi_H = -X'(z)r\frac{J}{H+K} - \frac{H+K}{r}\left(\frac{z}{r\alpha} - \frac{1}{2\alpha}\sqrt{\frac{zF}{rH}}\right) - \frac{1}{4}\left[\frac{(H+K+J)}{z} - (H+K)\right]\frac{\sqrt{rF}}{\sqrt{H}(\sqrt{H}-\sqrt{rF})^2} < 0$$

Q.E.D.

We have been unable to determine the effect of a change in the interest rate r on T^*. What can be shown is that z^* is increasing in r[22], but this is not sufficient to sign dT^*/dr.

All results parallel those obtained in the case of monopoly in the invention industry. The only new result is ii). When F decreases, the aggregate R & D effort X tends to increase and this allows society to shorten the patent grant's term. But we have seen that the equilibrium number of firms n increases when F decreases, so that F may be taken as a parameter measuring the degree of competition in R & D. Proposition 7 then says that more competition in the invention industry entails a shorter patent life. In the next section we study what happens when F becomes negligible and perfect competition prevails in the invention industry.

[22] It is easy to show that $\Psi_r > 0$. An increase in r shifts downward the marginal social cost curve and upward the marginal social benefit curve.

7. Perfect competition in the invention industry

A special case of the model analysed in the previous section arises when the fixed R & D cost F goes to zero. In this case it turns out that the equilibrium number of firms and the individual R & D effort are indeterminate. However, the aggregate R & D effort X is determinate and is given by:

(32)
$$X = \frac{zH}{r\alpha} - r$$

The equilibrium aggregate R & D effort is positive when $T = \infty$ provided:

$$\sqrt{\frac{H}{\alpha}} > r$$

This condition is exactly the same as (11), which guarantees that with monopoly in the invention industry the R & D effort can be made positive prolonging indefinitely the life of the patent.

Interestingly, the equilibrium condition (32) may be justified in a different way. Following Dasgupta and Stiglitz (1980b), one may define perfect competition in the R & D industry as a state where each firm maximises its profits taking the expected date of the innovation as given. In other words, firms do not take into account the effect of their R & D investment on the aggregate R & D effort X.

Under our assumptions, this implies that the profit function perceived by a representative firm i is linear in x, and its derivative is:

$$\frac{\partial \Pi_i}{\partial x_i} = \frac{V}{X + r} - \alpha$$

It follows that if there is an equilibrium in the invention industry with positive R & D effort, it must satisfy equation (32). This argument shows that Dasgupta and Stiglitz's (1980b) assumption of a perfectly competitive invention industry corresponds in our model to $F = 0$.

Substituting (32) into (8) we obtain the following expression for social welfare under perfect competition in the invention industry:

$$W = \left(1 - \frac{r^2\alpha}{zH}\right)\left[\frac{J}{r} + (1 - z)\frac{H + K}{r}\right]$$

which neglecting constants reduces to:

(33)
$$W = -\frac{H+K}{r}z - \frac{r\alpha}{H}(H+K+J)\frac{1}{z},$$

The solution to the social problem is straightforward and is described by the following Proposition.

Proposition 8. Under perfect competition in the R & D industry, the optimal patent's length is positive and is implicitly given by:

(34)
$$z^* = \min\left(1, r\sqrt{\frac{\alpha}{H}}\sqrt{1+\frac{J}{H+K}}\right)$$

When

(35)
$$r\sqrt{\frac{\alpha}{H}}\sqrt{1+\frac{J}{H+K}} \geq 1$$

the optimum life of the patent is infinite. In particular, since $r < \sqrt{H/\alpha}$ by (11), this case may occur only when J is large relative to $(H+K)$. When J=0, the optimum patent length is always finite and is given by:

$$T^* = -\frac{\log\left(1 - r\sqrt{\frac{\alpha}{H}}\right)}{r}$$

From (34) it is immediate to deduce the following comparative statics results.

Proposition 9. Under perfect competition in the invention industry, if the optimal lifetime of the patent is finite, then it is:

i) increasing in the cost of R & D α*;*

ii) decreasing in the deadweight loss associated with the patent K, provided $J > 0$*;*

iii) increasing in J, the difference between the social and private benefit from the innovation during the life of the patent;

iv) decreasing in the patentee's instantaneous profits H;

v) increasing in the rate of interest r.

Proof. The only result that does not follow directly by inspection of (34) is v). But using the definition of z and (18) one can easily show that:

$$\frac{dT}{dr} = \frac{1}{r}\left[\log(1-z^*) + \frac{z^*}{1-z^*} \right] > 0$$

Q.E.D.

8. A comparison between monopoly and free-entry in the invention industry

We have seen that with free entry in the invention industry the optimal patent length is decreasing in the size of the fixed R & D cost F. Since the equilibrium number of firms in the invention industry increases when the fixed R & D cost does down, there is a sense in which more competition in R & D allows society to reduce the length of patent grants.

This is not, however, the standard result that the optimal patent term is longer when the inventor is a monopolist than when inventors are competitive, which holds in Nordhaus-type models. Nor does the former result imply the latter, as could be conjectured noting that monopoly is nothing else than $n = 1$. The reason that this seemingly convincing argument is fault is that the negative relationship between the optimal patent life and the equilibrium number of firms n holds whithin the confines of the model with free entry. Even if the fixed R & D cost F were so high as to leave room for only one firm in the invention industry[23], the reaction of the industry to a change in the exogenous parameters under free entry would be quite different from the reaction of a true monopolist. This implies that the social incentives to change the life of a patent would also be different. In fact, in this section we shall show that comparing the optimal patent term with monopoly and with free entry in the invention industry, the former may be shorter than the latter, however, the optimal patent life is shortest in the limit case of perfect competition.

Consider again the marginal social cost and marginal social benefit curves. The former is proportional to the equilibrium aggregate R & D effort; however, the factor of proportionality is K/r under monopoly and $(H+K)/r$ under free entry. Even if we assume that the equilibrium aggregate R & D effort is the same, the marginal social cost is higher under free entry. The reason is that the privately appropriable part of the social gain from the innovation before the expiration of the patent, H, is dissipated under free entry but not under monopoly. In other words, monopoly avoids excessive duplication of R & D efforts.

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[23] That is, if $\sqrt{F} = \sqrt{H/r} - \sqrt{r\alpha}$. See footnote 15 above.

On the other hand, the marginal social benefit is $\frac{1}{2}[(H + K + J)/z - (H + K)]$ under monopoly, whereas under free entry the term inside square brackets is multiplied by a factor greater than 1. Thus the marginal social benefit is also higher under free entry. This is due to the same factors that make the marginal social cost higher under free entry: the duplication of efforts that characterises free entry implies that prolonging the patent duration provides a higher stimulus to invest in R & D than under monopoly.

To facilitate the comparison, divide both the marginal social cost and the marginal social benefit under free entry by the factor $\left(\sqrt{H}-\frac{1}{2}\sqrt{rF}\right)/(\sqrt{H}-\sqrt{rF})$, and multiply the corresponding two terms under monopoly by 2, so that the "adjusted" social marginal benefit with both free entry and monopoly equals $[(H + K + J)/z - (H + K)]$. Then, the optimal patent life will be shorter under free entry if and only if the "adjusted" marginal social cost under free entry is higher than the marginal social cost under monopoly, i.e.

(36)
$$\frac{H+K}{r}X_F > 2\frac{K}{r}\frac{\left(\sqrt{H}-\frac{1}{2}\sqrt{rF}\right)}{(\sqrt{H}-\sqrt{rF})}X_M$$

where X_F denotes the aggregate R & D effort under free entry and X_M that under monopoly.

In the model with free entry, we know that the optimal patent term is shortest when $F = 0$, i.e. under perfect competition. Let us therefore compare monopoly with perfect competition. The "adjusted" marginal social cost is

$$\frac{H+K}{r}\left(\frac{zH}{r\alpha}-r\right)$$

under perfect competition, and

$$\frac{2K}{r}\left(\sqrt{\frac{zH}{\alpha}}-r\right)$$

under monopoly. Using condition (11) - which must hold under both monopoly and perfect competition - it can be easily shown that the "adjusted" marginal social cost curve is higher under competition over the range $(\bar{z}, 1)$, where $\bar{z} = r^2\alpha/H < 1$. It follows that perfect competition entails a shorter patent life than monopoly. Figure 8 illustrates.

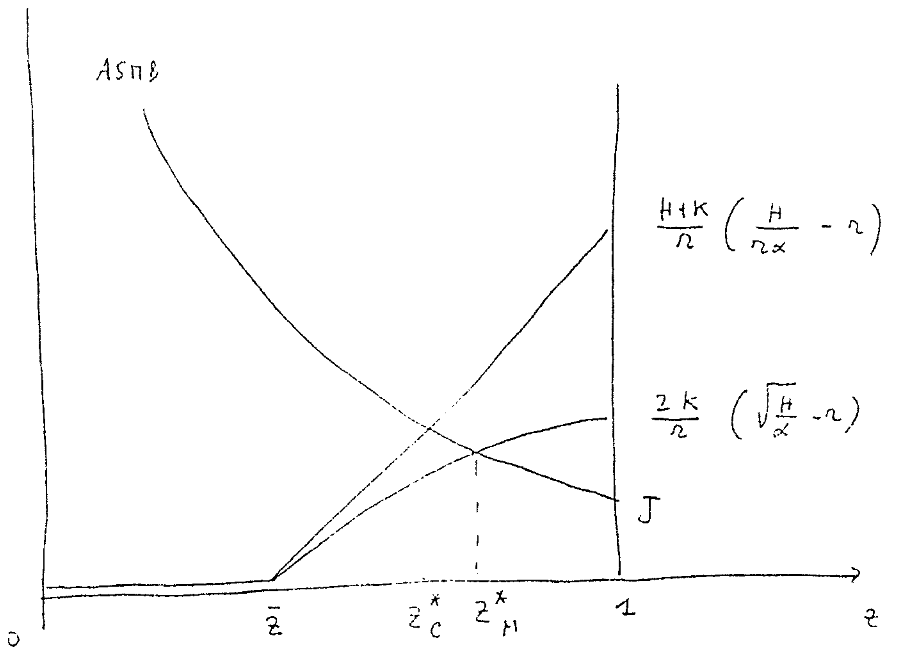

figure 8

Proposition 10. With perfect competition in the invention industry the optimal patent life is longer than with monopoly.

Consider now the case of a positive fixed R & D cost. Moreover, fix

(37)
$$\sqrt{F} = \sqrt{\frac{H}{r}} - \sqrt{r\alpha}$$

so that the equilibrium number of firms is 1. We compare the optimal patent length in this case with that under monopoly.

Since when (37) holds we have $X_F = X_M$, the adjusted social marginal benefit will be greater under monopoly if

$$\frac{2K}{r}\frac{\left(\sqrt{H}-\frac{1}{2}\sqrt{rF}\right)}{(\sqrt{H}-\sqrt{rF})} > \frac{H+K}{r}$$

which using (37) after some algebra reduces to:

$$K > r\sqrt{\alpha H}$$

This means that if we compare monopoly and oligopoly with free entry in the invention industry holding the aggregate R & D investment constant (that is, assuming that the R & D fixed cost F is such that in the equilibrium with free entry $n = 1$), the optimal patent life may be shorter under monopoly if K is sufficiently high relative to H. By continuity, the optimal patent life may continue to be shorter under monopoly even when F starts decreasing; however, Proposition 10 shows that for F sufficiently low the optimal patent life is necessarily shorter under free entry for all admissible parameters configurations.

9. Over-investment in R & D?

We now turn to a different problem. It is well known that with perfect competition in the invention industry there may be over-investment in R & D, due to a socially wasteful duplication of efforts. We now ask, whether over-investment can still occur once the life of the patent is optimally set in the way analysed in the previous sections.

Since this question involves a comparison between the socially optimal level of investment in R & D and the equilibrium one, to avoid spurious effects, we rule out fixed R & D costs [24]. Thus we limit our analysis to monopoly and perfect competition on the one hand, and the social optimum on the other hand.

Also, we consider the constrained social optimum, that is we assume that the policymaker fixes the level of R & D investment taking as given the life of the patent, and therefore taking into account that society must wait until T to appropriate the welfare triangle K. From (7), it is clear that the constrained socially optimal R & D investment is:

(38)
$$X_S = \sqrt{\frac{H + J + (1 - z)K}{\alpha}} - r$$

Of course, the unconstrained socially optimal level of investment corresponds to $z = 0$ and is therefore higher than the constrained one.

[24] Since there are constant returns to scale in R & D apart from the fixed cost, society would obviously pay the fixed cost F only once; this would give a socially managed R & D industry a technological advantage over an oligopolistic one.

Comparing X_S and the equilibrium R & D effort of a monopoly X_M, it is immediate that the latter is always not greater than the former: the two coincide only if $J = 0$ and $z = 1$, i.e. the patent has infinite duration. But from Proposition 1 we know that when $J = 0$ the optimal patent life can be infinite only if $K = 0$. Thus with monopoly in the invention industry there will always be under-investment in R & D; only when $K = J = 0$, the equilibrium R & D investment coincides with the socially optimal one.

Proposition 11. With monopoly in the invention industry, if the patent life is set optimally, there is under-investment in R & D unless $K = J = 0$.

This result is quite obvious as we know that monopoly is immune from the duplication of effrots problem, whereas if $K + J > 0$ it does not appropriate all the returns to R & D. The case of perfect competition in the invention industry is more interesting. In this case, for arbitrary patent length T, there may indeed be over-investment in R & D. Consider for instance the case $J = 0$ and $z = 1$. Comparing (38) and (32) one immediately confirms that there is over-investment in R & D (recall condition (11)). By continuity, there will still be over-investment if J is positive but small and if the patent life T is finite but large.

But what happens if T is set optimally according to (34)? Proposition 12 provides the answer.

Proposition 12. With perfect competition in the invention industry, if the patent life is set optimally, there is under-investment in R & D unless $K = J = 0$*; in this case, the equilibrium aggregate R & D investment coincides with the constrained socially optimal one.*

Proof. We must distinguish the case when the optimal patent life is finite and the case when it is infinite.

The latter case occurs when:

$$\frac{H + K + J}{H + K} > \frac{\alpha}{H} r^2$$

Then we have:

$$X_S + r = \sqrt{\frac{H + J + K}{\alpha}} > \frac{\sqrt{H}\sqrt{H + K}}{\alpha r} \geq \frac{H}{\alpha r} = X_C + r$$

so that the socially optimal level of R & D is greater than the R & D effort with perfect competition given by (32).

10. Perfect competition in the product market

Up to now we have black-boxed the product market, whose equilibrium is described by the three parameters H, K and J. We now turn to a more detailed analysis of the product market equilibrium, focussing on the case of perfect competition. Without aiming at dealing with all relevant issues, we concentrate on a few topics where the results follow most directly from our general treatment of the problem. Following Nordhaus, we assume that initially production is carried out at constant unit cost c, and that the innovation reduces the production cost to $c - d$. We also make the regularity assumption that the marginal revenue curve is downward sloping.

A) Non-drastic innovation with imitation

Patents not always confere complete protection over the invention. As we explained in sect. 2, if the patent is narrowly defined, a fraction γ of the cost reduction may spill out as freely available technology to the non innovating firms. Our first task is to analyse the effect on T^* of a change in the parameter γ, that can be taken as an inverse measure of the breadth of the patent.

When γ increases, the sum $(H + K + J)$ stays constant. However, K decreases and J increases. If the innovation is non drastic, H decreases too (for otherwise the innovator - or a unique licensee - could profitably engage in monopoly pricing[25]). The optimal patent life T^* would therefore tend to decrease.

Proposition 13. With perfect competition in the product market, non-drastic innovation and imperfect protection of the invention, T^* is decreasing in the breadth of the patent.

B) Non-drastic innovation, no imitation

Consider now the effect on the optimal patent life of a change in the cost improvement d. Clearly, as far as the invention is non drastic, an increase in d affects both H and K positively, and therefore leads to a reduction of the socially optimal life of the patent.

Proposition 14. With perfect competition in the product market, if the innovation is non drastic the optimal lifetime of the patent is decreasing in the size of the innovation d.

Indeed, it can be shown that for sufficiently small inventions the optimal duration of the patent becomes infinite. We have seen that when the aggregate R & D effort corresponding to infinite patent length tends to 0, the optimal patent length tends to infinity because the marginal social cost tends to vanish. Formally, if $\overline{T}$ tends to infinity, also T^* must tend to infinity because $T^* > \overline{T}$. But equations (10) and (25) show that $\overline{T}$ tends to infinity if H becomes sufficiently small

[25] Notice that with constant marginal cost and a downward sloping marginal revenue curve, the profit function of the monopoly is quasi-concave.

relative to r and α (and also F in the case of free entry). Now, since $H = Q_0 d$ (where Q_0 denote the pre-innovation equilibrium output), it is clear that for sufficiently small innovations the optimal patent length is infinite. More precisely, define $\bar{d}$ as an invention so small that conditions (11) and (26) hold as equalities. Then we have:

Proposition 15. With perfect competition in the product market, when d tends to $\bar{d}$ from the above, the optimal life of the patent tends to infinity.

Propositions 14 and 15 provide an answer in the negative to the question whether large innovations should be protected more than smaller ones. Small innovations should be protected for a longer period, for two reasons: first, large innovations are associated with large deadweight loss; second, small innovations require long patent duration to persuade firms in the invention industry to make a positive R & D effort (Scherer's Lebensraum effect).

Obviously, very small innovations (i.e. $d < \bar{d}$) will never command a positive R & D effort and therefore need not be protected.

We have thus shown that T^* is negatively related to the size of the innovation d, as long as the invention is non drastic. We now ask what happens when the innovation is so large as to become drastic. As we know, for drastic innovations J is positive. Moreover, as long as the marginal revenue curve is downward sloping, J increases with the size of the innovation. This observation led Nordhaus (1972) to conjecture that the optimal patent life is longer for drastic and product inventions. But this is not necessarily true, because H continues to increase with the size of the innovation even when the innovation is drastic, and the same may be true of K; these countervailing effects may prevail over the effect of a positive and increasing J.

In order to bring these points out more clearly it will prove useful to consider the case of a product invention with a linear demand function.

C) Product (or drastic) innovation with a linear demand

Assume that the demand function is $p = a - Q$, where p is price and Q is industry output and that the initial cost is $c = a$, so that before the innovation the good is not produced. In this case, $H = \frac{1}{4}d^2$ and $K = J = \frac{1}{2}H$.

Consider the model with free entry first. Divide both the marginal social cost and the marginal social benefit by H. The marginal social cost becomes:

$$MSC = \frac{3}{2r}\left(\frac{zH}{r\alpha} - \frac{1}{\alpha}\sqrt{\frac{zHF}{r}} - r\right)$$

and the marginal social benefit reduces to:

$$MSB = \left(\frac{2}{z} - \frac{3}{2}\right)\left(\frac{\sqrt{H} - \frac{1}{2}\sqrt{rF}}{\sqrt{H} - \sqrt{rF}}\right)$$

Then, it is clear that the marginal social cost is increasing in H and the marginal social benefit is decreasing in H. This implies that the optimal patent life decreases as d (and hence H) increases. A similar argument establishes the result for the case of monopoly in the invention industry.

Proposition 16. With a drastic innovation, if the initial size of the market is zero, the optimal patent life decreases as the size of the innovation increases.

By continuity, the same conclusion holds when the size of the market before the innovation, Q_0, is positive but small.

D) Uncertainty over the size of the cost reduction

Let us come back to the general case, assuming that the innovation in non drastic and there is no imitation, so that $J = 0$. Suppose that not only the timing, but also the size of the innovation d is stochastic [26]. If firms and society are risk neutral, and if d is distributed independently of the date of the innovation, they are interested only in the expected values of H and K, $E(H)$ and $E(K)$.

Consider now an increase in the uncertainty over the size of the innovation. By this we mean a mean preserving spread of the probability distribution of d. Clearly, $E(H)$ is unaffected by this change because H is a linear function of d, whereas $E(K)$ increases because K is a convex function of d [27]. Since an increase in the deadweight loss of the patent reduces the optimal patent lifetime, we have the following result.

Proposition 17. With perfect competition in the product market an increase in the uncertainty over the size of the innovation leads to a shorter optimal patent life.

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[26] To wit, when the research process is completed and the patent is awarded, it is not yet known the size of the cost reduction that the invention will bring about.

[27] The reason is that:

$$\frac{\partial K}{\partial d} = Q(c - d) - Q_0$$

where $Q(p)$ is the demand function and $Q_0 = Q(c)$. Since the demand function is decreasing, K is convex in d.

Notice that this result holds independently of the conditions prevailing in the invention industry, whereas Rafiquzzaman (1987, 1988) found that in the context of a Nordhaus-type model the effect of uncertainty is different under monopoly and perfect competition [28].

11. Summary and conclusions

We briefly summarise the main results of the paper.

1. *The optimal patent's length is increasing in the cost of R & D* α. The reason is that an increase in α reduces the marginal social cost of increasing the patent's duration, and does not affect the marginal social benefit.

2. *The optimal patent's length increases with J, the difference between the social and private benefit from the new technology before the term of the patent grant.* One may think of J as a positive externality of the R & D activity. The greater this externality, the greater the necessity of stimulating R & D.

3. *The optimal patent's length is decreasing in K, the deadweight loss of the patent.* This result is obvious, since K measures the social cost of the monopolistic power accorded to the patentee.

4. *The optimal patent's length is decreasing in H, the privately appropriable benefit from the innovation.* This result is quite general, but far from intuitive. It is clear that a higher H enhances the private incentive to do R & D. As it turns out, it enhances it more than is socially desirable, so that the policymaker reacts by lowering the patent duration.

5. *The optimal patent's length decreases with the degree of competition in the invention industry.* With more competition in the invention industry, the aggregate equilibrium R & D effort increases because of the "duplication of efforts" effect. Thus there is less need to incentivate R & D. However, the optimal patent length may be shorter under monopoly than under free entry, if K is high relative to H and if the fixed R & D cost F is sufficiently high.

6. *Whenever it could be privately convenient to do R & D, the patent's length should be sufficiently long so as to encourage a positive R & D effort.*

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[28] Clearly, a result similar to Proposition 17 would hold also in the case of drastic invention considered in subsection *C*), because in that case H is convex in d and therefore a mean preserving spread in the probability distribution of d would increase $E(H)$.

7. *With perfect competition in the product market, the optimal patent's length is decreasing in the size of the innovation.* A larger innovation entails a larger deadweight loss and a larger private incentive to innovate; both factors tend to reduce the optimal patent length.

8. *When the patent's length is set optimally, there is under-investment in R & D.* This result shows that patent policy always solves the duplication of effort problem. But, since there is under-investment, other policy instruments, such as R & D subsidies or direct public provision of basic research, should be used to regulate the R & D activity.

Indeed, the simultaneous use of different instruments of R & D policy is one of the highest-ranked issues in the agenda for future research. Patent's breadth and length is one relevant example; patent policy and R & D subsidies is another. Though there has been some research on the optimal patent breadth recently, much remains to be done in this area.

Our analysis has also side-stepped the cumulative nature of the innovative process. New inventions build on older ones, and many innovations are surpassed well before the expiration of the patent. This puts new demands on the patents system, which are discussed by Scotchmer (1991) and still await formal analysis[29].

Another limit of our model is the assumption that firms doing R & D are neatly separated from firms operating in the downstream product market. As a matter of fact, most R & D is done by firms that are themselves the users of their inventions, whereas licensing provides only a small fraction of the overall private benefits from R & D. But when the same firms compete in the patent race and in the product market, their strategic interaction is more complex and new problems arise.

These three issues are quite important to a better understanding of the functioning and the optimal structure of the patent system, and we plan to deal with them in future research.

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[29] See Green and Scotchmer (1995) and Chang (1995); these authors study some aspects of patent policy in models with two innovations in sequence; however, they do not address the issue of the optimal patent life.

REFERENCES

Barzel, Y., "The Optimal Timing of Innovations", *Review of Economic Studies*, 35 (1968), pp. 348-355.

Beck, R.L., "Patents, Property Rights, and Social Welfare: Search for a Restricted Optimum", *Southern Economic Journal*, 1976, pp. 1045-1055.

Berkowitz M.K. and Y. Kotowitz, "Patent Policy in an Open Economy", *Canadian Journal of Economics*, 15 (1982), pp.1-17.

Chang, H.F., "Patent Scope, Antitrust Policy, and Cumulative Innovation", *Rand Journal of Economics*, 26 (1995), pp. 34-57.

P. Dasgupta and J. Stiglitz, "Industrial Structure and the Nature of Innovative Activity", *Economic Journal*, 90 (1980a), pp. 266-293.

P. Dasgupta and J. Stiglitz, "Uncertainty, Industrial Structure and the Speed of R & D", *Bell Journal of Economics*, 11 (1980b), pp. 1-28.

De Brock, L.M., "Market Structure, Innovations and Optimal Patent Life", *Journal of Law and Economics*, 28 (1985), pp. 223-244.

Delbono F. and V. Denicolò, "Incentives to Innovate in a Cournot Oligopoly", *Quarterly Journal of Economics*, 105 (1991), pp. 951-961.

Denicolò, V., "Two-Dimensional Patent Policy", mimeo, Univeristy of Bologna, 1995.

Gallini, N., "Patent Policy and Costly Imitation", *Rand Journal of Economics*, 23 (1992), pp. 52-63.

Gilbert, R. and C. Shapiro, "Optimal Patent Length and Breadth", *Rand Journal of Economics*, 21 (1990), pp. 106-112.

Green J. and S. Scotchmer, "Novelty and Disclosure in Patent Law", *Rand Journal of Economics*, 21 (1990), pp. 131-146.

Green J. and S. Scotchmer, "On the Division of Profit in Sequential Innovation", *Rand Journal of Economics*, 26 (1995), pp. 20-33.

Kamien, M.I., "Patent Licensing", Ch. 11 in R.J. Aumann and S. Hart (eds.), *Handbook of Game Theory with Economic Applicatins*, vol. I, (1992), pp. 331-354, North-Holland, Amsterdam.

Kamien, M.I. and N.L. Schwartz, "Patent Life and R & D Rivalry", *American Economic Review*, 64 (1974), pp. 183-187.

Klemperer, P., "How Broad Should the Scope of Patent Protection Be?", *Rand Journal of Economics*, 21 (1990), pp. 113-130.

Lee, T. and L. Wilde, "Market Structure and Innovation: A Reformulation", *Quarterly Journal of Economics*, XCIV (1980), pp. 429-36.

Loury, G., "Market Structure and Innovation", *Quarterly Journal of Economics*, XCIII (1979), pp. 395-410.

Martin S., *Advanced Industrial Economics*, Blackwell, Oxford, (1993).

Nordhaus, W., *Invention, Growth and Welfare: A Theoretical Treatment of Technological Change*, Cambridge, MIT Press, (1969).

Nordhaus, W., "The Optimal Life of a Patent: Reply", *American Economic Review*, 62 (1972), pp. 428-431.

Rafiquzzaman, M., "The Optimal Patent Term under Uncertainty", *International Journal of Industrial Organization*, 5 (1987), pp. 233-246.

Rafiquzzaman, M., "Invention, Market Structure, Cost Reduction Uncertainty and the Optimal Patent Term", *Journal of Economics*, 48 (1988), pp. 303-312.

Reinganum J., "The Timing of Innovations", Ch. 14 in R. Schmalensee and R. Willig (eds.), *Handbook of Industrial Organization*, (1989), North-Holland, Amsterdam.

Scherer, F.M., "Research and Development Resource Allocation under Rivalry", *Quarterly Journal of Economics*, 81 (1967), pp. 359-394.

Scherer, F.M., "Nordhaus' Theory of Optimal Patent Life: A Geometric Reinterpretation", *American Economic Review*, 62 (1972), pp. 422-427.

Scotchmer, S., "Standing on the Shoulders of Giants: Cumulative Research and the Patent Law", *Journal of Economic Perspectives*, 5 (1991), pp. 29-41.

Tirole, J., *The Theory of Industrial Organization*, Cambridge, Mit Press, (1988).