AN IMPOSSIBILITY THEOREM ON EXTENDING ORDERS

by Vincenzo Denicolò
Dipartimento di Scienze Economiche
Università di Bologna
Strada Maggiore 45
I-40125 Bologna (Bo)
ITALY

November 1994

Abstract. The problem of extending an ordering on a set of alternatives to its power set is analysed. Kannai and Peleg (1984) and Barberà and Pattanaik (1984) have shown that no extension rule satisfies certain reasonable conditions. This paper proves a new impossibility result, using a condition, called Dominance principle, which states that if x is preferred to y, then $\{x,z\}$ must be preferred to $\{y,z\}$.
Journal of Economic Literature classification numbers: 025, 026

I. INTRODUCTION

In many different contexts - including voting theory, choice under uncertainty, the theory of plausible reasoning, etc. - the problem arises of extending an order over a set of alternatives to the set of all non-empty subsets of alternatives. Consider, for example, a voting situation where the outcome is determined by a social choice *correspondence* (so that the choice set is not necessarily a singleton). Then, in order to ascertain whether a given voter i has an incentive to manipulate, it is necessary to extend his/her honest preference ordering R_i over the universal set of alternatives X to an ordering $\mathbf{R}_i$ over *sets* of alternatives, i.e. an ordering over the power set 2^X.

Kannai and Peleg (1984) have shown that two plausible axioms for extending orders are mutually inconsistent. The two axioms are the *Gardenfors principle* (Gardenfors 1976, 1979) and a *monotonicity condition* (Kannai and Peleg, 1984). Barberà and Pattanaik (1984) have shown that the inconsistency disappears if the Gardenfors principle is weakened into the *Kelly principle* (Kelly, 1977). Since the *Barberà principle* (Barberà, 1977) is in turn weaker than the Kelly principle, Barberà and Pattanaik's theorem implies that the Kannai-Peleg monotonicity condition is compatible with the Barberà principle. However, Barberà and Pattanaik (1984) have also shown that even with the Barberà principle, the impossibility reappears if Kannai and Peleg's monotonicity condition is strengthened into *strict monotonicity.*

In this paper we obtain a new impossibility result adding to the Barberà principle and the Kannai and Peleg monotonicity condition another condition, called the *Dominance principle,* which is logically independent of the Gardenfors and the Kelly principles. The Dominance principle is implied by strict monotonicity. However, our result is not technically a strengthening of the Barberà and Pattanaik (1984) impossibility result as it requires a larger number of distinct alternatives and some regularity conditions on the extended order.

II. NOTATION AND DEFINITIONS

Let X be a set of alternatives, and let $L(X)$ be the set of all linear orderings over X (i.e., the set of complete, transitive and antisymmetric binary relations over X).

Denote by 2^X the set of all *non-empty* subsets of X, and by $\mathbf{W}(X)$ the set of all weak orderings (i.e. complete, reflexive and transitive binary relations) over 2^X. Let $R \in L(X)$ and $\mathbf{R} \in \mathbf{W}(X)$. Clearly, xRy implies $x \neq y$. Define the antisymmetric and symmetric components of $\mathbf{R}$ as $\mathbf{P}$ and $\mathbf{I}$, respectively. An *extension rule* is a function γ mapping $L(X)$ into $\mathbf{W}(X)$ such that for all $x,y \in X$, xRy implies $\{x\}\ \mathbf{P}\ \{y\}$, where $\mathbf{R} = \gamma(R)$.

We now define some well known conditions on extension rules and the new condition introduced in this paper.

DEFINITION 1. An extension rule satisfies *Kannai and Peleg's monotonicity condition* if for all $B,C \in 2^X$, $B\mathbf{P}C$ and $x \notin B \cup C$ imply $(B \cup \{x\})\ \mathbf{R}\ (C \cup \{x\})$.

DEFINITION 2. An extension rule satisfies the *strict monotonicity condition* if for all $B,C \in 2^X$, $B\mathbf{P}C$ and $x \notin B \cup C$ imply $(B \cup \{x\})\ \mathbf{P}\ (C \cup \{x\})$.

Strict monotonicity implies that if two sets of alternatives partially overlap, then their ordering should be based solely on their non overlapping parts. A very weak version of the strict monotonicity condition is the following condition, which may be called the Dominance principle.

DEFINITION 3. An extension rule satisfies the *Dominance principle* if for all distinct $x,y,z \in X$, xRy implies $\{x,z\}\ \mathbf{P}\ \{y,z\}$.

That is, if two pairs of alternatives have one common alternative, their ordering should coincide with the ordering of the distinct alternatives.

Let A be a finite subset of X. Given a linear order R over X, we denote by $\max(A)$ ($\min(A)$) the greatest (lowest) member of A in the order R.

DEFINITION 4. An extension rule satisfies:

i) the *Gardenfors principle* if for all finite $A \in 2^X$ and for all $x \in X\text{-}A$, $xR\max(A)$ implies $(A \cup \{x\})\ \mathbf{P}\ A$ and $\min(A)Rx$ implies $A\ \mathbf{P}\ (A \cup \{x\})$;

ii) the *Kelly principle* if for all $A,B \in 2^X$, xRy for all $x \in A$ and all $y \in B$ implies APB;
iii) the *Barberà principle* if for all $x,y \in X$, xRy implies $\{x\}$ $\mathbf{P}$ $\{x,y\}$ and $\{x,y\}$ $\mathbf{P}$ $\{y\}$.

It can be easily shown that the Gardenfors principle implies the Kelly principle, which in turns implies the Barberà principle. One can immediately confirm that the Gardenfors principle does not imply the Dominance principle, and the Dominance principle does not imply the Barberà principle.

III. RESULTS

The following results have been proved by Kannai and Peleg (1984) and Barberà and Pattanaik (1984), respectively [1].

THEOREM 1. *If* $|X| > 5$, *there is no extension rule satisfying Kannai and Peleg's monotonicity condition and the Gardenfors principle.*

THEOREM 2. *If X is finite, there exists an extension rule satisfying Kannai and Peleg's monotonicity condition and the Kelly principle.*

THEOREM 3. *If* $|X| > 3$, *there is no extension rule satisfying the strict monotonicity condition and the Barberà principle.*

In this paper we prove the following.

THEOREM 4. *If* $|X| > 5$, *there is no extension rule satisfying Kannai and Peleg's monotonicity condition, the Barberà principle, and the Dominance principle.*

[1] Actually, the result of Barberà and Pattanaik (1984) is stronger than theorem 3, as they show that there is no *binary relation* satisfying the Barberà principle and the strict monotonicity condition, without requiring that the binary relation be an extension rule (i.e., reflexive, transitive and complete).

The proof is based upon the following lemma, established by Barberà, Barrett and Pattanaik (1984).

LEMMA. *Suppose an extension rule satisfies Kannai and Peleg's monotonicity condition and the Barberà principle. Then, for all* $A \in 2^X$, $A\ \boldsymbol{I}\ \{\min(A), \max(A)\}$.

Proof of Theorem 4. Let $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, with $a_i R a_{i+1}$ for $i = 1,2,3,4,5$. We first show that

$$\{a_3\}\ \boldsymbol{I}\ \{a_2, a_4\} \qquad (i)$$

Suppose not. Then, two cases are possible: *a)* $\{a_3\}\ \boldsymbol{P}\ \{a_2, a_4\}$; *b)* $\{a_2, a_4\}\ \boldsymbol{P}\ \{a_3\}$. Consider case *a)* first. By monotonicity, $\{a_3, a_5\}\ \boldsymbol{R}\ \{a_2, a_4, a_5\}$. By the lemma, $\{a_3, a_5\}\ \boldsymbol{R}\ \{a_2, a_5\}$, but this violates the Dominance principle. Next consider case *b)*. By monotonicity, $\{a_1, a_2, a_4\}$ $\boldsymbol{R}\ \{a_1, a_3\}$. By the lemma, $\{a_1, a_4\}\ \boldsymbol{R}\ \{a_1, a_3\}$; again, the Dominance principle is violated.

By a similar argument, it can be proved that

$$\{a_3\}\ \boldsymbol{I}\ \{a_2, a_5\} \qquad (ii)$$

By *(i)*, *(ii)* and the transitivity of $\boldsymbol{R}$ it follows $\{a_2, a_4\}\ \boldsymbol{I}\ \{a_2, a_5\}$; this, however, violates the Dominance principle.

Q.E.D.

The minimum number of alternatives required for Theorem 4 to hold may be reduced to 4 if one is willing to accept one more axiom, which may be called the axiom of *Independence of Irrelevant Alternatives.*

Notice that all conditions defined in section 2 impose restrictions on the ordering of two sets A and B on the basis of the relative ordering of the alternatives in A and B only. The axiom of Independence of Irrelevant Alternatives extends and generalizes this requirement. It requires that the ordering of two sets of alternatives A and B depend only on the relative ordering of the alternatives in A and B.

To state this formally, istead of keeping fixed the ordering R over X we must explicitly consider how the extended order changes as the ordering over X changes. Let $R{:}A$ denote the restriction of the linear order R over X to the set $A \in 2^X$.

DEFINITION 4. An extension rule γ satisfies the condition of *Independence of Irrelevant Alternatives* if, for all $R,R' \in L(X)$ and all $A,B \in 2^X$, $R{:}(A \cup B) = R'{:}(A \cup B)$ implies that $A\mathbf{R}B$ if and only if $A\mathbf{R}'B$, where $\mathbf{R} = \gamma(R)$ and $\mathbf{R}' = \gamma(R')$.

We can now prove the following result.

THEOREM 5. *If* $|X| > 3$, *there is no extension rule satisfying Independence of Irrelevant Alternatives, Kannai and Peleg's monotonicity condition, the Barberà principle, and the Dominance principle.*

Proof. Let $B = \{x,y,z,t\}$. We first show that $xRyRz$ implies $\{y\}\ \mathbf{I}\ \{x,z\}$. Suppose not. If $\{y\}\ \mathbf{P}\ \{x,z\}$, consider an order R such that zRw. Then by monotonicity it follows $\{y,w\}\ \mathbf{R}\ \{x,z,w\}$. By the lemma, $\{y,w\}\ \mathbf{R}\ \{x,w\}$, but this violates the Dominance principle. If, on the other hand, $\{x,z\}\ \mathbf{P}\ \{y\}$, consider an order R' such that $wR'x$. Then by monotonicity it follows $\{w,x,z\}\ \mathbf{R}'\ \{w,y\}$. By the lemma, $\{w,z\}\ \mathbf{R}'\ \{w,y\}$; again, the Dominance principle is violated.

Next, let $B = \{b_1,b_2,b_3,b_4\}$, with b_iRb_{i+1} for $i = 1,2,3$. We have $\{b_2\}\ \mathbf{I}\ \{b_1,b_3\}$ and $\{b_2\}\ \mathbf{I}\ \{b_1,b_4\}$. By the transitivity of $\mathbf{R}$ it follows $\{b_1,b_4\}\ \mathbf{I}\ \{b_1,b_3\}$ which violates the Dominance principle.

Q.E.D.

The following observation is very obviuos: if there are no more than 3 alternatives, there exists an extension rule satisfying Independence of Irrelevant Alternatives, Kannai and Peleg's monotonicity condition, the Barberà principle, and the Dominance principle. Let $C = \{c_1,c_2,c_3\}$, with c_iRc_{i+1} for $i = 1,2$. Let $\{c_1\}\ \mathbf{P}\ \{c_1,c_2\}\ \mathbf{P}\ \{c_1,c_2,c_3\}\ \mathbf{I}\ \{c_1,c_3\}\ \mathbf{P}\ \{c_2\}\ \mathbf{P}\ \{c_2,c_3\}\ \mathbf{P}\ \{c_3\}$. This example satisfies the Gardenfors principle but violates strict monotonicity.

REFERENCES

S. BARBERA, "Manipulation of Social Decision Functions", *Journal of Economic Theory*, **15** (1977), 266-278.

S. BARBERA, C. R. BARRETT AND P. PATTANAIK, "On Some Axioms for Ranking Sets of Alternatives", *Journal of Economic Theory*, **33** (1984), 301-308.

S. BARBERA AND P. PATTANAIK, "Extending an Order on a Set to the Power Set: Some Remarks on Kannai and Peleg's Approach", *Journal of Economic Theory*, **32** (1984), 185-191.

P. GARDENFORS, "Manipulations of Social Choice Functions", *Journal of Economic Theory*, **13** (1976), 217-228.

P. GARDENFORS, "On Definitions of Manipulation of Social Choice Functions", in *Aggregation and Revelation of Preferences* (J. J. Laffont Ed.), North-Holland, Amsterdam, 1979.

Y. KANNAI AND B. PELEG, "A Note on the Extension of an Order on a Set to the Power Set", *Journal of Economic Theory*, **32** (1984), 172-175.

J. S. KELLY, "Strategy-proofness and Social Choice Functions without Single-valuedness", *Econometrica*, **45** (1977), 439-446.