Quality of Service in the Congestible Internet: a Differential Game with Capacity Investments

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Abstract

We take a differential game approach to study the dynamic market interaction between two Internet Service Providers (ISP) offering services characterized by different quality levels. Web congestion is accounted for, consisting in the fact that for a given network capacity, i.e. for given amount of resources to be shared, the quality of services decreases with the number of customers. ISP firms, by accumulating capital, may invest in order to increase their own network capacity. In contrast with the acquired wisdom, we prove that there exists an admissible intertemporal parameters subset wherein the low quality firm performs better than the high quality firm in terms of equilibrium profits. Furthermore, we establish conditions under which the low quality firm becomes a natural monopolist. Finally, we prove that consumers may be better off under cooperative rather than under non cooperative play.

JEL Classification: D43, D62, L13.

Key Words: differential games, Internet, quality of service, network externalities, congestion.

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1. Introduction

Many network goods, among which communication and information network goods, feature two characteristics: (i) there are positive network externalities, consisting in the fact that the utility an user can draw from the purchaise of such goods is increasing in the total number of others doing likewise; (ii) there is any intrinsic utility that can be justified by itself consumption, since being connected to the network when none can be reached is completely unuseful. Furthermore, some of them may suffer from congestion phenomena, to the extent to which users share a common technology. In this respect, think about phones, faxes and telegraphs. These are circuit-switched communication services, which means that a fixed percentage of network resources is reserved for the call, and no other call can use those resources until the original connection is closed. But emails, Internet telephony, Internet fax and video conferencing, to name a few, make use of the Internet, which is a shared-media technology. As a consequence, these are services subject to web congestion: for a given network capacity, i.e. for given amount of resources to be shared, the quality of service (QoS) decreases as the traffic flow increases.[1]

Nowadays, we register an exponential increase in the demand for communication services through the Internet, and we expect it to grow more and more in next years. The Federal Communication Commission reports that on the Internet the volume of traffic is doubling every 90-100 days, while the growth rate of telephone traffic is around 5 per cent a year! Moreover, we observe an increase in the demand for new services like internet telephony, video-conferencing and on-line multiplayer games, which require a much larger band-width than simple emails or web chats do. All these factors, make web congestion a no more negligeable danger as far as sustainable web growth is concerned.[2]

Ever since Rohlfs' contribution (1974), the economic literature dealing with network industries has dramatically grown.[3] Several pricing mechanism have been proposed in order to cope with congestion, from flat pricing to much more complex schemes based on auctions over packets or on priority pricing.[4] However, the problem of traffic "accountability" on the Web has prevented regulators from their practical implementation. On the supply side,

[1] The quality of service on the Web can be measured as the expected waiting time or the expected loss rate for data transfers.

[2] Furthermore, the transmission protocol adopted for the Internet was not engineered to manage congestion effects.

[3] An exhaustive survey on markets characterized by network effects is in Katz and Shapiro (1994). For a good exposition of many problems concerning competition and regulation in Telecommunication Industries see Laffont and Tirole (1999) and Shy (2001). A focus on information network goods is in Shapiro and Varian (1999).

[4] See DaSilva (2000) and Falkner (2000) et al. for surveys on congestion pricing.

the issue of capacity upgrades as a way to match the increasing demand has been exlored by Mackey-Mason and Varian (1995), within a static setting. As to the interplay between network effects and the provision of product quality, contributions have been made by Baake and Boom (2001), Bental and Spiegel (1995) and Lambertini and Orsini (2001), either in duopoly or in monopoly models. With reference to the Internet, Crémer *et al.* (2000) and Laffont *et al.* (2001) have focussed on positive network externalities, while Gibbons *et al.* (2000) and Mason (2000), have shed some light on congestion phenomena under competition.[5]

To the best of our knowledge, little has been done to gather both positive and negative network externalities in a unified framework. Yet, it seems to us that the comprehension of congestion effects be essential to completing the broad economic picture referred to network goods, both on positive and normative standpoint. In addition to this, despite their actual relevance, very few papers have studied the economic incentives for firms to invest in network capacity in relation with the choice of which variety to offer.

Aiming to provide a theoretical contribution in these directions, we study a differential duopoly game where firms, either non cooperatively or cooperatively, offer information/communication services characterized by different qualitity levels through congestible networks and, by accumulating capital, invest in order to improve their own transmission technologies. This is equivalent to building up network capacity: if for a given data package to be transferred less common resources are needed, new customers may enter without negatively affecting the conditions of others' usage.

We assume that heterogeneous consumers base their choice of which services subscribing to on the connection price and the quality of services offered. Consumers' decisions are also based on the expected number of subscribers to each networks, with expectations supposed to be rational. In line with the existing static literature on product quality provision in oligopoly (Gabszewicz and Thisse, 1979, 1980; Shaked and Sutton, 1982, 1983; Lehmann-Grube, 1997, *inter alia*), we show that the high quality firm serves more customers and invests more than the low quality firm. However, the order relationship between profits can take a different sign than the one we are accomplished with from the aforementioned literature. More precisely, there exists an admissible intertemporal parameters subset wherein the low quality firm performs better than the high quality firm in terms of equilibrium profits. This occurs whenever future profits matter enough. Moreover, we argue that it may be the case in which the low quality firm becomes a natural monopolist. As to the cooperative play, we

[5] More technical literature has studied several protocols to cope with web congestion. For an easy to read exposition of some relevant issues see Vorthman (1999) and the references therein. At a level more tailored to telecommunications engineers, see Kohler *et al.* (2003).

find that the multi-product cartel invests less, sells less, and provides lower quality levels of both varieties than the duopoly does.

Before turning to the model formulation, we would like to underline that, although our leading example is the Internet, the theoretical framework provided in this paper could be applied to the studying of all those markets for vertically differentiated goods characterized by positive network externalities and congestion phenomena, where firms invest so as to increase capacity.

The remainder of this paper is structured as follows: the model is laid out in section 2; section 3 deals with non cooperative play, while section 4 deals with cooperative play, i.e. a full cartelization. Concluding remarks are in section 5.

2. Model Formulation

Time is continous and, as usual, indicated by t. At each $t \in [0, \infty)$ a market for network services exists. Let this market be supplied by two single-product firms offering network services of on-net quality $q_i(t)$ in a number of units $x_i(t)$ at a connection price $p_i(t)$, with $i = \{H, L\}$, $\infty > q_H(t) \geq q_L(t) \geq 0\ \forall t$. For the sake of simplicity, we assume that the off-net quality is nil, meaning that consumers' satisfaction from joining network i is totally independent from network j. This is the same assumption as in Mackey-Mason and Varian (1995). Moreover, we abstract from the presence of switching costs, so customers that, as time goes by, switch from one variety to the other, bear any disutility.[6] Each consumer is characterized by a willingness to pay θ, uniformly distributed over the support $[\overline{\theta} - 1, \overline{\theta}]$, with $\overline{\theta} > 1$. Without any loss of generality, assume $f(\theta) = 1$, so that consumers' population is normalized to 1.[7]

The situation modelled here is one where consumers utility from subscribing to a given network service is positively affected by the expected number of others subscribing to the same. We define the instantaneous net surplus a consumer of type θ draws from the variety characterized by $q_i(t)$ as:

$$U_\theta(t) = \begin{cases} [\theta + q_H(t) - p_H(t)]\, x_H^e(t) & \text{if she subscribes to service } H \\ [s\theta + q_L(t) - p_L(t)]\, x_L^e(t) & \text{if she subscribes to service } L \\ 0 & \text{if she does not subscribe to any service} \end{cases} \tag{1}$$

where $x_i^e(t)$ is consumers' expected number of subscribers to service i at time t, and s is a time-invariant real parameter. It is worth noting that, in order to ensure a higher willingness

[6] For a good survey on consumers' switching costs see Klemperer (1995).

[7] At each point in time, each consumer buys at most one unit of the preferred quality. This rules out the use of second-degree price discrimination.

to pay for the high quality service, $s \in \left(0, x_H^e(t)/x_L^e(t)\right)$.[8] (1) is such that if a consumer subscribe to a service but only a negligible part of the population does likewise, her utility tends to zero. This amounts to saying that the intrinsic utility that can be justified by itself consumption is nil.[9]

In order to derive the expressions of market demands, we compute the threshold of θ which characterizes the consumer who is indifferent between subscribing to the high quality service and subscribing to the low quality service:

$$\widehat{\theta}(t) = \frac{x_H^e(t)\left(p_H(t) - q_H(t)\right) - x_L^e(t)\left(p_L(t) - q_L(t)\right)}{x_H^e(t) - s x_L^e(t)} \tag{2}$$

and the one which characterizes the consumer who is indifferent between subscribing to the low quality service and not subscribing at all:

$$\widetilde{\theta}(t) = \frac{p_L(t) - q_L(t)}{s} \tag{3}$$

The direct demand system follows:

$$x_H(t) = \overline{\theta} - \widehat{\theta}(t) \tag{4}$$

$$x_L(t) = \widehat{\theta}(t) - \widetilde{\theta}(t) \tag{5}$$

Notice that $\widetilde{\theta}(t)$ does not contain neither x_L^e nor x_H^e, implying that the individual decision whether to subscribe to service L, contrasted with the alternative not to subscribe to any service, does not depend on others' decisions. As to $\widehat{\theta}(t)$, things are more involved, in that it clearly becomes crucial the way consumers form their expectations. In this respect, let us make the following assumption:

Axiom 1 *Consumers have a perfect foresight. Formally:* $x_i^e(t) = x_i(t) \ \ \forall t, \ \ i = \{H, L\}$.

The above axiom guarantees that, in the fixed-point equilibrium, expectations about prices and network sizes turn out to be correct.

In order to deal with quantity competition, we need to explicitly solve the system (4-5) for prices. This can be done as long as partial market coverage prevails, which amounts to requiring that $\widetilde{\theta}(t) > 0$:

$$p_H(t) = \overline{\theta} + q_H(t) - x_H(t) - s\frac{\left[x_L(t)\right]^2}{x_H(t)} \tag{6}$$

[8] Parameter s is so introduced in order not to restrict the spectrum of admissible cases to $x_H^e \geq x_L^e$.
[9] Such an assumption is particularly appropriate for information and communication networks.

$$p_L(t) = q_L(t) + s\left[\overline{\theta} - x_H(t) - x_L(t)\right] \tag{7}$$

The above demand system is well specified iff $\widehat{\theta}(t) > \widetilde{\theta}(t)$. From a direct comparison between the two thresholds, it is easy to assess that last inequality holds for all $s \in (0, x_H(t)/x_L(t))$, that is, in the entire admissible parameter range. Notice also that, in this range, $p_H(t)$ and $p_L(t)$ are always positive.

On the supply side, production entails the following instantaneous cost, convex in the current quality level:

$$C_i(t) = cx_i(t) + [q_i(t)]^2 \tag{8}$$

Since (8) is separable in $x_i(t)$ and $q_i(t)$, quality improvements entail fixed costs.[10] Without any loss of generality, we normalize marginal costs to zero.

Instantaneous profits then write:

$$\Pi_i(t) = p_i(t)x_i(t) - [q_i(t)]^2 - [k_i(t)]^2 \tag{9}$$

where $[k_i(t)]^2$ is the instantaneous quadratic cost of investing to build up own network capacity, $k_i(t)$ being the amount of capital devoted by firm i at time t.

We assume that the evolution of QoS depends positively from the investment of a firm in its network capacity and negatively from the total amount of consumers using the same network. Consider the following linear state equation:

$$\frac{\partial q_i(t)}{\partial t} = \dot{q_i}(t) = k_i(t) - \delta x_i(t) \tag{10}$$

where $\delta > 0$ is the constant decay rate. (10) reflects the basic congestion property of communication and information networks: for a given network capacity, the higher the number of customers using the network (more precisely, the more data packages are transferred), the higher the expected delay or the loss rate. On the other hand, for a given number of customers using the network, the larger the network capacity, the lower the expected delay or the loss rate. It is worth considering the above kinematic equation together with (1). A given increase in $x_i^e(t)$ yields two opposite effects: (i) a direct increase in $U_\theta(t)$, i.e. a positive network effect; (ii) an indirect decrease in $U_\theta(t)$ due to a decrease in $q_i(t)$, i.e. a congestion effect.

The object of firm i is to maximize the present value of its profit stream over an infinite time horizon:

$$\Pi_i(t) = \int_0^\infty \pi_i(t)e^{-\rho t}dt \tag{11}$$

[10]For models where quality improvements entail fixed costs see Aoki and Prusa (1997) and Lehmann-Grube (1997). Motta (1993) provides a comparative evaluation between variable and fixed costs.

w.r.t. controls $x_i(t)$ and $k_i(t)$, under the constraint given by (10). The discount rate $\rho > 0$ is assumed to be constant and common to both firms.

In order to evaluate the industry performance in terms of welfare, we adopt the following welfare function:

$$W(t) = \Pi_H(t) + \Pi_L(t) + CS(t) \tag{12}$$

where $CS(t)$ is the instantaneous consumer surplus:

$$CS(t) = \int_{\widehat{\theta}}^{\overline{\theta}} U_H(t)\, d\theta + \int_{\widetilde{\theta}}^{\widehat{\theta}} U_L(t)\, d\theta = x_H s x_L^2 + \frac{1}{2}\left(x_H^3 + s x_L^3\right) \tag{13}$$

3. Markov Perfect Open-Loop Nash Equilibria

Firm i's current value Hamiltonian function writes:

$$\mathcal{H}_i(t) = e^{-\rho t}\left[\pi_i(t) + \lambda_{ii}(t)\, \dot{q}_i + \lambda_{ij}(t)\, \dot{q}_j\right] \tag{14}$$

First order conditions (FOCs) on controls are (henceforth, time index is omitted for brevity):[11]

$$\frac{\partial \mathcal{H}_H}{\partial x_H} = 0 \Rightarrow x_H = \frac{1}{2}\left(\overline{\theta} + q_H - \delta\lambda_{HH}\right) \tag{15}$$

$$\frac{\partial \mathcal{H}_L}{\partial x_L} = 0 \Rightarrow x_L = \frac{1}{2s}\left[s\left(\overline{\theta} + x_H\right) + q_L - \delta\lambda_{LL}\right] \tag{16}$$

$$\frac{\partial \mathcal{H}_i}{\partial k_i} = 0 \Rightarrow \lambda_{ii} = 2k_i \tag{17}$$

The above FOCs are such that the present game produces Markov perfect open-loop Nash equilibria, i.e. equilibria which are subgame perfect.[12] Notice also that conditions (15-16-17) do not contain λ_{ij} because of the separated dynamics assumed in the model. Therefore, we set $\lambda_{ij} = 0$ for all $t \in [0, \infty)$ and $j \neq i$, and specify only one co-state equation per firm:

$$\frac{\partial \mathcal{H}_H}{\partial q_H} = -2q_H + x_H = \rho\lambda_{HH} - \dot{\lambda}_{HH} \tag{18}$$

$$\frac{\partial \mathcal{H}_L}{\partial q_L} = -2q_L + x_L = \rho\lambda_{LL} - \dot{\lambda}_{LL} \tag{19}$$

[11] Second order conditions are in the Appendix.

[12] See, e.g., Mehlman and Willig (1983), Reinganum (1982), Dockner, Feichtinger and JØrgensen (1985) and Fershtman (1987). For an exhaustive discussion on the coincidence between open-loop and closed-loop memoryless solutions, see Dockner *et al.* (2000, ch.7).

along with the transversality condition:

$$\lim_{t \to \infty} \mu_i(t) q_i(t) = 0 \tag{20}$$

and the initial conditions $q_i(0) = q_{i0} > 0$, with $q_{H0} > q_{L0}$.

Now, using (17), (15) and (16), we obtain:

$$x_H = \frac{1}{2}\left(\overline{\theta} + q_H - \delta 2k_H\right) \tag{21}$$

$$x_L = \frac{1}{2s}\left(\frac{3}{2}s\overline{\theta} + \frac{1}{2}sq_H - s\delta k_H + q_L - 2\delta k_L\right) \tag{22}$$

which can be plugged into the state equations, simplifying as follows:

$$\dot{q}_H = \frac{1}{2}\left(2k_H + 2\delta^2 k_H - \delta\overline{\theta} - \delta q_H\right) \tag{23}$$

$$\dot{q}_L = \frac{1}{4s}\left(4k_L s - \delta s\overline{\theta} + \delta s q_H - 2s\delta^2 k_H - 2\delta q_L + 4\delta^2 k_L\right) \tag{24}$$

In order to characterize the dynamics of investment, we need to differentiate (17) w.r.t. time:

$$\dot{\lambda}_{ii} = 2\,\dot{k}_i \tag{25}$$

By plugging (25), (17), (21) and (22) into the co-state equations, we get:

$$\dot{k}_H = \frac{3}{4}q_H - \frac{1}{4}\overline{\theta} + \frac{1}{2}\delta k_H + \rho k_H \tag{26}$$

$$\dot{k}_L = \frac{3}{4}q_L - \frac{1}{8}\overline{\theta} + \frac{1}{2}\delta k_L + \rho k_L + \frac{1}{8}q_H - \frac{1}{4}\delta k_H \tag{27}$$

The system $\left\{\dot{k}_i = 0,\, \dot{q}_i = 0\right\}$ yields the following steady states:

$$q_H^* = \frac{\overline{\theta}\,(1 - 2\delta\rho)}{3 + 2\delta\,(2\delta + \rho)} \tag{28}$$

$$k_H^* = \frac{2\delta\overline{\theta}}{3 + 2\delta\,(2\delta + \rho)} \tag{29}$$

$$q_L^* = \frac{s\overline{\theta}\,(1 - 2\delta\rho)\,[1 + 2\delta\,(2\delta + \rho)]}{[3 + 2\delta\,(2\delta + \rho)]\,[4s - 1 + 2\delta\,(2\delta + \rho)]} \tag{30}$$

$$k_L^* = \frac{2s\delta\overline{\theta}\,[1 + 2\delta\,(2\delta + \rho)]}{[3 + 2\delta\,(2\delta + \rho)]\,[4s - 1 + 2\delta\,(2\delta + \rho)]} \tag{31}$$

Proposition 1 *The steady state defined by* $\{q_H^*, k_H^*, q_L^*, k_L^*\}$ *is stable along a saddle path.*

Proof. See the Appendix.

The above candidates are acceptable as steady state solutions if and only if they belong to the set of positive real numbers. By a direct inspection of the involved expressions, we can write:

Lemma 1 $q_i^* \geq 0$ *with* $i = \{H, L\}$ *if*

$$\rho \leq \frac{1}{2\delta}$$

while $k_i^* > 0$ *always.*

Proof. The proof is straightforward. It suffices to note that $4s - 1 + 2\delta(2\delta + \rho) > 0$, since it has to be true that $s > 1/4$ for second order conditions to hold (see the Appendix). □

Now, we are in a position to derive the expressions of equilibrium quantities:

$$x_H^* = \frac{2\overline{\theta}}{3 + 4\delta^2 + 2\delta\rho} \tag{32}$$

$$x_L^* = \frac{2s\overline{\theta}\left(1 + 4\delta^2 + 2\delta\rho\right)}{\left(3 + 4\delta^2 + 2\delta\rho\right)\left(4s - 1 + 4\delta^2 + 2\delta\rho\right)} \tag{33}$$

which are both positive in the entire admissible parameter range. Once noted that

$$\frac{q_H^*}{q_L^*} = \frac{k_H^*}{k_L^*} = \frac{x_H^*}{x_L^*} \tag{34}$$

the following can immediately be established:

Lemma 2 *At the steady state, the high quality firm invests more in network capacity and obtains a larger market share than the low quality firm.*

The fact that $q_H^* > q_L^* \Rightarrow x_H^* > x_L^*$ in the entire admissible parameter range, allows us to simplify the algebra by setting $s = 1$. As to the comparison between quality levels, we have:

Lemma 3 $q_H^* \geq q_L^*$ *always.*

Proof. $q_H^* - q_L^* \propto 3s - 1 + 4\delta^2 + 2\delta\rho - 4s\delta^2 - 2s\delta\rho$. With $s = 1$ this expression simplifies to 2. □

Unlike the conventional wisdom coming from analyses based on multi-stage games, in our dynamic framework the order relationship between equilibrium profits is ambiguous.[13]

[13] A differential duopoly game where the low quality firm may perform better than the rival in terms of equilibrium profits is in Colombo and Lambertini (2003).

To see this, we compute the steady state level of profits accruing to firm H and to firm L, respectively:

$$\pi_H^* = -\frac{\{13 + 4\delta\,[\Delta]\}\,\overline{\theta}^2}{[3 + 2\delta\,(2\delta + \rho)]^4} \tag{35}$$

with

$$\Delta = -20\rho + \delta\left\{5 - 64\delta\rho - 22\rho^2 + 16\delta^3\left[\delta\left(-1 + \rho^2\right) - \rho\left(4 - \rho^2\right)\right] + 4\delta^2\left(-1 - 7\rho^2 + \rho^4\right)\right\}$$

$$\pi_L^* = \frac{\overline{\theta}^2\,[1 + 2\delta\,(2\delta + \rho)]^2\left[3 + 4\delta\rho + 4\delta^2\,(1 - \rho^2)\right]}{[3 + 2\delta\,(2\delta + \rho)]^4} \tag{36}$$

By substracting π_L^* from π_H^* we get:

$$\pi_H^* - \pi_L^* = \frac{16\left\{-1 + \delta\left[4\rho + \delta\left(-3 + 10\delta\rho + 8\delta^3\rho + 4\rho^2 + 4\delta^2\,(-1 + \rho^2)\right)\right]\right\}}{[3 + 2\delta\,(2\delta + \rho)]^4} \tag{37}$$

which, *a priori*, can take either sign.

Now, let us define $\overline{\rho}$ the admissible value of ρ such that $\pi_H^* = \pi_L^*$. From (37) we obtain:

$$\overline{\rho} = \frac{-2 - 5\delta^2 - 4\delta^4 + \sqrt{8 + 36\delta^2 + 69\delta^4 + 56\delta^6 + 16\delta^8}}{4\delta\left(1 + \delta^2\right)} \tag{38}$$

wich is always admissible, being $0 < \overline{\rho} < 1/\,(2\delta)$. Moreover, let us define $\widetilde{\rho}$ the value of ρ such that $\pi_H^* = 0$.

The sustainability of either the monopoly or the duopoly regime depends upon the non-negativity of profits. In this respect, we have:

Lemma 4 *$\pi_L^* > 0$ always; $\pi_H^* > 0$ for all $\rho > \widetilde{\rho}$, with $\widetilde{\rho} \in (0, \overline{\rho})$ $\forall\ \delta \in (0, 0.81832)$ and $\widetilde{\rho} < 0$ $\forall\ \delta \in (0.81832, 1)$.*

Proof. From (36):

$$\pi_L^* > 0 \text{ if } \rho < \frac{1}{2\delta}\left(1 + 2\sqrt{(1 + \delta^2)}\right)$$

Since $1/\,(2\delta)\,(1 + 2\sqrt{1 + \delta^2}) > 1/\,(2\delta)$, and provided that, from Lemma 1, the admissible range for ρ is $\rho < 1/\,(2\delta)$, we have the result. The analytical expresion of $\widetilde{\rho}$ is cumbersome, therefore omitted. By taking $\lim_{\rho \to 0}(\pi_H^*) = \overline{\theta}^2\left(-13 - 20\delta^2 + 64\delta^6 + 16\delta^4\right) / \left(3 + 4\delta^2\right)^4$ and by evaluating the sign of the numerator, we can assess that for all $\delta \in (0.81832, 1)$ $\pi_H^* < 0$ iff $\rho > \widetilde{\rho} < 0$, therefore in this regime we have $\pi_H^* > 0$ always. $\square$

The above discussion yields our:

Proposition 2 *For all $\delta \in (0.81832, 1)$ the market is a natural duopoly; if $\rho < \overline{\rho}$ we have $0 < \pi_H^* < \pi_L^*$, otherwise we have $\pi_H^* > \pi_L^* > 0$. For all $\delta \in (0, 0.81832)$ the market is served by both firms iff $\rho > \widetilde{\rho}$, otherwise the low quality firm becomes a natural monopolist; if $\rho \in (\widetilde{\rho}, \overline{\rho})$ we have $0 < \pi_H^* < \pi_L^*$, otherwise we have $0 < \pi_L^* < \pi_H^*$.*

The following figure illustrates the above proposition:

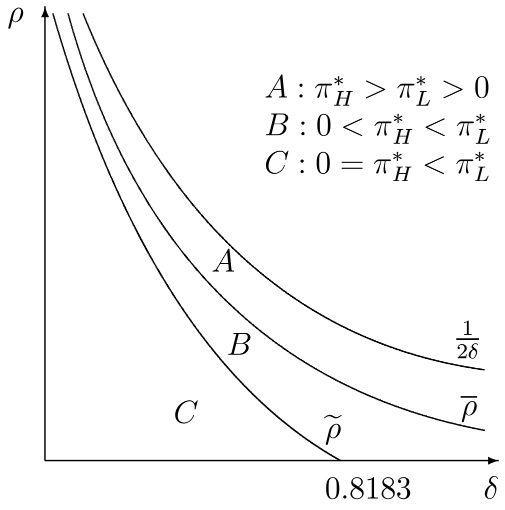

Figure 1 : Parameter Space

Within area A, we find the traditional static result on product quality provision in oligopoly; within area B, although the market is still served by both firms, the order relationship between profits is reversed; finally, what we register within area C, is that the market becomes a natural monopoly, with only the low quality firm being active. This, is in contrast with the so called *finitess property* (Shaked and Sutton, 1983), according to which if only one firm can earn positive profits while supplying a vertically differentiated good, such a firm will be the one providing the highest (technically producible) quality level. Not surprisingly, in our dynamic model, the sustainability of a duopoly regime crucially depends upon the intertemporal parameters, δ and ρ. If the decay rate is sufficiently high, implying that congestion phenomena are sufficiently important, we expect the market to be a natural duopoly regardless of discounting. If, on the other hand, congestion effects are very small, we expect the market to be served only by the low quality firm. Therefore, we can conclude that, in the long run, congestion is pro-competitive.

4. Full Cooperative Play

In this section, we assume that the two competitors decide to implement a full cartelization rule, that is, to act so as to maximize the present value of joint profits[14] The relevant current value Hamiltonian function writes:

$$\mathcal{H}(t) = e^{-\rho t}\left[\pi_H(t) + \pi_L(t) + \lambda_H(t)\,\dot{q}_H + \lambda_L(t)\,\dot{q}_L\right] \tag{39}$$

By applying Pontryagin's Maximum Principle, first order conditions are:

$$\frac{\partial \mathcal{H}_H}{\partial x_H} = 0 \Rightarrow x_H = \frac{1}{2}\overline{\theta} + \frac{1}{2}q_H - \frac{1}{2}x_L - \frac{1}{2}\lambda_H\delta \tag{40}$$

$$\frac{\partial \mathcal{H}_H}{\partial x_L} = 0 \Rightarrow x_L = \frac{1}{4}q_L + \frac{1}{4}\overline{\theta} - \frac{1}{4}x_H - \frac{1}{4}\lambda_L\delta \tag{41}$$

$$\frac{\partial \mathcal{H}_H}{\partial k_H} = 0 \Rightarrow \lambda_H = 2k_H \tag{42}$$

$$\frac{\partial \mathcal{H}_H}{\partial k_L} = 0 \Rightarrow \lambda_L = 2k_L \tag{43}$$

$$\frac{\partial \mathcal{H}_H}{\partial q_H} = x_H - 2q_H = \rho\lambda_H - \dot{\lambda}_H \tag{44}$$

$$\frac{\partial \mathcal{H}_H}{\partial q_H} = x_L - 2q_L = \rho\lambda_L - \dot{\lambda}_L \tag{45}$$

along with the same transversality and initial conditions as in duopoly. From the above conditions, we obtain the following dynamic system:

$$\dot{k}_H = \rho k_H + q_H - \frac{x_H}{2} \tag{46}$$

$$\dot{k}_L = \rho k_L + \frac{6}{7}q_L - \frac{1}{14}\overline{\theta} + \frac{1}{14}q_H + \frac{2}{7}k_L\delta - \frac{1}{7}k_H\delta \tag{47}$$

$$\dot{q}_H = k_H - \delta\frac{3}{7}\delta\overline{\theta} - \frac{4}{7}\delta q_H + \frac{1}{7}\delta q_L - \frac{1}{7}2k_L\delta^2 + \frac{4}{7}2k_H\delta^2 \tag{48}$$

$$\dot{q}_L = k_L - \frac{2}{7}\delta q_L - \frac{1}{7}\delta\overline{\theta} + \frac{1}{7}\delta q_H + \frac{4}{7}k_L\delta^2 - \frac{2}{7}k_H\delta^2 \tag{49}$$

whose steady state point is defined as follows:

$$k_H^{**} = \frac{2\delta\overline{\theta}\left(4\delta^2 + 2\delta\rho + 5\right)}{20\delta\rho + 40\delta^2 + 17 + 4\delta^2\rho^2 + 16\delta^3\rho + 16\delta^4} \tag{50}$$

[14]For models where R&D in product quality improvements is activated by joint ventures, see Motta (1992) and Rosenkranz (1995).

$$k_L^{**} = \frac{2\delta\overline{\theta}\left(4\delta^2+1+2\delta\rho\right)}{20\delta\rho+40\delta^2+17+4\delta^2\rho^2+16\delta^3\rho+16\delta^4} \tag{51}$$

$$q_H^{**} = \frac{\overline{\theta}\left(1-2\delta\rho\right)\left(4\delta^2+5+2\delta\rho\right)}{20\delta\rho+40\delta^2+17+4\delta^2\rho^2+16\delta^3\rho+16\delta^4} \tag{52}$$

$$q_L^{**} = \frac{\overline{\theta}\left(1-2\delta\rho\right)\left(4\delta^2+1+2\delta\rho\right)}{20\delta\rho+40\delta^2+17+4\delta^2\rho^2+16\delta^3\rho+16\delta^4} \tag{53}$$

From Lemma 1, the above solutions are always non negative in the admissible parameter range.

Proposition 3 *The steady state defined by $\{k_H^{**}, k_L^{**}, q_H^{**}, q_L^{**}\}$ is stable along a saddle path.*

Proof. See the Appendix.

As to equilibrium quantities, we have:

$$x_H^{**} = 2\overline{\theta}\frac{4\delta^2+2\delta\rho+5}{20\delta\rho+40\delta^2+17+4\delta^2\rho^2+16\delta^3\rho+16\delta^4} \tag{54}$$

$$x_L^{**} = 2\overline{\theta}\frac{4\delta^2+1+2\delta\rho}{20\delta\rho+40\delta^2+17+4\delta^2\rho^2+16\delta^3\rho+16\delta^4} \tag{55}$$

Notice that, as in duopoly, $k_H^{**}/k_L^{**} = x_H^{**}/x_L^{**} = q_H^{**}/q_L^{**}$, implying that $k_H^{**} > k_L^{**}$ and $x_H^{**} > x_L^{**}$. Therefore, Lemma 2 can be extended to the case in which firms behave cooperatively. By comparing the above solutions with those arising in duopoly, we can write:

Proposition 4 *$k_i^{**} < k_i^*$, $x_i^{**} < x_i^*$, $q_i^{**} < q_i^*$, with $i = \{H, L\}$, provided that the duopoly regime be sustainable.*

Proof. It suffices to make a direct comparison between the involved expressions. As to qualities, we have $(q_i^{**} - q_i^*) \propto (-1 + 2\delta\rho)$. Since, from Lemma 1, $\rho < 1/(2\delta)$, the sign is negative. $\square$

Corollary 1 *$x_i^{**} < x_i^* \Rightarrow CS^{**} < CS^*$, $i = \{H, L\}$. At the steady state, consumers are better off under duopoly than under full cartelization regime, provided that the duopoly regime be sustainable.*

When the duopoly regime is not sustainable (see Proposition 2), only the low quality firm survives and (13) becomes $CS(t) = x_L^3/2$. Under full cartelization, using equilibrium quantities, the equilibrium level of consumers' surplus turns out to be:

$$CS^{**} = 32\overline{\theta}^3\frac{64\delta^4+50\delta^2+25\delta\rho+16\delta^2\rho^2+64\delta^3\rho+48\delta^5\rho+32\delta^6+24\delta^4\rho^2+4\delta^3\rho^3+17}{\left(20\delta\rho+40\delta^2+17+4\delta^2\rho^2+16\delta^3\rho+16\delta^4\right)^3} \tag{56}$$

while, under non cooperative play (in case of natural monopoly), we obtain:

$$CS^{*}=\frac{1}{2}\left[\frac{2\overline{\theta}\left(1+4\delta^{2}+2\delta\rho\right)}{\left(3+4\delta^{2}+2\delta\rho\right)^{2}}\right]^{3} \tag{57}$$

By a direct comparison between (56) and (57) we can establish our:

Proposition 5 *When the duopoly regime is not sustainable, there exists an admissible subset of parameters* $\{\overline{\theta},\delta,\rho\}$ *wherein* $CS^{**}>CS^{*}$*.*

As an illustration, a numerical example is provided in the Appendix.

5. Concluding Remarks

We have investigated a differential duopoly game with vertical differentiation and network externalities and focussed on the steady state properties of the system. Unlike multi-stage games, differential games are particularly suitable to shed light on the deep nature of investments, which is inherently a dynamic one. Another novelty of our approach has been to jointly deal with both the positive and the negative side of network externalities. It seems to us that these two main departures from the existing relevant literature, represents a step ahead in the comprehension of the nature and the effects of firms investments on the Web.

The main result obtained in our paper is that, in contrast with the acquired wisdom based on static models, the low quality firm may earn higher equilibrium profits than the high quality firm. The intuition of this result lies in the fact that, whenever active, the high quality firm sells always more than the rival, leading to more relevant congestion effects on its infrastructures. In order not to let its quality decreasing, the firm supplying the superior variety has to devote more resources to building up network capacity. Indeed, in the long run equilibrium, it may be more profitable to provide the market with the inferior quality than otherwise. Furthermore, in contrast with the so called *finitess property* (Shaked and Sutton, 1983), we have shown the low quality firm may become a natural monopolist.

The duopoly regime has been compared with the full cartelization regime. In this respect, we have shown that the multi-product cartel invests less, sells less, and provides lower quality levels of both varieties than the duopoly does, implying that consumers are better off under non cooperative play. However, one important remark is in order. The superiority of the non cooperative play in terms of consumers well-being requires that duopoly be sustainable. When congestion effects are very small, only the low quality firm may survive. In this case, the relevant comparison between non cooperative and cooperative play involves a single-product monopolist and a multi-product monopolist, respectively. Quite surprisingly, we have found that it may be socially desirable to let the market be served by the multi-product cartel.

Appendix

Stability

Proof of Proposition 1. We consider the system composed by (10) in combination with (26) and (27).The resulting dynamic system can be written in matrix form as follows:

$$\begin{bmatrix} \dot{q}_H \\ \dot{k}_H \\ \dot{q}_L \\ \dot{k}_L \end{bmatrix} = \begin{bmatrix} -\frac{\delta}{2} & \left(1+\delta^2\right) & 0 & 0 \\ \frac{3}{4} & \frac{1}{2}\left(\delta+2\rho\right) & 0 & 0 \\ \frac{\delta}{4} & -\frac{\delta^2}{2} & -\frac{\delta}{2} & 1+\delta^2 \\ \frac{1}{8} & -\frac{\delta}{4} & 1-\frac{1}{4} & \rho+\frac{\delta}{2} \end{bmatrix} \begin{bmatrix} q_H \\ k_H \\ q_L \\ k_L \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\overline{\theta}}{4} \\ -\frac{\delta\overline{\theta}}{4} \\ -\frac{\overline{\theta}}{8} \end{bmatrix}$$

By computing the four eigenvalues, it is easy to assess that two eigenvalues are positive while two eigenvalues are negative. Hence the equilibrium is a saddle point.

Proof of Proposition 3. We consider the dynamic system (46-49), which can be written in matrix form as follows:

$$\begin{bmatrix} \dot{q}_H \\ \dot{k}_H \\ \dot{q}_L \\ \dot{k}_L \end{bmatrix} = \begin{bmatrix} -\frac{4\delta}{7} & \left(1+\frac{8\delta^2}{7}\right) & \frac{\delta}{7} & -\frac{2\delta^2}{7} \\ 1-\frac{2}{7} & \rho+\frac{4}{7}\delta & \frac{1}{14} & -\frac{1}{7}\delta \\ \frac{1\delta}{7} & -\frac{2\delta^2}{7} & -\frac{2\delta}{7} & 1+\frac{4\delta^2}{7} \\ \frac{1}{14} & -\frac{1\delta}{7} & \frac{6}{7} & \rho+\frac{2\delta}{7} \end{bmatrix} \begin{bmatrix} q_H \\ k_H \\ q_L \\ k_L \end{bmatrix} + \begin{bmatrix} -\frac{3\delta\overline{\theta}}{7} \\ -\frac{3\overline{\theta}}{14} \\ -\frac{\delta\overline{\theta}}{7} \\ -\frac{\overline{\theta}}{14} \end{bmatrix}$$

By computing the four eigenvalues, as before, we find that two are negative while two are positive, implying saddle path stability.

Sufficient Conditions

Sufficiency (Arrow) for the duopoly:

Using Arrow's sufficiency theorem:

$$\frac{\partial^2 \mathcal{H}_H\left(x_H^O, k_H^O\right)}{\partial^2 q_H} = -\frac{3}{2}$$

$$\frac{\partial^2 \mathcal{H}_L\left(x_L^O, k_L^O\right)}{\partial^2 q_L} = \frac{1-4s}{2s}$$

where x_i^0 and k_i^0 denote optimal control expressions. While s.o.c. are always satisfied for firm H, as to firm L we need $s > 1/4$.

Sufficiency (Arrow) for the full cartelization:

The relevant Hessian metrix is

$$\begin{bmatrix} \frac{\partial^2 \mathcal{H}\left(x_i^O, k_i^O\right)}{\partial^2 q_H} = -\frac{10}{7} & \frac{\partial^2 \mathcal{H}\left(x_i^O, k_i^O\right)}{\partial q_H q_L} = -\frac{1}{7} \\ \frac{\partial^2 \mathcal{H}\left(x_i^O, ki\right)}{\partial q_L q_H} = -\frac{1}{7} & \frac{\partial^2 \mathcal{H}\left(x_i^O, k_i^O\right)}{\partial^2 q_L} = -\frac{12}{7} \end{bmatrix}$$

which is negative definite.

Numerical Examples

Duopoly regime:

$$\delta = \frac{1}{10},\ \rho = 2,\ \overline{\theta} = \frac{11}{10}$$

$\pi_L^* = 0.06594$, $\pi_H^* = 0.05921$, $q_H^* = 0.19186$, $q_L^* = 0.08031$, $x_H^* = 0.63953$, $x_L^* = 0.26771$, $k_H^* = 0.06395$, $k_L^* = 0.02677$, $p_H^* = 0.54027$, $p_L^* = 0.27307$, $\overline{\rho} = 1.39045$, $\widetilde{\rho} = 2.08458$.

$$\delta = \frac{15}{100},\ \rho = 1,\ \overline{\theta} = \frac{12}{10}$$

$\pi_L^* = 0.07584$, $\pi_H^* = 0.01598$, $q_H^* = 0.24779$, $q_L^* = 0.1016$, $x_H^* = 0.70796$, $x_L^* = 0.29029$, $k_H^* = 0.10619$, $k_L^* = 0.04354$, $p_H^* = 0.6208$, $p_L^* = 0.30335$, $\overline{\rho} = 1.40118$, $\widetilde{\rho} = 0.9124$.

Welfare comparison between single- and multi-product monopolist:

$$\delta = \frac{5}{100},\ \rho = \frac{1}{2},\ \overline{\theta} = \frac{11}{10}$$

$\pi_L^* = 0.04819$, $CS^* = 0.00515$, $CS^{**} = 0.1387$, $W^* = 0.05334$.

$CS^{**} > W^* \Rightarrow W^{**} > W^*$.

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