CHOOSING ROLES IN A DUOPOLY FOR ENDOGENOUSLY DIFFERENTIATED PRODUCTS

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Abstract

The choice of the roles by firms in a differentiated duopoly is analysed, under both the assumption of full and non full market coverage. Under the first, it turns out that, due to the endogeneity of product differentiation, both firms would prefer to be price leader, contrarily to the results obtained by previous literature. Under the latter, it is possible to analyse both price and quantity competition. The consequences in terms of social welfare are also outlined.

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1. Introduction

The choice of the respective roles by duopolists in a market for homogeneous goods has been analysed by Ono (1978, 1980, 1982), Gal-Or (1985) and Dowrick (1986), showing that, under quantity competition, *SL>N>SF*, while under price competition, *SF>SL>N*, where *SL*, *SF* and *N* stand for Stackelberg leader, Stackelberg follower and simultaneous Nash equilibrium, respectively. According to Bulow, Geanakoplos and Klemperer (1985), we can also say that both firms would prefer to be the leader (follower) if goods are strategic substitutes (complements) in the relevant strategic variable, or if the slope of the reaction functions is negative (positive) in the relevant space. These results have been confirmed by Gal-Or (1985) and Boyer and Moreaux (1987) under exogenuos product differentiation.

Our purpose is to investigate the same issue under endogenous product differentiation within the address approach, i.e., either horizontal or vertical product differentiation under price competition. Under the assumption of non-full market coverage, it is possible to investigate firms' preferences under both price and quantity competition. Our results point to the conclusion that, if firms can choose the degree of differentiation before competing on the market, then, under price competition, *SL>SF>N*, so that the sequence according to which the outcomes of the sequential game are ordered is reversed.

2. The horizontal model

We adopt the quadratic horizontal differentiation model described by D'Aspremont et al. (1979). The duopolists sell the same physical good. Consumers are uniformly distributed along a linear city whose length can be normalized to 1 without loss of generality, and their total density is 1. They have unit demands, and consumption yields a positive constant surplus s; each consumer buys if and only if the following condition is met:

$$U = s - tx^2 - p_i \geq 0, \quad 0 \leq x \leq 1, \quad t > 0, \quad i = 1,2; \qquad (1$$

where tx^2 is the transportation cost incurred by a consumer living at distance x from store i, and p_i is the price of good i. We assume that s is large enough for total demand to be always equal to 1. Firm 1 is located at a, while firm 2 is located at $1-b \geq a$, with $a,b \in R$. The demand functions are, respectively:[1]

$$y_1 = a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \qquad (2$$

$$if \quad 0 \leq a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \leq 1;$$

$$y_1 = 0 \qquad (2'$$

$$if \quad a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \leq 0;$$

$$y_1 = 1 \qquad (2''$$

$$if \quad a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \geq 1;$$

$$y_2 = 1 - y_1 = b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \qquad (3$$

1. The following specification ensures that the first and second order conditions are sufficient for a global maximum. See Lambertini (1993, Appendix).

$$if \quad 0 \leq b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \leq 1;$$

$$y_2 = 0 \tag{3'}$$

$$if \quad b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \leq 0;$$

$$y_2 = 1 \tag{3''}$$

$$if \quad b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \geq 1.$$

Clearly, for a=1-b, i.e., when sellers locates at the same point, the demand functions are not determined and profits are nil as a consequence of the Bertrand paradox. Since unit costs are constant and normalized to zero, the two objective functions are then

$$\pi_1 = p_1 y_1; \tag{4}$$

$$\pi_2 = p_2 y_2. \tag{5}$$

2.1. Price Leadership

Assume firm 1 acts as a Stackelberg leader in the price stage, i.e., maximizes profits under the constraint given by the rival's reaction function. Firms play simultaneously in the location stage. The outcome of such a game is summarized by the following:

Proposition 1: If the price stage is played sequentially, while the location stage is played simultaneously, then

i) the degree of differentiation at equilibrium is greater than that which is obtained when both stages are played simultaneously;

ii) both firms sell the same quantity;

iii) the leader's profit is greater than the follower's.

Proof.

The objective of seller 1 is:

$$\max_{p_1} \quad \pi_1 = p_1 y_1 \tag{6}$$

$$s.t. \quad R_2(p_1) = p_2 = \frac{p_1 + t - 2at + a^2t - b^2t}{2}. \tag{7}$$

The first order condition is:

$$\frac{\delta\pi_1}{\delta p_1} = \frac{2p_1 - 3t + 2at + a^2t + 4bt - b^2t}{4t(a+b-1)} = 0, \tag{8}$$

which yields the following equilibrium prices:

$$p_1^* = \frac{t}{2}(a-b+3)(1-a-b); \tag{9}$$

$$p_2^* = \frac{t}{4}(a - b - 5)(a + b - 1). \quad (10$$

Substituting expressions (9-10) into the objective functions (4-5), we obtain:

$$\pi_1 = \frac{t}{16}(a - b + 3)^2 (1 - a - b); \quad (11$$

$$\pi_2 = \frac{t}{32}(a - b - 5)^2 (1 - a - b). \quad (12$$

The first order conditions relative to the location stage of the game, which is played noncooperatively, are:

$$\frac{\delta \pi_1}{\delta a} = \frac{t}{16}(b - a - 3)(3a + b + 1) = 0; \quad (13$$

$$\frac{\delta \pi_2}{\delta b} = \frac{t}{32}(a - b - 5)(a + 3b + 3) = 0. \quad (14$$

The system (13-14) has the following critical points: (-3,0); (0,-1); (1,-4). By inspection of the second order conditions,

$$\frac{\delta^2 \pi_1}{\delta a^2} = \frac{t}{8}(b - 3a - 5) \leq 0; \quad (15$$

$$\frac{\delta^2\pi_2}{\delta b^2}=\frac{t}{16}(a-3b-9)\leq 0, \tag{16}$$

it turns out that these are simultaneously satisfied only in (0,-1), which identifies the Nash equilibrium of the location stage. The equilibrium locations of the two-stage simultaneous game being given by $a=b=-\frac{1}{4}$,[2] the degree of differentiation of the game in which the price stage is played sequentially is larger. Equilibrium profits are $\pi_1=2t$, $\pi_2=t$; equilibrium prices are $p_1=4t$, $p_2=2t$, while quantities are obviously $y_1=y_2=\frac{1}{2}$. *Q.E.D.*

As a consequence, under endogenous horizontal differentiation, both firms would prefer to be the leader, despite of the strategic complementarity in prices.[3] This contrasts with the opposite results derived by Gal-Or (1985) and Boyer and Moreaux (1987), in whose contributions, yet, the degree of differentiation (or substitutability) is exogenous. Furthermore, it is easily shown that, if the location space is bounded, as in the model proposed by D'Aspremont et al. (1979), the conclusions reached by Gal-Or (1985) and Boyer and Moreaux (1987) hold.[4]

2.2. Location leadership

Assume firm 1 is leader in the location stage, while the price stage is played noncooperatively. The outcome of the two-stage game can be summarized as follows:

Proposition 2: If the price stage is played simultaneously, while the location stage is played sequentially, then

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2. Cfr. Lambertini (1993).

3. The concept of strategic complementarity or substitutability is due to Bulow et al. (1985). See Appendix A.

4. See Appendix B.

i) the degree of differentiation at equilibrium is lower than that obtained when both stages are played simultaneously;

ii) the quantity sold by the leader is greater than the quantity sold by the follower;

iii) the leader's profit is greater than the follower's.

Proof.

Since we are looking for the perfect subgame equilibrium of the two-stage game in locations and prices, let's proceed by backward induction, maximizing (4) and (5) w.r.t. p_1 and p_2. The reaction functions in prices are:

$$R_1(p_2) = p_1 = \frac{p_2 + t - a^2 t - 2bt + b^2 t}{2}; \qquad (17$$

$$R_2(p_1) = p_2 = \frac{p_1 + t + a^2 t - 2at - b^2 t}{2}. \qquad (7'$$

The equilibrium prices are:[5]

$$p_1^* = t(1-a-b)\left(1 + \frac{a-b}{3}\right); \qquad (18$$

$$p_2^* = t(1-a-b)\left(1 + \frac{b-a}{3}\right). \qquad (19$$

If we substitute (18) and (19) into the profit functions (4) and (5), we obtain:

5. Cfr. D'Aspremont et al. (1979, p.1149) and Tirole (1988, p.281).

$$\pi_1 = \frac{t}{18}(a-b+3)^2(1-a-b); \qquad (20$$

$$\pi_2 = \frac{t}{18}(a-b-3)^2(1-a-b); \qquad (21$$

the first order conditions w.r.t. locations are:

$$\frac{\delta\pi_1}{\delta a} = \frac{t}{18}(b-a-3)(1+3a+b) = 0; \qquad (22$$

$$\frac{\delta\pi_2}{\delta b} = \frac{t}{18}(a-b-3)(1+a+3b) = 0. \qquad (23$$

The leader's problem consists in the maximization of (20) under the constraint given by the follower's reaction function (23). The following second order conditions must be met:

$$\frac{\delta^2\pi_1}{\delta a^2} = b-3a-5 \leq 0; \qquad (24$$

$$\frac{\delta^2\pi_2}{\delta b^2} = a-3b-5 \leq 0. \qquad (25$$

The SOCs and the constraint $a \leq 1-b$ define the region in which the Stackelberg equilibrium lies, i.e., along b=-(a+1)/3. The critical points of the system (20-23) are $\left(\frac{1}{2},-\frac{1}{2}\right)$ and $\left(-\frac{5}{2},\frac{1}{2}\right)$. The SOCs (24-25) are simultaneously met only in $\left(\frac{1}{2},-\frac{1}{2}\right)$, yielding $\pi_1 = \frac{8}{9}t$ and $\pi_2 = \frac{2}{9}t$, as equilibrium profits, while prices are $p_1 = \frac{4}{3}t$ and $p_2 = \frac{2}{3}t$ and quantities $y_1 = \frac{2}{3}$ and $y_2 = \frac{1}{3}$. *Q.E.D.*

2.3. Repeated leadership

The outcome of the game in which the same firm acts as a Stackelberg leader in both stages is summarized by the following:

Proposition 3: If both stages are played sequentially, with the same firm being appointed the leader's role, then

i) the degree of differentiation at equilibrium is lower than the one observed when both stages are played simultanously;

ii) the quantity sold by the leader is greater than the quantity sold by the follower;

iii) the leader's profit is greater than the follower's.

Proof.

The equilibrium for the price stage is shown in section 3 above. The leader's problem in the location stage is:

$$\max_{a} \quad \pi_1 = \frac{t}{16}(a-b+3)^2(1-a-b) \tag{26}$$

$$s.t. \quad \frac{\delta\pi_2}{\delta b} = \frac{t}{32}(a-b-5)(a+3b+3) = 0, \tag{27}$$

i.e., $b=a-5$ or $b=-(3+a)/3$. The SOCs are given by (15) and (16). The latter solution turns out to be the only acceptable, and substituted into (26) gives the following FOC:

$$\frac{\delta\pi_1}{\delta a} = \frac{2}{9}t(1-a)(a+3) = 0. \tag{28}$$

The solutions to (28) are a=-3, which doesn't satisfy the SOCs, and a=1, which yields $b=-\frac{4}{3}$ as the follower's optimal location. The perfect subgame equilibrium of the game in which the same firm acts as a Stackelberg leader in both stages is then characterized by $\left(a=1, b=-\frac{4}{3}\right)$, $\left(p_1=\frac{32}{9}t, p_2=\frac{8}{9}t\right)$. Equilibrium profits are $\pi_1=\frac{64}{27}t$, $\pi_2=\frac{8}{27}t$, and demands $y_1=\frac{2}{3}$, $y_2=\frac{1}{3}$. *Q.E.D.*

2.4. Alternate leadership

The outcome of the game in which firms are alternatively appointed the leader's role is summarized by the following:

Proposition 4: If both stages are played sequentially, with one firm acting as a leader in the location stage, and the other acting as a leader in the price stage, then

i) the degree of differentiation at equilibrium is lower than the one observed when both stages are played simultaneously;

ii) the quantity sold by the location leader is greater than the quantity sold by the price leader;

iii) the location leader's profit is larger than the price leader's.

Proof.

Assume firm 2 acts as a Stackelberg leader in the location stage, while firm 1 is leader in the price stage. The equilibrium prices are given by expressions (9-10) above. Again, the SOCs are given by expressions (15) and (16). The objective of seller 2 in the location stage is then:

$$\max_{b} \quad \pi_2=\frac{t}{32}(a-b-5)^2(1-a-b) \qquad (29$$

$$s.t.\quad \frac{\delta\pi_1}{\delta a}=\frac{t}{18}(b-a-3)(1+3a+b)=0, \tag{30}$$

i.e., $a=b-3$, which is not acceptable, or $a=-(1+b)/3$. Substituting the latter into (29) and differentiating w.r.t. b, we obtain the following FOC:

$$\frac{\delta\pi_2}{\delta b}=-\frac{bt(b+4)}{9}=0. \tag{31}$$

The solutions to (31) are $b=-4$, which doesn't satisfy the SOCs, and $b=0$, yielding $a=-\frac{1}{3}$ as the follower's optimal location. The perfect subgame equilibrium of the Stackelberg game with alternate leadership is then defined by $\left(a=-\frac{1}{3}, b=0\right)$ and $\left(p_1=\frac{16}{9}t, p_2=\frac{16}{9}t\right)$. Equilibrium profits are $\pi_1=\frac{16}{27}t$ and $\pi_2=\frac{32}{27}t$, whereas demands are $y_1=\frac{1}{3}$ and $y_2=\frac{2}{3}$. *Q.E.D.*

3. The vertical model: The symmetric setup

Duopolists supply a vertically differentiated good, whose quality is denoted by q_i, $i=H,L$, $q_H \geq q_L > 0$. They adopt the same technology, represented by the following cost function:

$$C_i = tq_i x_i, \quad t>0, \quad i=H,L, \tag{32}$$

where x_i denotes firm i's quantity.

Consumers are uniformly distributed over the interval $[\underline{\theta}, \overline{\theta}]$, with $\underline{\theta} > 0$ and $\overline{\theta} = \underline{\theta} + 1$. Their total density is one. Parameter θ represents each consumer's marginal willingness to pay for quality, or the reciprocal of her marginal utility of income. Assume the market is covered, and each consumer buys one unit of the good which maximizes the following indirect utility function:

$$U = \theta q_i - p_i \geq 0. \tag{33}$$

Consumers may thus be divided in two groups: those buying the high-quality good, and those buying the low-quality good, so that the demands for the two commodities are, respectively:

$$x_H = \overline{\theta} - \frac{(p_H - p_L)}{q_H - q_L}, \quad iff \quad 0 < \overline{\theta} - \frac{(p_H - p_L)}{q_H - q_L} < 1, \tag{34}$$

$$x_H = 1, \quad iff \quad \overline{\theta} - \frac{(p_H - p_L)}{q_H - q_L} \geq 1, \tag{34'}$$

$$x_H = 0, \quad iff \quad \overline{\theta} - \frac{(p_H - p_L)}{q_H - q_L} \leq 0; \tag{34''}$$

$$x_L = \frac{(p_H - p_L)}{q_H - q_L} - \underline{\theta} \quad iff \quad 0 < \frac{(p_H - p_L)}{q_H - q_L} - \underline{\theta} < 1, \tag{35}$$

$$x_L = 1 \quad iff \quad \frac{(p_H - p_L)}{q_H - q_L} - \underline{\theta} \geq 1, \tag{35'}$$

$$x_L = 0 \quad iff \quad \frac{(p_H - p_L)}{q_H - q_L} - \underline{\theta} \leq 0. \tag{35''}$$

The profit functions are then:

$$\pi_H = (p_H - tq_H^2)x_H, \tag{36}$$

$$\pi_L = (p_L - tq_L^2)x_L, \tag{37}$$

Firms play a two-stage game, in which they first compete in quality, and then in prices. The solution concept is a subgame perfect equilibrium obtained through backward induction (Selten, 1975).

3.1. Simultaneous moves

The outcome of the game in which firms move simultaneously in both stages can be summarized by the following:

Proposition 5: If both stages are played simultaneously, then

i) their respective equilibrium locations in the quality spectrum are $q_L^* = \frac{4\bar{\theta}-5}{8t}$, $q_H^* = \frac{4\bar{\theta}+1}{8t}$;

ii) the degree of differentiation at equilibrium is $\frac{3}{4t}$;

iii) both firms sell the same quantity;

iv) both firms obtain the same profit.

Proof.

Differentiating (36) and (37) w.r.t. p_H and p_L, respectively, we obtain the following first order conditions (FOCs):

$$\frac{\delta\pi_H}{\delta p_H} = \bar{\theta} - \frac{(p_H - p_L)}{q_H - q_L} - \frac{(p_H - tq_H^2)}{q_H - q_L} = 0, \tag{38}$$

$$\frac{\delta\pi_L}{\delta p_L} = \frac{(p_H - p_L)}{q_H - q_L} - \frac{(p_L - tq_L^2)}{q_H - q_L} - \underline{\theta} = 0. \tag{39}$$

Solving (38-39) w.r.t. p_H and p_L, we obtain the following equilibrium prices:

$$p_H^* = \frac{2\overline{\theta} q_H - 2\overline{\theta} q_L + t q_L^2 + 2t q_H^2 + t q_L - t q_H}{3} \qquad (40$$

$$p_L^* = \frac{\overline{\theta} q_H - \overline{\theta} q_L + 2t q_L^2 + t q_H^2 + 2t q_L - 2t q_H}{3} \qquad (41$$

Substituting (40-41) into (36-37) and differentiating w.r.t. q_H and q_L, respectively, we obtain the FOCs for the first stage of the game:

$$\frac{\delta \pi_H}{\delta q_H} = \frac{(\overline{\theta} + t q_L - 3t q_H + 1)(\overline{\theta} - t q_L - t q_H + 1)}{9} = 0, \qquad (42$$

$$\frac{\delta \pi_L}{\delta q_L} = \frac{(\overline{\theta} + t q_H - 3t q_L + 1)(t q_L + t q_H + 2 - \overline{\theta})}{9} = 0. \qquad (43$$

The system (42-43) has the following critical points: $\left(q_L = \frac{4\overline{\theta}-5}{8t}; q_H = \frac{4\overline{\theta}+1}{8t}\right)$; $\left(q_L = \frac{2\overline{\theta}-1}{4t}; q_H = \frac{2\overline{\theta}+5}{4t}\right)$; $\left(q_L = \frac{2\overline{\theta}-7}{4t}; q_H = \frac{2\overline{\theta}-1}{4t}\right)$. Since the first is the only meeting the second order conditions (SOCs), the equilibrium of the first stage is defined by $q_L^* = \frac{4\overline{\theta}-5}{8t}$, $q_H^* = \frac{4\overline{\theta}+1}{8t}$, so that the degree of differentiation is $\frac{3}{4t}$. The equilibrium prices are $p_H^* = \frac{49}{64t}$ and $p_L^* = \frac{25}{64t}$; equilibrium profits are $\pi_H^* = \pi_L^* = \frac{3}{16t}$, while demands are $x_H^* = x_L^* = \frac{1}{2}$. *Q.E.D.*

It is worth noting that, under the assumption that the market is covered, the model is symmetric the mark-up of price over cost is independent of the quality supplied by each firm, so that the vertical model is symmetric with respect to both equilibrium demands and profits, exactly as the horizontal one. This conclusion was firstly reached by Cremer and Thisse (1991); they also showed that, if $t = \frac{1}{2}$ and $\overline{\theta} = 1$, the two models are equivalent, and the horizontal model with convex transportation costs can be considered as a special case of vertical differentiation models. As a consequence, we could expect this correspondence to hold in any sequential setting as well.

3.2. Price leadership

Assume now one firm acts as a Stackelberg leader in prices, while the quality stage is played simultaneously. The outcome of such a game is summarized by:

Proposition 6: If the price stage is sequential, while the quality stage is simultaneous, then

i) the degree of differentiation at equilibrium is $\frac{1}{t}$;

ii) both firms sell the same quantity;

iii) the leader's profit is greater than the follower's.

Proof.

Let firm H be the price leader. Then, the objective of firm H at the second stage of the game is:

$$\max_{p_H} \quad \pi_H = (p_H - tq_H^2)x_H, \tag{44}$$

$$s.t.: \quad R_L(p_H) = p_L = \frac{\overline{\theta}q_L - q_L + q_H - \overline{\theta}q_H + p_H + tq_L^2}{2} \tag{45}$$

Solving the constrained maximum problem defined by (44-45), we obtain:

$$p_H^* = \frac{\overline{\theta}q_H + q_H - \overline{\theta}q_L - q_L + tq_L^2 + tq_H^2}{2} \tag{46}$$

After simple albeit tedious substitutions, we get:

$$\pi_H = \frac{(q_H - q_L)(3 - \bar{\theta} + tq_H + tq_L)^2}{16} \quad (47$$

$$\pi_L = \frac{(q_H - q_L)(tq_L + tq_H - \bar{\theta} - 1)^2}{16} \quad (48$$

Differentiating the above profit functions w.r.t. q_H and q_L, respectively, we derive the FOCs relative to the first stage of the game:

$$\frac{\delta\pi_H}{\delta q_H} = \frac{(\bar{\theta} + 1 + tq_L - 3tq_H)(\bar{\theta} + 1 + tq_L - tq_H)}{8} = 0 \quad (49$$

$$\frac{\delta\pi_L}{\delta q_L} = \frac{(3 - \bar{\theta} + tq_L + tq_H)(\bar{\theta} - 3 - 3tq_L + tq_H)}{8} = 0 \quad (50$$

The system (49-50) has three critical points: $\left(q_L = \frac{\bar{\theta}-2}{2t}; q_H = \frac{\bar{\theta}}{2t}\right)$; $\left(q_L = \frac{\bar{\theta}-5}{2t}; q_H = \frac{\bar{\theta}-1}{2t}\right)$; $\left(q_L = \frac{\bar{\theta}-1}{2t}; q_H = \frac{\bar{\theta}+3}{2t}\right)$. The SOCs are met only by the first, defining thus the equilibrium of the quality stage; thus, the distance between products within the quality spectrum at equilibrium is $\frac{1}{t}$.. The equilibrium profits are $\pi_H^* = \frac{1}{2t}$; $\pi_L^* = \frac{1}{4t}$, while demands are $x_H^* = x_L^* = \frac{1}{2}$. *Q.E.D.*

Thus, as in the horizontal setup (see subsection 2.1 above), under endogenous vertical differentiation both firms would prefer to be price leader.

3.3. Quality leadership

Assume now firms compete simultaneously in the price stage, while sequentially in the quality stage. The outcome of this game is described by:

Proposition 7: If firms move simultaneously in the price stage, while they move sequentially in the quality stage, then

i) the degree of differentiation at equilibrium is $\frac{1}{2t}$;

ii) the quantity sold by the leader is greater than that sold by the follower;

iii) the leader's profit is greater than the follower's.

Proof.

By symmetry, we can confine to the case in which the high-quality firm is appointed the leader's role.

Price competition in the second stage is described by conditions (38-41) above. Then, the leader's problem at the first stage of the game consists in maximizing (36) under the constraint given by (43). Solving the latter, we obtain the roots $q_{L1} = \frac{\bar{\theta}-2+tq_H}{3t}$, and $q_{L2} = \frac{\bar{\theta}-2-tq_H}{3t}$. Substituting q_{L1} into (36) and differentiating w.r.t. q_H, we obtain the FOC relative to the leader's problem. Solving this latter condition, we have $q_{H1} = \frac{2\bar{\theta}-1}{4t}$ and $q_{H2} = \frac{2\bar{\theta}+5}{4t}$. The SOCs turn out to be satisfied only by the pair (q_{H1}, q_{L1}).[6] Consequently, the equilibrium of the quality stage is defined by $\left(q_H^* = \frac{2\bar{\theta}-1}{4t}; q_L^* = \frac{2\bar{\theta}-3}{4t}\right)$. It is easily checked that, if the low-quality firm is appointed the leader's role, the equilibrium at the first stage is given by $\left(q_H^* = \frac{2\bar{\theta}+1}{4t}; q_L^* = \frac{2\bar{\theta}-1}{4t}\right)$. Thus, it appears that (i) the leader's equilibrium quality does not depend on her identity, and (ii) the absolute distance between goods at equilibrium in the quality spectrum is $\frac{1}{2t}$ in either case. Besides, The leader's profit amounts to $\frac{2}{9t}$, while the follower's amounts to $\frac{1}{18t}$; the quantity sold by the leader is $x_l^* = \frac{2}{3}$, while the quantity sold by the follower is $x_f^* = \frac{1}{3}$; subscripts l and f stand for *leader* and *follower*, respectively. *Q.E.D.*

6. The SOCs are not met by any of the critical points obtained by substituting q_{L2} into (36).

3.4. Repeated leadership

The outcome of the game in which the same firm is being appointed the leader's role in both stages is summarized by the following:

Proposition 8: Let the same firm act as a leader both in qualities and in prices; then

i) the degree of differentiation at equilibrium is $\frac{2}{3t}$;

ii) the quantity sold by the leader is greater than the quantity sold by the follower;

iii) the leader's profit is greater than the follower's.

Proof.

Let the high-quality firm act as a Stackelberg leader in both stages. The leader's problem in the price stage is described by (44-46) above. Then in the quality stage, the high-quality firm maximizes (47) under the constraint (50), whose roots are $q_{L1} = \frac{\bar{\theta}-3+tq_H}{3t}$ and $q_{L2} = \frac{\bar{\theta}-3-tq_H}{3t}$. Let us substitute q_{L1} in (47) and differentiate w.r.t. q_H. Solving the resulting FOC, we get $q_{H1} = \frac{\bar{\theta}-1}{2t}$ and $q_{H2} = \frac{\bar{\theta}+3}{2t}$. The SOCs are met only by the pair (q_{H1}, q_{L1}); the equilibrium is then defines by $\left(q_H^* = \frac{\bar{\theta}-1}{2t}; q_L^* = \frac{3\bar{\theta}-7}{6t}\right)$. One can quickly check that, under the reverse angle, i.e., when the low-quality firm acts as a leader in both stages, the equilibrium qualities are $\left(q_L^* = \frac{\bar{\theta}}{2t}; q_H^* = \frac{3\bar{\theta}+4}{6t}\right)$, and the distance between them is $\frac{2}{3t}$, in either case. The leader's equilibrium profit is $\pi_l^* = \frac{16}{27t}$, while the follower's is $\pi_f^* = \frac{2}{27t}$. Finally, the quantities sold at equilibrium are $x_l^* = \frac{2}{3}$ and $x_f^* = \frac{1}{3}$, respectively. *Q.E.D.*

3.5. Alternate leadership

It remains to be investigated the case in which leadership is assigned to a different firm in each stage. The outcome of this last game is described by the following:

Proposition 9: If one firm is appointed the leader's role in the quality stage, while the other is appointed the leader's role in the price stage, then

i) the degree of differentiation at equilibrium is $\frac{2}{3t}$;

ii) the quantity sold by the quality leader is greater than that sold by the price leader;

iii) the quality leader's profit is greater than the price leader's.

Proof.

Let the low-quality firm act as a quality leader, and the high-quality firm as a price leader. The price stage is described by (44-46). Then, in the quality stage, the low-quality firm maximizes (48) under the constraint given by (49). From the latter, we obtain the following roots: $q_{H1} = \frac{\bar{\theta}+1+tq_L}{3t}$ and $q_{H2} = \frac{\bar{\theta}+1-tq_L}{3t}$. Substituting q_{H1} into (48) and differentiating w.r.t. q_L, we get $q_{L1} = \frac{\bar{\theta}-1}{2t}$ and $q_{L2} = \frac{\bar{\theta}-5}{2t}$. In (q_{H1}, q_{L1}) - and only in this point - the profit functions turn out to be concave, so that the equilibrium of the quality stage is described by the pair $\left(q_L^* = \frac{\bar{\theta}-1}{2t}; q_H^* = \frac{3\bar{\theta}+1}{6t}\right)$. Profits are $\pi_H^* = \frac{4}{27t}$ and $\pi_L^* = \frac{8}{27t}$, while quantities are $x_H^* = \frac{1}{3}$ and $x_L^* = \frac{2}{3}$. As in the previous cases, geiven the symmetry of the model, the reverse perspective yields the same results. *Q.E.D.*

As shown in Cremer and Thisse (1991), we have seen that if (i) the market is covered, and (ii) production costs are conves in quality, then (a) the equilibrium of the simultaneous game is unique and symmetric; and (b) the unit mark-up is the same for both firms. Furthermore, the above analysis allows us to claim that any pair of Stackelberg equilibria is observationally equivalent as for profits, i.e., both leader's and seller's equilibrium profits depend on the role being playied but not on their respective positions in the quality spectrum.

4. A taxonomy of Stackelberg equilibria

We are now able to order the results obtained throughout the preceding sections, together with the outcome of the strictly simultaneous Hotelling game.[7] The payoffs are shown in table 1 (location) and table 2 (quality).

	N	ll_1	pl_1	rl_1	$al_{2,1}$
π_1	$\frac{3}{4}t$	$\frac{8}{9}t$	$2t$	$\frac{64}{27}t$	$\frac{16}{27}t$
π_2	$\frac{3}{4}t$	$\frac{2}{9}t$	t	$\frac{8}{27}t$	$\frac{32}{27}t$

Table 1

N = simultaneous equilibrium

ll_1 = firm 1 is leader in locations

pl_1 = firm 1 is leader in prices

rl_1 = firm 1 is leader in both stages

$al_{2,1}$ = firm 2 is leader in locations, firm 1 in prices

7. Cfr. Lambertini (1993).

	N	ql_H	pl_H	rl_H	$al_{L,H}$
π_H	$\frac{3}{16t}$	$\frac{2}{9t}$	$\frac{1}{2t}$	$\frac{16}{27t}$	$\frac{4}{27t}$
π_L	$\frac{3}{16t}$	$\frac{1}{18t}$	$\frac{1}{4t}$	$\frac{2}{27t}$	$\frac{8}{27t}$

Table 2

N = simultaneous equilibrium

ql_H = the high-quality firm is leader in qualities

pl_H = the high-quality firm is leader in prices

rl_H = the high-quality firm is leader in both stages

$al_{L,H}$ = the low-quality firm is leader in qualities, the high-quality firm is leader in prices

Since the first stage of both games is devoted to define the respective positions of firms in the product space, we can simply refer to it as to the "location" stage, adopting the Hotelling jargon.[8] The payoffs can thus be ordered as follows:

$$\pi_{rl} > \pi_{pl} > \pi_{ll,pf} > \pi_{pf} > \pi_{ll} > \pi_N > \pi_{lf,pl} > \pi_{rf} > \pi_{lf}.$$

A few comments are now in order. First, and obvious, the possibility of overlapping leaderships in both stages enhances the leader's profit. Second, the profits generated by price leadership turn out to be greater than those associated to location leadership. While this may seem counterintuitive at first sight, it can be given an interpretation taking into account the fact that total demand is always equal to one by assumption; consequently, the advantage in terms

8. It is easily verified that, if t=1/2, the payoffs shown in the two tables coincide.

of market share implicit in the location leadership is limited. Third, in contrast with Gal-Or (1985) and Boyer and Moreaux (1987), we have that the price leader gains larger profits than the follower, if the location stage is simultaneous. This result relies on the endogenous choice of their respective locations in the first stage. Fourth, as it frequently happens, some Stackelberg equilibria can be characterized as quasi-cooperative solutions. Specifically, the seller who accepts to act as a follower in prices, regardless of what happens in the location stage, is strictly better off than both in the noncooperative case and in the case in which she acts as a follower in locations and then as a leader in prices. Finally, the leadership in locations appears advantageous if price leadership is then attributed to the rival. This means that, if firms are alternatively appointed the leader's role in the two stages of the game, then the one acting as a leader in the long run variable, i.e., location, is strictly better off.

5. The vertical model: the asymmetric setup

This section is devoted to the analysis of the non-full market coverage case, i.e., in which the poorest consumers cannot afford to buy either the high-quality or the low-quality good.

In order to simplify calculations, let t=1, and $\overline{\theta} = 1$, so that $\underline{\theta} = 0$.[9] Furthermore, let θ_H and θ_L denote the marginal willingness to pay characterizing, respectively, the consumer who is indifferent between buyng either the high-quality or the low-quality good and the one who is indifferent between buying the low-quality good and not to buy at all:

9. Notice that this condition would be sufficient to imply non-full market coverage, since a consumer whose marginal willingness to pay for quality is nil cannot afford to buy any good being offered at a positive price.

$$\theta_H = \frac{p_H - p_L}{q_H - q_L}, \qquad (51$$

$$\theta_L = \frac{p_L}{q_L}. \qquad (52$$

Then, the demands for the two goods are, respectively:

$$x_H = 1 - \theta_H = \frac{(q_H - q_L) - (p_H - p_L)}{(q_H - q_L)} \quad \textit{iff} \quad 0 < 1 - \theta_H < 1; \qquad (53$$

$$x_H = 0 \quad \textit{iff} \quad 1 - \theta_H \leq 0; \qquad (53'$$

$$x_H = 1 \quad \textit{iff} \quad 1 - \theta_H \geq 1; \qquad (53''$$

$$x_L = \theta_H - \theta_L = \frac{q_L p_H - q_H p_L}{q_L(q_H - q_L)} \quad \textit{iff} \quad 0 < \theta_H - \theta_L < 1; \qquad (54$$

$$x_L = 0 \quad \textit{iff} \quad \theta_H - \theta_L \leq 0; \qquad (54'$$

$$x_L = 1 \quad \textit{iff} \quad \theta_H - \theta_L \geq 1; \qquad (54''$$

Thus, the profit functions look as follows:

$$\pi_H = (p_H - q_H^2)\frac{[(q_H - q_L) - (p_H - p_L)]}{(q_H - q_L)} \qquad (55$$

$$\pi_L = (p_L - q_L^2)\frac{(p_H q_L - p_L q_H)}{q_L(q_H - q_L)} \qquad (56$$

Inverting the demand system (53-54) it is possible to investigate also quantity competition.[10] The inverse demand functions are, respectively:

$$p_H = q_H - q_L x_L - q_H x_H; \tag{57}$$

$$p_L = q_L(1 - x_L - x_H), \tag{58}$$

while the profit functions are:

$$\pi_H = x_H(q_H - q_H^2 - q_L x_L - q_H x_H); \tag{59}$$

$$\pi_L = q_L x_L(1 - q_L - x_L - x_H). \tag{60}$$

Obviously, the solution concept is a subgame perfect equilibrium in qualities and in the relevant market variable, i.e., prices or quantitites. Given the non-full coverage of the market, we must resort to numerical analysis. However, since the underlying logic is the same followed in the previous sections, we can confine ourselves to the description of the results, as we are going to do in the next section.

6. A taxonomy under non-full coverage

We can summarize the results of the non-full coverage model in which firms compete either in prices or in quantities through tables 3-4 and 5-6, respectively. Tables 3 and 5 refer to firm H while tables 4 and 6 to firm L.

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10. Under the full ccoverage assumption, the demand functions cannot be inverted, since total demand is not a function of prices. See Motta (1993, p.116).

	N	ql_H	pl_H	rl_H	$al_{L,H}$
π_H	0.0164	0.0190	0.0151	0.0172	0.0102
π_L	0.0121	0.0075	0.0141	0.0096	0.0148

Table 3

N = simultaneous equilibrium

ql_H = the high-quality firm is leader in qualities

pl_H = the high-quality firm is leader in prices

rl_H = the high-quality firm is leader in both stages

$al_{L,H}$ = the low-quality firm is leader in qualities, the high-quality firm is leader in prices

	N	ql_L	pl_L	rl_L	$al_{H,L}$
π_H	0.0164	0.0102	0.018	0.0128	0.0200
π_L	0.0121	0.0129	0.012	0.0116	0.0079

Table 4

N = simultaneous equilibrium

ql_L = the low quality firm is leader in qualities

pl_L = the low quality firm is leader in prices

rl_L = the low quality firm is leader in both stages

$al_{H,L}$ = the high quality firm is leader in qualities, the low quality firm is leader in prices

The profits associated with Bertrand competition can be ordered as follows:

$$qlpf_H > ql_H > pf_H > rl_H > N_H > pl_H > qlpf_L > pf_L > ql_L >$$
$$rl_L > N_L > pl_L > rf_H > qf_H > qfpl_H > rf_L > qfpl_L > qf_L.$$

The subscripts are needed since the game is not symmetric. It is immediate to notice that $pf>pl$ for both firms, as in Gal-Or (1985) and Boyer and Moreaux (1987), with a relevant caveat: both duopolists are strictly better off in the simultaneous game than in the sequential one, should they happen to be the leader. Furthermore, repeated leadership is not the first best; instead, both firms would prefer to act as a leader in location and then as a follower in prices, and this is likely due to the assumption of non-full coverage of the market.

The symbology adopted for the quantity competition case is analogous, but for the "x" standing for quantity.

	N	ql_H	xl_H	rl_H	$al_{L,H}$
π_H	0.0176	0.0176	0.0206	0.0197	0.0207
π_L	0.0175	0.0174	0.0131	0.0147	0.0130

Table 5

N = simultaneous equilibrium

ql_H = the high quality firm is leader in qualities

xl_H = the high quality firm is leader in quantities

rl_H = the high quality firm is leader in both stages

$al_{L,H}$ = the low quality firm is leader in qualities, the high quality firm is leader in quantities

	N	ql_L	xl_L	rl_L	$al_{H,L}$
π_H	0.0176	0.0168	0.0123	0.0134	0.0123
π_L	0.0175	0.0172	0.0197	0.0194	0.0198

Table 6

N = simultaneous equilibrium

ql_L = the low quality firm is leader in qualities

xl_L = the low quality firm is leader in quantities

rl_L = the low quality firm is leader in both stages

$al_{H,L}$ = the high quality firm is leader in qualities, the low quality firm is leader in quantities

The payoffs displayied in tables 5 and 6 can be ordered according to the following sequence:

$$qfxl_H > xl_H > qfxl_L > xl_L > rl_H > rl_L > ql_H = N_H > N_L >$$
$$qf_L > ql_L > qf_H > rf_L > rf_H > xf_L > qlxf_L > xf_H = qlxf_H.$$

In this case, the main result reached by the existing literature is confirmed, as *xl>xf* for both firms. As compared to Bertrand competition, under Cournot competition both firms would prefer to be a follower in the quality stage and a leader in the quantity stage. Furthermore, if both stages are simultaneous, both would prefer the market stage to be played in quantitites. Yet, if the quality stage is simultaneous while the market stage is sequential, firm *H* orders her payoffs according to the sequence *xl>pf>pl>xf*, while firm *L* according to *xl>pf>xf>pl*; thus, the choice of the strategic variable does not dominate the choice of the role, as claimed by Boyer and Moreaux (1987).[11] This is due to the fact that, while in their model the substitutability between products is exogenuosly set, in ours the degree of differentiation is endougenuosly determined by firms before they compete in the market variable.

11. See their Proposition III, p.223.

7. Welfare evaluation

We are now going to assess the consumers' (or social planner's) ranking of the market settings considered so far. Fist, we evaluate the welfare levels associated with the hypothesis of full coverage of the market.

7.1. Full coverage of the market

Since any Hotelling model with convex transportation costs can be considered as a special case of a model of vertical differentiation, we confine ourselves to the analysis of the welfare levels associated with the latter. The social welfare function is:

$$SW = \int_{\underline{\theta}}^{i} (\theta q_L - tq_L^2)d\theta + \int_{i}^{\overline{\theta}} (\theta q_H - tq_H^2)d\theta, \qquad (61$$

where i identifies the consumer who is indifferent between the two goods. The equilibrium values of (61) for all the relevant market configuration are shown in table 7 below.

SW_{SP}	SW_N	SW_{QL}	SW_{PL}	SW_{RL}	SW_{AL}
$\frac{16\overline{\theta}^2 - 16\overline{\theta} + 5}{64t}$	$\frac{16\overline{\theta}^2 - 16\overline{\theta} - 1}{64t}$	$\frac{36\overline{\theta}^2 - 36\overline{\theta} + 5}{144t}$	$\frac{2\overline{\theta}^2 - 2\overline{\theta} - 1}{8t}$	$\frac{27\overline{\theta}^2 - 27\overline{\theta} - 20}{108t}$	$\frac{27\overline{\theta}^2 - 27\overline{\theta} + 4}{108t}$

Table 7

SP stands for social planning, or public monopoly. The remaining terminology has the usual meaning. These results can be ordered as follows:

$$SW_{SP} > SW_{AL} > SW_{QL} > SW_{N} > SW_{PL} > SW_{RL}.$$

The setting in which firms are alternatively appointed the leader's role appears to be the closest to social planning. On the contrary the poorest performance is offered by repeated leadership by the same firm. Furthermore, the welfare level yielded by quality leadership is higher than the one yielded by price leadership, if the remaining stage is simultaneous. This is due to the fact that in the first case both qualities are higher than the corresponding qualities offered in the latter.

7.2. Non-full coverage of the market

We can now analyse the behaviour of social welfare under the non-full coverage assumption. The social welfare function is the following:

$$SW = \int_{k}^{h} (\theta q_L - q_L^2) d\theta + \int_{h}^{1} (\theta q_H - q_H^2) d\theta, \qquad (62$$

where k and h identify, respectively, the marginal willingness to pay of the consumer who is indifferent between the low-quality good and not to buy at all, and the marginal willingness to pay of the consumer who is indifferent between the two goods.

First, we investigate the case in which in the second stage of the game firms compete in prices. The results are shown in table 8.

SW_{SP}	SW_{QLH}	SW_{PLH}	SW_{RLH}	SW_{ALHL}
0.08	0.071	0.0749	0.0711	0.0713
SW_{N}	SW_{QLL}	SW_{PLL}	SW_{RLL}	SW_{ALLH}
0.0756	0.0762	0.075	0.0752	0.0756

Table 8

The same results can be ranked according to the following sequence of inequalities:

$$SW_{SP} > SW_{QLL} > SW_{N} > SW_{ALLH} > SW_{RLL} > SW_{PLL} > SW_{PLH} > SW_{ALHL} > SW_{RLH} > SW_{QLH}.$$

Unlike under the full coverage assumption, here quality leadership by firm L appears to be the closest to social planning, while we find quality leadership by firm H at the last position of the ranking. Again, however, alternate leadership is preferable to repeated leadership.

If firms compete in quantities, the welfare levels are those shown in table 9.

SW_{SP}	SW_{QLH}	SW_{XLH}	SW_{RLH}	SW_{ALHL}
0.08	0.0682	0.0687	0.0716	0.071
SW_{N}	SW_{QLL}	SW_{XLL}	SW_{RLL}	SW_{ALLH}
0.0683	0.0667	0.071	0.0714	0.0714

Table 9

We can order these results as follows:

$$SW_{SP} > SW_{RLH} > SW_{RLL} = SW_{ALLH} > SW_{ALHL} = SW_{XLL} > SW_{XLH} > SW_{N} > SW_{QLH} > SW_{QLL}.$$

In this setting, repeated leadership is preferred to alternate leadership. Besides, under both price and quantity competition, the social welfare is higher if the leader's role in the market stage is appointed to the low-quality firm, provided that the quality stage is simultaneous. Comparing the results shown in tables 8 and 9, price competition is globally preferred to price competition, with the exception represented by repeated leadership by the high-quality firm. This conclusion thus largely confirms the results achieved by the existing literature.

8. Conclusions

In this paper, we investigated the nature and consequences of Stackelberg leadership in a duopolistic model of endogenous differentiation under the address approach. As it could be expected from the outset, if the market is completely served, the largest advantage is attached to repeated leadership by the same firm. This also represents the only case, within the horizontal setting, leading to an equilibrium outcome which is typically associated to vertical differentiation. Moreover, due to demand rigidity, price leadership turns out to be preferable to location leadership, if the remaining stage is played simultaneously. If, instead, the remaining stage sees the rival playing the leader's role, this result is reversed. Contrarily to the conclusions reached by previous research, the role of price leader is preferred to the role of price follower, since firms can endogenously differentiate their products before competing in the market variable. Under the non-full coverage assumption, if firms compete *à la* Bertrand, both would prefer to act as a leader in qualities and as a follower in prices. If instead firms compete *à la* Cournot, both would prefer to be the follower in qualities and the leader in quantitites. Our results does not yield any evidence in favour of the dominance of the strategy space chosen for the market stage over the distribution of roles. From a social standpoint, price competition is generally preferable to quantity competition.

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