# INTERTEMPORAL COURNOT AND WALRAS EQUILIBRIUM: AN ILLUSTRATION

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# Intertemporal Cournot and Walras Equilibrium: An Illustration

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**ABSTRACT**

In a simple dynamic general equilibrium model, we introduce the concept of an intertemporal Cournot equilibrium. We show that if the number of strategic agents increases without limit, the intertemporal Cournot equilibrium converges to the intertemporal Walras equilibrium *only* when the time horizon for the agent is finite. If the time horizon is infinite, each strategic agent is able to exert nonnegligible market power, no matter how large their number is.

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# 1 Introduction

The equilibrium concept introduced by Augustin Cournot (1838) is one of the most widely used notion of noncooperative equilibrium in modern game theory and its applications in economics. In the partial equilibrium context, it is a combination of outputs, one for each firm, all producing the same good such that no firm can gain from unilateral deviation. It is also known as a Cournot-Nash equilibrium. A strategic agent in the Cournotian framework perceives the influence of his output decisions on the market-clearing price while a Walrasian competitive agent does not. Perfect competition or the Walrasian general equilibrium theory is an analysis of coordination between decentralized economic agents under the hypothesis that prices are treated as exogenous by these agents. It is well-known that, in the static Cournot model, the behaviour of the agents become more competitive (or, have less market power) as their number becomes larger[1].

In this paper, we show that this property of convergence need not hold if the strategic agents live forever. With the help of an example we show that, a Cournot-Nash equilibrium does not converge to the competitive Walrasian equilibrium, even when the economy is replicated unboundedly. However, it is interesting to note that if agents live for a finite time, their strategic effect vanishes as their number becomes very large.

Much work has been done on the relationship between the set of perfectly competitive equilibria and the set of Cournot-Nash equilibria in large economies. Many important contributions related to this topic are gathered in the Symposium issue of the *Journal of Economic Theory* (1980, vol. 22, no. 2) on noncooperative approaches to the theory of perfect competition. Dubey (1980) and Dubey-MasColell-Shubik (1980) are concerned with static noncooperative equilibrium in a general equilibrium framework of "market games". Green (1980) and Radner (1980) consider a dynamic setting and analyze the noncooperative equilibrium of repeated games for partial equilibrium models. MasColell (1980) presents an overview of the results. Most of the work look explicitly at perfect competition as a limit case of various types of market imperfections. It is generally expected that every Cournot equilibrium is approximately competitive.

The questions we attempt to answer are in the tradition of the above-mentioned literature, but the framework we consider is different. The exam-

[1] "S'il y avait 3, 4,...$n$ producteurs en concurrence [...] la valeur de $p$, qui en résulte, diminuerait indéfiniment par l'accroissement indéfini du nombre $n$", Cournot (1838), page 63.

ple we present is a dynamic version of the Gabszewicz-Michel (1992) general equilibrium oligopoly model. There are two types of agents and two goods. Each agent produces one good but consumes both. The type is identified with the good produced by the agent. There is a spot market in every time period in which the agents trade. No trade occurs across time-periods. There are no futures market for goods to be delivered at a later date[2]. Production has a time lag: inputs in the previous period generate outputs at the current period. A part of the current output is used for current consumption and the rest used as input to produce future output. Consumption generate utility and agents maximize the discounted sum of utilities over the relevant time horizon. We analyze two cases: first, in which the agents are finitely-lived and, second, they are infinitely-lived.

A Walrasian agent is a price-taker. In the Cournotian framework, the agent is strategic and takes into account the effect of its quantity decisions on the market clearing prices. We contrast the outcomes in the two cases. Every Cournotian agent perceives a market power, since it controls a share of the market in the good it produces and supplies. There is one such market in every period of its life. We need to solve a dynamic game to find a Cournot equilibrium. We show that consumption of own good per unit of production is higher under Cournot-Nash equilibrium compared to that under the Walrasian equilibrium. The agents manipulate the market price in their favour by restricting the supply, in the former case.

A strategic agent creates a (finite) distortion in every market he participates. When the agents are finitely-lived, the total oligopolistic distortion generated by each agent is finite. As the number of agents becomes very large, the strategic effect of each agent becomes negligible compared to the economy as a whole. On the contrary, an infinitely-lived agent generates an unbounded distortion, no matter how large the economy (or, how small the agent) is. We show that as the economy is replicated, the Cournot-Nash equilibrium does not converge to the Walrasian equilibrium.

Some instances of similar nonconvergence property are Green (1980), Guesnerie-Hart (1985) and Chari-Kehoe (1990). In the presence of mutually strategic threats, Green (1980) shows that a noncooperative dynamic equilibrium do not converge to the price-taking equilibrium in a sequence of replicating dynamic markets. In a static framework, Guesnerie-Hart (1985) find that the asymptotic behaviour of welfare loss is sensitive to whether firms' average cost curves are U-shaped or everywhere declining. Chari-

[2]See Allaz-Vila (1993) for an analysis of strategic existence of futures market.

Kehoe (1990) consider a model in which the governments play the role of strategic agents and perceive the influence of taxation policy on the competitive private agents whose utility they maximize, given the policies of other governments. The welfare loss associated with the noncooperative equilibrium disappears as the number of countries increase only if the taxation policy is not distortionary. Our model differs from Green (1980) because we do not consider trigger strategies. Also, we have a general equilibrium model in which actions in one period affect the feasible set in the next period. In contrast to Guesnerie-Hart (1985) and Chari-Kehoe (1990), we have a dynamic model and the strategic agent is involved in both consumption and production.

It is well known that outcomes of games may be very sensitive to the information structure, we note that we have a model of no uncertainty and full information. See Friedman (1990) and Fudenberg-Tirole (1991) for detailed analyses of the role of information in oligopoly games. As noted by Fudenberg-Tirole (1986), "The state-space perfectness concept of dynamic games is related to perfectness in games in extensive form. While perfectness is typically explained in terms of ruling out threats, it can also (equivalently) be motivated as the natural extension of dynamic programming games with more than one player. Perfectness is just a many-player version of the principle of optimality." We use the technique of Fischer-Mirman (1992) in order to solve for a closed-loop, subgame perfect Cournot-Nash equilibrium in a dynamic programming game.

The paper is arranged in the following way. We present our model in section 2 and define intertemporal "Cournot" and Walras equilibrium. In section 3, we explicitly solve for the equilibria, when agents live for a finite time. By comparing the two equilibria, we note that in the finite horizon case, the strategic behaviour of agents does not affect the investment dynamics. However, the consumption allocations are different. Also, if the number of agents increases without limit, the intertemporal "Cournot" equilibrium converges to the intertemporal Walrasian equilibrium. In section 4, we consider an infinite time horizon. Here, the strategic behaviour affects the dynamics of the model: the "Cournotian" agents consume more and invest less in each period than the Walrasian agents. We demonstrate that "Cournotian" agents maintain a nonnegligible market power even when their number becomes very large. The intertemporal "Cournot" equilibrium does not converge to the intertemporal Walras equilibrium. Concluding thoughts are summarized in the last section.

## 1.1 Notation

The two goods are denoted by subscripts $i$, $k = 1, 2$, $k \neq i$ and time by the subscript $t$ where $t = 0, 1, \cdots, T$, $T \leq \infty$. There are two types of agents. Every agent is identified with a commodity and superscript $ij$ stands for the $j$-th agent of type $i$ where $j = 1, 2, \cdots, n$. We also denote an agent by $l = 1, 2, \cdots, n$ but in this case the convention is $l \neq j$. If the commodity subscript unambiguously identifies the type of the agent, then instead of superscript $ij$ we only use $j$. The superscript $j$ is omitted when agents of one type are identical.

$z_{it}^j$ is the stock of good $i$ held by the agent $j$ in period $t$. Only an agent of type $i$ holds commodity $i$. $y_{it}^j$ is the amount of good $i$ utilized for consumption by the agent $j$ in period $t$ and $(z_{it}^j - y_{it}^j)$ is the amount of good $i$ used for investment by him in period $t$. $(c_{it}^{ij}, c_{kt}^{ij})$ is the bundle of goods consumed by the agent $ij$ in period $t$. The price of commodity $i$ in period $t$ is given by $p_{it}$. $p_t = \frac{p_{2t}}{p_{1t}}$ is the relative price in period $t$.

$y_{it}$ is the $n$-dimensional vector with the typical element $y_{it}^j$ and, $y_{it}^{-j}$, the $(n-1)$-dimensional vector with the typical element $y_{it}^l$ with $l \neq j$. Finally, $\sigma_{it}^{-j} = \sum_{l \neq j}^{n}(y_{it}^l - c_{it}^{il})$ and $\{a_t\}$ for the time sequence of the variable $a$.

# 2 The Model

We consider an economy with two goods (1 and 2) and $2n$ agents. There are two types of agents ($i = 1, 2$). Each agent owns and produces one good. The types are identified with the goods being produced. There are $n$ agents of each type. In the initial period, ($t = 0$), they have an endowment of the same good. In all subsequent periods, $t = 1, 2, \cdots, T$, $T \leq \infty$, the goods are produced.

$z_{it}^j$ denotes the stock of good the $j$th agent of type $i$ holds in period $t$. The $j$th agent of type $i$ is called the agent $ij$. A fraction of this stock is utilized for consumption and the rest is invested. There is no borrowing or lending. Each agent derives utility from the consumption of both goods. Investment is needed to support future consumption. Let $y_{it}^j$ be the amount of good the agent $ij$ uses for consumption in period $t$. A part of this is consumed directly and the remainder is traded for the other good. We assume agents of each type are identical. They have the same initial endowment, preference and technology. The technology of the type $i$ agent is given by:

$$z^j_{i(t+1)} = (z^j_{it} - y^j_{it})^{\alpha^i}, \tag{1}$$

where $\alpha^i > 0$. Let $(c^{ij}_{it}, c^{ij}_{kt})$, $i, k = 1, 2$ and $k \neq i$ be the bundle of goods $i$ and $k$ consumed by $ij$ in period $t$. This yields utility,

$$lnc^{ij}_{it} + lnc^{ij}_{kt}.$$

The agents maximize the discounted sum of utilities subject to the budget constraint and technical feasibility every period. The choice of the agent $ij$ is dictated by the following optimization problem:

$$\begin{aligned}
&\max_{\{c^{ij}_{it}, c^{ij}_{kt}, y^j_{it}\}} \sum_{t=0}^{T} (\delta^i)^t (lnc^{ij}_{it} + lnc^{ij}_{kt}), \\
&\text{subject to} \\
&p_{it}c^{ij}_{it} + p_{kt}c^{ij}_{kt} = p_{it}y^j_{it}, \\
&0 \leq y^j_{it} \leq z^j_{it} = (z^j_{i(t-1)} - y^j_{i(t-1)})^{\alpha^i} \\
&\text{and } c^{ij}_{it} \geq 0, c^{ij}_{kt} \geq 0, \text{ given } z^j_{i0} > 0.
\end{aligned} \tag{P1}$$

Here $p_{it}$ is the price of good $i$ in period $t$ which is taken as given *only* when agents behave competitively and $\delta^i$ in (0,1) is the discount factor.

Notice that the consumption choice $(c^{ij}_{it}, c^{ij}_{kt})$ cannot solve the intertemporal optimization problem unless the $t$th period utility is maximized subject to the budget in that period. Thus, for every period $t$ the following maximization has to be solved:

$$\begin{aligned}
&\max_{c^{ij}_{it}, c^{ij}_{kt}} lnc^{ij}_{it} + lnc^{ij}_{kt}, \\
&\text{subject to} \\
&p_{it}c^{ij}_{it} + p_{kt}c^{ij}_{kt} = p_{it}y^j_{it}, \\
&\text{and } c^{ij}_{it} > 0, c^{ij}_{kt} > 0.
\end{aligned}$$

Or, in other words,

$$\max_{c^{ij}_{it}} lnc^{ij}_{it} + ln[\frac{p_{it}}{p_{kt}}(y^j_{it} - c^{ij}_{it})]. \tag{2}$$

We analyze the behaviour of the agents in two different cases. In the first case, the agents behave as price-takers and we call them Walrasian agents. In the second case, the agents perceive the influence of their supply on the market-clearing price and we call them "Cournotian" agents. Since the utility function is intertemporally separable and we do not allow trade across different time periods, the equilibrium price depends only on the demand and supply in that period. In subsections 2.1 and 2.2, we explain in detail the behaviour of a Walrasian and a "Cournotian" agent and define the corresponding equilibria.

## 2.1 The Walrasian Behavior

When the agents behave as price-takers, in the "Walrasian" tradition, the solution to the $t$th period utility maximization problem (2) is,

$$c_{it}^{ij} = \frac{y_{it}^j}{2}, \tag{3}$$

and using the $t$th period budget constraint,

$$c_{kt}^{ij} = \frac{y_{it}^j}{2}\frac{p_{it}}{p_{kt}}. \tag{4}$$

The agent $ij$'s indirect utility function is,

$$ln(\frac{y_{it}^j}{2}) + ln(\frac{y_{it}^j p_{it}}{2p_{kt}}) = 2ln(\frac{y_{it}^j}{2}) + ln(\frac{p_{it}}{2p_{kt}}). \tag{5}$$

We may rewrite the problem (P1) as,

$$\begin{aligned} &\max_{\{y_{it}^j\}} \sum_{t=0}^{T} (\delta^i)^t [ln(\frac{y_{it}^j}{2}) + ln(\frac{y_{it}^j p_{it}}{2p_{kt}})] \\ &\text{subject to} \\ &0 \leq y_{it}^j \leq z_{it}^j = (z_{i(t-1)}^j - y_{i(t-1)}^j)^{\alpha^i} \text{ and given } z_{i0}^j > 0. \end{aligned} \tag{6}$$

The relative price, $p_t \equiv \frac{p_{2t}}{p_{1t}}$, that clears the market in period $t$ is,

$$p_t = \frac{\sum_{j=1}^n y_{1t}^j}{\sum_{j=1}^n y_{2t}^j}. \tag{7}$$

An intertemporal Walras equilibrium (IWE) is defined as a competitive equilibrium of the $2n$-agent economy in which each agent maximizes the discounted sum of utilities and markets clear.

**Definition 1** *An intertemporal Walras equilibrium is a (2n+1)-tuple of sequences* $(\{y_{1t}^{1\star}(z_{1t}^1)\}, \{y_{1t}^{2\star}(z_{1t}^2)\}, \cdots, \{y_{1t}^{n\star}(z_{1t}^n)\}, \{y_{2t}^{1\star}(z_{2t}^1)\}, \{y_{2t}^{2\star}(z_{2t}^2)\}, \cdots, \{y_{2t}^{n\star}(z_{2t}^n)\}; \{p_t^\star\})$ *such that for all* $i, j$, $i = 1, 2$ *and* $j = 1, 2, \cdots, n$, $\{y_{it}^{j\star}\}$ *solves the problem (6) given* $\{p_t^\star\}$ *and*

$$p_t^\star = \frac{\sum_{j=1}^n y_{1t}^{j\star}}{\sum_{j=1}^n y_{2t}^{j\star}}.$$

In section 3.1, we solve for an intertemporal Walras equilibrium (IWE) for the case in which time horizon is finite and in section 4.1, for the case in which time horizon is infinite.

## 2.2 The "Cournotian" Behaviour

Should the agents behave strategically they do not act as price takers but perceive the influence of their individual supply on the equilibrium exchange rate of goods. If this is the case, each agent will try to improve his term of trade by reducing the supply of the good he produces. Following the oligopoly equilibrium concept proposed by Codognato-Gabszewicz (1991) and Gabszewicz-Michel (1992) we suppose that, in each period $t$, the strategic agent $ij$ brings to the market place an amount $(y_{it}^j - c_{it}^{ij})$ of good $i$, while consuming directly $c_{it}^{ij}$. Total supply of good $i$ is thus given by $\sum_{j=1}^n (y_{1t}^j - c_{1t}^{1j})$. Moreover, if we assume that agents do not buy back from the market the good they produce[3], they will spend their total income in the other good. The aggregate demand for good $k$ is $p_{it}[\sum_{j=1}^n (y_{it}^j - c_{2t}^{ij})]/p_{kt}, i, k = 1, 2, i \neq k$. Thus, the price system which clears the market in period $t$ must satisfy,

$$p_t = \frac{\sum_{j=1}^n (y_{1t}^j - c_{1t}^{1j})}{\sum_{j=1}^n (y_{2t}^j - c_{2t}^{2j})}. \tag{8}$$

By substituting equation (8) in problem (2), the one-period utility maximization reduces to,

$$\max_{c_{it}^{ij} \in [0, y_{it}^j]} \; lnc_{it}^{ij} + ln[\sum_{j=1}^n (y_t^{kj} - c_{kt}^{kj})] + ln(y_{it}^j - c_{it}^{ij}) - ln[\sum_{j=1}^n (y_{it}^j - c_{it}^{ij})]. \tag{9}$$

[3]It can, nevertheless, be shown easily, with an argument similar to the one developed by Gabszewicz-Michel (1992), that even if we allow this possibility the final outcome would be the same.

From the first-order condition of problem (9), we have,

$$\frac{y_{it}^j - c_{it}^{ij} + \sigma_{it}^{-j}}{c_{it}^{ij}} = \frac{\sigma_{it}^{-j}}{y_{it}^j - c_{it}^{ij}}, \tag{10}$$

Or,

$$c_{it}^{ij} = y_{it}^j + \sigma_{it}^{-j} - \sqrt{\sigma_{it}^{-j}(\sigma_{it}^{-j} + y_{it}^j)}. \tag{11}$$

where $\sigma_{it}^{-j} = \sum_{l \neq j}^{n}(y_{it}^l - c_{it}^{il})$. We note that once $y_{it}^j$ has been chosen, the optimal consumption decision of an agent $ij$, $(c_{it}^{ij}, c_{kt}^{ij})$, are uniquely determined.

By substituting equation (11) into the objective function of the problem (2) and simplifying, we express the maximum possible $t$-th period utility that can be derived by the agent $ij$, $w^{ij}(y_{it}^j, \sigma_{it}^{-j}, y_{kt}, c_{kt}^k)$, as a function of $y_{it}^j$ and the choice of the other agents. Here $y_{kt}$ is the $n$-dimensional vector with the typical element $y_{kt}^j$, $c_{kt}^k$is the $n$-dimensional vector with the typical element $c_{kt}^{kj}$. $y_{kt}$ and $c_{kt}^k$ contain the choice of all agents of type $k$ and $\sigma_{it}^{-j}$ contain the choice of all agents of type $i$, except $ij$.

$$w^{ij}(y_{it}^j, \sigma_{it}^{-j}, y_{kt}^j, c_{kt}^{kj}) = ln[\sum_{j=1}^{n}(y_{kt}^j - c_{kt}^{kj})] + 2ln[\sqrt{\sigma_{it}^{-j} + y_{it}^j} - \sqrt{\sigma_{it}^{-j}}]. \tag{12}$$

The same argument can be made for every agent. If all agents make their consumption choices optimally, then $c_{it}^{il}$ is a function of $y_{it}^l$ and $c_{kt}^{kj}$ is a function of $y_{kt}^j$ in a way similar to equation (11). Thus, we may write[4],

$$w^{ij}(y_{it}^j, \sigma_{it}^{-j}, y_{kt}, c_{kt}^k) = w^{ij}(y_{1t}^1, \cdots, y_{1t}^n, y_{2t}^1, \cdots, y_{2t}^n).$$

This allows us to reduce the optimization problem (P1) to the following:

$$\begin{aligned} &\max_{\{y_{it}^j\}} \sum_{t=0}^{T} (\delta^i)^t w^{ij}(y_{1t}^1, \cdots, y_{1t}^n, y_{2t}^1, \cdots, y_{2t}^n) \\ &\text{subject to} \\ &0 \leq y_{it}^j \leq z_{it}^j = (z_{i(t-1)}^j - y_{i(t-1)}^j)^{\alpha^i} \text{ and given } z_{i0}^j > 0. \end{aligned} \tag{13}$$

[4]Because we are only considering cases in which agents make their choice optimally, we do not worry about what happens out of equilibrium.

The strategy of the agent $j$ of type $i$ is a sequence $\{y_{it}^j(z_{it}^j)\}$ for $t = 0, 1, \cdots, T$, and the payoff,

$$\Pi^{ij}(y_{1t}^1, \cdots, y_{1t}^n, y_{2t}^1, \cdots, y_{2t}^n) = \sum_{t=0}^{T} (\delta^i)^t w^{ij}(y_{1t}^1, \cdots, y_{1t}^n, y_{2t}^1, \cdots, y_{2t}^n).$$

With these strategies and payoffs, we have a $2n$-agent dynamic game. We define an intertemporal "Cournot" equilibrium as a Nash equilibrium of this game.

**Definition 2** *An intertemporal Cournot equilibrium is a 2n-tuple of strategies* $(\{y_{1t}^{1\star}(z_{1t}^1)\}, \{y_{1t}^{2\star}(z_{1t}^2)\}, \cdots, \{y_{1t}^{n\star}(z_{1t}^n)\}, \{y_{2t}^{1\star}(z_{2t}^1)\}, \{y_{2t}^{2\star}(z_{2t}^2)\}, \cdots, \{y_{2t}^{n\star}(z_{2t}^n)\})$ *such that for all* $i, j, i = 1, 2$ *and* $j = 1, 2, \cdots, n$,

$$\Pi^{1j}(y_{1t}^{1\star}(z_{1t}^1), \cdots, y_{1t}^{j\star}(z_{1t}^j), \cdots, y_{1t}^{n\star}(z_{1t}^n), y_{2t}^{1\star}(z_{2t}^1), \cdots, y_{1t}^{j\star}(z_{1t}^j), \cdots, y_{2t}^{n\star}(z_{2t}^n)) \geq$$
$$\Pi^{1j}(y_{1t}^{1\star}(z_{1t}^1), \cdots, y_{1t}^{j}, \cdots, y_{1t}^{n\star}(z_{1t}^n), y_{2t}^{1\star}(z_{2t}^1), \cdots, y_{2t}^{j\star}(z_{2t}^j), \cdots, y_{2t}^{n\star}(z_{2t}^n)),$$

*and*

$$\Pi^{2j}(y_{1t}^{1\star}(z_{1t}^1), \cdots, y_{1t}^{j\star}(z_{1t}^j), \cdots, y_{1t}^{n\star}(z_{1t}^n), y_{2t}^{1\star}(z_{2t}^1), \cdots, y_{1t}^{j\star}(z_{1t}^j), \cdots, y_{2t}^{n\star}(z_{2t}^n)) \geq$$
$$\Pi^{2j}(y_{1t}^{1\star}(z_{1t}^1), \cdots, y_{1t}^{j\star}(z_{1t}^j), \cdots, y_{1t}^{n\star}(z_{1t}^n), y_{2t}^{1\star}(z_{2t}^1), \cdots, y_{2t}^{j}, \cdots, y_{2t}^{n\star}(z_{2t}^n)),$$

*for all* $y_{1t}^j$ *and* $y_{2t}^j$ *feasible.*

In section 3.2, we compute a closed-loop intertemporal "Cournot" equilibrium (ICE) for the case of finite-lived agents and in section 4.2, for the case in which agents are infinitely-lived.

# 3 The Finite Horizon Case

## 3.1 The Intertemporal Walras Equilibrium

In this section, we find an intertemporal Walras equilibrium, when the time horizon is finite and we show that equilibrium is unique. We need to solve problem (6) for $T < \infty$ to find the IWE. Since the agent treats prices, $\{p_{it}, p_{kt}\}$, for $t = 1, 2, \cdots, T$, as given, problem (6) is a familiar one-sector optimal growth problem with a unique solution. In the terminal period, $T$,the agent $ij$ has incentive to save nothing and consume the $T$-th period stock entirely. Thus,

$$y_{iT}^{j\star} = z_{iT}^{j}.$$

For all other time periods, $t < T$, we have an interior solution given by the following first-order condition which equates the marginal benefit of using an additional unit of own good towards consumption in period $t$ and the marginal benefit of consumption from the additional stock in period $(t+1)$ resulting from that unit of investment in period $t$:

$$\frac{1}{y_{it}^{j\star}} = \frac{\delta^i \alpha^i z_{i(t+1)}^{j}}{(z_{it}^{j} - y_{it}^{j\star}) y_{i(t+1)}^{j\star}}.$$

On rearrangement, we get,

$$y_{i(t+1)}^{j\star} = \frac{\delta^i \alpha^i y_{it}^{j\star} z_{i(t+1)}^{j}}{z_{it}^{j} - y_{it}^{j\star}}. \tag{14}$$

Consequently, from equations (3) and (4), the optimal consumption choices are,

$$c_{it}^{ij\star} = \frac{y_{it}^{j\star}}{2}, \qquad c_{kt}^{ij\star} = \frac{y_{it}^{j\star} p_{it}^{\star}}{2p_{kt}^{\star}}. \tag{15}$$

The equilibrium price ratio can be found by substituting $y_{1t}^{j\star}$ and $y_{2t}^{j\star}$ in equation (7). Since $y_{it}^{j\star}$ is uniquely determined, Walras equilibrium is also unique.

### 3.2 An Intertemporal Cournot Equilibrium

In this section, we compute an intertemporal "Cournot" equilibrium when the time horizon is finite and compare it with the intertemporal Walrasian equilibrium derived in the previous section. In order to derive an ICE, we solve problem (13) for $T < \infty$.

In the terminal period, T, there is no incentive to save or invest and the agent $ij$ consumes the entire stock,

$$y_{iT}^{j\star} = z_{iT}^{j}.$$

For $t < T$, agent $ij$ equates the marginal benefit of using additional unit $y_{it}^{j}$ towards consumption in period $t$ and the marginal benefit of investing it:

$$\frac{1}{(\sigma_{it}^{-j}+y_{it}^{j\star})-\sqrt{\sigma_{it}^{-j}(\sigma_{it}^{-j}+y_{it}^{j\star})}}=$$

$$\frac{\delta^i\alpha^i z_{i(t+1)}^j}{(z_{it}^j-y_{it}^{j\star})[\sigma_{i(t+1)}^{-j}+y_{i(t+1)}^{j\star}-\sqrt{\sigma_{i(t+1)}^{-j}(\sigma_{i(t+1)}^{-j}+y_{i(t+1)}^{j\star})}]}.$$

Since agents of type $i$ are identical we have,

$$\sigma_{it}^{-j}=(n-1)(y_{it}^{j\star}-c_{it}^{j\star}). \tag{16}$$

¿From equations (16) and (11) we get,

$$\sigma_{it}^{-j}=\frac{(n-1)^2 y_{it}^{j\star}}{2n-1},$$

and the first order condition reduces to,

$$\frac{2n-1}{ny_{it}^{j\star}}=\frac{2n-1}{ny_{i(t+1)}^{j\star}}\frac{\delta^i\alpha^i z_{i(t+1)}^j}{z_{it}^j-y_{it}^{j\star}},$$

or,

$$y_{i(t+1)}^{j\star}=\frac{\delta^i\alpha^i y_{it}^{j\star} z_{i(t+1)}^j}{z_{it}^j-y_{it}^{j\star}}. \tag{17}$$

The consumption choices are,

$$c_{it}^{ij\star}=\frac{ny_{it}^{j\star}}{2n-1},\qquad c_{kt}^{ij\star}=\frac{ny_{it}^{j\star}p_{it}^{\star}}{(2n-1)p_{kt}^{\star}}, \tag{18}$$

and using equation (8) we get the equilibrium price.

By comparing equations (14), (15) with (17) and (18) we note that the investment decisions are the same in the "Cournotian" and the Walrasian framework while the consumption decisions are not. Proposition 1 states that as the number of strategic agents become very large, their consumption choice approach the Walrasian allocations.

**Proposition 1** *If time horizon is finite and the number of strategic agents, $n$, goes to infinity,the consumption and investment choice that correspond to the intertemporal "Cournot" equilibrium converges to the consumption and investment allocation in the intertemporal Walrasian equilibrium, respectively.*

**Proof:** Take the limit as $n \to \infty$ in equation (18). □

In the finite horizon set-up, the strategic behaviour of the agents does not affect the dynamics of the investment-saving decision. Oligopolistic distortion is only generated by the consumption allocations in every period and becomes negligible as the number of agents becomes very large. The same result is demonstrated by Gabszewicz-Michel (1992) in a static framework.

# 4 The Infinite Horizon Case

## 4.1 The Intertemporal Walras Equilibrium

In this section, we find an intertemporal Walras equilibrium when the time horizon is infinite and we show that it is unique. As in section 3.1, we solve problem (6). The agents treat prices as exogenously given. To avoid the case in which every solution is trivially optimal, we assume the relative price to be bounded, in particular, $\sum_{t=0}^{\infty}(\delta^i)^t ln(\frac{p_{it}}{p_{ik}})$ is finite. Note that the dynamic programming problem (6) is not stationary if the sequence of relative price is not a constant sequence. However, for our simple linear-logarithmic economy, we may solve the following equivalent stationary dynamic optimization:

$$\begin{aligned} &\max_{\{y_{it}^j\}} \sum_{t=0}^{\infty}(\delta^i)^t[2ln(\frac{y_{it}^j}{2})] \\ &\text{subject to} \\ &0 \leq y_{it}^j \leq z_{it}^j = (z_{i(t-1)}^j - y_{i(t-1)}^j)^{\alpha^i} \text{ and given } z_{i0}^j > 0. \end{aligned} \tag{19}$$

This is a well-known one-sector optimal growth model in the tradition of Koopmans (1963) and Cass (1965) in which the unique solution is,

$$y_{it}^{j\star} = (1 - \delta^i\alpha^i)z_{it}^j, \qquad \text{for all } t. \tag{20}$$

From equations (3) and (4), the optimal consumption choices are,

$$c_{it}^{ij\star} = \frac{y_{it}^{j\star}}{2}, \qquad c_{kt}^{ij\star} = \frac{y_{it}^{j\star}p_{it}^\star}{2p_{kt}^\star}. \tag{21}$$

Since the agents of each type are identical, we omit the superscript $j$ and from equation (7), we derive the unique relative price, $p_t^\star$, that clears the market in period $t$,

$$p_t^\star = \frac{(1-\delta^1\alpha^1)z_{1t}}{(1-\delta^2\alpha^2)z_{2t}}. \tag{22}$$

Equations (1) and (20) summarize the relation between stocks of two consecutive time-periods. We have,

$$z_{i(t+1)} = (z_{it} - y_{it}^\star)^{\alpha^i},$$

That is,

$$z_{i(t+1)} = (\delta^i \alpha^i z_{it})^{\alpha^i}. \tag{23}$$

Taking logarithms on both sides of equation (23) we get,

$$lnz_{i(t+1)} = \alpha^i ln\delta^i\alpha^i + \alpha^i lnz_{it}.$$

The solution for this difference equation [see, for example, Baumol (1970)] is,

$$lnz_{it} = \frac{\alpha^i}{1-\alpha^i}ln(\delta^i\alpha^i) + [lnz_{i0} - \frac{\alpha^i}{1-\alpha^i}ln(\delta^i\alpha^i)](\alpha^i)^t, \tag{24}$$

which gives the time-path of stock of goods in the Walrasian economy. This implies that for $\alpha^i < 1$, the stocks converge to a steady-state, in the long run. Consequently, the relative price also converges. The steady-state stocks and prices are,

$$\lim_{t\to\infty} z_{it} = (\delta^i\alpha^i)^{\frac{\alpha^i}{1-\alpha^i}}, \lim_{t\to\infty} p_t^\star = \frac{(1-\delta^1\alpha^1)(\delta^1\alpha^1)^{\frac{\alpha^1}{1-\alpha^1}}}{(1-\delta^2\alpha^2)(\delta^2\alpha^2)^{\frac{\alpha^2}{1-\alpha^2}}}. \tag{25}$$

## 4.2 An Intertemporal Cournot Equilibrium

In this section, we solve for an intertemporal "Cournot" equilibrium for the infinite horizon case. An ICE is a Nash equilibrium of the dynamic game described in section 2.2. We adapt the technique of Fischer-Mirman (1992) to compute a subgame perfect Nash equilibrium of this game.

By extrapolating from the form of the value function of finite period problem, we assume that the value function for the agent $ij$, $W^{ij}(z_i^1, \cdots, z_i^n; z_k^1, \cdots, z_k^n)$ is

$$W^{ij}(z_i^1,\cdots,z_i^n;z_k^1,\cdots,z_k^n)=\sum_{j=1}^{n}(A_i^j lnz_i^j)+\sum_{j=1}^{n}(B_k^j lnz_k^j)+D^{ij} \qquad (26)$$

where $A_i^j, B_k^j$, and $D^{ij}$ are constants for all $i,k = 1,2, k \neq i$ and $j = 1,2,\cdots,n$. By applying Bellman's principle, we have the functional equation,

$$W^{ij}(z_i^1,\cdots,z_i^n;z_k^1,\cdots,z_k^n)=\max_{0\le y_i^j\le z_i^j}\{w^{ij}(y_i^1,\cdots,y_i^n,y_k^1,\cdots,y_k^n)+$$
$$\delta^i W^{ij}[(z_i^1-y_i^1)^{\alpha^i},\cdots,(z_i^n-y_i^n)^{\alpha^i};(z_k^1-y_k^1)^{\alpha_k},\cdots,(z_k^n-y_k^n)^{\alpha_k}]\} (27)$$

Using equation (26), the maximand of the functional equation (27) can be written as,

$$\max_{0\le y_i^j\le z_i^j}\{w^{ij}(y_i^j,y_i^{-j},y_k)+\delta^i\alpha^i\sum_{j=1}^{n}[A_i^j ln(z_i^j-y_i^j)]+\delta^i\alpha_k\sum_{j=1}^{n}[B_k^j ln(z_k^j-y_k^j)]+\delta_i D^{ij}\}, \qquad (28)$$

which has the following first-order condition,

$$\frac{\sqrt{\sigma_i^{-j}+y_i^{j\star}}}{\sqrt{\sigma_i^{-j}+y_i^{j\star}}-\sqrt{\sigma_i^{-j}}}=\frac{\delta^i\alpha^i A_i^j}{z_i^j-y_i^{j\star}}. \qquad (29)$$

Since all the agents of type $i$ are identical, from equation (16) we have,

$$\sigma_i^{-j}=\frac{(n-1)^2 y_i^{j\star}}{2n-1}, \qquad (30)$$

which along with the first-order condition (29) suggest an equilibrium in linear strategies. That is, a strategy of using a constant proportion of stock every period for consumption is an equilibrium. In fact, it is easy to check that the strategies of the form $y_i^j = \gamma_i^j z_i^j$ where $0 \le \gamma_i^j \le 1$ and the value function (26) satisfy the Bellman's equation (27) for all $ij$, if

$$A_i^j=\frac{1}{1-\delta^i\alpha^i}, \qquad (31)$$
$$B_k^j=\frac{1}{1-\delta^i\alpha^k}. \qquad (32)$$

Equations (29), (30) and (31) imply an equilibrium strategy,

$$y_i^{j\star} = \frac{(2n-1)(1-\delta^i\alpha^i)z_i^j}{(2n-1)(1-\delta^i\alpha^i)+n\delta^i\alpha^i}. \tag{33}$$

Now, from equation (11), we find the consumption choices:

$$c_{it}^{ij\star} = \frac{ny_{it}^{j\star}}{2n-1}, \qquad c_{kt}^{ij\star} = \frac{ny_{it}^{j\star}p_{it}^{\star}}{(2n-1)p_{kt}^{\star}}. \tag{34}$$

The market-clearing price is,

$$p_t^{\star} = \frac{(1-\delta^1\alpha^1)}{(1-\delta^2\alpha^2)}\frac{[(2n-1)(1-\delta^2\alpha^2)+n\delta^2\alpha^2]}{[(2n-1)(1-\delta^1\alpha^1)+n\delta^1\alpha^1]}\frac{z_{1t}}{z_{2t}}, \tag{35}$$

from equations (8), (33) and (34). We may omit the superscript $j$ since all agents of each type are identical. The stocks for agents of type $i$ in two consecutive periods are related as,

$$z_{i(t+1)} = [z_{it} - \frac{(2n-1)(1-\delta^i\alpha^i)z_i^j}{(2n-1)(1-\delta^i\alpha^i)+n\delta^i\alpha^i}]^{\alpha^i},$$

which gives[5] the time-path of stocks,

$$\begin{aligned} ln z_{it}^{\star} &= \frac{\alpha^i}{1-\alpha^i} ln(\frac{n\delta^i\alpha^i}{(2n-1)(1-\delta^i\alpha^i)+n\delta^i\alpha^i}) + \\ &[ln z_{i0} - \frac{\alpha^i}{1-\alpha^i} ln(\frac{n\delta^i\alpha^i}{(2n-1)(1-\delta^i\alpha^i)+n\delta^i\alpha^i})](\alpha^i)^t. \end{aligned} \tag{36}$$

It is easy to see that, for $\alpha^i < 1$, the output of the $i$th good converges to a steady state in the long run. The longrun steady-state output of good $i$ is,

$$\lim_{t\to\infty} z_{it}^{\star} = [\frac{n\delta^i\alpha^i}{(2n-1)(1-\delta^i\alpha^i)+n\delta^i\alpha^i}]^{\frac{\alpha^i}{1-\alpha^i}}. \tag{37}$$

When the stocks reach a steady-state, it is clear from equation (35), that the equilibrium price will also be stationary. It is given by,

$$\begin{aligned} \lim_{t\to\infty} p_t^{\star} &= \frac{(1-\delta^1\alpha^1)}{(1-\delta^2\alpha^2)}\frac{[(2n-1)(1-\delta^2\alpha^2)+n\delta^2\alpha^2]}{[(2n-1)(1-\delta^1\alpha^1)+n\delta^1\alpha^1]} \\ &[\frac{n\delta^1\alpha^1}{(2n-1)(1-\delta^1\alpha^1)+n\delta^1\alpha^1}]^{\frac{\alpha^1}{1-\alpha^1}}[\frac{n\delta^2\alpha^2}{(2n-1)(1-\delta^2\alpha^2)+n\delta^2\alpha^2}]^{\frac{-\alpha^2}{1-\alpha^2}}. \end{aligned} \tag{38}$$

[5] By solving the difference equation, see, for example, Baumol (1970).

By comparing equations (20) and (33), we note that the agents consume more and invests less per unit of stock every period, when they behave strategically in the "Cournotian" framework than in the Walrasian model. Proposition 2 summarizes our result.

**Proposition 2** *The proportion of stock utilized for consumption, in each period, is higher in the intertemporal "Cournot" equilibrium than the corresponding ratio in the intertemporal Walras equilibrium.*

**Proof:** Follows directly from equations (20) and (33) and noting that $n > 1$.□

The intuition for the result is the following: when agents behave strategically they have interest in restricting their supply in the market in the current and all the future periods and they do so by consuming relatively more and investing relatively less compared to the Walrasian agent.

We compare the equilibria derived in section 4.1 and 4.2. We replace the superscript $\star$ by the superscripts Walras and Cournot, respectively, for the equilibrium price and quantities derived in sections 4.1 and 4.2. Equations (22) and (35) allow us to compare the steady-state market-clearing prices in the "Cournotian" and the Walrasian case. We have,

$$\lim_{t\to\infty} p_t^{Cournot} = \lim_{t\to\infty} p_t^{Walras} \frac{[(2n-1)(1-\delta^2\alpha^2)+n\delta^2\alpha^2]}{[(2n-1)(1-\delta^1\alpha^1)+n\delta^1\alpha^1]} \tag{39}$$
$$\left(\frac{n}{(2n-1)(1-\delta^1\alpha^1)+n\delta^1\alpha^1}\right)^{\frac{\alpha^1}{1-\alpha^1}}\left(\frac{n}{(2n-1)(1-\delta^2\alpha^2)+n\delta^2\alpha^2}\right)^{\frac{-\alpha^2}{1-\alpha^2}}.$$

Notice that if the agents have different intertemporal time-preference but are equally productive, i.e., $\alpha^1 = \alpha^2$ and $\delta^1 \neq \delta^2$, the steady state "Cournotian" and Walrasian prices are related as follows:

$$\lim_{t\to\infty} p_t^{Cournot} < \lim_{t\to\infty} p_t^{Walras}, \tag{40}$$

whenever, $\delta^1 < \delta^2$, and the reverse inequality holds for $\delta^1 > \delta^2$. A stronger preference for the present amplifies the strategic distortion, since, the inequality (39) is equivalent to,

$$\lim_{t\to\infty} \frac{y_{2t}^{Cournot}}{y_{2t}^{Walras}} < \lim_{t\to\infty} \frac{y_{1t}^{Cournot}}{y_{1t}^{Walras}}.$$

The strategic behaviour of the Cournotian agents is reflected even in the long-run, in the inequality of the prices and in the steady-state outputs[6]. We conclude by observing that even when the number of agents becomes very large the differences in the investment decisions in an ICE and the IWE does not become negligible[7] and neither does the steady-state "Cournotian" price converge to the Walrasian price[8]. This allows us to state,

**Proposition 3** *If the time horizon is infinite and the number of agents, $n$, increases without limit, the consumption, investment and the market-clearing prices in the intertemporal "Cournot" equilibrium do not converge to the intertemporal Walras equilibrium consumption, investment and price.*

A Cournotian agent creates inefficiency in the model by manipulating prices every period. In the finite horizon case, the total amount of inefficiency generated by each agent is finite which becomes negligible as $n$ tends to infinity. An agent $ij$ controls the supply of good $i$ and the proportion of own stock of good which is consumed directly is the strategic variable. The difference between the corresponding ratios in the case of a Walrasian and a Cournotian agent is a measure of the strategic effect generated by the agent of the latter type. By comparing the results summarized by equations (14), (15) with those of (17) and (18), it is easy to see that Cournot-Nash consumption per unit of output of own good converges to the Walrasian ratio as the number of strategic agents becomes very large. Or, in other words, when the agents become too small to exert their strategic influence in the market they behave almost like price-takers.

In the infinite horizon case, however, this is not true because each agent, no matter how small, generates an unbounded amount of inefficiency since there are infinite number of periods in which an agent comes to the market. From equations (33) and (34), own consumption per unit of output is,

$$\frac{(1-\delta^i\alpha^i)}{(2-\frac{1}{n})((1-\delta^i\alpha^i)+\delta^i\alpha^i}.$$

This is larger than the corresponding ratio in the Walrasian equilibrium [equations (20) and (21)] even when $n$ becomes very large. In fact, in the limit, the difference is,

[6]This can be seen by comparing equations (24) and (36), except in the uninteresting case in which we have $\delta^1 = \delta^2$, $\alpha^1 = \alpha^2$.

[7]In equation (33), take limit as n tends to infinity, and compare the investment per unit of stock in the "Cournotian" case to the same ratio in the Walrasian case given by equation (20).

[8]Take the limit as n tends to infinity in equation (37).

$$\frac{\delta^i \alpha^i}{2},$$

which can be interpreted as the magnitude of strategic market manipulation by an infinitesimal "Cournotian" agent, every time he participates in the process of trading. This is a constant fraction independent of $t$, which summed over an infinite time horizon is unboundedly large.

# 5 Conclusions

In this paper, we present a dynamic version of the general equilibrium oligopoly model introduced by Codognato-Gabszewicz (1991) and generalized by Gabszewicz-Michel (1992). By means of an example, we have shown that the equilibrium allocations and the effects of replication are qualitatively different in the finite and infinite horizon economies. The crucial difference is that in the last period of his life an agent does not invest and in every other period he does. Of course, there is no last period, if the agent lives forever. The linear-logarithmic functional forms of preferences and technologies are chosen to facilitate computation of a closed-loop subgame perfect equilibrium. In general, it is hard to characterize the properties of closed-loop solutions of a differential game. Perhaps, one could argue that, the specific functional forms chosen also simplify the strategic interplay between the agents, by introducing "linearity" in equilibrium policies. Nonetheless, the strategic influence of a very small economic agent cannot be neglected, in the case in which the agent is infinitely-lived.

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