Technological Innovation and Diffusion, Fluctuations and Growth (II): Deterministic and Stochastic Laws of Motion*

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June 1993

Classification JEL nuber: 030, 040

Abstract
In the following Part II the deterministic and stochastic laws of motion arising from the processes depicted in Part I (particularly Section 2), are analyzed in detail. In Section 4 we study the typical non-linear logistic model emerging as the deterministic equivalent of the diffusion processes of Sections 2.3 and 2.4. The interaction of firm size and firm number are also studied within the same Section. In Section 5 we analyze the (asymptotic) stochastic laws of motion of the system. In particular, we study the Langevin equation and the approximations of the Fokker-Planck equations equivalent of the master equations of the same stochastic processes. We see that stochastic laws of motion may not be equivalent (not even asymptotically) to deterministic ones (e.g. due to variance effects which determine increasingly larger fluctuations).

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*This work is part of an ongoing research on technological innovation and growth, whose preliminary results have been presented at seminars held at the University of Trento, the Center for Economic Dynamics and Institutional Change of the University of Bologna, and at the University of Ancona, whose organizers Massimo Egidi and Pier Luigi Sacco, Gilberto Antonelli and Patrizio Bianchi, and Giorgio Fuà, respectively, and qualified audiences we thank wholeheartedly.

4. Growth and fluctuations: deterministic laws of motion

4.1. Firm size growth

Under the assumptions made in the previous Section we can design the following simple process for the firm maximized present value growth. In equilibrium, maximized cash flows imply that net profits are zero and that the firm present value sticks at the stationary steady state rate R. Yet, investment in R&D induces technological advances in the form of productivity increases and the perspective of positive, albeit temporary, profits. Once a new technology is implemented, a higher individual level of output is attained, together with a higher share of aggregate output. As cash flows increase, the maximized firm equity capital (given by the firm's market value) raises. If diffusion of the new technology takes place, either by means of adoption by imitation or by true innovation, profits will be slowly eroded. Thus, as the number of innovator increases, cash flows decrease and so does the firm market value.

More formally, let:

$$q(t) = \phi(t)\ l(t) \tag{4.1}$$

be a production function linear in the labour input, where $\phi(t)$ denotes labour productivity at time t: the latter is not a constant as it represents technology, which evolves over time. Suppose innovations arrive at a mean rate λ. Then, the average growth rate of productivity is $\pi\lambda$, corresponding to the differential equation:

$$\frac{\dot{\phi}(t)}{\phi(t)} = \pi\lambda\ , \tag{4.2}$$

which implies that the mean growth rate of output, for a given labour demand $l(t)=l$, is:

$$\frac{\dot{q}(t)}{q(t)} \equiv \frac{\frac{dq(t)}{dt}}{q(t)} = \frac{\frac{d\phi(t)l}{dt}}{\phi(t)l} = \pi\lambda , \tag{4.3}$$

a constant. Also, if prices are expressed as an inverse demand equation (as in Section 3 above) we will have that:

$$\begin{aligned} \dot{P}(t) &\equiv \frac{dP(q(t))}{dt} = \frac{dP}{dQ}\dot{q}(t) \\ &= -b[m+(\pi-1)n]\pi\lambda q(t) \end{aligned} \tag{4.4}$$

Finally, assume that $q(t)$ is a function of $A(t)$ and approximate such a function by a linear one (we might consider that it is certainly so in the neighborhood of the equilibrium but, for the sake of simplicity, we assume it holds everywhere):

$$q(t) = q(A(t)) \cong \eta A(t)|_{A=A^*} . \tag{4.5}$$

Thus:

$$\dot{q}(t) = \pi\lambda\eta A(t) . \tag{4.6}$$

Also:

$$\dot{P}(t) = -b[m+(\pi-1)n]\pi\lambda\eta A(t) . \tag{4.7}$$

Consider once more equation (3.6) substituting for $P(t)$ from (3.19):

$$\dot{A}(t) = RA(t) - (1-\psi)\big[a-b[m+(\pi-1)n]q(t)\big]q(t) + (1-\psi)\,Wl(t) \tag{4.8}$$

and differentiate it with respect to time:

$$\begin{aligned} \ddot{A}(t) = R\dot{A}(t) &- (1-\psi)\big[a-b[m+(\pi-1)n]q(t)\big]\dot{q}(t) \\ &+ (1-\psi)\,b[m+(\pi-1)n]\dot{q}(t)q(t) + (1-\psi)\,w\dot{l}(t) \end{aligned}$$

that is

$$\ddot{A}(t) = R\dot{A}(t) - (1-\psi)\big[a-2b[m+(\pi-1)n]q(t)\big]\dot{q}(t) + (1-\psi)w\dot{l}(t) \; . \qquad (4.9)$$

Substituting from (4.6) and (4.7) and assuming a given (fixed) *demand* for labor, i.e. $l(t)=l$, we have that:

$$\begin{aligned} \ddot{A}(t) &= R\dot{A}(t) - (1-\psi)\big[a-2b[m+(\pi-1)n]\eta A(t)\big]\eta\pi\lambda A(t) \\ &= R\dot{A}(t) - (1-\psi)a\eta\pi\lambda A(t) + (1-\psi)2b[m+(\pi-1)n]\eta^2\pi\lambda A^2(t) \; . \end{aligned} \qquad (4.10)$$

This is a second-order non-linear differential equation in $A(t)$ which can be solved by letting:

$$\begin{aligned} x(t) &= \dot{A}(t) \, ; \\ y(t) &= A(t) \, ; \\ \alpha &= R \, ; \\ \beta &= (1-\psi)a\eta\pi \, ; \\ \gamma &= 2(1-\psi)b[m+(\pi-1)n]\eta^2\pi \, ; \end{aligned} \qquad (4.11)$$

so that we can write (4.10) as the first-order system in x and y:

$$\begin{cases} \dot{x}(t) = \alpha x(t) - \beta\lambda y(t) + \gamma\lambda y^2(t) \\ \dot{y}(t) = x(t) \; . \end{cases} \qquad (4.12)$$

which is has two singular points at:

$$i) \begin{cases} \bar{x} = 0 \\ \bar{y} = 0 \end{cases} \qquad ii) \begin{cases} \bar{x} = 0 \\ \bar{y} = \beta/\gamma \end{cases} \qquad (4.13)$$

The Jacobian matrix of the linearized system can be easily computed to be:

$$\begin{aligned} J &= \begin{bmatrix} \alpha & -\beta\lambda + 2\lambda\gamma y \\ 1 & 0 \end{bmatrix}_{x=\bar{x},\, y=\bar{y}} \\ &= \begin{bmatrix} \alpha & \beta\lambda \\ 1 & 0 \end{bmatrix} \quad \text{for } (\bar{x}=0;\ \bar{y}=\beta/\gamma) \\ &= \begin{bmatrix} \alpha & -\beta\lambda \\ 1 & 0 \end{bmatrix} \quad \text{for } (\bar{x}=0;\ \bar{y}=0) \end{aligned}$$

The determinant of **J** is equal to $\beta\lambda$ and so is positive for $x=0$ and $y=0$, whereas it is equal to $-\beta\lambda$ and so is negative for $x=0$ and $y=\beta/\gamma$. The trace of **J** is equal to α in both cases and so is positive. In the first case, the characteristic roots are complex conjugates with positive real parts (as $\alpha>0$) if:

$$\alpha^2 - 4\beta < 0 \quad \Rightarrow \quad \alpha^2 < 4\beta \quad \Rightarrow \quad \alpha < 2\sqrt{\beta}$$

that is, if:

$$R^2 < 4(1-\psi)a\eta\pi \quad \Rightarrow \quad \frac{4a\eta\pi}{R^2}\psi < \frac{4a\eta\pi}{R^2} - 1 \quad \Rightarrow \quad \psi < 1 - \frac{R^2}{4a\eta\pi} \qquad (4.15)$$

Just looking at the parameters it appears that (4.15) looks reasonably likely[1]. Hence, the first singular point is an *unstable spiral point*: all trajectories will diverge from it according to an oscillating exploding motion[2]. In the second case, the characteristic roots are both real and positive: the equilibrium is a *unstable node*. All trajectories will diverge from it monotonically. Therefore, the two singular points are both *unstable*.

4.2. Firm growth (in number): aggregate growth

Insofar we have considered the deterministic law of motion of the maximized firm market value, given by the present value of the future stream of cash flows, in terms of the parameters of the model, taking the number of innovative firms as fixed. We are now going to consider explicitly how the laws of motion of the second state variable, namely the number of innovative firms, affects the dynamic behavior of the other variable, the firm market value (which proxies for size).

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[1] Recall that a is the intercept of the inverse demand function. The product of a times π times η is likely to be very large, so that the second member on the r.h.s. of (4.15) is bound to be small. Thus, the conditions for having complex conjugates roots are verified.

[2] If $\alpha^2 > 4\beta\lambda$, we would still have an unstable singular point, albeit of a *node* type. Thus, we would have a monotonically divergent motion away from the equilibrium.

Let us consider, first, the models of innovation and diffusion introduced in Section 2, as they all follow deterministic laws of motion, about which the random component hinges in a fluctuating manner. Those are, in fact, all Poisson processes, whose deterministic equivalent can be derived (but only under some assumptions, as we will see below in Section 5) when coinciding with their expectations. Consider, for instance, the diffusion process in a total population of firms of unlimited size (Section 2.2). The quantity λ_n in (2.8) could be called the average growth rate of a population of innovative firms of size *n*, and in the special case of (2.8) it is proportional to the actual size of the population of innovative firms. If the growth of the number of innovative firms would not be subject to random fluctuations and had a rate of increase proportional to the instantaneous size *n*(*t*), then the latter would vary in accordance to the deterministic differential equation:

$$\dot{n}(t) = \lambda\, n(t)\,, \tag{4.16}$$

implying:

$$n(t) = n(0)\, e^{\lambda t}\,, \tag{4.17}$$

namely, exponential growth. In this case, the expectation of *n*(*t*), the random variable following a Poisson process, coincides with the average growth rate of the deterministic process: this is why the latter is the deterministic *equivalent* of the former. Thus, *n*(*t*) describes not only a deterministic growth process, but also the expected population size at time *t* (see Figures 4.1 and 4.2).

In a similar manner we can treat the case of a diffusion process in a total population of firms of limited size *m* (Section 2.3). If we focus on the number of innovators only, as we have done in Section 2.3, by keeping the size of the total population fixed, then λ_n in (2.9) indicates, again, the average growth rate of a population of innovators of size *n*, whose growth process starts at time 0 with one individual and ends when it reaches the given maximum size m. Again, if growth would not be to random fluctuations, the instantaneous population size $n(t)$ would vary in accordance to the deterministic differential equation:

$$\dot{n}(t) = (m - n(t))\lambda n(t) \tag{4.18}$$

a well-known differential equation of the *logistic* type. This equation has two singular points in $n=0$ and $n=m$, but since $\dot{n}(t) \geq 0$ for $n(t) \geq 0$, it moves asymptotically toward m unless $n(t)$ is exactly zero. Thus, zero is an *unstable* state, while m is asymptotically stable (Figure 4.3). The higher the *diffusion rate* λ, the faster the convergence toward the equilibrium (Figure 4.4).

The case of multiple technological paradigms can also be treated accordingly. As we have seen in Section 2.4, a succession of technologies can be treated as a *birth-death* process, where the state of the sistem is defined by the number of firms adopting a given technology. Thus, the state n is defined as the state in which there are n firms who have already switched from an old technology to a new one but have not found yet a newer technology to switch to. In a deterministic treatment the variable $n(t)$ satisfies the equation

$$\dot{n}(t) = \lambda(n(t)) - \mu(n(t)),$$

or

$$\dot{n}(t) = \lambda[m - n(t)]n(t) - \mu n(t), \tag{4.19}$$

with possible equilibria at the roots of:

$$\lambda(\bar{n}) = \mu(\bar{n}) \Rightarrow \lambda(m-\bar{n})\bar{n} = \mu\bar{n} \tag{4.20}$$

which are

$$\bar{n} = 0; \qquad \bar{n} = m - \frac{\mu}{\lambda} . \tag{4.21}$$

By solving the linearized approximation of (2.19) and evaluating it at the two equilibrium points, we find that, for $n=0$, the system has unstable behavior, of a monotonically divergent type, whereas for $n=m-\lambda/\mu$ the system is asymptotically stable (see Figure 4.5).

4.3. Aggregate growth by means of firm growth (in size and number)

The examples given in Section 2 have displayed a variety of cases in which the number of innovative firms follows a Poisson process with different mean rates, depending on the model specification. In Section 2.1, we have just considered the case of firms who do not interact, so that their ability to innovate is just a function of their research effort and of the probability of being successful. In Sections 2.2, 2.3, and 2.4 we have considered simple *diffusion* processes, where the mean rates of arrival of new technologies depend on the number of firms already adopting. Those examples are basically intended to capture the imitating attitude involved in the process of adoption of new technologies. Yet, as we have seen above, we have not taken into account explicitly the interaction between the process of technological innovation and diffusion and the dynamic behavior of the *single* firm who innovates. Let us start, then, by analyzing how does the number $n(t)$ of innovative firms of value $A(t)$ behave. Let $z(t)$ denote the number of firms of a given size, i.e. $z(t)=n(t)A(t)$ at time t. Here we take into account the interaction of the value of the firm and the number of firms. In fact:

$$\dot{z}(t) = \dot{n}(t)A(t) + n(t)\dot{A}(t) . \tag{4.22}$$

In this way, $A(t)$ becomes a *scale* factor for $n(t)$, while at the same time $n(t)$ is a *scale* factor for $A(t)$.

An interesting example can be given by considering the logistic birth-death model of Section 2.4 for $n(t)$ together with the present-value model equation for $A(t)$:

$$\begin{aligned} \dot{n}(t) &= [m - n(t)]\lambda n(t) - \mu n(t) \\ \dot{A}(t) &= \left[R - a(1-\psi)\eta\right]A(t) + b(1-\psi)\eta^2\left[m + (\pi - 1)n(t)\right]A^2(t) + (1-\psi)Wl \end{aligned} \tag{4.23}$$

which implies that:

$$\begin{aligned} \dot{z}(t) &= \dot{n}(t)A(t) + n(t)\dot{A}(t) \Rightarrow \\ \dot{z}(t) &= \left[[m - n(t)]\lambda n(t) - \mu n(t)\right]A(t) \\ &\quad + \left[\left[R - a(1-\psi)\eta\right]A(t) + b(1-\psi)\eta^2\left[m + (\pi - 1)n(t)\right]A^2(t) + (1-\psi)Wl\right]n(t) \end{aligned} \tag{4.24}$$

Thus, when $n(t)$ does not grow, $z(t)$ grows because of the firm-size effect, whereas when $A(t)$ does not grow, $z(t)$ grows because of the firm-number effect. So, $z(t)$ does not grow if the growth rate of $n(t)$ equals the negative of the growth rate of $A(t)$. Rewriting (4.24):

$$\left\{ \begin{aligned} \dot{z}(t) &= \left[[m - n(t)]\lambda z(t) - \mu z(t)\right] \\ &\quad + \left[[R - a(1-\psi)\eta]z(t) + b(1-\psi)\eta^2[m + (\pi - 1)n(t)]z(t)A(t) + (1-\psi)Wln(t)\right] \\ \dot{n}(t) &= [m - n(t)]\lambda n(t) - \mu n(t) \\ \dot{A}(t) &= [R - a(1-\psi)\eta]A(t) + b(1-\psi)\eta^2[m + (\pi - 1)n(t)]A^2(t) + (1-\psi)Wl \end{aligned} \right. \tag{4.25}$$

a first-order system in three differential equations whose last one is in the form of a *Bernoulli* equation.

A different setting could be put forward as follows (here we look at the interaction between firm growth and growth of the number of firms by a different perspective). By considering simultaneously *true innovation* (as in Section 2.1) and *innovation by imitation* (as in Section 2.2 and 2.3), and abstracting by random fluctuations, we can posit the following deterministic law of motion for $n(t)$, the number of innovative firms at time t:

$$\dot{n}(t) = p + \lambda n(t) + \mu \dot{A}(t) n(t) \tag{4.26}$$

Equation (4.26) is the combination of three different processes. The first is the deterministic equivalent of a Poisson process whose mean rate indicates the rate at which new technologies are discovered, independently of the number of existing firms and the research effort: its expectation coincides with p. This is the *innovation* component of the whole process, and it represents a "gain" in the balance. The second is the deterministic equivalent of a Poisson process describing an ever increasing number of firms who switch to a new technology proportionally to the number of innovators already present (as in Section 2.2): its average rate is equal to λn, meaning that its growth rate is constant and equal to λ. This represents the *diffusion* component, and it adds to the "gain" in the balance, too. The third captures the interaction between the rate of change of value of the firm at t and the number of innovative firms, $n(t)$[3]. On one hand, this interaction is certainly positive: the higher the firm value, the higher R&D investment, the higher expected profits. On the other hand, this interaction is negative: the more innovative firms are present, the lower are profits. Also, there can be a negative effect due to the narrowing of market shares for the non-innovators, which makes difficult for other firms to innovate altogether. Hence, the rate of the process, given by μ, can certainly be either positive (in which case it would be a "gain" in the balance) or negative (in which case it would be a "loss"). Therefore, the last component represents the *interaction* component between *firm growth* and *aggregate growth.*

Now, a way to get around the problem and solving simultaneously for the

[3] The reason why in (4.26) we have the rate of change of $A(t)$ rather than $A(t)$ itself is that there is a stationary long-run component in the growth trend of $A(t)$ which is independent of technological change. That is, as $A(t)$ grows anyway (albeit at a constant rate), we need to capture the change in its rate of change, and this comes about exactly because of productivity improvements that lead to higher output and higher profits. Hence, the number of innovative firms is related to the rate of change of $A(t)$ rather than to its level.

deterministic laws of motion of the value of the firm (the growth of the firm) and the number of innovators (the growth of the economy) could be the following. If we put together the first-order differential equation (4.26) with the first-order differential system (4.12), by letting

$$\begin{aligned} x(t) &= \dot{A}(t)\,; \\ y(t) &= A(t)\,; \\ \alpha &= R\,; \\ \beta &= (1-\psi)a\eta\pi; \\ \gamma &= 2(1-\psi)b\eta^2\pi\,; \end{aligned} \tag{4.27}$$

we then have:

$$\begin{cases} \dot{n}(t) = p + \lambda n(t) + \mu x(t)n(t) \\ \dot{x}(t) = \alpha x(t) - \beta\lambda y(t) + \gamma\lambda m y^2(t) + \gamma(\pi-1)\lambda n(t)y^2(t) \\ \dot{y}(t) = x(t)\ . \end{cases} \tag{4.28}$$

whose singular points are given by:

$$i)\begin{cases} \bar{n} = -p/\lambda \\ \bar{x} = 0 \\ \bar{y} = 0 \end{cases} \qquad ii)\begin{cases} \bar{n} = -p/\lambda \\ \bar{y} = \dfrac{\beta}{\gamma[m-(\pi-1)p]} \\ \bar{x} = 0 \end{cases} \tag{4.29}$$

The Jacobian of the linearized system can be computed to be:

$$\boldsymbol{J} = \begin{bmatrix} \lambda+\mu x & \mu n & 0 \\ \gamma(\pi-1)\lambda y^2 & \alpha & -\beta\lambda+2\gamma\lambda m y+2\gamma(\pi-1)\lambda n y \\ 0 & 1 & 0 \end{bmatrix}_{(n=\bar{n},x=\bar{x},y=\bar{y})} \tag{4.30}$$

Now, the trace of this matrix is positive for $x=0$: hence, we would certainly have that some of the roots are positive. The first principal minor is positive ($\lambda>0$), while the second can be negative if $\alpha\lambda+[\beta\mu p(\pi-1)/(m-(\pi-1)p)]<0$: this happens if $\mu<0$ and $\alpha\lambda< -[\beta\mu p(\pi-1)/(m-$

$(\pi-1)p)]$. In any case, the system is unstable (maybe with oscillating behavior). A more detailed analysis of this system, which is not carried through here, may reveal interesting dynamics, although it is rather complicated and subject to a difficult interpretation.

5. Growth and fluctuations: stochastic laws of motion

5.1. Deterministic and stochastic forces at work: the Langevin equation

As we know, the existence of macroscopic deterministic laws is a very significant result, although there is often a limit in which the solution of a master equation (like the Chapman-Kolmogorov equation) can be approximated asymptotically by a deterministic part (which is the solution of a deterministic differential equation), plus a fluctuating part, describable by a stochastic differential equation, whose coefficients are given by the original master equation. There are cases in which the deterministic laws of motion approximate quite satisfactorily the dynamics of a system, through the *deterministic equivalent* of the stochastic process. However, these descriptions are generally unable to capture the intrinsically random, and therefore fluctuating, component of the process (an example will be discussed below in Section 5.2). Yet, the description of a dynamic process in terms of a deterministic component with which a stochastic part interacts, whether additively or multiplicatively, is of interest for some purposes. The Langevin equation, which combines a deterministic law of motion and a stochastic process, is one such case. The Langevin equation can be described heuristically as an ordinary differential equation on which a rapidly and irregularly fluctuating random function of time acts. Here, a deterministic law is assumed first and then a random component is added, under some additional albeit restrictive assumptions. Although generally dismissed as a good approximation of a stochastic law of motion, it provides an interesting example on the way to a completely stochastic dynamic modeling.

We first consider a *completely deterministic motion* and treat equation (4.19), rewritten here as

$$\dot{n}(t) = \lambda(m - n(t))\, n(t) - \mu\, n(t) = (\lambda m - \mu)\, n(t) - \lambda\, n^2(t) \tag{5.1}$$

which may be interpreted as the equation of overdamped motion of a population (in our case, the number of innovative firms) under the two "opposite" *forces* λ and μ and an upper bound m. Since we want to derive equations capable of describing both deterministic and random processes we treat the motion of the population according to a different formalism. We suppose that our population follows the deterministic law given in (5.1). In the course of time, the population proceeds along a path in the n-t plane. Thus, if we pick out a fixed time t, we can ask for the probability of finding the population at a certain coordinate n. We know, in fact, that (5.1) describes the growth as *proportional* to the population size and *inversely proportional* to the square of the population size. The probability of finding the population at a certain coordinate n is evidently zero if $n \neq n(t)$, where $n(t)$ is the solution of (5.1). We know that one such a solution is $n=m-\lambda/\mu$ (the other being $n=0$). What kind of probability function yields 1 if $n=n(t)$ and 0 otherwise? This can be achieved by introducing a probability density equal to the δ-function:

$$P(n, t) = \delta(n - n(t)) \tag{5.2}$$

(see Figure 5.1). We know that an integral over a function $\delta(n-n_0)$ vanishes if the integration interval does not contain n_0 and that it yields 1 if that interval contains a surrounding of n_0:

$$\int_{n_0-\epsilon}^{n_0+\epsilon} \delta(n - n_0) = 1 \tag{5.3}$$

$$= 0 \quad \text{otherwise.}$$

We now want to derive an equation for the probability distribution P. Thus, let us differentiate P with respect to time to get:

$$\dot{P}(n, t) = -\frac{d}{dn}[\delta(n - n(t))]\, \dot{n}(t) = -\frac{d\,P(n, t)}{d\,n}\, \dot{n}(t) \tag{5.4}$$

and, substituting from (5.1):

$$\dot{P}(n, t) = -\frac{d\,P(n, t)}{d\,n} (\lambda m - \mu - \lambda n(t))\, n(t) \tag{5.5}$$

Consider now that, as n can follow several different paths, the probability distribution for a given path, i, is

$$P_i(n, t) = \delta(n - n_i(t)) \tag{5.6}$$

If we take the average over all paths we can write the function

$$f(n, t) = \langle P(n, t) \rangle \tag{5.7}$$

and if we denote the probability of occurrence of a path i with p_i, this probability distribution can be written in the form

$$f(n, t) = \sum_i p_i\, \delta(n - n_i(t)) = \langle \delta(n - n(t)) \rangle \;. \tag{5.8}$$

Here fdn gives the probability of finding the population at position n in the interval dn of time t. To this end, we can investigate the change of f in a time interval Δt

$$\Delta f(n, t) \equiv f(n, t + \Delta t) - f(n, t) \tag{5.9}$$

which, by use of (5.8) takes the form

$$\Delta f(n,t) = \langle \delta(n - n(t+\Delta t)) \rangle - \langle \delta(n - n(t)) \rangle \ . \tag{5.10}$$

Now put

$$n(t+\Delta t) = n(t) + \Delta n(t) \tag{5.11}$$

and expand the δ-function with respect to powers of Δn

$$\Delta f(n,t) = \langle [-\frac{d}{dn}\delta(n - n(t))]\,\Delta n(t) \rangle + \frac{1}{2}\langle [\frac{d^2}{dn^2}\delta(n - n(t))]\,(\Delta n(t))^2 \rangle \ . \tag{5.12}$$

We are now ready to introduce the Langevin equation. The simplest form of Langevin equation can be written as (see Gardiner (1983, p. 80))

$$\frac{dx}{dt} = a(x,t) + b(x,t)\,\xi(t)\,, \tag{5.13}$$

where x is the variable of interest, $a(x,t)$ and $b(x,t)$ are certain known functions and $\xi(t)$ is a *rapidly* fluctuating random term. It is assumed that for $t \neq t'$, $\xi(t)$ and $\xi(t')$ are statistically independent, that $\langle \xi(t) \rangle = 0$, since any non-zero mean component can be absorbed into the definition of $a(x,t)$, and that

$$\langle \xi(t)\,\xi(t') \rangle = \delta(t - t') \tag{5.14}$$

which satisfies the requirement of no correlation at different times (a good example is an uncorrelated white noise). Also, and most importantly, it is assumed that $x(t)$ and $\xi(t)$ are uncorrelated at all times[4]. Hence, from (5.1) the Langevin equation for $n(t)$ can be written as:

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[4] We will discuss the implications of such assumptions later in Section 5.3, together with the possible extensions which might overcome the shortcomings of the Langevin equation.

$$\dot{n}(t) = (\lambda m - \mu - \lambda n(t))\, n(t) + F(t) \tag{5.15}$$

where $F(t)$ is a zero-mean random component *independent* of $n(t)$ as of time t. Thus, we can find Δn by integration over the time interval Δt

$$\int_t^{t+\Delta t} \dot{n}(t')\, dt' = n(t+\Delta t) - n(t) \equiv \Delta n$$

$$= \int_t^{t+\Delta t} (\lambda m - \mu) n(t')\, dt' - \int_t^{t+\Delta t} \lambda n^2(t')\, dt' + \int_t^{t+\Delta t} F(t')\, dt'$$

which gives

$$\Delta n = [\lambda m - \mu - \lambda n(t)]\, n(t)\, \Delta t + \Delta F(t) \ . \tag{5.16}$$

Consider now the first term on the right-hand side of (5.12). By inserting in it the right-hand side of (5.16) we get

$$- \frac{d}{dn}\left[\langle \delta(n - n(t))\, [(\lambda m - \mu - \lambda n(t)) n(t)]\, \Delta t \rangle + \langle \delta(n - n(t)) \rangle \langle \Delta F(t) \rangle \right] \tag{5.17}$$

The splitting of the average containing ΔF in the product of two averages is due to the fact that ΔF contains the shocks which have occurred after time t, whereas $n(t)$ is determined by all the shocks prior to this time. Due to the independence of the shocks (this is a Poisson process), we may split the total average into the product of the averages as in (5.17). Since the average of F vanishes (the process is *on average* along the growth path) and so does ΔF, (5.17) reduces to

$$- \Delta t \frac{d}{dn} \langle \delta(n - n(t))\, (\lambda m - \mu - \lambda n) n \rangle \tag{5.18}$$

where $n(t)$ has been replaced by n. Consider now the second term on the right-hand side of (5.12). By the same arguments it can be split as

$$\frac{1}{2}\frac{d^2}{dn^2}\langle\delta(n-n(t))\rangle \quad \langle[\Delta n(t)]^2\rangle \ . \tag{5.19}$$

When substituting for Δn from (5.16) we have terms containing Δt^2, terms containing $\Delta t\Delta F$ and $(\Delta F)^2$. Now, it turns out that $\langle(\Delta F)^2\rangle$ goes with Δt. In fact, the average of ΔF vanishes (and so does the average of Δt^2) and thus $\langle(\Delta F)^2\rangle$ is the only contribution to (5.19) which is linear in Δt. Evaluating $\langle(\Delta F)^2\rangle$:

$$\langle\Delta F(t)^2\rangle \ \equiv \ \langle\Delta F(t)\,\Delta F(t)\rangle \ = \ \int_t^{t+\Delta t}\int_t^{t+\Delta t} dt'\,dt''\ \langle F(t')\,F(t'')\rangle \ . \tag{5.20}$$

Assuming that the correlation function between the F's is equal to

$$\langle F(t)\,F(t')\rangle \ = \ Q\delta(t-t') \tag{5.21}$$

allows the evaluation of (5.20), yielding

$$Q\,\Delta t \tag{5.22}$$

Thus, (5.19) can be written as

$$\frac{1}{2}\frac{d^2}{dn^2}\langle\delta(n-n(t))\rangle \ \ Q(\Delta t) \ . \tag{5.23}$$

Hence, taking the original equation (5.12), dividing it by Δt, substituing from (5.18) and (5.23), and considering (5.8), we obtain

$$\frac{\Delta f(n,t)}{\Delta t} \underset{\Delta t\to 0}{=} \frac{df(n,t)}{dt} = -\frac{d}{dn}(\lambda m-\mu-\lambda n)\,n f(n,t) + \frac{1}{2}\frac{d^2}{dn^2}Q f(n,t) \tag{5.24}$$

This is the so-called *Fokker-Planck equation*, describing the change of the probability distribution of n over time (see Figure 5.2). The first term on the right, which multiplies f, namely $(\lambda m\text{-}\mu\text{-}\lambda n)n$, is called *drift coefficient*, while the Q term is the *diffusion coefficient*.

The *stationary* solution of the FPE is found by equating (5.24) to zero, that is, by finding those values of (5.24) for which f is time-independent. From (5.24) we have that

$$- \frac{d}{dn}[(\lambda m - \mu - \lambda n) n f(n,t)] + \frac{1}{2} Q \frac{d^2 f(n,t)}{dn^2} = 0$$
$$\Rightarrow \quad \frac{d}{dn}\left[(\lambda m - \mu - \lambda n) n f(n,t) - \frac{1}{2} Q \frac{df(n,t)}{dn}\right] = 0 \tag{5.25}$$

which implies that

$$\frac{df(n,t)}{dn} = \frac{2}{Q}(\lambda m - \mu - \lambda n) n f(n,t) \tag{5.26}$$

which is solved by

$$f(n) = \mathbb{N}\, e^{-2V(n)/Q} \tag{5.27}$$

where

$$V(n) = -\int_{n_0}^{n} (\lambda m - \mu - \lambda n)\, n\, dn = -\frac{1}{2}(\lambda m - \mu)\, n^2 + \frac{1}{3}\lambda\, n^3 \tag{5.28}$$

(assuming n_0=0) has the meaning of a *potential*, and

$$\mathbb{N} \text{ is determined by } \int_{-\infty}^{+\infty} f(n)\, dn = 1 \tag{5.29}$$

(a normalization constant). The case is illustrated in Figure 5.3.

What happens to our population of innovators with coordinate n can be described as follows. The random force $F(t)$ (the stochastic component of growth) pushes the population up the *potential slope*, which stems from the *systematic force*, $(\lambda m\text{-}\mu\text{-}\lambda n)n$, namely, the systematic component. After each random "push", namely a shock induced by a random arrival or departure, the population falls down the slope towards the equilibrium state where $n=m-\mu/\lambda$. Therefore, the most probable position is $n=m-\mu/\lambda$, but other positions are also possible, due to the random arrivals and departures. Since many random shocks are necessary to drive the population far from $n=m-\mu/\lambda$, the probability of finding it in those regions decreases rapidly. When we let λ become smaller, the restoring force becomes weaker. As a consequence, the potential curve becomes flatter and the probability density $f(n)$ is more spread out.

5.2. Approximations of the Fokker-Planck-Equations equivalent of Master Equations

Even though macroscopic deterministic laws can show to exist, a limit is known to exist at which the solution of a master equation (like the Chapman-Kolmogorov equation) can be approximated asymptotically in terms of a parameter describing the system size by a deterministic part (which is the solution of a deterministic differential equation), plus a fluctuating part, describable by a stochastic differential equation, whose coefficients are given by the original master equation. The results of these expansions is the development of simple Fokker-Planck equations equivalent (in an asymptotic approximation) of master equations. Even though there are different ways of formulating the first-order approximate Fokker-Planck equation (see e.g. Gardiner (1983)), we will follow here the system size expansion of van Kampen.

Consider the case of multiple technological paradigms, as modeled by a *birth and death* process where the state of the system is defined by the number of firms, n, adopting a given technology. Here, the random variable is N, which denotes the number of innovative firms, and the values it can take are denoted with n (a non-negative integer). To model such a process we need to consider the conditional probability $P(n,t|n',t')$ of N being equal to n

at time t given that N was equal to n' at time t' and its corresponding master equation. Let λ denote the (generic) *birth-rate* of the process (the transition probability $p^+(n)$ per unit time of going from n-1 to n, that is, of entering state n) and μn denote the *death-rate* of the process (the transition probability $p^+(n)$ of going from n to n-1, that is, of exiting state n). Notice that the *mean* of n satisfies

$$\frac{d}{dt}\langle n(t)\rangle = \langle p^+(n)\rangle - \langle p^-(n)\rangle \tag{5.30}$$

whose corresponding deterministic equation is the one which would be obtained by neglecting fluctuations, that is

$$\frac{d}{dt}n(t) = p^+(n) - p^-(n)\,. \tag{5.31}$$

Notice also that a stationary steady state occurs deterministically when

$$p^+(n) = p^-(n)\,. \tag{5.32}$$

The *generating function* of such process is, in general:

$$G(s,\,t) = F[(s-1)e^{-\mu t}]\,e^{(s-1)\lambda/\mu} \tag{5.33}$$

where normalization requires $G(1,t)=1$, and hence $F(0)=1$. The initial condition determines F, and the conditional probability $P(n,\,t\,|\,\bar{n},\,0)$ can be calculated, but is very complicated and of little practical use. It turns out that it is better to work with the generating function from which we get:

$$G(s,\,0) = s^{\bar{n}} = F(s-1)\,e^{(s-1)\lambda/\mu} \tag{5.34}$$

and

$$G(s,t) = e^{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})}\,[1 + (s-1)\,e^{-\mu t}]^{\bar{n}}. \tag{5.35}$$

From the generating function we can compute the *mean*, the *variance* and the *correlation* of $n(t)$:

$$\langle n(t)\rangle = \frac{\lambda}{\mu}(1 - e^{-\mu t}) + \bar{n}\,e^{-\mu t} \tag{5.36}$$

$$\begin{aligned}\langle n(t)\,[n(t)-1]\rangle &= \langle n(t)\rangle^2 - \bar{n}\,e^{-2\mu t}\\ \mathrm{VAR}[n(t)] &= \left[\bar{n}\,e^{-\mu t} + \frac{\lambda}{\mu}\right](1-e^{-\mu t})\end{aligned} \tag{5.37}$$

As t tends to infinity, we find from (5.33) the generating function:

$$G(s, t\to\infty) = e^{(s-1)\lambda/\mu} \tag{5.38}$$

corresponding to the *Poissonian solution*, namely the *stationary* solution $P^*(n)$:

$$P^*(n) = \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!}\,e^{-\frac{\lambda}{\mu}} \tag{5.39}$$

Since the equation of time evolution for $\langle n(t)\rangle$ is linear, we can apply the *regression theorem* (Gardiner (1983)), which states that the stationary autocorrelation function has the same time dependence as the mean, and its value at $t=0$ is the stationary variance. Hence:

$$\langle n(t)\rangle^* = \frac{\lambda}{\mu} \tag{5.40}$$

$$\begin{aligned}\langle n(t)\,n(0)\rangle^* &= e^{-\mu t}\,\frac{\lambda}{\mu}\\ \mathrm{VAR}[n(t)]^* &= \frac{\lambda}{\mu}\end{aligned} \tag{5.41}$$

These were time-independent solutions, but there also exist *Poissonian time-dependent* *solutions*. Suppose we choose:

$$P(n,0) = \frac{e^{-\alpha_0}\alpha_0^n}{n!} \tag{5.42}$$

then

$$G(s,0) = e^{(s-1)\alpha_0} \tag{5.43}$$

and

$$G(s,t) = e^{(s-1)(\alpha_0 e^{-\mu t} + \frac{\lambda}{\mu})} \tag{5.44}$$

corresponding to

$$P(n,t) = \frac{e^{-\alpha(t)}\alpha(t)^n}{n!} \tag{5.45}$$

with

$$\alpha(t) = \alpha_0 e^{-\mu t} + \frac{\lambda}{\mu} \tag{5.46}$$

where $\alpha(t)$ is the solution of the deterministic equation:

$$\dot{n}(t) = \lambda - \mu\, n(t) \tag{5.47}$$

with the initial condition $n(0)=\alpha_0$. We will come back to the qualitative effects of this time-dependence in a moment, as it is the dependence on time which is the distinctive characteristic of this approach.

Let us first introduce a *system-size parameter* Ω (in our case it can be m, the total number of firms in the economy) such that the transition probabilities can be written in terms of the *intensive* variables n/m. In van Kampen's notation (Gardiner (1983, p. 250ff), let n denote the *extensive* variable (number of innovative firms) and $z=n/m$ the *intensive* variable (concentration of innovative firms). The limit of interest is large Ω (i.e. large m) at fixed z, which corresponds to a macroscopic system. Thus, we can choose the new variable, z, so that

$$n = \Omega\,\phi(t) + \Omega^{1/2} z \equiv m\,\phi(t) + m^{1/2} z \tag{5.48}$$

where $\phi(t)$ is a function to be determined. We can define the Kramers-Moyal *dummy* variables (Gardiner (1983, p. 249)) $\alpha_q(x)$ to be proportional to Ω, so that

$$\alpha_q(n) = \Omega\,\tilde{\alpha}_q(n) \equiv m\,\tilde{\alpha}_q(n) \tag{5.49}$$

In our case

$$\begin{aligned} \alpha_1(n) &= m\,(\frac{\lambda}{m} - \mu\, z) \\ \alpha_2(n) &= m\,(\frac{\lambda}{m} + \mu\, z) \end{aligned} \tag{5.50}$$

By taking the Kramers-Moyal expansion and changing the variable one gets:

$$\frac{\partial P(z,t)}{\partial t} - \Omega^{1/2}\,\phi'(t)\,\frac{\partial P(z,t)}{\partial z} = \sum_{q=1}^{\infty} \frac{\Omega^{1-q/2}}{q!} \left[-\frac{\partial}{\partial z}\right]^{q} \tilde{\alpha}_q\,[\phi(t) + \Omega^{-1/2} z]\, P(z,t) \tag{5.51}$$

where the terms of order $\Omega^{1/2}$ on either side will cancel if $\phi(t)$ obeys

$$\phi'(t) = \tilde{\alpha}_1\,[\phi(t)] \tag{5.52}$$

which is the *deterministic equation* expected. As (5.50) shows, this holds true in our case. Then, expanding $\tilde{\alpha}_q[\phi(t)+\Omega^{-1/2}]$ in powers of $\Omega^{-1/2}$ and rearranging, one finds

$$\frac{\partial P(z,t)}{\partial t} = \sum_{r=2}^{\infty} \frac{\Omega^{-(r-2)/2}}{r!} \sum_{q=1}^{\infty} \frac{r!}{q!(r-q)!}\, \tilde{\alpha}_q^{r-q}[\phi(t)] \left[-\frac{\partial}{\partial z}\right]^{q} z^{r-q}\, P(z,t)\ . \tag{5.53}$$

Taking the limit for large Ω only the r=2 term survives, thus giving

$$\frac{\partial P(z,t)}{\partial t} = -\tilde{\alpha}_1^1[\phi(t)]\frac{\partial}{\partial z}z\,P(z,t) + \frac{1}{2}\tilde{\alpha}_2[\phi(t)]\frac{\partial^2}{\partial z^2}P(z,t)\ . \tag{5.54}$$

In our case, the deterministic equation is of the form

$$\dot{\phi}(t) = \frac{\lambda}{m} - \mu\,\phi(t) \tag{5.55}$$

whose solution is

$$\phi(t) = \phi(0)e^{-\mu t} + \frac{\lambda}{m\mu}(1 - e^{-\mu t})\ . \tag{5.56}$$

The Fokker-Planck equation is thus

$$\frac{\partial P(z)}{\partial t} = \mu\frac{\partial}{\partial z}zP(z) + \frac{1}{2}\frac{\partial^2}{\partial z^2}[\frac{\lambda}{m} + \mu\,\phi(t)]\,P(z) \tag{5.57}$$

and

$$\langle z(t)\rangle = z(0)\,e^{-\mu t}\ . \tag{5.58}$$

We can assume that $z(0)=0$, since the initial condition can be fully dealt with by the initial condition on ϕ. Then we find

$$\mathrm{VAR}[z(t)] = [\frac{\lambda}{m\mu} + \phi(0)\,e^{-\mu t}]\,(1 - e^{-\mu t}) \tag{5.59}$$

so that

$$\langle n(t)\rangle = m\,\phi(t) = m\,\phi(0)\,e^{-\mu t} + \frac{\lambda}{m\mu}(1 - e^{-\mu t}) \tag{5.60}$$

and

$$\mathrm{VAR}[n(t)] = m\,\mathrm{VAR}[z(t)] = [\frac{\lambda}{m\mu} + m\phi(0)\,e^{-\mu t}]\,(1 - e^{-\mu t}) \;. \qquad (5.61)$$

If we identify $m\phi(0)$ with $\bar{n}$, we see that these are exactly the same as the exact solutions (5.36) and (5.37) above. The *stationary* solution of (5.51) is

$$P^{*}(z) = \mathbb{N}\,e^{-\frac{\mu m z^2}{2\lambda}} \qquad (5.62)$$

which is the Gaussian approximation to the exact Poissonian. The stationary solution of the Kramers-Moyal equation is

$$P^{*}(n) = \mathbb{N}\,(\lambda + \mu n)^{-1+4\lambda/\mu} e^{-2n} \qquad (5.63)$$

It must be noticed that the existence of a system-size expansion as outlined above depend on the fact that $\tilde{\alpha}_1'(n)$ does not vanish. Consider again (5.48), rewritten as

$$n = \Omega\,\phi(t) + \Omega^{\sigma} z \qquad (5.64)$$

so that (5.51) becomes

$$\frac{\partial P(z,t)}{\partial t} - \Omega^{1-\sigma}\phi'(t)\,\frac{\partial P(z,t)}{\partial z} = \sum_{q=1}^{\infty}\frac{\Omega^{1-\sigma q}}{q!}\left(-\frac{\partial}{\partial z}\right)^{q}\tilde{\alpha}_q\,[\phi(t) + \Omega^{\sigma-1} z]\,P(z,t) \;. \qquad (5.65)$$

Suppose now that the first r-1 derivatives of $\tilde{\alpha}_1(\phi^{*})$ vanish, where ϕ^{*} is a stationary solution of the deterministic equation. Then, if we replace $\phi(t)$ with ϕ^{*}, (5.65) becomes to lowest order:

$$\frac{\partial P(z,t)}{\partial t} = -\frac{1}{r!}\tilde{\alpha}_1^{(r)}(\phi^{*})\,\Omega^{(1-r)(1-\sigma)} + \frac{1}{2}\alpha_2(\phi^{*})\,\Omega^{1-2\sigma}\frac{\partial^2}{\partial z}P(z,t) + \text{higher terms} \;. \qquad (5.66)$$

To make sure z remains of order unity, set

$$(1 - r)(1 - \sigma) = (1 - 2\sigma) \quad \Rightarrow \quad \sigma = \frac{r}{r+1} \tag{5.67}$$

so that the result is

$$\frac{\partial P(z,t)}{\partial t} = \Omega^{-\frac{r-1}{r+1}} \left[- \frac{1}{r!} \tilde{\alpha}_1^{(r)} z^r P(z,t) + \frac{1}{2} \tilde{\alpha}_2 \frac{\partial^2}{\partial z^2} P(z,t) \right]$$

where $\tilde{\alpha}_1^{(r)}$ and $\tilde{\alpha}_2$ are evaluated at ϕ^*. This is very interesting, because now the fluctuations vary on a slower time scale τ given by

$$\tau = t\,\Omega^{-\frac{r-1}{r+1}} \tag{5.68}$$

and the equation for the average is

$$\frac{d\langle n \rangle}{d\tau} = \frac{1}{r!} \tilde{\alpha}_1(r) \langle n^r \rangle \tag{5.69}$$

which is no longer the one associated with the linearized deterministic equation. Obviously, stability depends on the sign of $\tilde{\alpha}_1^{(r)}$ and whether r is odd or even. Conversely, the long-time scale is given by

$$\tau = t\,\Omega^{-\frac{1}{2}} \tag{5.70}$$

We can see that for large Ω (large m, which means a large economy), ther system's time dependence is given as a function of τ, where τ is given by (5.70). Only for times

$$t \gtrsim \Omega^{\frac{1}{2}}$$

does τ become a significant size, and thus it is only for very long times t that any significant time development of the system takes place. Thus, the interesting conclusion is that *the motion of the system becomes very slow at large* Ω. With a large initial number of firms, m, the motion of the system is therefore *very slow*.

5.3. Stochastic processes and the description of fluctuations and growth

We have seen in Section 5.1 the derivation of a Langevin equation and the corresponding description of the dynamics which comes out of that picture. As we mentioned, the assumptions that lay behind the Langevin equation are quite restrictive, so that we may just consider it as a first approximation toward a fully stochastic description of a dynamic process. Yet, as Gardiner (1983) has pointed out, although a strict version of the Langevin equation is not consistent, the corresponding *integral equation*

$$x(t) - x(0) = \int_0^t a[x(s), s]\, ds + \int_0^t b[x(s), s]\, \xi(s)\, ds$$

can be interpreted consistently by making the replacement, which follows directly from the interpretation of the integral of $\xi(t)$ as the Wiener process $W(t)$, that

$$dW(t) \equiv W(t + dt) - W(t) = \xi(t)$$

and thus writing the second integral as

$$\int_0^t b[x(s), s]\, d\,Ws(s)$$

which is a kind of stochastic Stieltjes integral with respect to a sample function $W(t)$ that can be solved by stochastic integration.

A different (and more complicated) formalism is therefore called for here. Notice that the kind of stochastic integral that most satisfactorily one would use, the Ito integral, is not a natural choice in this case. As ξ is most likely *not* a white noite but just a *real noise* with finite correlation time, the Stratonovich integral would be a better choice, albeit more difficult to treat. This issue points to another and maybe more relevant issue: the nature of the random force we assume it affects the growth path of our variable of interest. In general we are interested in a limit of a differential equation

$$\frac{dx}{dt} = a(x) + b(x)\,\alpha_0(t)$$

where $a(x)$ can be zero and $\alpha_0(t)$ is a stochastic source with some nonzero correlation time. If $\alpha_0(t)$ is a Markov process, in the limit it becomes a *delta-correlated* process, and the differential equation becomes a *Stratonovich stochastic differential equation* with the same coefficients:

$$dx = a(x)\,dt + b(x)\,dW(t)$$

which is equivalent to the Ito equation

$$dx = [a(x) + \tfrac{1}{2}\,b(x)\,b'(x)]\,dt + b(x)\,dW(t)\,.$$

It can be shown that for a Markov process all that is needed is a mean equal to zero and stationarity to have an evolution equation of the form

$$\frac{\partial p(\alpha)}{\partial t} = L\,p(\alpha)$$

where L is a linear operator. A Fokker-Planck equation for the pair (x, α) can also be derived.

More general fluctuation equations can be derived along the same lines, as well as time-nonhomogeneous systems and time-dependent systems. All this leads to consider the general case of two variables x and α which are coupled together in such a way that each affects the other: for instance, the *adiabatic elimination* of fast variables, the *slaving principle* introduced by Haken (1983), and so forth. But all this is left to further research.

What is interesting to notice here is that the formalism we have introduced above really fits a lot of different cases and situations. So far in this Section we have focused on the dynamics of the number of innovative firms in an industry or an economy. We have left the dynamics of the firm value out, although nothing prevents us to consider it, as we did in the treatment of the deterministic laws of motion in Section 4. The Langevin equation and the Stratonovich stochastic differential equations can be used as well in this case. For if we let the firm value be our "deterministic" variable $x(t)$ and the number of innovators be the "stochastic" force, we would actually have a heuristically better model. But, as this would need the introduction of further approximation methods for diffusion processes, we leave the development of it for future research.

Before we conclude, let us go back, for a moment, to the solution of the birth-and-death process we found above in Section 5.2. We saw that there exist a Poissonian stationary solution, corresponding to the steady state, which coincides with the equilibrium solution of the deterministic equivalent equation. And yet, we saw that there also exist time-dependent solution. Now, while the stationary solution implies that the mean and the variance of the process do not depend on time, the latter solution does not. When then one solution applies instead of the other? When the *local* aspects of the phenomenon, for instance, fluctuations in small *volumes*, are to be neglected, namely when we are mainly interested in the macroscopic dimension of the problem, then the asymptotic solution may be obtained. However, although the initial variances can be set to zero and the system is macroscopically at a steady state, the variances increase over time and deviate immediately from the initial values. It is impossible for the variances to reach a new steady state. Thus, the steady state *describes the equilibrium at a macroscopical level which nothing prevents to leave once we let the system move*. The steady state is *atemporal*, but once we take time into account we will "automatically" depart from it without never coming back. Stochastically, the macroscopic equilibrium is just meaningless even in the limit of small fluctuations. The system exihibits abnormal fluctuations that increase linearly over time and eventually these fluctuations cannot be neglected. As a result, the average values are driven by the fluctuations to a time dependent regime far from the steady state, which implies that the fluctuations play a decisive role by qualitatively altering the prediction of a macroscopic analysis.

The possibility of spontaneous deviations from the equilibrium regime of the fluctuations provides a striking illustration of the breakdown of the law of large numbers. As pointed out by Nicolis and Prigogine (1977), this is a consequence of the *coupling*, as a result of which a transition undergone by the stochastic variables are not statistically independent events, even in the limit of a large system.

To conclude, we see that there are quite a lot of suggestions for further research as several issues remain open. We have seen that stochastic laws of motion may not be equivalent (not even asymptotically) to deterministic ones (variance effects which determine increasing fluctuations). The effect of "noise" on nonlinear stochastic systems close to critical points remains to be analyzed, as bifurcations may occur in such systems. This means that the solutions of these systems starting with slightly different initial values in the vicinity of a critical state may have completely different paths so that adding fluctuations to such systems may lead to completely different paths from those determined by the corresponding deterministic equations.

But several other issues may be addressed if we extend the horizon of our basic setup. A number of these can be listed, as an example, but several others can be called for: a) a simultaneous growing number of innovators and a decreasing number of non-innovators; b) the interaction between firms by means of interacting particle systems; c) slow variables versus fast variables; d) macroscopic effects versus microscopic effects; e) hysteresis effects (memory and catastophes); f) all the possible limit cycles of any interest, bifurcations and chaos. Let the valuable and faithful reader be sure that further research on these is already in progress and will become publicly available as soon as we can.