Endogenous Choice between Horizontal and Vertical Product Differentiation*

Luca Lambertini

Dipartimento di Scienze Economiche
Università degli Studi di Bologna

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Abstract

The endogenous choice between two alternative kinds of product differentiation is addressed in a duopoly model where firms are free to locate along the real axis, while consumers are distributed along a linear city of finite length. It turns out that the nature of differentiation may be heavily affected by the sequence of decisions. If firms simultaneously choose first locations and then prices, product differentiation at equilibrium is horizontal. If instead one firm acts as a Stackelberg leader in both stages, product differentiation at equilibrium is vertical.

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INTRODUCTION

One of the major issues on product differentiation is whether firms prefer horizontal to vertical differentiation. This question can be given a partial answer, which builds on a well established stream of literature.

In their pathbreaking paper, D'Aspremont, Gabszewicz and Thisse (1979) criticize the minimum differentiation principle derived by Hotelling (1929). They showed that Hotelling's linear transportation cost game may fail to reach a pure-strategy Nash equilibrium in prices when firms are too close to each other, due to the lack of quasi-concavity of the profit functions.[1] Resorting to a quadratic transportation cost function, they showed that the existence of a price equilibrium is ensured for any pair of locations, and the Nash equilibrium in the location stage of the game implies maximum differentiation.

Gabszewicz and Thisse (1986) extended the analysis to the case of a generalized transportation cost function in which both a linear and a quadratic component are present, investigating two models which capture the essential feature of horizontal versus vertical product differentiation. In the first, two sellers can choose their location within the linear city; in the second, both sellers are compelled to locate either to the left or to the right of the city limits. Within the latter framework, all consumers agree as to the quality ranking of available products, making this case a prototype of vertical differentiation. It turns out that under vertical product differentiation there always exists a stable market outcome in terms of prices and locations, while the horizontal differentiation model may fail to reach the same result, due to the presence of a linear component in the transportation cost function.

In this paper, I will confine myself to a framework in which the existence of a stable outcome in pure strategies is warranted. I will show that, if sellers are allowed to choose whatever location along the real axis, then in a simultaneous setting they endogenously choose

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1. While a mixed-strategy Nash equilibrium in prices always exists (Dasgupta and Maskin, 1986).

to play a game of horizontal product differentiation. For an equilibrium closely recalling vertical differentiation to emerge, the same firm must enjoy a Stackelberg leadership both in prices and in locations.

I. THE MODEL

Two firms supply a physically homogeneous good at different locations on the real axis. Production costs are nil. Consumers are uniformly distributed along a linear city whose length can be normalized to 1 without loss of generality. Their total density is 1. They have unit demands, and consumption yields a positive constant surplus s; each consumer buys if and only if the following condition is met:

$$U = s - tx^2 - p_i \geq 0, \quad t > 0, \quad i = 1,2; \tag{1}$$

where tx^2 is the transportation cost incurred by a consumer living at distance x from store i, and p_i is the price of good i. We assume that s is large enough for total demand to be always equal to 1. Firm 1 is located at a, while firm 2 is located at $1 - b \geq a$, with $a, b \in R$. Clearly, if we allow both a and b to be negative, firms are located outside the city boundaries. The demand functions are, respectively:

$$y_1 = a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \tag{2}$$

$$\text{if} \quad 0 < a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} < 1;$$

$$y_1 = 0 \tag{2'}$$

$$if \quad a+\frac{1-a-b}{2}+\frac{p_2-p_1}{2t(1-a-b)}\leq 0;$$

$$y_1=1 \qquad (2''$$

$$if \quad a+\frac{1-a-b}{2}+\frac{p_2-p_1}{2t(1-a-b)}\geq 1;$$

Demand to firm 2 is obtained by replacing a by b in equations (2-2"):

$$y_2=1-y_1=b+\frac{1-a-b}{2}+\frac{p_1-p_2}{2t(1-a-b)} \qquad (3$$

$$if \quad 0<b+\frac{1-a-b}{2}+\frac{p_1-p_2}{2t(1-a-b)}<1;$$

$$y_2=0 \qquad (3'$$

$$if \quad b+\frac{1-a-b}{2}+\frac{p_1-p_2}{2t(1-a-b)}\leq 0;$$

$$y_2=1 \qquad (3''$$

$$if \quad b+\frac{1-a-b}{2}+\frac{p_1-p_2}{2t(1-a-b)}\geq 1.$$

For a=1-b, i.e., when sellers locate at the same point, the demand functions are not determined and profits are nil as a consequence of the Bertrand paradox. Since there are no production costs, the two profit functions are then

$$\pi_1 = p_1 y_1; \qquad (4$$

$$\pi_2 = p_2 y_2. \qquad (5$$

Firms play a noncooperative two-stage game in locations (first stage) and prices (second stage). The solution concept is subgame perfect equilibrium in the sense of Selten (1965, 1975).

In the remainder of the paper, I will refer to the taxonomy of equilibria outlined by the following definitions:

DEFINITION 1: *a Nash equilibrium in locations is strictly horizontal if, at equal prices, the indifferent consumer lies in (0, 1).*

DEFINITION 2: *a Nash equilibrium in locations is weakly vertical if, at equal prices, the indifferent consumer lies in 0 (or 1) while the others strictly prefer the right (or left) firm.*

DEFINITION 3: *a Nash equilibrium in locations is strictly vertical if, at equal prices, one firm is strictly preferred by all consumers.*

II. SIMULTANEOUS MOVES

Sellers move simultaneously in both stages, the first being played in the location space, while the second in the price space. The outcome is summarized by the following proposition

PROPOSITION 1: *if (i) sellers choose their locations in R and (ii) consumers' preferences are described by (1), both firms set* $z=-\frac{1}{4}$, *z=a,b; then, the Nash equilibrium in locations is strictly horizontal. Then, the unique subgame perfect equilibrium of the simultaneous two-stage game implies that sellers choose to play a game of horizontal product differentiation.*

PROOF. Let us proceed by backward induction, maximizing (4) and (5) w.r.t. p_1 and p_2. The equilibrium prices are:[2]

$$p_1^* = t(1-a-b)\left(1+\frac{a-b}{3}\right) \quad (6$$

$$if \quad t(1-a-b)\left(1+\frac{a-b}{3}\right) > 0;$$

$$p_1^* = 0 \quad if \quad t(1-a-b)\left(1+\frac{a-b}{3}\right) \leq 0; \quad (6'$$

$$p_2^* = t(1-a-b)\left(1+\frac{b-a}{3}\right) \quad (7$$

$$if \quad t(1-a-b)\left(1+\frac{b-a}{3}\right) > 0;$$

$$p_2^* = 0 \quad if \quad t(1-a-b)\left(1+\frac{b-a}{3}\right) \leq 0 \quad (7'$$

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2. Cfr. D'Aspremont et al. (1979, p.1149) and Tirole (1988, p.281).

If we substitute (6) and (7) into the profit functions (4) and (5), we obtain:

$$\pi_1 = \frac{t}{18}(a - b + 3)^2 (1 - a - b); \qquad (8$$

$$\pi_2 = \frac{t}{18}(a - b - 3)^2 (1 - a - b); \qquad (9$$

the first order conditions (FOCs) w.r.t. locations are:

$$\frac{\delta\pi_1}{\delta a} = \frac{t}{18}(b - a - 3)(1 + 3a + b) = 0; \qquad (10$$

$$\frac{\delta\pi_2}{\delta b} = \frac{t}{18}(a - b - 3)(1 + a + 3b) = 0. \qquad (11$$

If sellers are constrained to locate within the city limits, the sign of (10-11) is negative, as noted by D'Aspremont et al. (1979, p.1149) and Tirole (1988, p.281), since the sign of the first parenthesis is negative, while, within the unit interval, the sign of the second is positive. Outside the city, the sign of the FOCs changes, due to the fact that outside the unit interval the sign of the expression within the second parenthesis in (10-11) may become negative. Moreover, following Bulow et al. (1985, p.494), the incentive to locate outside the city boundaries can be pointed out by inspection of the following mixed derivative:

$$\frac{\delta\pi_i^2}{\delta a \delta b} = \frac{t}{9}(a + b - 1), \qquad (12$$

which is everywhere negative, if the restriction $a \leq 1 - b$ is imposed, except for a=1-b, in which it is nil. This means that products act everywhere as strategic complements.[3]

Let us now turn to the Nash equilibrium. The system (10-11) has the following critical points: $\left(a = -\frac{1}{4}; b = -\frac{1}{4}\right)$; $\left(a = \frac{1}{2}; b = -\frac{5}{2}\right)$; $\left(a = -\frac{5}{2}; b = \frac{1}{2}\right)$. It can be easily verified that the Nash equilibrium is given by $\left(a = b = -\frac{1}{4}\right)$, since the second order conditions (SOCs) are not simultaneously satisfied by the remaining critical points, so that both profit functions are single-peaked over the whole interval in which conditions (2-7') are simultaneously satisfied. In order to show this, let us check the second derivatives of (8) and (9) w.r.t. a and b, respectively. It must be that:

$$\frac{\delta^2 \pi_1}{\delta a^2} = b - 3a - 5 \leq 0; \tag{13}$$

$$\frac{\delta^2 \pi_2}{\delta b^2} = a - 3b - 5 \leq 0. \tag{14}$$

For $\left(a = b = -\frac{1}{4}\right)$ both (13) and (14) are respected; for $\left(a = \frac{1}{2}; b = -\frac{5}{2}\right)$ (13) is satisfied, while (14) is not: this means that in $b = -\frac{5}{2}$, π_2 is being minimized. The reverse is true for $\left(a = -\frac{5}{2}; b = \frac{1}{2}\right)$. Thus, it turns out that the only Nash equilibrium is given by $a = b = -\frac{1}{4}$, which implies that both firms locate outside the city. The equilibrium profits are $\pi_1 = \pi_2 = \frac{3}{4}t$, while demands are

3. Notice that the location of firm 2 is given by 1-b; thus, as a increases, b decreases according to (12), and 1-b increases. Furthermore, if we define $\frac{\delta \pi_i^2}{\delta a \delta b} = S_i$, we obtain:

$$\frac{\delta S_i}{\delta z} = \frac{t}{9} > 0, \quad z = a, b;$$

which means that the degree of strategic complementarity given by (12) increases as z increases, i.e., both firms' marginal profit decreases at an increasing rate as distance shrinks.

$y_1 = y_2 = \frac{1}{2}$, obviously. As a consequence, the Nash equilibrium for the location stage is strictly horizontal, and the simultaneous game yields horizontal differentiation as an equilibrium outcome. *Q.E.D.*[4]

The equilibrium can be characterized in the following way: if seller 1 chooses $a = -\frac{1}{4}$, seller 2 can't do any better than choosing the same value for b; otherwise, she would either loose demand by increasing the degree of differentiation (since transportation costs would rise) or intensify price competition by decreasing the degree of differentiation. Accordingly, it can be quickly verified that the profits associated to the pair $\left(-\frac{1}{4}, -\frac{1}{4}\right)$ Pareto-dominate those associated to the corner configuration in which one seller locates accordingly to her Nash strategy, while the other locates at the border of the residential area which is closer to the rival.

III. SEQUENTIAL MOVES

III(i). Location leadership

The outcome of the game in which the price stage is played simultaneously, while the location stage is played sequentially is summarized by the following

PROPOSITION 2: *if (i) sellers choose their locations in R, (ii) consumers' preferences are described by (1) and (iii) one seller acts as Stackelberg leader in the location stage, while firms*

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4. In a related paper, Economides (1986) has shown that, if transportation cost are defined by tx^{α}, where $\alpha \geq 1.26$, the equilibrium locations are given by $a = b = \frac{5}{4} - \frac{3}{4}\alpha$. This allows to extend Proposition 1 to a wide class of models with convex transportation costs.

move simultaneously in the price stage, then the leader locates in the middle of the residential area while the follower locates outside the city, at unit distance from the leader. Consequently, the Nash equilibrium in locations is weakly vertical.

PROOF. See the Appendix.

III(ii). Price leadership

The outcome of the game in which the price stage is played sequentially, while the location stage is played simultaneously is summarized by the following

PROPOSITION 3: *if (i) sellers choose their locations in R, (ii) consumers' preferences are described by (1) and (iii) one seller acts as Stackelberg leader in the price stage, while firms move simultaneously in the location stage, then the leader sets* $w=0$, $w=a,b$, *while the follower sets* $z=-1$, $z=a,b$, $w \neq z$, *so that the Nash equilibrium in locations is weakly vertical.*

PROOF. See the Appendix.

III(iii). Alternate leadership

The outcome of the game in which one seller is leader in the price stage, while the other is leader in the location stage is summarized by the following

PROPOSITION 4: *if (i) sellers choose their locations in R, (ii) consumers' preferences are described by (1) and (iii) one seller acts as Stackelberg leader in the location stage, while the other acts as Stackelberg leader in the price stage, then the price leader sets* $w=-\frac{1}{3}$, $w=a,b$, *while the location leader sets* $z=0$, $z=a,b$, $w \neq z$,, *so that the Nash equilibrium in locations is strictly horizontal.*

PROOF. See the Appendix.

III(iv). Repeated leadership

Let us now assume that firm 1 acts as Stackelberg leader both in prices and in locations, i.e., maximizes profits under the constraint given by the rival's relevant reaction function. The outcome is summarized by the following

PROPOSITION 5: *if (i) sellers choose their locations in R, (ii) consumers' preferences are described by (1) and (iii) the Stackelberg leadership is assigned to the same seller in both stages, then the leader sets w=1, w=a,b, while the follower locates outside the city, so that the Nash equilibrium in locations is strictly vertical and the unique subgame perfect equilibrium of the sequential two-stage game implies that sellers choose to play a game of vertical product differentiation.*

PROOF. The objective of seller 1 in the price stage is:

$$\max_{p_1} \quad \pi_1 = p_1 y_1 \tag{15}$$

$$s.t. \quad R_2(p_1) = p_2 = \frac{p_1 + t - 2at + a^2 t - b^2 t}{2}. \tag{16}$$

The first order condition is:

$$\frac{\delta \pi_1}{\delta p_1} = \frac{2p_1 - 3t + 2at + a^2 t + 4bt - b^2 t}{4t(a + b - 1)} = 0, \tag{17}$$

which yields the following equilibrium prices:

$$p_1^* = \frac{t}{2}(a - b + 3)(1 - a - b); \qquad (18$$

$$p_2^* = \frac{t}{4}(a - b - 5)(a + b - 1). \qquad (19$$

Substituting expressions (18-19) into the objective functions (4-5), we obtain:

$$\pi_1 = \frac{t}{16}(a - b + 3)^2 (1 - a - b); \qquad (20$$

$$\pi_2 = \frac{t}{32}(a - b - 5)^2 (1 - a - b). \qquad (21$$

The leader's problem in the location stage is thus:

$$\max_a \quad \pi_1 = \frac{t}{16}(a - b + 3)^2 (1 - a - b) \qquad (22$$

$$s.t. \quad \frac{\delta \pi_2}{\delta b} = \frac{t}{32}(a - b - 5)(a + 3b + 3) = 0, \qquad (23$$

i.e., $b=a-5$ or $b=-(3+a)/3$. By inspection of the SOCs, the latter solution turns out to be the only acceptable, and substituted into (22) gives the following FOC:

$$\frac{\delta\pi_1}{\delta a} = \frac{2}{9}t(1-a)(a+3) = 0. \qquad (24$$

The critical points of (24) are a=-3, which doesn't satisfy the SOCs, and a=1, which yields $b = -\frac{4}{3}$ as the follower's optimal location. The perfect subgame equilibrium of the game in which the same firm acts as a Stackelberg leader in both stages is then characterized by $\left(a = 1, b = -\frac{4}{3}\right)$, $\left(p_1 = \frac{32}{9}t, p_2 = \frac{8}{9}t\right)$. Equilibrium profits are $\pi_1 = \frac{64}{27}t$, $\pi_2 = \frac{8}{27}t$, and demands $y_1 = \frac{2}{3}$, $y_2 = \frac{1}{3}$. In this case the equilibrium configuration of the market typycally yields vertical differentiation, since all consumers rank the goods in the same order. *Q.E.D.*

IV. CONCLUSIONS

One of the main conclusions drawn by Gabszewicz and Thisse (1986, p.167) from the analysis of two separate models of horizontal and vertical product differentiation based on a generalized transportation cost function was that more stability should be expected in the vertical framework.

The foregoing analysis sheds some new light on this issue. Once the existence of equilibrium in pure strategies is ensured, and firms are free to choose between horizontal and vertical differentiation, their decision depends on the sequence of moves. When both stages are played simultaneously, product variety at equilibrium is the outcome of a process of horizontal differentiation. If, instead, one stage is played sequentially, the equilibrium locations mimic vertical differentiation. Finally, if the game is played sequentially both in locations and in prices, the equilibrium clearly exhibits vertical differentiation, although only in logistical terms.

This result can be given an intuitive explanation on two different grounds. First, the higher stability characterising the equilibrium yielded by the game of vertical differentiation described by Gabszewicz and Thisse (1986) is simply due to the assumption that firms' strategies have a lower (or upper) bound in the product space.[5] If the strategy space is unbounded, this result does not apply. Second, and more relevant, the choice of horizontal differentiation can be viewed as an attempt by both firms to avoid playing the underdog role typically attached to the low-quality firm in a vertical differentiation framework.

The outcome of the sequential game should be considered with some caveat. On the one hand, the result of strictly vertical differentiation depends on the same firm being leader in both stages; on the other, the highest quality firm is ranked at the top by all consumers, suggesting that she could plausibly enjoy a first-mover advantage in one stage or - possibly - in both.

The theoretical framework adopted in this paper allows us to compare alternative models of product differentiation otherwise largely heterogeneous. The extension of these results to a wider class of models remains a field open to further research.

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APPENDIX

PROOF OF PROPOSITION 2. The price stage is described by (6-9). By symmetry, I can confine myself to the case in which firm 1 is the leader. She aims at maximizing (8) w.r.t. a, under the constraint given by (11). The critical points are $\left(-\frac{5}{2};\frac{1}{2}\right)$ and $\left(a=\frac{1}{2};b=-\frac{1}{2}\right)$; the SOCs (13-14) are simultaneously met only by the second. The equilibrium profits are then $\pi_1=\frac{8}{9}t$ and $\pi_2=\frac{2}{9}t$. Demands are $y_1=\frac{2}{3}$ and $y_2=\frac{1}{3}$. Since, at equal prices, the indifferent consumer lies at the right end of the city, the equilibrium is weakly vertical. *Q.E.D.*[6]

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6. Notice that, if sellers were to locate inside the city, to look for a Stackelberg equilibrium would be economically meaningless. For the analysis of Stackelberg leadership within the original Hotelling model with linear transportation costs, see Anderson (1987).

PROOF OF PROPOSITION 3. The price stage is described by (15-21). Then, seller 1 maximizes (20) w.r.t. a while seller 2 maximizes (21) w.r.t. b. The FOCs relative to this stage are

$$\frac{\delta\pi_1}{\delta a}=\frac{t}{16}(b-a-3)(3a+b+1)=0; \qquad (25$$

$$\frac{\delta\pi_2}{\delta b}=\frac{t}{32}(a-b-5)(a+3b+3)=0. \qquad (26$$

The system (25-26) has the following critical points: (-3,0); (0,-1); (1,-4). By inspection of the second order conditions,

$$\frac{\delta^2\pi_1}{\delta a^2}=\frac{t}{8}(b-3a-5)\leq 0; \qquad (27$$

$$\frac{\delta^2\pi_2}{\delta b^2}=\frac{t}{16}(a-3b-9)\leq 0, \qquad (28$$

it turns out that these are simultaneously satisfied only in (0,-1), which identifies the Nash equilibrium of the location stage. Equilibrium profits are $\pi_1=2t$, $\pi_2=t$; equilibrium prices are $p_1=4t$, $p_2=2t$, while quantities are $y_1=y_2=\frac{1}{2}$. Since, at equal prices, the indifferent consumer lies at the right end of the city, the equilibrium is weakly vertical. *Q.E.D.*

PROOF OF PROPOSITION 4. Assume firm 2 acts as a Stackelberg leader in the location stage, while firm 1 is leader in the price stage. The price stage is described by (15-21). The objective of seller 2 in the location stage is then:

$$\max_{b} \quad \pi_2 = \frac{t}{32}(a-b-5)^2(1-a-b) \tag{29}$$

$$s.t. \quad \frac{\delta \pi_1}{\delta a} = \frac{t}{18}(b-a-3)(1+3a+b) = 0, \tag{30}$$

i.e., $a=b-3$, which is not acceptable, or $a=-(1+b)/3$. Substituting the latter into (29) and differentiating w.r.t. b, we obtain the following FOC:

$$\frac{\delta \pi_2}{\delta b} = -\frac{bt(b+4)}{9} = 0. \tag{31}$$

The solutions to (31) are $b=-4$, which doesn't satisfy the SOCs, and $b=0$, yielding $a=-\frac{1}{3}$ as the follower's optimal location. The perfect subgame equilibrium of the Stackelberg game with alternate leadership is then defined by $\left(a=-\frac{1}{3}, b=0\right)$ and $\left(p_1=\frac{16}{9}t, p_2=\frac{16}{9}t\right)$. Equilibrium profits are $\pi_1=\frac{16}{27}t$ and $\pi_2=\frac{32}{27}t$, whereas demands are $y_1=\frac{1}{3}$ and $y_2=\frac{2}{3}$. Since, at equal prices, the indifferent consumer lies in $\frac{1}{3}$, the equilibrium is strictly horizontal. *Q.E.D.*

Optimal Taxation
in a Spatial Duopoly

Luca Lambertini
Dipartimento di Scienze Economiche
Università degli Studi di Bologna

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Abstract

The aim of this paper is to investigate a horizontally differentiated duopoly in which a public authority imposes taxes on firms, in order to induce duopolists to choose the socially optimal locations. It is shown that there exists a proper tax scheme which warrants the optimal differentiation degree at equilibrium, as well as a net transfer of surplus to consumers.

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1. Introduction

The purpose of this paper is to analyse the impact of taxation on the behaviour of firms as well as on social welfare in a horizontally differentiated industry.

The recent literature on imperfect competition has taken into consideration the issue of regulating oligopolistic markets for homogeneous goods through public firms (Cremer et al., 1989; De Fraja and Delbono, 1989). Cremer, Marchand and Thisse (1991) study a horizontally differentiated oligopoly, i.e., an oligopoly in which at least one public firm operates, showing, *inter alia*, that in the duopoly case the presence of a public firm minimizing social costs is sufficient to yield the first best locational configuration.[1]

I will show that this result can also be obtained in a private duopoly, provided that the public authority imposes a proper taxation scheme, which builds on what might be called a Bertrand threat, i.e., a tax schedule such that firms can avoid paying taxes if and only if they accept to act as perfect competitors.

2. The model

Our starting point is the well known framework described by D'Aspremont et al. (1979). The duopolists sell a physically homogeneous good at zero marginal cost. Consumers are uniformly distributed along a unit interval, and their total density is 1. They have unit demands, and consumption yields a positive constant surplus s. Then, each consumer buys if and only if the net utility derived from consumption is non-negative:

$$U = s - tx^2 - p_i \geq 0, \quad t > 0, \quad i = 1,2; \qquad (1$$

1. The analysis of a mixed duopoly under vertical product differentiation is in Delbono et al. (1991).

where tx^2 is the transportation cost incurred by a consumer living at distance x from store i, and p_i is the price of good i. We assume that s is large enough for total demand to be always equal to 1. Firm 1 is located at a, while firm 2 is located at $1-b \geq a$, with $a, b \in R$. The demand functions are, respectively:

$$y_1 = a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \qquad (2$$

$$if \quad 0 < a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} < 1;$$

$$y_1 = 0 \qquad (2'$$

$$if \quad a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \leq 0;$$

$$y_1 = 1 \qquad (2''$$

$$if \quad a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \geq 1;$$

Demand to firm 2 is obtained by replacing a by b in equations (2-2"):

$$y_2 = 1 - y_1 = b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \qquad (3$$

$$if \quad 0 < b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} < 1;$$

$$y_2 = 0 \qquad (3'$$

$$if \quad b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \leq 0;$$

$$y_2 = 1 \qquad (3''$$

$$if \quad b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \geq 1.$$

Clearly, for a=1-b, i.e., when sellers locate at the same point, the demand functions are not determined and profits are nil as a consequence of the Bertrand paradox. In the absence of taxation, firms' profit functions are then

$$\pi_i = p_i y_i; \quad i = 1,2. \qquad (4$$

From a social standpoint, the optimal locations are obtained through the following minimization:

$$\min_{a,b} SC = t\left[\int_0^{\alpha_{1,2}} (x-a)^2 dx + \int_{\alpha_{1,2}}^{1} (1-b-x)^2 dx\right], \qquad (5$$

where $\alpha_{1,2}$ denotes the position of the consumer who is indifferent between the two firms. The minimization of social costs given by expression (5) is achieved setting $a = b = \frac{1}{4}$, which amounts to saying that the maximum distance between any consumer and the nearest firm is being minimized. It is well known that the quadratic transportation cost version of Hotelling's duopoly yields excess differentiation at equilibrium, as compared to the social optimum.[2] Consequently, a public authority could impose a tax on firms, increasing in the distance between the duopolistic equilibrium locations and the socially optimal ones.

The most intuitive tax schedule is the following:

$$T_i = k\left(l - \frac{1}{4}\right)^2, \quad k > 0, \quad l = a, b, \quad i = 1, 2, \tag{6}$$

i.e., firms pay no taxes if and only if they locate at the socially optimal points along the linear city.

Alternatively, the public authority could adopt the following tax schedule:

$$T_i = k\left(l - \frac{1}{2}\right)^2, \quad k > 0, \quad l = a, b, \quad i = 1, 2. \tag{7}$$

According to (7), firms pay no taxes if and only if they locate in the middle of the city. This would obviously entail zero profit at equilibrium, due to the Bertrand paradox. However, I will show that this is indeed the socially optimal solution, given a proper value of the ratio $\frac{k}{t}$, i.e., the relative weight of taxes and transportation costs. The next section is devoted to the analysis of the case in which rule (6) is adopted.

2. For an exhaustive survey of location models, see Gabszewicz and Thisse (1992).

3. Duopoly equilibrium when taxation starts at the socially optimal locations

If the public authority adopts rule (6), each duopolist's profit function is given by:

$$\pi_i = p_i y_i - k\left(l - \frac{1}{4}\right)^2, \quad l = a, b, \quad i = 1,2 \tag{8}$$

Firms noncooperatively maximize (8) in a two-stage game in which first they simultaneously choose locations, and then they simultaneously compete in prices. The game is solved through backward induction. The solution concept is a subgame perfect equilibrium in locations and prices (Selten, 1975).

The equilibrium prices are:[3]

$$p_1^* = t(1 - a - b)\left(1 + \frac{a - b}{3}\right), \tag{9}$$

$$p_2^* = t(1 - a - b)\left(1 + \frac{b - a}{3}\right), \tag{10}$$

Substituting (9-10) into the profit functions (8), we obtain:

$$\pi_1 = \frac{t}{18}(a - b + 3)^2(1 - a - b) - k\left(a - \frac{1}{4}\right)^2, \tag{11}$$

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3. Notice that the introduction of such a taxation scheme bears no consequence on the equilibrium prices. Cfr. D'Aspremont et al. (1979).

$$\pi_2 = \frac{t}{18}(a-b-3)^2(1-a-b) - k\left(b-\frac{1}{4}\right)^2. \qquad (12$$

Differentiating (11) and (12) w.r.t. a and b, respectively, we get the following first order conditions (FOCs):

$$\frac{\delta\pi_1}{\delta a} = \frac{1}{18}(9k - 36ak - 3t - 10at - 3a^2t - 2bt + 2abt + b^2t) = 0, \qquad (13$$

$$\frac{\delta\pi_2}{\delta b} = \frac{1}{18}(9k - 36bk - 3t - 10bt - 3b^2t - 2at + 2abt + a^2t) = 0, \qquad (14$$

The equilibrium locations are obtained by solving the FOCs (13-14) w.r.t. a and b:

$$a^* = b^* = \frac{1 - \frac{2t^3}{3kt^2 + t^3}}{4}, \qquad (15$$

which reduces to

$$a^* = b^* = \frac{3r-1}{12r+4}, \qquad (16$$

where $r = \frac{k}{t}$.[4] Notice that:

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4. Both the second order and the border conditions are met. Explicit calculations are available upon request.

$$\lim_{r \to \infty} \frac{3r-1}{12r+4} = \frac{1}{4}. \qquad (17$$

This means that this taxation scheme forces duopolists to choose the socially optimal locations only if the ratio between tax and transportation cost rates tends to infinity. Under this condition, the taxes accruing to the public authority are nil, while firms' equilibrium profits are $\pi_1^* = \pi_2^* = \frac{t}{4}$.

4. Duopoly equilibrium when taxation starts at the middle of the city

We are now going to investigate the alternative scheme defined by (7). Each duopolist now maximizes the following profit function:

$$\pi_i = p_i y_i - k\left(l - \frac{1}{2}\right)^2, \quad l = a, b, \quad i = 1,2 \qquad (18$$

The price equilibrium of the second stage is again represented by (9-10). In the location stage, firms maximize, respectively:

$$\pi_1 = \frac{t}{18}(a-b+3)^2(1-a-b) - k\left(a - \frac{1}{2}\right)^2, \qquad (19$$

$$\pi_2 = \frac{t}{18}(a-b-3)^2(1-a-b) - k\left(b - \frac{1}{2}\right)^2. \qquad (20$$

The FOCs are the following:

$$\frac{\delta\pi_1}{\delta a}=\frac{1}{18}(18k-36ak-3t-10at-3a^2t-2bt+2abt+b^2t)=0, \qquad (21$$

$$\frac{\delta\pi_2}{\delta b}=\frac{1}{18}(18k-36bk-3t-10bt-3b^2t-2at+2abt+a^2t)=0, \qquad (22$$

Solving the system (21-22) w.r.t. a and b, we get

$$a^*=b^*=\frac{2-\frac{3t^3}{3kt^2+t^3}}{4}, \qquad (23$$

which reduces to

$$a^*=b^*=\frac{6r-1}{12r+4}, \qquad (24$$

where $r=\frac{k}{t}$. Again, notice that:

$$a^*=b^*=\frac{1}{4} \quad iff \quad r=\frac{k}{t}=\frac{2}{3}. \qquad (25$$

Under this condition, the taxes collected by the public authority amount to $\frac{t}{12}$, while firms' equilibrium profits are $\pi_1^*=\pi_2^*=\frac{5t}{24}$.

A remark is now in order. Although we confine ourselves to the explicit analysis of the two polar cases described by (6) and (7), it appears that:

$$\sum_i \pi_i + \sum_i T_i = \frac{t}{2}, \quad \forall \frac{k}{t} \quad \ni \quad a^* = b^* = \frac{1}{4}. \qquad (26$$

This implies that, provided that the total transportation costs are being minimized by the choice of the socially optimal locations by the duopolists, the public authority finds it advantageous to adopt rule (7). This is made clear by inspection of the following table:

INSERT TABLE 1

The entries by column refers to the starting point of the quadratic taxation scheme available to the public authority. The first row shows the optimal value of the ratio $\frac{k}{t}$ corresponding to each of the four taxation schemes included in the table. The second row shows the total tax revenue accruing to the public authority at equilibrium. It can be seen the latter magnitude is higher, the closer to the middle of the city taxation starts.

5. Conclusions

We analysed the behaviour of a horizontally differentiated duopoly subject to taxation by the public authority. The main result is that the nearer to the socially optimal locations the authority starts taxing firms, the higher must be the ratio between tax and transportation cost rates in order for the differentiation degree to be socially optimal at equilibrium.

As a consequence, both the public authority and consumers prefer to adopt the tax scheme starting from the middle of the city, since this allows the highest surplus transfer from producers to consumers through taxation, simultanously ensuring the optimal degree of differentiation for a reasonably low value of the ratio between tax and transportation cost rates.