Stackelberg equilibria and horizontal differentiation

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Abstract

This paper proposes a taxonomy of the Stackelberg equilibria emerging from a standard game of horizontal differentiation à la Hotelling in which the strategy set of the sellers in the location stage is the real axis. Repeated leadership appears the most advantageous position. Furthermore, this endogenously yields vertical differentiation between products at equilibrium.

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1. Introduction

In their remarkable article, D'Aspremont, Gabszewicz and Thisse (1979) criticize Hotelling's *minimum differentiation principle*,[1] showing that, when firms are sufficiently close to each other, both demand and reaction functions are discontinuous and, consequently, Hotelling's linear transportation cost game fails to reach a pure-strategy equilibrium in prices.[2] Adopting a quadratic representation of the transportation cost function, they also show that the game had a unique Nash equilibrium in locations, implying *maximum differentiation.*

In a related contribution, Gabszewicz and Thisse (1986), adopting a generalized transportation cost function, allow sellers to locate outside the residential area, although on the same side. The strategy set at the location stage is thus $[1,\infty]$ for both sellers. This captures the essential feature of vertical differentiation, since it generates an asymmetry in favour of the nearer seller, who can always undercut the rival's price, which can turn out to face no demand at equilibrium. If sellers are allowed to choose whatever location along the real axis, this asymmetry vanishes, as well as the kink in demand.

We aim at analysing the nature of all possible Stackelberg equilibria within the quadratic transportation cost framework, under the assumption that duopolists are free to locate also outside the city limits, which seems to be consistent with casual observation. The paper is structured as follows: in section 2, the demand and the profit functions of the sellers are defined; section 3 is devoted to to the analysis of price leadership case; section 4 deals with location leadership, whereas section 5 with the case in which the same firm acts as Stackelberg leader in both stages. Section 6 investigates the case in which leadership is assigned to one firm in the location stage, while it is assigned to the rival in the price stage. Finally, section 7 presents a taxonomy of the equilibria outlined in the previuos ones.

2. The model

As a starting point, we adopt the quadratic horizontal differentiation model described by D'Aspremont et al. (1979). The duopolists sell a good which is physically homogeneous.

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1. See Hotelling (1929).

2. Dasgupta and Maskin (1986) showed that a mixed-strategy Nash equilibrium in prices always exists.

Consumers are uniformly distributed along a linear city whose length can be normalized to 1 without loss of generality, and their total density is 1. They have unit demands, and consumption yields a positive constant surplus s; each consumer buys if and only if the following condition is met:

$$U = s - tx^2 - p_i \geq 0, \quad 0 \leq x \leq 1, \quad t > 0, \quad i = 1,2; \tag{1}$$

where tx^2 is the transportation cost incurred by a consumer living at distance x from store i, and p_i is the price of good i. We assume that s is large enough for total demand to be always equal to 1. Firm 1 is located at a, while firm 2 is located at $1 - b \geq a$, with $a, b \in R$. Clearly, if we allow both a and b to be negative, firms are located outside the city boundaries. The demand functions are, respectively:

$$y_1 = a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \tag{2}$$

$$if \quad 0 \leq a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \leq 1;$$

$$y_1 = 0 \tag{2'}$$

$$if \quad a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \leq 0;$$

$$y_1 = 1 \tag{2''}$$

$$if \quad a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)} \geq 1;$$

$$y_2 = 1 - y_1 = b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \tag{3}$$

$$if \quad 0 \leq b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \leq 1;$$

$$y_2 = 0 \qquad (3'$$

$$if \quad b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \leq 0;$$

$$y_2 = 1 \qquad (3''$$

$$if \quad b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \geq 1.$$

Clearly, for $a=1-b$, i.e., when sellers locate at the same point, the demand functions are not determined and profits are nil as a consequence of the Bertrand paradox. Since unit costs are constant and normalized to zero, the two objective functions are then

$$\pi_1 = p_1 y_1; \qquad (4$$

$$\pi_2 = p_2 y_2. \qquad (5$$

Firms play a noncooperative two-stage game in locations and prices. The game is solved through backward induction, and the solution concept is subgame perfect equilibrium *à la Selten* (1965, 1975).

3. Price Leadership

Let us assume that firm 1 acts as Satackelberg leader in the price stage, i.e., maximizes profits under the constraint given by the rival's reaction function. Firms play simultanously in the location stage. The objective of seller 1 is thus:

$$\max_{p_1} \quad \pi_1 = p_1 y_1 \tag{6}$$

$$s.t. \quad R_2(p_1) = p_2 = \frac{p_1 + t - 2at + a^2 t - b^2 t}{2}. \tag{7}$$

The first order condition is:

$$\frac{\delta \pi_1}{\delta p_1} = \frac{2p_1 - 3t + 2at + a^2 t + 4bt - b^2 t}{4t(a + b - 1)} = 0, \tag{8}$$

which yields the following equilibrium prices:

$$p_1^* = \frac{t}{2}(a - b + 3)(1 - a - b); \tag{9}$$

$$p_2^* = \frac{t}{4}(a - b - 5)(a + b - 1). \tag{10}$$

Substituting expressions (9-10) into the objective functions (4-5), we obtain:

$$\pi_1 = \frac{t}{16}(a - b + 3)^2 (1 - a - b); \tag{11}$$

$$\pi_2 = \frac{t}{32}(a - b - 5)^2 (1 - a - b). \tag{12}$$

The first order conditions relative to the location stage of the game, which is played simultanously, are:

$$\frac{\delta\pi_1}{\delta a}=\frac{t}{16}(b-a-3)(3a+b+1)=0; \qquad (13$$

$$\frac{\delta\pi_2}{\delta b}=\frac{t}{32}(a-b-5)(a+3b+3)=0. \qquad (14$$

The system (13-14) has the following critical points: (-3,0); (0,-1); (1,-4). By inspection of the second order conditions,

$$\frac{\delta^2\pi_1}{\delta a^2}=\frac{t}{8}(b-3a-5)\leq 0; \qquad (15$$

$$\frac{\delta^2\pi_2}{\delta b^2}=\frac{t}{16}(a-3b-9)\leq 0, \qquad (16$$

it turns out that these are simultaneously satisfied only in (0,-1), which identifies the Nash equilibrium of the location stage. Equilibrium profits are $\pi_1 = 2t$, $\pi_2 = t$; equilibrium prices are p_1=$4t$, p_2=$2t$, while quantities are $y_1 = y_2 = \frac{1}{2}$.

4. Location leadership

Assume firm 1 is leader in the location stage, while the price stage is played simultanously. Since we are looking for the perfect subgame equilibrium of the two-stage game in locations and prices, let's proceed by backward induction, maximizing (4) and (5) w.r.t. p_1 and p_2. The reaction functions in prices are:

$$R_1(p_2)=p_1=\frac{p_2+t-a^2t-2bt+b^2t}{2}; \qquad (17$$

$$R_2(p_1) = p_2 = \frac{p_1 + t + a^2t - 2at - b^2t}{2}. \qquad (7')$$

The equilibrium prices are:[3]

$$p_1^* = t(1 - a - b)\left(1 + \frac{a - b}{3}\right); \qquad (18)$$

$$p_2^* = t(1 - a - b)\left(1 + \frac{b - a}{3}\right). \qquad (19)$$

If we substitute (18) and (19) into the profit functions (4) and (5), we obtain:

$$\pi_1 = \frac{t}{18}(a - b + 3)^2(1 - a - b); \qquad (20)$$

$$\pi_2 = \frac{t}{18}(a - b - 3)^2(1 - a - b); \qquad (21)$$

the first order conditions of firm 2 w.r.t. locations is:

$$\frac{\delta\pi_2}{\delta b} = \frac{t}{18}(a - b - 3)(1 + a + 3b) = 0. \qquad (22)$$

The leader's problem consists in the maximization of (20) under the constraint given by the follower's reaction function (22). The following second order conditions must be met:[4]

$$\frac{\delta^2\pi_1}{\delta a^2} = b - 3a - 5 \le 0; \qquad (23)$$

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3. Cfr. D'Aspremont et al. (1979, p.1149) and Tirole (1988, p.281).

4. We could alternatively substitute (22) into (20), and then calculate the first and second order conditions. As it can be quickly verified, the two methods yield the same outcome.

$$\frac{\delta^2\pi_2}{\delta b^2} = a - 3b - 5 \leq 0. \qquad (24$$

The SOCs and the constraint $a \leq 1 - b$ define the region in which the Stackelberg equilibrium lies, i.e., along b=-(a+1)/3. This is shown in figure 1.

INSERT FIGURE 1

The critical points of the maximum problem defined by system (20-22) are $\left(\frac{1}{2},-\frac{1}{2}\right)$ and $\left(-\frac{5}{2},\frac{1}{2}\right)$. The SOCs (23-24) are simultaneously met only in $\left(\frac{1}{2},-\frac{1}{2}\right)$, yielding $\pi_1 = \frac{8}{9}t$ and $\pi_2 = \frac{2}{9}t$, as equilibrium profits, while prices are $p_1 = \frac{4}{3}t$ and $p_2 = \frac{2}{3}t$ and quantities $y_1 = \frac{2}{3}$ and $y_2 = \frac{1}{3}$. Notice that, given the sign of the FOCs within the unit interval, to look for a Stackelberg equilibrium inside the city is economically meaningless, since, given the lower bound to her strategy space in the location stage, the follower must confine herself to locate at the boundary of the residential area, yielding a suboptimal degree of differentiation, which in turn implies that both profits are lower than those emerging from both the unconstrained Stackelberg equilibria outlined above and from the simultaneous constrained Nash equilibrium identified by D'Aspremont, Gabszewicz and Thisse (1979).

5. Repeated leadership

The equilibrium for the price stage is shown in section 3 above. The leader's problem in the location stage is:

$$\max_a \quad \pi_1 = \frac{t}{16}(a-b+3)^2(1-a-b) \qquad (25$$

$$s.t. \quad \frac{\delta\pi_2}{\delta b} = \frac{t}{32}(a-b-5)(a+3b+3) = 0, \qquad (26$$

i.e., $b=a-5$ or $b=-(3+a)/3$. The SOCs are given by (15) and (16). The latter solution turns out to be the only acceptable, and substituted into (25) gives the following FOC:

$$\frac{\delta\pi_1}{\delta a}=\frac{2}{9}t(1-a)(a+3)=0. \tag{27}$$

The solutions to (27) are $a=-3$, which doesn't satisfy the SOCs, and $a=1$, which yields $b=-\frac{4}{3}$ as the follower's optimal location. The perfect subgame equilibrium of the game in which the same firm acts as a Stackelberg leader in both stages is then characterized by $\left(a=1, b=-\frac{4}{3}\right)$, $\left(p_1=\frac{32}{9}t, p_2=\frac{8}{9}t\right)$. Equilibrium profits are $\pi_1=\frac{64}{27}t$, $\pi_2=\frac{8}{27}t$, and demands $y_1=\frac{2}{3}$, $y_2=\frac{1}{3}$. Interestingly, this is the only case in which the equilibrium configuration of the market recalls vertical differentiation, although only in a logistic sense.

6. Alternate leadership

Assume firm 2 acts as a Stackelberg leader in the location stage, while firm 1 is leader in the price stage. The equilibrium prices are given by expressions (9-10) above. Again, the SOCs are given by expressions (15) and (16). The objective of seller 2 in the location stage is then:

$$\max_b \quad \pi_2=\frac{t}{32}(a-b-5)^2(1-a-b) \tag{28}$$

$$s.t. \quad \frac{\delta\pi_1}{\delta a}=\frac{t}{18}(b-a-3)(1+3a+b)=0, \tag{29}$$

i.e., $a=b-3$, which is not acceptable, or $a=-(1+b)/3$. Substituting the latter into (28) and differentiating w.r.t. b, we obtain the following FOC:

$$\frac{\delta\pi_2}{\delta b}=-\frac{bt(b+4)}{9}=0. \tag{30}$$

The solutions to (30) are $b=-4$, which doesn't satisfy the SOCs, and $b=0$, yielding $a=-\frac{1}{3}$ as the follower's optimal location. The perfect subgame equilibrium of the Stackelberg game with alternate leadership is then defined by $\left(a=-\frac{1}{3}, b=0\right)$ and $\left(p_1=\frac{16}{9}t, p_2=\frac{16}{9}t\right)$. Equilibrium profits are $\pi_1=\frac{16}{27}t$ and $\pi_2=\frac{32}{27}t$, whereas demands are $y_1=\frac{1}{3}$ and $y_2=\frac{2}{3}$.

7. A taxonomy of Stackelberg equilibria

We are now able to order the results obtained throughout the preceding sections, together with the outcome of the strictly noncooperative game.[5] The payoffs are shown in table 1.

INSERT TABLE 1

The payoffs can thus be ordered as follows:

$$\pi_{rl} > \pi_{pl} > \pi_{ll,pf} > \pi_{pf} > \pi_{ll} > \pi_{N} > \pi_{lf,pl} > \pi_{rf} > \pi_{lf}.$$

A few comments are now in order. First, the possibility of overlapping leadership in both stages enhances the leader's profit. Second, the profits generated by price leadership turn out to be greater than those associated to location leadership. While this may seem counterintuitive at first sight, it can be given an interpretation taking into account the fact that total demand is always equal to one by assumption; consequently, the advantage in terms of market share implicit in the location leadership is limited. Third, as it frequently happens, some Stackelberg equilibria can be characterized as pseudo-cooperative solutions. Specifically, the seller who accepts to act as a follower in prices, regardless of what happens in the location stage, is strictly better off than both in the simultaneous case and in the case in which she acts as a follower in locations and then as a leader in prices. Finally, the leadership in locations appears advantageous if price leadership is then attributed to the rival.

5. Cfr. Lambertini (1993).

8. Conclusions

In this paper, we investigated the nature and consequences of Stackelberg leadership in a standard duopolistic model of horizontal differentiation *à la Hotelling*, where both sellers are allowed to locate outside the residential area. As it could be expected from the outset, the largest advantage is attached to repeated leadership by the same firm. This also represents the only case leading to an equilibrium outcome which is typically associated to vertical differentiation. Moreover, due to demand rigidity, price leadership turns out to be preferable to location leadership, if the remaining stage is played noncooperatively. If, instead, the remaining stage sees the rival playing the leader's role, this result is reversed.