# A Game-Theoretic Analysis of a Welfare Maximizing Oligopolist *

GIOVANNI DE FRAJA

FLAVIO DELBONO

Marzo 1986

N. 25

Linacre College, Oxford

and

University of Siena, Istituto di Scienze Economiche

* Thanks are due to Giacomo Bonanno, William James and John Vickers for helpful comments on an earlier draft.

# A Game-Theoretic Analysis of a Welfare Maximizing Oligopolist *

GIOVANNI DE FRAJA

FLAVIO DELBONO

Linacre College, Oxford
and
University of Siena, Istituto di Scienze Economiche

March 1986

* Thanks are due to Giacomo Bonanno, William James and John Vickers for helpful comments on an earlier draft.

**Abstract**

This paper deals with the optimal behaviour of a single public firm in an oligopolistic market where there are also n private firms. The public firm aims at maximizing a social welfare function depending on producers' and consumers' surplus. In section 3 it is shown that there exists an optimal strategy for the public firm when it is Stackelberg leader. When, instead, the public firm has no move advantage and the game follows Cournot-Nash rules, we show that the outcome is generically Pareto-inefficient. These two regimes are then compared with two extreme cases: a nationalized industry and a pure oligopoly (all firms maximize profit) in the example provided in section 5, where the equilibria considered in the previous sections are fully characterized. Amongst other results, one seems rather paradoxical: when the number of firms is sufficiently large, the optimal strategy of a welfare maximizing firm is to act as if it wanted to maximize its profit.

## 1. Introduction

An oligopolistic market is socially inefficient. So is a monopoly. Both situations, therefore, leave some scope for public intervention. The literature has mainly focused on forms of intervention in which the public authority exploits its legal and institutional power, e. g., regulatory constraints, taxation, subvention. [1] It seems to us that the very idea of nationalization is to let a publicly owned enterprise compete with the rest of the economy without using non-market power. In principle, the only difference between a private and a public firm lies in its aim. This interpretation is broad enough to capture the nature of a public firm in a natural monopoly as well as in a competitive market. The case of natural monopoly is well analysed in the literature (Boiteux 1956 is a classic paper); the case of perfect competition is of course of little interest as there is no room for improvement upon the competitive outcome.

In this paper we want to study the intermediate case of an oligopolistic market where a socially managed firm is given only the market instruments. This approach has been adopted in very few papers (Merrill and Schneider 1966, Harris and Wiens 1980, Beato and Mas-Colell 1984). Even though there are many cases in which the public firm is a monopolist, there are many other industries in which a public firm competes with other producers. Our main interest

1 See, for instance, Baumol et al. 1982, and Bös 1986.

is the comparison among the three cases of "pure oligopoly" (profit maximizer firms only), of a wholly public industry and of interaction between firms with different goals.

The plan of the paper is as follows. In the next section we will discuss the main theoretical features of our model and we will stress the alternative behaviour which can be adopted by a public enterprise; a rough distinction will be set between two cases which we christen Stackelberg and Cournot-Nash (CN hereafter). These cases are modelled formally in sections 3 and 4, respectively, and some general results are provided.

In section 5 a particular example is presented allowing us to obtain more precise characterizations of the general outcomes of the previous sections. There we show that under reasonable circumstances, the welfare is higher in a "pure oligopoly" than when the public firm strives to maximize welfare. Concluding remarks and suggested developments are in section 6.

## 2. The setting

We want to model a public firm playing an oligopolistic game. Obviously, stated in this way, it is a formidable task, therefore we need a large number of important restrictions.

First of all we adopt a partial equilibrium point of view, we consider a static framework, a homogeneous good market, and we assume complete knowledge of everything by everybody. In other words, we put our work in the same context as that of the simplest Cournot-Nash oligopolistic models. As Harris and Wiens (1980, p. 127) have argued, the assumption that the public authority knows all other firms' technology seems more defendable when a public firm is one of the producers.

Secondly, we rule out any "principal-agent" complication: when we say that the public firm aims at maximizing social welfare, we assume that no problem arises in implementing this objective function at the managerial level (see Rees 1984). In other words, public managers do have the same objective as the public authority, for instance because their actions are perfectly monitored.

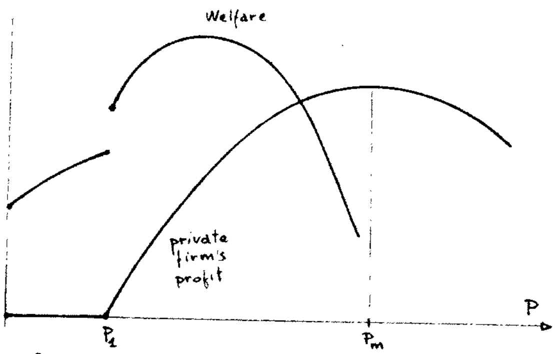
The problem therefore is that of determining the optimal behaviour of a public enterprise acting in a market where there are n other firms [2]. Some heuristic considerations suggest to us that the technology of the industry under examination must be such that the average cost functions are U-shaped; i.e., there is a fixed cost and the marginal cost functions are increasing. In fact, if there are no fixed costs and there is a finite number of firms achieving positive profits, it means that there are some non-technological entry barriers, and improvement in efficiency could be reached by the public authority by weakening these barriers. On the other hand, when the

2 A similar problem is investigated by Rees (1976, section 7.1), where the public firm produces a good "which is a close substitute to that produced by a single monopoly" (p. 130).

marginal cost is constant, two cases can be considered. The marginal cost is the same for all firms: here the public firm prices at the marginal cost, and supplies the amount not produced by the private firms. If the marginal costs are different across the firms and the public firm has the lowest one then there is no problem. When there is a more efficient private firm, then the public firm can announce that it will sell at a price which maximizes the social welfare and allows the most efficient firm to have positive profits. The resulting outcome will be the most efficient firm satisfying the entire demand [3].

The established model of oligopolistic market is the game theoretic one; in the discussion of the last paragraph we have touched on the game theoretic interaction of the various agents. In the rest of this paper we will distinguish two different kinds of game theoretic behaviour. The distinction between the two cases lies in the move order of the game. In the third section we will examine

Figure 1




3 The most efficient firm does not produce when the price is below $p_1$ in figure 1, and without the public firm it would produce at a price $p_m$. If the welfare function depends on profits, then it is discontinuous in $p_1$, as, below $p_1$, only the public firm is producing, and it has a "big" negative profit; above $p_1$, the most efficient firm wants to satisfy the whole demand, as it has small positive unit profits, and its total profit is increasing with the quantity sold. An analogous result, in a different framework, is reached by Beato and Mas-Colell (1984, prop. 8).

a Stackelberg leader public firm. It decides an output level and announces it; the other firms (Stackelberg follower) take that output level as given, and maximize their profits. We assume that the public firm's decision is credible: the other firms believe it, and they accept their role as followers. This move advantage is dropped out in the game considered in the fourth section where the only difference between firms is in the payoff function. We think that there are circumstances in which the first model is more realistic, and others in which the second seems closer to reality.

A further question we rule out deals with entry and exit of firms in and from our market. We assume that there are an exogenously given number of firms; the proportion of firms which are active (produce a positive output) is determined by the condition that a private firm will not have negative profit, because it would prefer not to produce.

This rather unsatisfactory hypothesis can be justified by the presence of prohibitive sunk costs for the outsiders. In both next sections we restrict our attention to the one-shot noncooperative solutions of the games proposed.

In the example provided in section 5, we compare the outcome obtained when the public firm behaves in the ways we have just described with the outcome which would emerge when all firms maximize their profits and when they all maximize social welfare (nationalized industry).

## 3. The public firm as Stackelberg leader

In this section we will consider a rather general model formalizing the ideas illustrated above. We assume that the public firm enjoys an advantage in that it can commit itself to an action and announce it in a "credible" way. As it will be shown in the next section, without this "move advantage", the outcome is not socially optimal. This fact seems to support the credibility of the commitment, for it gives the public authority a reason to commit itself and the other oligopolists a reason to believe the commitment. We are therefore going to draw a simple Stackelberg-like model where the public firm is the leader and the other firms are followers.

Formally, we model a static situation in which there are (n+1) firms labelled by $i=0,1,\dots,n$, where 0 denotes the public firm, producing a homogeneous commodity, the inverse demand function of which is given by $p=D(Q)$, Q being the aggregate supply: $Q=\sum_{i=0}^{n} q_i$. Each firm chooses its output level $q_i$ in order to maximize its payoff function. We adopt the following assumptions:

A.1 D(Q) is a continuously differentiable function $D:\mathbf{R}_+ \rightarrow \mathbf{R}_+$, $dD/dp<0$, satisfying :

A.1.1 $\int_0^{\infty} D(Q)dQ=M<+\infty$

A.2 For all $i=0,1,\dots,n$:

$C_i: R_+ \longrightarrow R_+$

$C_i|_{R_{++}}$ is twice continuously differentiable

$C_i(0)=0$

$\lim_{q_i \to 0} C_i(q_i)=c_i>0$

$dC_i(q_i)/dq_i>0$ for all $q_i \in R_{++}$

$d^2C_i(q_i)/dq_i^{\,2}>0$ for all $q_i \in R_{++}$

In words A.2 says that for all firms producing nothing costs nothing, there is a positive fixed cost for all nonzero output and that the marginal cost is positive and increasing; A.2 captures the features which justify the kind of presence of a public firm in this context as we have argued in section 2.

A.3 private firms' payoff is given by profit:

$$\pi_i = q_i D(Q) - C_i(q_i) \qquad \text{for } i=1,\dots,n.$$

Clearly $\pi_i : R_+^{n+1} \to R$.

A.4 Public firm's payoff is given by a social welfare function:

$W: R^{n+1} \times R_+ \longrightarrow R$

(1) $$W:(\pi_0,\dots,\pi_n,p) \mapsto W\left(\pi_0,\dots,\pi_n, \int_0^Q (D(t)-p)dt\right).$$

which satisfies:

A.4.1 $W$ is increasing in all its arguments.

A.4.2 $\lim_{\pi_0 \to -\infty} W(x) = -\infty$

We have assumed that the public goal firm aims at maximizing a function of producers' and consumers' surplus, thus implying the following simplifying hypotheses: i) redistributive effects among consumers are ruled out, it is as if there were a single consumer, whose utility is given simply by the area between the price and the demand curve. ii) adopting a partial equilibrium viewpoint, we assume that the social welfare function does not vary with all indirect effects, such as, for example, income effects, occupational effects, cross substitution effects, and the like. A.4.2 is needed in the proof of Proposition 1; a much weaker hypothesis could do as well but even as it stands, A.4.2 does not seem implausible.

Some definitions are in order. Let $\hat{q}_i : R_+^n \rightarrow R_+$ the reaction correspondence of firm i, i=1,...,n; i.e., $\hat{q}_i : \bar{q}_i \mapsto \{q_i \in R_+ | \pi_i(q_i, \bar{q}_i) \geq \pi_i(q_i', \bar{q}_i)$ for all $q_i' \in R_+\}$. Let $G_i \subset R_+^{n+1}$ be the graph of $\hat{q}_i$, i.e.: $G_i = \{q \in R_+^{n+1} | q_i = \hat{q}_i(\bar{q}_i)\}$. Let $G = \bigcap_{i=1}^{n} G_i$. G is the set of (n+1)-dimensional output vectors which are <u>viable</u> in the sense that given the public firm's output, if the other firms produce a vector in G then every firm is maximizing its own profit given $q_0$ and the production of the other (n-1) firms.

**Lemma** 1: G is closed and nonempty.

**Proof**: G is nonempty as there exists $q_0$ large enough such that all other firms cannot make positive profits. $G_i$ is closed for all i=1,...,n, as the graphs of the reaction correspondences are closed, therefore G is closed too. ∎

Let $g:\mathbf{R}_+^{n+1}\to\mathbf{R}^{n+2}$, $g:q\mapsto(\pi_0,\ldots,\pi_n,p)$; $g$ associates an output vector with the resulting profits for the $(n+1)$ firms and the resulting market price. $g$ is a well defined function. The composition $U=W\circ g$ sends an output vector into the social welfare which it yields.

**Lemma 2**: $\lim_{q_0\to+\infty} U(q_0,\ldots,q_n)= -\infty$.

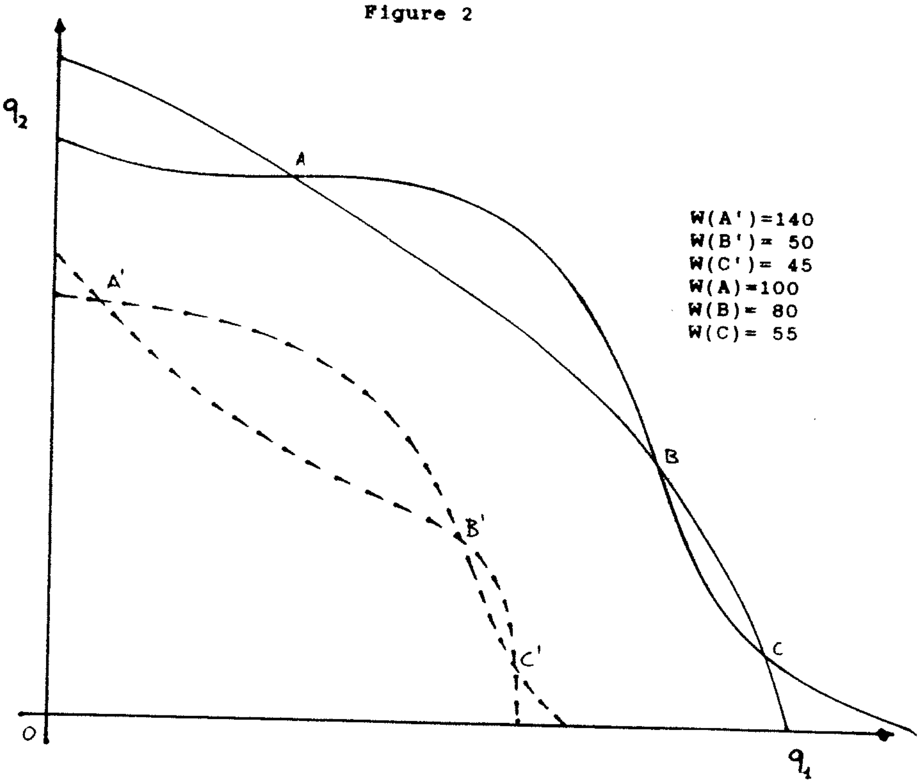
**Proof**: Follows easily from A.4.2, A.1.1, and A.2.■

**Proposition 1**: Under A.1-A.4 there exists $q^*\in G$ such that $U(q^*)\geq U(q)$, for all $q\in G$: i.e., there exists a viable output vector which is a social optimum.

**Proof**: By Lemma 1 G is nonempty; pick a $q\in G$, let $U(q)=w$. By Lemma 2 there exists $q_0'$ such that $\sup_{(q_0,\bar{q}_0)\in G}\{(q_0,\bar{q}_0))\}<w$, for all $q_0>q_0'$. Now consider $G\cap[0,q_0']\times\mathbf{R}_+^n=G^*$. $G^*$ is compact and obviously nonvoid. $U:\mathbf{R}_+^{n+1}\to\mathbf{R}$ is the composition of two continuous functions and its restriction to a subset of its domain is continuous too. The maximum of $U|_{G^*}$ is the sought social optimum.■

This result is less interesting than it might appear: summarizing its proof in less formal words, we can say that the public firm is a Stackelberg leader in the sense that it calculates the social welfare for all levels of its own output assuming that the other firms take this level as given, and among themselves act like Cournot players. Thus we have a Stackelberg game in which the public firm is the leader and the set of the other n firms is the follower, and a CN game is played among the n private firms. It could seem that the policy of the public firm would simply be to choose the level of $q_0$ in correspondence of which the welfare is maximum; but the problem of uniqueness of the CN equilibrium of the n-player game arises. Formally, let $pr: R^{n+1} \rightarrow R$ be the projection which sends $(x_0, \ldots, x_n) \in R^{n+1}$ into $x_0 \in R$, and consider the set $pr^{-1}(pr(q^*))$, where $q^*$ is the socially optimum output vector shown to exist in Proposition 1; this set is the set of the CN equilibria of the game played by the n firms, given $q_0^*$. In general the CN equilibrium of a noncooperative game is not unique, therefore $pr^{-1}(pr(q^*))$ need not be a singleton, so the public firm has no guarantee that its preferred point will actually be chosen. This problem is illustrated in Figure 2 which shows a plausible couple of pairs of reaction functions when n=1, given two different levels of $q_0$. The intersections of the pairs of curves are the CN equilibria . The dashed lines are the reactions functions of the private firms for the level $q_0'$, the full curves the reactions functions for the level $q_0$. If the levels of welfare associated with the intersections points of the pairs of reaction functions are those besides the figure, the choice of public firm between $q_0'$ and $q_0$ is intriguing, unless it has some way of forcing the outcome to A'.

Figure 2




The non-uniqueness of Nash equilibrium depends on the parameters of demand and cost functions. We give below a condition ensuring that the CN equilibrium is unique (Theorem 7.11 in Friedman 1977):

A.5 (Diagonal Dominance): For all i=1,...,n:

$$\partial^2\pi_i(q)/\partial q_i^2+\sum_{i\neq j}|\partial^2\pi_i(q)/\partial q_i\partial q_j|<0$$

for all $q\in\{q_0^*\}\times R_+^n$

Where $q_0^*$ is the first element of the output vector q* which maximizes U in G. Unfortunately, this condition can be violated for non-pathological examples of cost and demand functions.

When there is more than one possible outcome associated to the various levels of $q_0$ the analysis requires specifications which are beyond the scope of this paper. We will limit ourselves to a few hints. If the public policy can impose one of the CN equilibria, for instance through taxation and/or redistribution of profits and consumers' surplus, then the public firm will choose the "maximax", $q_0^*$. If external intervention is not allowed, the public firm might adopt a "minimax" strategy, which guarantees a minimum target in terms of welfare. Of course, in many situations, it seems plausible that the various CN equilibria are not equally probable; the public firm could act in such a way to maximize the expected value of the welfare function.

## 4. Cournot-Cournot behaviour

In this section we relax the assumption which attributes a move advantage to the public firm. In other words we will consider a (n+1)-player noncooperative game, where each player maximizes their payoff function given the strategies of the other n players. Formally, we model the following game:

(H)

Strategy sets: $S_i=\{q_i \in R_+\}$ for all $i=0,1,\dots,n$

Payoff functions: $P_i(q)=\pi_i(q)=D(q)q_i-C_i(q_i)$

$q \in S=S_0 \times \dots \times S_n = R_+^{n+1}$ for all $i=1,\dots,n$

$P_0(q)=U=W \circ g$

where $W$ is defined in A.3, and $g$ is defined between Lemma 1 and Lemma 2.

We are interested in Nash equilibrium of the game (H). Let $\hat{q}_i$ be the reaction correspondence of firm i, $i=0,1,\ldots,n$, as defined above. A Nash equilibrium is simply a fixed point for

$$\hat{q}:R_+^{n+1}\to R_+^{n+1},\ \hat{q}:(q_0,q_1,\ldots,q_n)\mapsto(\hat{q}_0(\bar{q}_0),\hat{q}_1(\bar{q}_1),\ldots,\hat{q}_n(\bar{q}_n))$$

The structure of the cost functions gives us payoff functions which are both discontinuous and not quasi-concave [4]; therefore a Nash equilibrium may not exist. We do not look for conditions ensuring the existence of a Nash equilibrium, which appear to be rather cumbersome; we prefer to characterize a Nash equilibrium in the case it exists. For the sake of simplicity, we assume $W$ is differentiable.

**Proposition 2:** Let $q^*=(q_0^*,q_1^*,\ldots,q_n^*)$ be a Nash equilibrium for the game (H). Let $I=\{i\in\{0,1,\ldots,n\}\,|\,q_i^*>0\}$; $I$ is the set of the firms the output of which is strictly positive at the Nash equilibrium $q^*$. If $I\neq\{0\}$, then, apart from exceptional cases, there exists $q'\in R_+^{n+1}$, such that $P_i(q')>P_i(q^*)$ for all $i\in I$; i.e., $q^*$ is not Pareto efficient.

**Proof:** If $0\notin I$, we are in the classical oligopoly, and the assert is known to be true (Friedman 1977, Theorem

4 As the payoff functions are not quasi-concave, Dasgupta and Maskin's (1986) results and, more particularly, Novshek's (1985) results do not apply. We feel that considering mixed strategy equilibria is not appropriate in our simple model.

2.2). Suppose $0 \in I$. We want to show that the determinant of the following matrix is nonzero:

$$J(q^*) = \begin{bmatrix} \frac{\partial U(q^*)}{\partial q_0} & \frac{\partial U(q^*)}{\partial q_1} & \cdots & \frac{\partial U(q^*)}{\partial q_m} \\ \frac{\partial \pi_1(q^*)}{\partial q_0} & \frac{\partial \pi_1(q^*)}{\partial q_1} & \cdots & \frac{\partial \pi_1(q^*)}{\partial q_m} \\ \vdots & & & \vdots \\ \frac{\partial \pi_m(q^*)}{\partial q_0} & \cdots & \cdots & \frac{\partial \pi_m(q^*)}{\partial q_m} \end{bmatrix}$$

Where $m = \# I$, and, to lighten the notation, we have relabelled the firms in such a way that $I = \{0, 1, \ldots, m\}$. This amounts to showing that, generically, for all $x \in R^{n+1}$, if $x \neq 0$, then $J(q^*)x \neq 0$. From $J(q^*)x = 0$ we obtain:

$$\frac{\partial U}{\partial q_1} x_1 + \cdots + \frac{\partial U}{\partial q_m} x_m = 0$$

$$\frac{\partial \pi_i}{\partial q_0}(x_0 + x_1 + \ldots + x_{i-1} + x_{i+1} + \ldots + x_n) = 0 \qquad i = 1, \ldots, m$$

From which: $x_i = x_1 \quad i = 2, 3, \ldots, m$, $x_0 = -(m-1)x_1$, and

(2) $\quad x_1\left(\frac{\partial U}{\partial q_1} + \cdots\cdots + \frac{\partial U}{\partial q_m}\right) = 0$

Generically, the term in brackets will be nonzero, so $x_1$ and all the other $x_i$'s must be zero if $J(q^*) = 0$. ∎

The exceptional cases occur when the partial derivatives of the welfare function with respect to the private firms' output sum to zero. In this case, there exists a slight perturbation of the demand, cost and welfare functions such that this Nash equilibrium either does not exist or is Pareto improvable. It may not exist if one firm is just indifferent whether to produce or not; a slight perturbation of the structure of the model can render it convenient for this particular firm to switch to the other decision (for instance, if it was producing at zero profit, after the perturbation it may find convenient not to produce, and viceversa).

Coming back to the expression in brackets in (2), it can be written

$$\sum_{i=1}^{m} \frac{\partial W}{\partial p} \frac{\partial p}{\partial q_i} + \sum_{i=1}^{m} \sum_{j=0}^{m} \frac{\partial W}{\partial \pi_j} \frac{\partial \pi_j}{\partial q_i}$$

When this is zero, it means that the total impact of consumers' surplus variation on welfare due to a change in the output of private firms is exactly offset by the impact on welfare of the variation in firms' profit due to this change. When all the terms in the expression in brackets in (2) are zero, we find a critical point of the maximand function of public firm when it has the move advantage we discussed in the previous section, provided that it is possible to express other firms' output as a function of $q_0$. More precisely, suppose that for all $q_0$ there exists at most one point in G; let $T:pr(G) \to R_+$ be the composition $T: q_0 \mapsto U(q_0, q_1(q_0), \ldots, q_n(q_0))$; then:

$$\frac{dT}{dq_0} = \frac{\partial U}{\partial q_0} + \frac{\partial U}{\partial q_1} \frac{\partial q_1}{\partial q_0} + \ldots + \frac{\partial U}{\partial q_n} \frac{\partial q_n}{\partial q_0}$$

Notice that the Nash equilibrium condition imposes the first term in the RHS to be zero. There could clearly be higher maxima for $T(q_0)$, therefore even when the Nash outcome is Pareto efficient it can be a worse outcome for the public welfare.

In the traditional Stackelberg model, the leader chooses its preferred point on the reaction function of the other player, whereas in the Cournot duopoly the outcome is given by the intersection of the reaction functions. Analogously here, G, the intersection of the n reaction functions of the n private firms can be seen as the reaction functions of a Stackelberg follower in a traditional duopoly. Generically, G is one-dimensional and the set of the CN equilibria of game (H) is a zero-dimensional subset of G; therefore the maximum welfare in G will be a CN equilibrium only with probability zero.

## 5. An example

In this section we provide an example [5] under alternative behaviour of the public firm; in particular, along the lines of the considerations of section 2, we compare four possible regimes, as far as the behaviour of the public firm is concerned, labelled as follows:

S, for Stackelberg: the public firm acts as a Stackelberg leader, and maximizes the social welfare;

5 This section draws heavily on our paper (De Fraja and Delbono 1986).

N, for Nash: we look for a Nash equilibrium of the (n+1)-player game (H), as described in section 4;

E, for entrepreneur (or, if the reader prefers, egoist): here the public firm maximizes its own profit, as any private firm; again we restrict our attention to the Nash equilibria. This framework is an oligopolistic market without any public intervention;

M, for public monopoly : the public authority "nationalizes" the all sector, and maximizes its welfare function.

For the sake of simplicity, we consider the inverse demand function:

$$p = a - Q \qquad a>0$$

where p is the price, Q is the total output, and the cost function, assumed to be the same for all firms:

$$c_i(q_i) = c + kq_i^2/2 \qquad i=0,1,\dots,n$$

This cost function is a simple functional form that fulfills A.2. Thus we have n identical firms, and a zero-th firm which has the same technology as the others, and, except in case E, a different payoff function. While aware that a non-symmetric outcome may arise, we only consider identical production for the n private firms. We specify the social welfare function as the sum of consumers' and producers' surplus:

$$W = \int_0^Q D(t)dt - \sum_{i=0}^{n} C(q_i) = \quad [a^2-p^2-2(n+1)c-kq_0^2-nkq^2]/2$$

We want to compare $q_0$ (the public firm's output), $q$ (the private firm's output), p (the price), Q (the total output), $\pi_0$ (the public firm's profit), $\pi$(the private firm's profit), W (the social welfare), in the various cases. Table 1 reports the values of the various magnitudes in the four cases. These values are obtained in the following way:

S) The public authority calculates $q_i$ and p as functions of $q_0$, and then constructs the function $T(q_0)$ by substituting $q_i$ and p in the welfare function; eventually it maximizes T with respect to $q_0$.

N) A Nash equilibrium is obtained as solution of the couple of equations :

$$\frac{\partial W}{\partial q_0} = 0 \qquad \frac{\partial \pi}{\partial q} = 0$$

E) As in N), with a different payoff function for the public firm: $\partial\pi_0/\partial q_0=0$.

M) Nationalization can mean two different things, which, given the simple model we are using, are equivalent. First, deciding the optimum number of "firms", and each of their output levels. Second, again deciding the optimum number of firms, and imposing them as objective the maximization of welfare [6].

---

6 In the first case it sets the gradient of the welfare function to zero and (n+1) linear equations in q are obtained, the only solution of which is the symmetric one, which is also the first order condition for a welfare maximizing firm, in a game where all other firms want to maximize welfare too. Of course the optimal number is chosen by calculating the welfare with different values of n.

# Table 1

| | S | N | E | M |
|---|---|---|---|---|
| $q_0$ | $\frac{at}{t+k\beta^2}$ | $\frac{a(k+1)}{t}$ | $\frac{a}{1+\beta}$ | $n^* = a\sqrt{\frac{k}{2c}} - (k+1)$ |
| $q$ | $\frac{ak\beta}{t+k\beta^2}$ | $\frac{ak}{t}$ | $\frac{a}{1+\beta}$ | $\sqrt{\frac{2c}{k}}$ |
| $Q$ | $\frac{a(nk\beta+t)}{t+k\beta^2}$ | $\frac{a(1+k+kn)}{t}$ | $\frac{a(n+1)}{1+\beta}$ | $a-\sqrt{2ck}$ |
| $P$ | $\frac{ak(1+k)\beta}{t+k\beta^2}$ | $\frac{ak(k+1)}{t}$ | $\frac{a(1+k)}{1+\beta}$ | $\sqrt{2ck}$ |
| $\pi_0$ | $\frac{a^2k[\beta^2(1+k)^2-n^2]}{2(t+k\beta^2)^2} - c$ | $\frac{a^2k(k+1)^2}{2t^2} - c$ | $\frac{a^2}{(1+\beta)^2}\left(1+\frac{k}{2}\right) - c$ | $0$ |
| $\pi$ | $\frac{a^2k^2\beta^2}{(t+k\beta^2)^2}\left(1+\frac{k}{2}\right) - c$ | $\frac{a^2k^2}{t^2}\left(1+\frac{k}{2}\right) - c$ | $\frac{a^2}{(1+\beta)^2}\left(1+\frac{k}{2}\right) - c$ | $0$ |
| $W$ | $\frac{t+kn(1+\beta)}{2(t+k\beta^2)}a^2 - (n+1)c$ | $\frac{(1+k)^3+nk(nk+2+4k+k^2)}{2t^2}a^2 - (n+1)c$ | $\frac{(3+k)+(4+k)n+n^2}{2(1+\beta)^2}a^2 - (n+1)c$ | $\frac{1}{2}\left(a-\sqrt{2ck}\right)^2$ |

$t = (1+k)^2 + nk \qquad \beta = 1+k+n$

Simple remarks can be drawn looking at the Table 1. The various magnitudes $q_0$, $q$, $p$, $Q$, $\pi_0$, $\pi$, $W$, and $C'_0$ and $C'$ (marginal cost of public firm and private firms respectively) have been labelled M, S, N, E, in the four cases.

[1a] $W_M > W_S > \max\{W_N, W_E\}$

[1b] there exists $m \in \mathbb{R}_+$, such that $W_E < W_N$ if $n<m$ and viceversa.

As expected, $W_M$ is greater than the welfare in all other cases, and $W_S$ is greater than cases E and N. From result [1b] it can be inferred that if the public firm cannot have the move advantage, then, if the market is competitive enough, it is socially better for the public firm to try to maximize its own profit instead of pursuing a social goal.

[2] $q_{ON} > q_{OS} > q_E > q_S > q_N$

In both cases the public firm produces more and the private firm less than the individual output of an oligopolistic market when all firms are profit maximizers.

[3] $p_E > p_S > p_N > p_M$

[4] $Q_M > Q_N > Q_S > Q_E$

[5a] $\pi_E > \pi_S > \pi_N$

[5b] $\pi_{OS} > \pi_S$;     $\pi_{ON} > \pi_N$

As the consumers' surplus is a decreasing function of the price, consumers are better off with the Nash regime, and worst off with the regime labelled E. For private firms it is the other way round.

[6a] $p_S > C'_{OS} > C'_S$

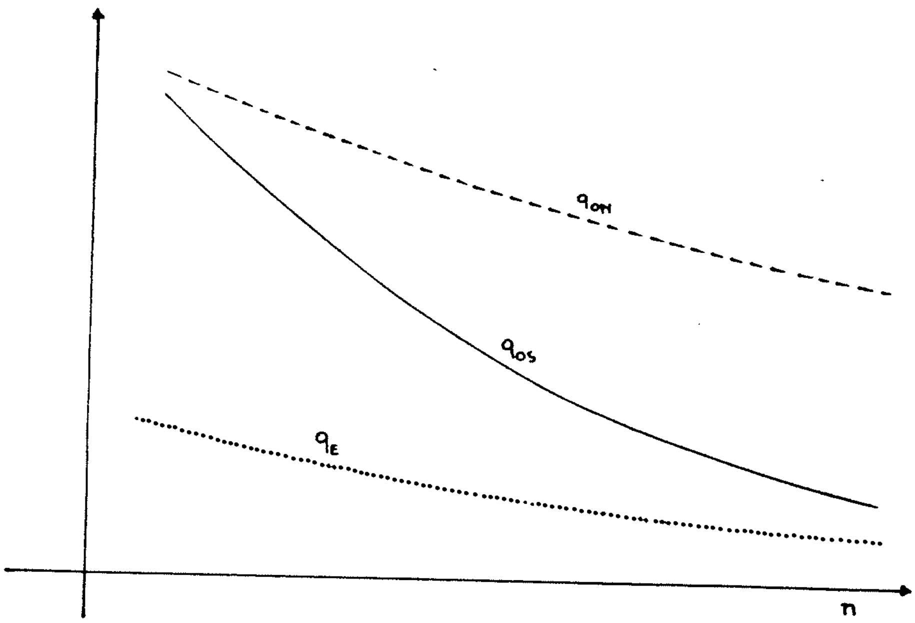
[6b] $p_N = C'_{ON} > C'_N$

[6c] $p_E > C'_E$

[6d] $p_M = C'_M$

Nobody ever produces at a marginal cost higher than the market price.

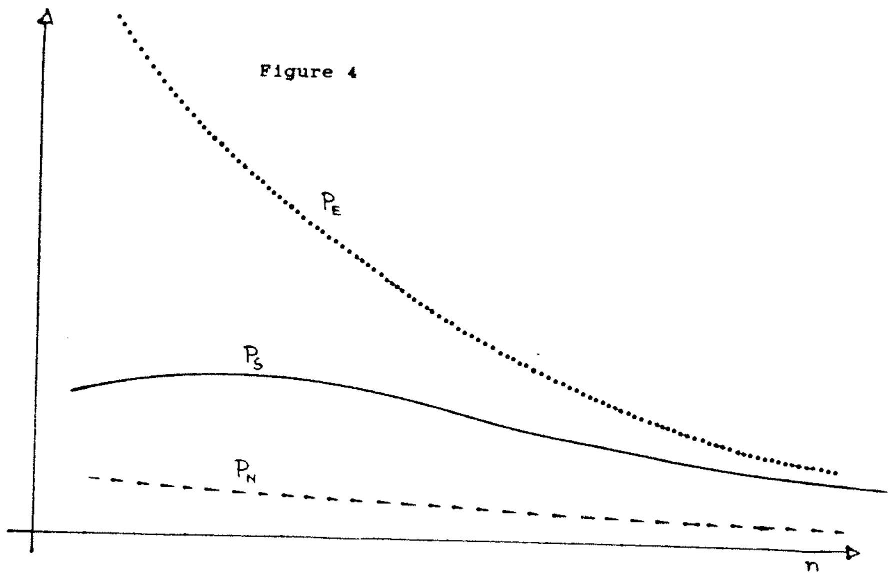
Figure 3



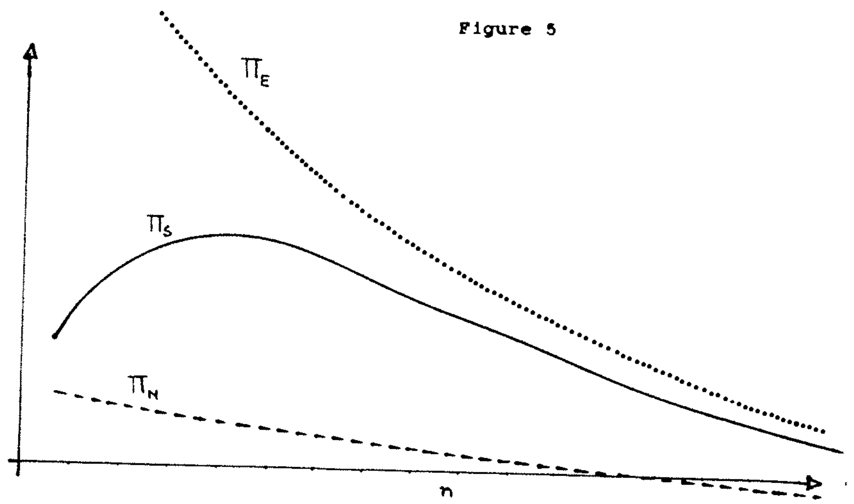

$q_{ON}$: public firm's optimal output when it is a Nash player and maximizes welfare.
$q_{OS}$: public firm's optimal output when it is a Stackelberg leader.
$q_E$ : public firm's optimal output when it is a profit maximizer.

Figure 3 visualizes the public firm's optimal output as a function of n, under the three regimes; it can be used to explain the paradoxical result [1b]. With a large number of firms (large with the respect to demand and technology), in the Nash case, the public firm must produce a very high level of output; therefore private profits are driven to a very low level; given the high $q_{ON}$, and that the public firm's marginal cost is equal to market price, the higher consumers' surplus under Nash regime, is not higher enough to compensate the lower private profits.

Figures 4 and 5 illustrate the trend of the two components of the social welfare as functions of the number of firms.




$P_N$: price when public firm is a Nash player and maximizes welfare.
$P_S$: price when public firm is a Stackelberg leader.
$P_E$: price when public firm is a profit maximizer.




$\pi_N$: total profit when public firm is a Nash player and maximizes welfare.
$\pi_S$: total profit when public firm is a Stackelberg leader.
$\pi_E$: total profit when public firm is a profit maximizer.

As far as the public firm's profit is concerned, the ordering of the three cases is not independent of the parameters. Some numerical examples show that in some circumstances the public firm gets a higher profit in the Nash regime than when it strives to maximizes its own profit [7]. This seems more likely to occur in a more competitive market (n large). Coupled with the consideration regarding the ordering of welfares, this seems rather paradoxical. The switching points between welfare and public firm's profit in the two cases do not coincide. There are circumstances in which trying to maximize social welfare, not only leads to a lower welfare, but also to lower profit for the public firm than being profit maximizer (e.g.: with a=160, k=4, c=40, n=6 $\pi_{ON}<\pi_E$ and $W_N<W_E$).

Coming back to the Stackelberg case we can calculate $q_0^{max}$ defined as the value of $q_0$ which maximizes $\pi_0$; i.e., the output of a profit maximizer Stackelberg leader; $q_0^{max}=a(k+1)/(k^2+k(3+n)+2)$. It turns out that for $n>\sqrt{2k^2+3k+4+1/k}$ , $q_{0S}<q_0^{max}$, i.e. when n is large, if firm 0 decides an output higher than $q_{0S}$, the consumers' surplus and its own profit increases, but the reduction in private profit more than compensates the effects on welfare. When n is small, the market is then less competitive, if firm 0 drives $q_0$ below $q_{0S}$ $(>q_0^{max})$, it leads to a lower consumer

7 This result parallels the conclusion of Vickers (1984), where a firm obtains a lower profit when its payoff function is the profit than when it is an average of profit and sales.

surplus which is not compensated by the positive effect of higher aggregate profit.

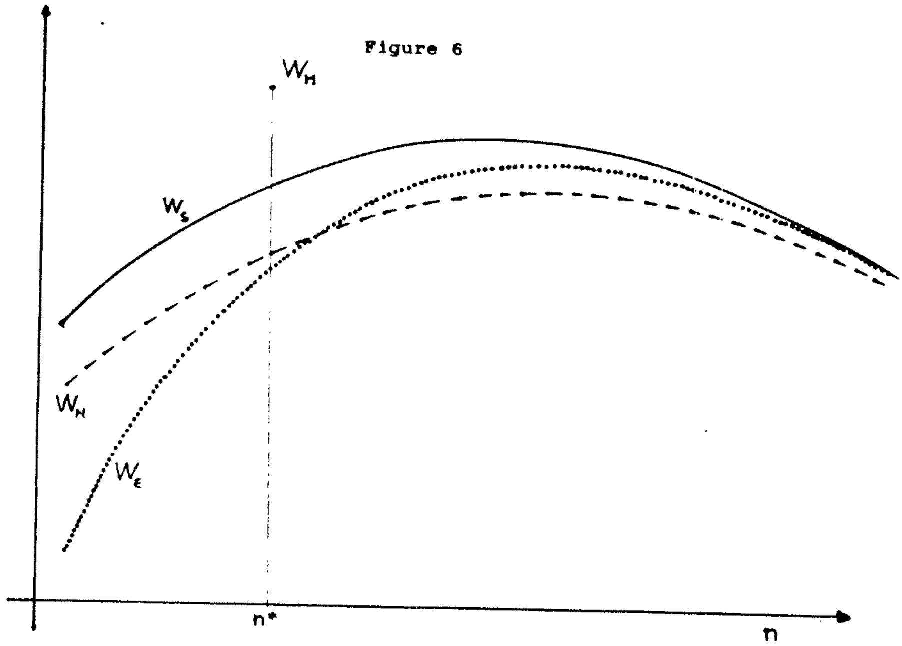
Another interesting magnitude is the level of $q_0$ beyond which private firms make zero profit: $q_0^0 = a-(1+k+n)\sqrt{2c/(k+2)}$. It is easily seen that an increase in a will increase the distance between $q_0^0$ and $q_{0S}$: if an equilibrium with n firms is viable, then, when the demand is higher, n firms are still making positive profits.

Let us examine now the case where the public authority nationalizes the whole industry. As it appears from table 1, each firm produces the quantity that minimizes the average cost [8]; the optimal number of firms is that one which satisfies the whole demand. It is also worth stressing that in this simple case the optimal policy for the public authority can be implemented by imposing a price given by the minimum cost on the productive units. It is worth noticing that the optimal number of firms is lower when the industry is nationalized than in the other cases (Figure 6).

This fact may be interpreted by saying that when there are profit maximizer firms, the public policy faces a trade-off: a higher number of firms, while reducing the distance between price and marginal cost of the various firms, due to greater competition, has a negative effect on welfare due to the presence of fixed costs.

8 Baumol and Fisher 1978, section 6, reach a conclusion with the same flavour in a slightly different context.

Figure 6




$W_N$: social welfare when public firm is a Nash player and maximizes welfare.
$W_S$: social welfare when public firm is a Stackelberg leader.
$W_E$: social welfare when public firm is a profit maximizer.

## 6. Concluding remarks

There could be reasons which render the nationalization of a whole industry impossible, even when, as we have shown, this would lead to the maximum welfare in the sector. However, nationalization is by no means the only form of intervention which is used by public authority. Notwithstanding all the limitations of the model we have considered, some provisional conclusions might be drawn about the effect of and scope for the existence of a socially managed firm in an oligopolistic market. Especially if the market is well away from its optimal structure, in terms of the number of active firms, the action of a welfare maximizing firm seems to be highly positive as far as welfare is concerned. Moreover, among the alternative types of behaviour we studied, the one we defined Stackelberg leader is far superior: if the public firm is given a move advantage it does succeed in driving the actions of the private firms towards a socially preferred situation. When the market is not "too oligopolistic", when there are a number of firms close to the optimal one, then, if public firm has not, for whatever reason, the leadership, trying to improve the social welfare leads to a situation worse than if it had acted to maximize profit. Asked whether they would like a public intervention in their market, private entrepreneurs would answer negatively: but if really there is no way of avoiding a competition with a public firm, they would prefer it to behave as a Stackelberg leader. The level of output of a welfare maximizer firm without any move advantage would be so high that very little room would be left for them.

We are aware that the models we proposed are extremely simple and that they do not represent any real market. However, we believe they might constitute a starting point for deeper insights into the analysis of the interaction between private and public firms in unregulated markets. Amongst the several other questions, two points which we have not pursued seem worth exploration. The first is the relaxation of the hypothesis that the product is homogeneous: again we refer to the recent literature on contestable markets (Baumol et al. 1982). The second extension should be the presence of asymmetric and/or incomplete information on the structure of the industry (e.g., the public authority does not know precisely the other firms' technology or the demand function), extending to the oligopolistic market the approach of Baron and Myerson (1984). We hope to tackle some of these questions in further research.

## References


BARON, DAVID P. and ROGER B. MYERSON, 1982, "Regulating a Monopoly with Unknown Cost", Econometrica, 50, 911-30.

BAUMOL, WILLIAM J. and DIETRICH FISCHER, 1978, "Cost-Minimizing Number of Firms and Determination of Industry Structure", Quarterly Journal of Economics, 92, 439-67.

BAUMOL, WILLIAM J. , JOHN C. PANZAR, ROBERT D. WILLIG, 1982, Contestable Markets and the Theory of Industry Structure, Harcourt Brace Jovanovich, New York.

BEATO, PAULINA and ANDREU MAS-COLELL, 1984, "The Marginal Cost Pricing as a Regulation Mechanism in Mixed Markets", in M. MARCHAND, P. PESTIEAU and H. TULKENS (eds.), The Performance of Public Enterprises, North-Holland, Amsterdam.

BOITEUX, MARCEL, 1956, "Sur la gestion des monopoles public astreints à l'équilibre budgétaire", Econometrica, 24, 22-40.

BÖS, DIETER, 1986, Public Enterprise Economics, North-Holland, Amsterdam.

DASGUPTA, PARTHA and ERIC MASKIN, 1986, "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory", Review of Economic Studies, 53, 1-26.

DE FRAJA, GIOVANNI and FLAVIO DELBONO, 1986, "Alternative Behaviour of a Public Firm in Oligopoly", mimeo, Linacre College.

FRIEDMAN JAMES W., 1977, Oligopoly and Theory of Games, North-Holland, Amsterdam.

HARRIS, RICHARD G. and ELMER G. WIENS, 1980, "Government Enterprise: An Instrument for the Internal Regulation of Industry", Canadian Journal of Economics, 13, 125-32.

MERRILL, WILLIAM C. and NORMAN SCHNEIDER, 1966, "Government Firms in Oligopoly Industries: A Short Run Analysis", Quarterly Journal of Economics, 80, 400-12.

NOVSHEK, WILLIAM, 1985, "On the Existence of Cournot Equilibrium", Review of Economic Studies, 52, 85-98.

REES, RAY, 1976, Public Enterprise Economics, Weidenfeld and Nicolson, London, II ed. 1984.

REES, RAY, 1984, "The Public Enterprise Game", Economic Journal, Conference Papers, 94, 109-23.

VICKERS, JOHN, 1984, "Delegation and the Theory of the Firm", Economic Journal, Conference Papers, 94, 138-47.