Monetary and Fiscal Policy Interactions: the Impact on the Term Structure of Interest Rates.

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Abstract

In this paper I consider a General Equilibrium framework to evaluate the role and the importance of the interactions between Monetary and Fiscal policies in the sense outlined by Leeper (1991) and the Fiscal Theory of the Price Level, in the determination of nominal and real term structure. The main results show that the term structure will depend not only upon monetary and technological factors - as in the traditional models - but also upon the fiscal policy reaction function parameters. This aspect is very useful in explaining the recent experience of some countries, like Italy, whose term structure shifted down after the fiscal retrenchement imposed by Fiscal Rules of Maastricht Treaty. In this paper the term structure is derived by using a very general approach, starting from implicit assumptions on the driving stochastic processes of the model, rather than imposing a specific structure right from the beginning. The results are consistent with the more recent results outlined by Fiscal Theory of the Price Level and by monetary models of the term structure.

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1 Introduction

Recently, the growing importance of public debt has forced many governments to adopt severe fiscal restrictions through cutting of expenses and/or increase of taxes. At the same time, we observed a marked reduction in nominal interest rates paid on the outstanding level of public debt. The reduction of interest rates can be observed for several maturities, implying a downward shift of the entire term structure equation. Starting from these simple observations, it becomes natural to ask if nominal term structure of interest rates could depend upon fiscal policy parameters.

The traditional models of the Term Structure of interest rate, like for example, Cox, Ingersoll and Ross (1985), Longstaff and Schwartz (1992), makes the nominal structure as dependent upon only two essential factors: monetary and technological factors. In particular, the monetary factors - generally modelled as relating with the pattern of the stochastic processes of the price level, determines the structure of nominal rates, while the technological factors will determine the structure of real rates. Usually, the technological factors are represented through a simple stochastic process for the rate of technological progress, while the nominal structure is inserted by considering a stochastic process for the inflation rate or the price level (as in Cox, Ingersoll and Ross (1985a,b), for example), or by specifying a stochastic process for the growth rate of nominal money supply.

The limit of this type of modelling is evident because it does not take into consideration any new progress made on the literature on monetary policy rules and especially on the role of fiscal policy in the stabilization of the inflation rate, as discussed by the recent contributions on the Fiscal Theory of the Price Level (FTPL, henceforth).

The present paper tries to fill in this gap, by presenting a framework where through the explicit modelling of the Government Budget constraint, it is possible to introduce a fiscal policy reaction function making the total fiscal revenue as reacting to the outstanding level of public debt, in order to ensure the solvability of government and of preserve the explosion of the price level, as proposed by Leeper (1991), Sims (1994), Woodford (1996, 2000) and more recently, by Cochrane (1998, 1999, 2000). This type of policy is often defined as 'Passive' (see Leeper (1991)) or as 'Ricardian' fiscal policy (see, in particular Woodford (1996, 2000)), even if this terms does not have very much to do with the traditional Ricardian equivalence term, or.

Apart from fiscal policy, the model builds a general equilibrium framework similar to what has been proposed by Bakshi and Chen (1996), where a representative agent optimally chooses among shares, nominal and real bonds. The economy is an endowment-economy type, where the output follows an exogenous stochastic process. To keep the model simple and at the same time comparable with most of the existing literature on the term structure of interest rate, monetary policy is assumed to follow an exogenous stochastic process for the growth rate of money supply.

The choice problem for the representative agent is represented in discrete time, while the main equilibrium relationships are obtained in continuous time. The general equilibrium price of a zero coupon bond is the found together with the final equation for the nominal and real term structure equation. All the main results are obtained in closed-form solution.

From the equilibrium relationships of the model, it is possible to check that the position of the nominal term structure depends upon both monetary and fiscal policy parameters. Thus, the mean of the stochastic process of the expected money growth supply and the parameter which indicates the stance of fiscal policy. With a 'tight' fiscal policy, i.e. a fiscal policy where taxes react to the stock of real debt, the price of bonds are high and the nominal interest rates are low. With a loose fiscal policy - a fiscal policy which sets a level of taxes independent upon the stock of public debt (nominal or real) - the price of a zero coupon bond will be lower than in the case with a tight fiscal policy and the nominal interest rates will be much higher.

The key channel for these results is the determination of the stochastic process for the price level, whose drift is defined as the inflation rate. Given the structure of the model, it is shown that the stochastic process of the price level entirely depends upon both elements affecting the nominal equilibrium of the economy: the drift of the money supply process and the parameter describing the fiscal policy stance.

An innovative aspect of the present paper is given by the solution method adopted to obtain the term structure. Differently, from the approach followed by Bakshi and Chen (1996) and Balduzzi (2000), the motion law for both nominal and real interest rates are derived following a fairly general approach where the coefficients of the main stochastic processes are time-variant, by directly applying the results from Feyman-Kač Theorem . The closed form solution is then obtained from the general case, by imposing a set of time invariant coefficients, as considered by the current literature.

To show the empirical performance of the model, I report the results from simulations of the term structure equations for given values of the parameters derived from Balduzzi (2000). The pictures confirm the presumption by which the position of the term structure in the plane depends upon both monetary, fiscal and technological factor. Fiscal policy appears to be important as much as monetary policy in determining the position of the term structure of interest rates. This will explain the performance of the term structure characterizing the experience of the industrialized countries which experienced episodes of fiscal retrenchment.

The rest of the paper is organized as follows. The next section introduces the reader to framework employed in the derivation of the results, including the description of monetary and fiscal policy aspects. Section derives the equilibrium in discrete time, while its continuous time limit is reported in the following section. The analysis of an example is conducted in Section 5. Then the following sections contain a description of the steps to obtain the real spot interest rate, the price of a real and nominal zero coupon bond, the term structure of real and nominal interest rate. Finally, a section with comments to the simulation results together with a brief set of concluding remarks will close the paper.

2 The Model

2.1 The Representative Agent

The economy is populated by a representative agent who optimally decides on the allocation of his (her) portfolio across a wide number of assets, including nominal and real bonds and a set of shares of stocks describing a set of property rights on private companies. The choice setup is described in discrete time intervals of length Δt.

The representative investor allocates his (her) wealth by maximizing the following utility function:

$$\sum_{t=0}^{\infty} e^{-\beta t} E_0 \left\{ u\left(C_t, \frac{M_t}{P_t}\right)\right\} \Delta t \tag{1}$$

The utility under (1) depends upon consumption C_t over the interval $[t, t+\Delta t]$. Moreover, M_t indicates the nominal money stock delivering utility to the representative agent over the interval $[t-\Delta t, t]$ and P_t is the price of the consumption good, here considered as the Consumer Price Index (CPI). The discount factor is β, so that the instantaneous discount rate is given by $e^{-\beta t}$. In (1) real money balances M_t/P_t bring directly utility to the representative agent, as in Bakshi and Chen (1996). Alternative formulation would consider either the transaction cost approach, like in Sims (1994) and Leeper and Sims (1994), or the analysis of Cash in Advance constraints, as has been done by Balduzzi (2000) in a framework similar to the present one, in order to simplify algebra when we are going to derive the term structure equation. Feenstra (1990) showed the functional equivalence between the various approaches followed to include money in a general equilibrium model.

I assume also that the utility function is twice continuously differentiable and concave in both consumption and real balances. Formally, this means that:

$$u_c > 0,\ u_m > 0,\ u_{cc} < 0,\ u_{mm} < 0,\ u_{cm} < 0,\ u_{cc}u_{mm} - (u_{cm})^2 > 0 \tag{2}$$

where $m_t = \frac{M_t}{P_t}$ is the demand for real money balances. In (2) the subscript to u indicates the partial derivative.

The optimal choice problem of the representative agent consists in maximizing the utility function (1) subjected to the following intertemporal budget constraint:

$$M_t + \left(P^a_{e,t} + P_t Y_t \Delta t\right) e_t + P_t a_{1,t} + a_{2,t} + \sum_{i=3}^{N} P^a_{i,t} a_{i,t} = P_t C_t \Delta t + M_{t+\Delta t} + P^a_{e,t} e_{t+\Delta t} +$$

$$+ P_t \frac{a_{1,t+\Delta t}}{1+R_t\Delta t} + \frac{a_{2,t+\Delta t}}{1+i_t\Delta t} + \sum_{i=3}^{N} P^a_{i,t} a_{i,t+\Delta t} \tag{3}$$

From equation (3) we see that each investor can choose among a wide range of assets traded on the market, as described by Grossmann and Shiller (1983), Marshall (1992), Sun (1992), Bakshi and Chen (1996) and Balduzzi (2000). At each time t, the agent demands M_t for cash, C_t, for consumption (in real terms), and equity holdings e_t (shares). The financial holdings are represented by vector $a_t = (a_{1,t},\ a_{2,t},\ \cdots, a_{N,t})'$ where $a_{i,t}$, for $i = 1, \ldots, N$ indicates the number of units of financial assets i held from $(t-\Delta t)$ to t. In particular, $a_{1,t}$ is the number of units of risk-free real bonds paying a real interest rate R_t, issued at time t and maturing at $t+\Delta t$. Similarly, $a_{2,t}$ is the number of units of risk-free nominal bonds paying a nominal interest rate i_t, issued at time t and maturing at $t+\Delta t$. Moreover, each agent can invest $a_{i,t}$ number of units in risky financial activities (stocks) whose nominal price (including dividend payments) is given by $P_{i,t}$, for $i = 3, \ldots, N$. Also, each representative agent is allowed to invest in one (single) equity share e_t which gives to the holder the property right on all the output produced through a single technology.

In what follows I assume a stochastic endowments economy so that the stochastic process governing the technology output is given by:

$$\frac{\Delta Y_t}{Y_t} = \frac{Y_{t+\Delta t} - Y_t}{Y_t} = \mu_{y,t}\Delta t + \sigma_{y,t}\Omega_{y,t}\sqrt{\Delta t} \tag{4}$$

where $\mu_{y,t}$ and $\sigma_{y,t}$ are, respectively, the conditional expected value and the standard deviation of the output growth per unit of time, and $\{\Omega_{y,t} : t = 0, \Delta t, \dots\}$ is an i.i.d. standard normal random process. From (4) $\mu_{y,t}$ and $\sigma_{y,t}$ can be time variants, as in Cox, Ingersoll and Ross (1995a,b) and Sun (1992).

Define now the real price in terms of the consumption goods of asset i at time t as $p^a_{i,t} = \frac{P^a_{i,t}}{P_t}$. Thus, as it is customary in the literature on asset prices (Merton (1971), Grossmann and Shiller (1983)), I assume that the real asset prices follow a vector diffusion process of the following type:

$$\frac{\Delta p^a_{i,t}}{p^a_{i,t}} = \mu^a_{i,t}\Delta t + \sigma^a_{i,t}\Omega^a_{i,t}\sqrt{\Delta t} \tag{5}$$

where $\mu^a_{i,t}$ and $\sigma^a_{i,t}$ are, respectively, the conditional expected value and the standard deviation of real return on asset i per unit of time and $\left\{\Omega^a_{i,t} : t = 0, \Delta t, \dots\right\}$ is an i.i.d. standard normal random process.

For explanatory reasons, let us define the following stochastic process for the price level (CPI), as follows:

$$\frac{\Delta P_t}{P_t} = \mu_{p,t}\Delta t + \sigma_{p,t}\Omega_{p,t}\sqrt{\Delta t} \tag{6}$$

For the moment, the drift term $\mu_{p,t}$ and the standard deviation $\sigma_{p,t}$ are taken as given, and will be derived later as functions of the core parameter of the economy. In (6) $\Omega_{p,t}$ is an i.i.d. standard normal random process.

Given these assumptions, the representative agent maximizes his (her) utility function (1) subjected to the constraint (3). The first order conditions for C_t, $a_{1,t}$, $a_{2,t}$, M_t and $a_{i,t}$ are respectively given by:

$$u_c\left(C_t, m_t\right) = \lambda_t P_t \tag{7}$$

$$E_t\left[e^{-\beta\Delta t}\lambda_{t+\Delta t}P_{t+\Delta t}\left(1 + R_t\Delta t\right)\right] = \lambda_t P_t \tag{8}$$

$$E_t\left[e^{-\beta\Delta t}\lambda_{t+\Delta t}\left(1 + i_t\Delta t\right)\right] = \lambda_t \tag{9}$$

$$E_t\left[e^{-\beta\Delta t}\lambda_{t+\Delta t} + \frac{u_m\left(C_{t+\Delta t}, m_{t+\Delta t}\right)}{P_{t+\Delta t}}\right] = \lambda_t \tag{10}$$

$$E_t\left[e^{-\beta\Delta t}\lambda_{t+\Delta t}P_{i,t+\Delta t}\right] = \lambda_t P_{i,t} \tag{11}$$

where λ_t is the stochastic Lagrange multiplier. The first order conditions $(7) - (11)$ are derived from a dynamic portfolio choice and they form the basis of the equilibrium characterization of this economy.

2.2 Monetary Policy

The crucial novelty of this paper is represented by the joint role of monetary and fiscal policy in the determination of the price level and, by consequence of that, of the explicit risk structure of the economy. In what follows, the interactions between monetary and fiscal policy are designed according to the literature on the 'Fiscal Theory of the Price Level' (FTPL), elaborated by Leeper (1991), Sims (1994), Woodford (1996, 2000) and Cochrane (1998, 1999, 2000).

Basically, FTPL states that monetary policy, without any additional constraints on fiscal policy cannot achieve the goal of price stability by himself. The fiscal policy constraints are essentially solvency constraints on the government budget constraint. These type of constraints are often described as 'Ricardian' fiscal policy. In practice, this implies that in order to ensure solvency and to guarantee the existence and the stability of a price level path, fiscal policy must be set such that taxes should be react proportionally to the existing stock of real (or nominal) debt[1]. The inclusion of fiscal policy variables in the determination of the term structure of interest rates, is done by exploiting the role of the Government Budget Constraint.

In this section I will introduce the role of monetary policy. In the economy here described suppose that there exists a Central Bank which keeps under control the aggregate amount of Money Supply M_t^s, defined as follows:

$$M_t^s = H_t + F_t \tag{12}$$

From (12) we see that money supply is composed by two distinct components: monetary base (or high powered money), given by H_t, and F_t, which is the amount of money needed by the Government to budget its balance. Apart from fiscal revenue and bonds, F_t represents an additional source of financing for the Government. To avoid confusion, it is worth to stress that in this framework, money demand is not divided, but only money supply is notational divided into two components.

As pointed out by Walsh (1998), monetary aggregate F_t is not often well detected by national accounts. Basically, it represents a sort of money financing of the government deficits, a monetization of the deficit up to a small amount. It is true, indeed, that in the most part of advanced economy this fraction of government deficit is very small and in many cases is also forbidden to allow monetary financing of the public debt. In practice, however, there exists always a margin - though small - where Central Bank transfers money to the Government under some forms. In US, for example, the Federal Reserve rebates to the Government the interest rate proceedings derived from bond holdings in her portfolio[2]. It is clear that the set up here described considers a Central Bank separated from the Government sector. This represents a case where there might be possibility for the born of conflicts between the two authorities. However, in what follows we abstract from the coordination debate, by assuming that Central Bank and Government are perfectly correlated among them, even if they are functionally separated. This also means that the Government Budget Constraint we are going to consider is not the consolidated public sector balance.

We can now specify the stochastic processes for both aggregate money supply H_t and F_t. To this

[1] To be precise, Leeper (1991) defines this type of policy as 'passive', while Woodford (1996, 2000) calls such policy as Ricardian, even if this does not have very much to do with the notion of Ricardian Equivalence, as clearly pointed out by Woodford (1996, 2000) and Sims (1999).

[2] The amount of F in U.S. is \$20.1 billion in 1996, as documented by Walsh (1998, page 132). For a careful discussion on the role of F in the construction of a monetary model (though different from the present one) see Walsh (1998).

goal I assume that the growth rate for H_t is given by:

$$\frac{\Delta H_t}{H_t} = \frac{H_{t+\Delta t} - H_t}{H_t} = \mu_{H,t}\Delta t \tag{13}$$

where $\mu_{H,t}$ is the mean of money growth rate. For simplicity, from (13) we have that monetary base contains only a deterministic component: this does not affect dramatically the main conclusion of the model. From the practical point of view, the assumption under (13) is equivalent to imagine a steady growth rate for money supply, *à la* Friedman. From the terminology proposed by FTPL, the above assumption is equivalent to say that monetary authority sets a 'passive' monetary policy rule[3]. The second part of the paper will include a generalization of the process considered in (13), by including a stochastic trend into (13) in order to make the results more comparable with the existing empirical literature.

The only stochastic source to the money supply comes from the motion law specified for F_t, given by:

$$\frac{\Delta F_t}{F_t} = \frac{F_{t+\Delta t} - F_t}{F_t} = \mu_{F,t}\Delta t + \sigma_{F,t}\Omega_{F,t}\sqrt{\Delta t} \tag{14}$$

where $\mu_{F,t}$ and $\sigma_{F,t}$ are, respectively, the mean and the standard deviation of the stochastic process leading the growth rate of the money supply term F_t and $\{\Omega_{F,t} : t = 0, \Delta t, \dots\}$ is again an i.i.d. standard normal random process. Combining (13) and (14) with the definition given to money supply (12) we get the stochastic process leading money supply given by:

$$\frac{\Delta M_t^s}{M_t^s} = \frac{M_{t+\Delta t}^s - M_t^s}{M_t^s} = \mu_{M,t}\Delta t + \sigma_{M,t}\Omega_{M,t}\sqrt{\Delta t} \tag{15}$$

where:

$$\mu_{M,t} = \mu_{H,t} + \mu_{F,t} \tag{16}$$

$$\sigma_{M,t}\Omega_{M,t} = \sigma_{F,t}\Omega_{F,t} \tag{17}$$

Monetary policy rule (15) is widely employed in the literature on the term structure of interest rates. In this framework it is clear that the trend component given by (16) is the sum of the two components of the monetary policy rule, and the stochastic component is exclusively given by the stochastic fluctuations in the transfers from Central Bank to the Government.

For convenience it will be instructive to express to above relationships in real terms. Thus, let me define the real (aggregate) money supply as $m_t^s = \frac{M_t^s}{P_t}$, where P_t is the price level (or CPI) whose stochastic process is defined according to (6). The other component of money supply H_t, and F_t are defined by $h_t = H_t/P_t$ and $f_t = F_t/P_t$. Thus, by taking the limit for $\Delta t \to 0$ and by applying Ito's Lemma to the stochastic processes of nominal money supply (13) and (14) we get that the real money

[3] It will be not difficult to include a different monetary policy rule, like, for example, an interest-rate rule in the present setting. This is an extention that would change the drift of the monetary base specificiaction in (13). This is a subject of investigation.

supply processes are given by:

$$\frac{dm_t^s}{m_t^s} = \mu_{m,t}dt + \sigma_{m,t}\Omega_{m,t}\sqrt{dt} \tag{18}$$

$$\frac{dh_t}{h_t} = \mu_{h,t}dt + \sigma_{h,t}\Omega_{h,t}\sqrt{dt} \tag{19}$$

$$\frac{df_t}{f_t} = \mu_{f,t}dt + \sigma_{f,t}\Omega_{f,t}\sqrt{dt} \tag{20}$$

where the drift terms are given by (after having dropped time subscript, in order to save notation):

$$\mu_m = \mu_M - \mu_p + 2\sigma_p^2 - \sigma_{pM} \tag{21}$$

$$\mu_h = \mu_H - \mu_p + 2\sigma_p^2 \tag{22}$$

$$\mu_f = \mu_F - \mu_p + 2\sigma_p^2 - \sigma_{pF} \tag{23}$$

where the stochastic terms are given by:

$$\sigma_m = \sigma_M - \sigma_p \tag{24}$$

$$\sigma_h = -\sigma_p \tag{25}$$

$$\sigma_f = \sigma_F - \sigma_p \tag{26}$$

Equipped with the above relationships, we can now describe the role of fiscal policy, in order to fully appreciate the importance of the interactions between monetary and fiscal policy in the present model.

2.3 Fiscal Policy

Since the work by Sargent and Wallace (1981) it has been recognized the importance of a joint determination of monetary and fiscal policy. This ins not a new idea, however, as witnessed by Cochrane (1999). Old writings, especially Lerner (1948), document the fact that the price level is jointly determined by monetary and fiscal policy. This is beaches the monetary base *together with* nominal debt is a residual claimant to government surpluses. In a crude way, FTPL says that if market participants are afraid that future surpluses will not be sufficient to pay back the outstanding level of public debt, the government is then expected to default or to inflate away the debt. Therefore, the price level can be simply determined through the simple valuation equation for government debt:

$$\frac{B}{P} = DPV \text{ of future gov. surpluses}$$

where B is the total values of nominal government debt, P is the price level and DPV is the Discounted Present Values of future government surpluses. In a truly idealized world, it could be possible to imagine a framework in which monetary policy is useless in the determination of the price level, which is entirely determined by the role of fiscal policy[4]. Even this is not a new idea: the famous economist Knut Wiksell affirmed that as long as the monetary authority is able - through an interest rate pegging regime - to modify the 'natural rate of interest' then, the control of a monetary aggregate is a redundant tool to ensure the control of the price level. In any case, in the following paper I do not take this extremal view

[4] A very good description of the presence in old economic writings of reasoning similar to what has been formalized by FTPL theorists is on Cochrane (1999).

of FTPL, but I consider a joint role for both monetary aggregate and fiscal policy in the determination of the price level.

To fix the ideas, let us introduce now the Government Budget Constraint as follows:

$$\Delta b_t + \Delta f_t = \Delta R_t b_t - \Delta T_t \tag{27}$$

In (27) I indicated with b_t the stock of real public debt issued in period $t - \Delta t$ and maturing in t, ΔT_t indicates the change in the fiscal revenue, while f_t is the amount of money which goes in financing the government deficit. The term $\Delta R_t b_t$ indicates the burden of interest payments maturing on the outstanding debt. The government Budget Constraint in (27) is a dynamic stochastic equation. The source of the stochastic volatility are three: changes in the conditions underlying the stochastic process specified for f_t, (see equation (20)), shocks to the fiscal revenue ΔT_t, and shock to the interest payment, due to changes in the underlying conditions of the process specified for the real interest rate R_t. All these three factors determine changes in the structure of public debt[5].

In order to guarantee solvency of the government and the existence of a price level, we can specify a fiscal policy rule, according the FTPL criteria such that:

$$\Delta T_t = \phi_1 b_t \Delta t + \phi_1 b_t \sigma_{T,t} \Omega_{T,t} \sqrt{\Delta t} \tag{28}$$

According to (28) we observe that Government set taxes by making the total fiscal revenue reacting to the total amount of real public debt. Taxes are raised in proportion to the stock of real debt b_t and with respect to the increase of the stochastic part of the tax process. In (28) we have that $\{\Omega_{T,t} : t = 0, \Delta t, \dots\}$ is a standard normal random process. A rule like (28) identifies a set of 'Ricardian' fiscal policies, i.e. fiscal policies which promptly react to the existing level of public debt, expressed in real terms.

A set of author proved how important is to set fiscal policy according to the above rule, in order to respect the full stability of price level. Among the others, Leeper (1991), Sims (1994) and Woodford (1996, 2000) have discussed the importance of such rule within a fully microfounded model for the monetary business cycle. Parameter ϕ_1 measures the extent to which a monetary policy is tight (high ϕ_1) or loose (low ϕ_1). A bound for ϕ_1 can be established by setting ϕ_1 less than (or at most equal) the inverse of the rate of intertemporal preference $\beta : 0 < \phi_1 \leq \beta^{-1}$. As long as this condition is met, we get full price stability. It should be noted, however, that the goal here is not to evaluate the conditions under which we get full price stability, but simply to detect the consequences on the term structure and the stochastic process of the price level of adopting a fiscal policy rule like (28), as suggested by the literature on FTPL. A set of condition for full price stability should involve also some restrictions on the monetary policy rule. It is clear, that the result of this paper can be extended by including a monetary policy rule of the interest rate pegging rule, à la Taylor, where the nominal interest rate reacts to inflation and output (an 'Active' Monetary policy rule, by using the Leeper's language).

To complete the setup, let us assume the following process for the real spot rate:

$$\Delta R_t = \mu_{R,t} \Delta t + \sigma_{R,t} \Omega_{R,t} \sqrt{\Delta t} \tag{29}$$

[5] In equation (27) it should be remembered that $\Delta b_t = b_{t+\Delta t} - b_t$, $\Delta f_{t+\Delta t} = f_{t+\Delta t} - f_t$.

In (29) I specified a generic exogenous stochastic process for the real rate, with drift $\mu_{R,t}$ and stochastic component $\Omega_{R,t}$, as i.i.d. normal variable $\{\Omega_{R,t} : t = 0, \Delta t, \dots\}$. The drift term and the stochastic component will be determined as function of the core parameter of the model.

Combining (28) and (29) with (27) we get the following expression for the motion law of the public debt[6]:

$$\Delta b_t + \Delta f_t = \left(\mu_{R,t} - \phi_1\right) b_t \Delta t + b_t \sigma_{R,t} \Omega_{R,t} \sqrt{\Delta t} - \phi_1 b_t \sigma_{T,t} \Omega_{T,t} \sqrt{\Delta t} \tag{30}$$

In order to get a semi-closed solution for the parameter of money supply, let us specify a generic motion law for real public debt also. In particular, I assume that the growth of real public debt follows a simple deterministic process given by:

$$\frac{\Delta b_t}{b_t} = \frac{b_{t+\Delta t} - b_t}{b_t} = \mu_{b,t} \Delta t \tag{31}$$

In (31) $\mu_{b,t}$ indicates the expected growth rate of public debt. Here I assumed a deterministic component in order to simplify the model. Thus, the only stochastic sources to public debt fluctuations come only from real interest rate shocks and tax revenue shocks.

Following the approach by Turnovksy (...), assume also that the Government aims to maintain a constant ratio of real bonds to money[7], i.e., $\psi = b/f$. In Woodford (1996) the parameter ψ indicates the importance of bonds over money and is set equal to $\psi = 0.1$.

To get a reduced form of the model, let us consider all the above relationships in continuous time, by taking the limit for $\Delta t \to 0$. Thus, by applying Ito's lemma to the definition of ψ and using the stochastic process for m_t, f_t, and b_t into (30) - after having equated the deterministic part of the RHS of (30) and the stochastic part - we get:

$$\mu_b = \mu_f - \sigma_f^2 \tag{32}$$

Moreover by using the definition of the stochastic process of b_t and f_t into (30), we also get (after having dropped the time index, to save notation):

$$\mu_f = \frac{\left(\mu_R - \phi_1 + \psi \sigma_f^2\right)}{1 + \psi} \tag{33}$$

$$\sigma_f = \psi \left(\sigma_R - \phi_1 \sigma_T\right) \tag{34}$$

Therefore, equations (32) - (34) indicates a set of equilibrium relationships among the drift terms of the relevant stochastic process. For the discussion which follows here the key relationship is given by equation (33). In fact, from (16) the conditional mean of the diffusion process for the real money supply growth (equation (15)) is given by:

$$\mu_m = \mu_f + \frac{\left(\mu_R - \phi_1 + \psi \sigma_F^2\right)}{1 + \psi} \tag{35}$$

[6] The main results of this paper are obtained after a manipulatio of the Government Budget Constraint, according to the methodology illustrated in Turnovsky (1995).

[7] The assumption of ψ constant is compatible with a constant set of shares invested in financial assets, as discussed by Turnovsky (xxx).

In nominal terms we can rewrite the equation (35) as follows:

$$\mu_M = \mu_F + \sigma_{pM} + \frac{\left(\mu_R - \phi_1 + \psi\sigma_f^2\right)}{1+\psi} \tag{36}$$

where σ_{pM} indicates the covariance between the price level and the nominal money supply.

Equation (35) is crucial in what follows and highlights the link between fiscal policy and the drift of the stochastic process of money supply. If fiscal policy is tightened (i.e. if ϕ_1 raises), then, the conditional mean of the stochastic process of the monetary aggregate f (or F, in nominal terms), and the conditional mean of the money supply process (m in real terms, M in nominal terms) are reduced. Thus, there is a negative relationship between μ_M and ϕ_1 (through the effect of ϕ_1 on μ_F, from (33)). This terminates the discussion on fiscal policy aspects. The next step consists in studying the equilibrium for the economy at hand.

3 The Equilibrium in Discrete Time

I assume that the present economy is populated by a plurality of identical agents. Thus, the behavior of the representative agent is a good proxy of all the agents living in this economy. Moreover, in a representative agent economy optimal consumption, money demand and portfolio holdings must adjust such that the following equilibrium conditions are verified:

$$C_t = Y_t \tag{37}$$

$$M_t = M_t^s \tag{38}$$

$$e_t = 1 \tag{39}$$

$$a_{i,t} = 0, \quad \forall \ \ i = 1, 2, ...N \tag{40}$$

Equation (37) simply states that in a pure endowment economy the total amount of consumption must equate the total output endowment Recall that in this case output is a random variable, so consumption is also a random variable. Equation (38) is just the equilibrium condition on the money market, by which money demand is equal to its supply in every instant. Equation (39) is just an equilibrium condition which states the equality between demand and supply. In the same way, in equilibrium it the demand of each agent for financial assets should equal the supply, which is zero, as showed[8] in (40).

Using the equilibrium conditions (37) - (40), we can rewrite the first order conditions (7) - (11) by getting rid of the Lagrange multiplier to get:

$$u_c\left(Y_t, m_t\right) = e^{-\beta\Delta t} E_t\left[u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)\left(1 + R_t\Delta t\right)\right] \tag{41}$$

$$u_c\left(Y_t, m_t\right) = e^{-\beta\Delta t} E_t\left\{\left[u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right) + u_m\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)\right]\frac{P_t}{P_{t+\Delta t}}\right\} \tag{42}$$

[8] Recall that in this case the financial assets of this economy are in zero net supply.

$$\frac{u_c(Y_t, m_t)}{P_t} = e^{-\beta \Delta t} E_t \left[\frac{u_c(Y_{t+\Delta t}, m_{t+\Delta t})}{P_{t+\Delta t}} (1 + i_t \Delta t) \right] \tag{43}$$

$$u_c(Y_t, m_t) = e^{-\beta \Delta t} E_t \left[u_c(Y_{t+\Delta t}, m_{t+\Delta t}) \frac{p^a_{i,t+\Delta t}}{p^a_{i,t}} \right] \tag{44}$$

where with m_t I indicated the real cash balances. Recall also that $p^a_{i,t}$ is the real price in terms of the consumption good of asset i at time t.

As it is customary in the current literature, equations (41) - (44) define a set of Euler equations derived from the optimal portfolio allocation of the representative agent. In particular, equations (41) - (42) together state that the representative investor should be indifferent between investing an amount of money equal to P_t in a real risk-free bond, and holding the same amount of money in terms of an additional amount of cash. The same should be true for the choice between nominal and real bonds: the no arbitrage condition between the returns form real and nominal bond is jointly determined by equations (41) and (43). Finally, from equation (42) and (44) we get an indifference relation between investing one more amount of cash of size P_t in asset i and holding the same amount in a pure cash form.

In other words, equations (41) and (42) establish the demand for real money balances, while (43) and (42) make possible the derivation of a nominal money demand. In fact, to be more specific, equation (42) alone states that the agent in equilibrium should be indifferent between holding an amount of cash of size P_t at time t and consuming one extra unit of the good, because both actions produce the same marginal utility. Moreover, equation (42) establishes the link between the price level and monetary and fiscal policy. The link between monetary policy and the asset market is provided by equations (44) and (42): jointly considered these two equations ensures the internal consistency between money supply and asset market equilibrium. Finally, the link between monetary policy and the goods market is represented by the joint role of equations (41) and (42) . An important fact to be noted, which will be exploited later in the solution of the model is given by the fact that equation (44) must hold also for the equity share e_t when we replace $\frac{\Delta p^a_{i,t+\Delta t}}{p^a_{i,t}}$ with $\frac{p^a_{e,t+\Delta t}+Y_{t+\Delta t}}{p^a_{e,t}}$ where $p^a_{e,t} = \frac{P^a_{e,t}}{P_t}$ is the real price of the equity share.

In addition to first order conditions (41) - (44), to guarantee the existence of an interior optimum the following sufficient conditions must be satisfied:

$$\lim_{T \to \infty} E_t \left\{ e^{-\beta \Delta t} \frac{u_c(Y_T, m_T)}{u_c(Y_t, m_t)} p^a_{i,t} \right\} = 0 \tag{45}$$

$$\lim_{T \to \infty} E_t \left\{ e^{-\beta \Delta t} \frac{u_c(Y_T, m_T)}{u_c(Y_t, m_t)} \frac{1}{P_t} \right\} = 0 \tag{46}$$

The Transversality condition (45) rules out bubbles in the price level of any risky asset. At the same time, the Transversality condition (46) rules out bubbles in the general price level. The intuition behind these two additional conditions is the following: if (45) is violated, the agent is willing to sacrifice actual consumption in favor of future consumption derived from the proceeds from investments in risky assets, without any bound at all. A similar interpretation works also for condition (46): the agent will be willing to reduce consumption today in exchange of an increased future monetary service, without bound when (46) is violated. Note that the bound on the utility function is crucial in determining the validity of both (45) and (46). This helps to exclude situation in which an equilibrium path where the utility of the agent increases (decreases) without bound for an excess (low level) of consumption derived from a steady increase (decrease) of real money balances, causes by a steady decline (rise) in the price level P_t. This motivates the assumptions on the bound from below and above of the utility function considered in (2). It also worth to mention that the stability of the price level is assured by the type of fiscal policy considered here. In fact, if fiscal policy is 'Ricardian', - at least locally - from the results discussed in Sims (1994), we have that the determinacy of the price level is guaranteed. The next step is represented by the analysis of the present model in the continuous time limit.

4 The Equilibrium in the Continuous time

In what follows I will derive in a general equilibrium framework all the most important relationships to be held in the continuous time limit. This is a necessary step in order to analyze the main equilibrium relationships of the model. Under several respects, the results considered here are general and do not strictly depend upon the assumptions considered so far about the role of fiscal and monetary policies. The reader can find a lot of complementarities and similarities with the results from Bakshi and Chen (1996) Balduzzi (2000), Breeden (1979,1986) and Stulz (1986). The main results are collected here in order to facilitate the discussion in what follows. Let us start with the following Proposition

Proposition 1 *The equilibrium risk premium for any risky asset over the real interest rate is given by:*

$$\mu_{i,t}^a - R_t = -\frac{Y_t u_{cc}}{u_c} cov_t \left(\frac{dp_{i,t}^a}{p_{i,t}^a}, \frac{dY_t}{Y_t} \right) - \frac{m_t u_{cm}}{u_c} cov_t \left(\frac{dp_{i,t}^a}{p_{i,t}^a}, \frac{dm_t}{m_t} \right) \tag{47}$$

Proof: See Appendix.

From equation (47) we observe that both production and monetary policy risk matters for the determination of the risk premium of asset i over the real interest rate. The interesting fact is represented by the fact that the risk premium is linear in both the covariance between the price of real assets and output and the covariance between the price of real asset and the money. Some interesting properties of the risk premium, implicit in equation (47). By using the following fact:

$$\frac{dm_t}{m_t} = \frac{dM_t}{M_t} - \frac{dP_t}{P_t} \tag{48}$$

we can rewrite equation (47) as follows:

$$\mu_{i,t}^a - R_t = -\frac{Y_t u_{cc}}{u_c} cov_t \left(\frac{dp_{i,t}^a}{p_{i,t}^a}, \frac{dY_t}{Y_t} \right) - \frac{m_t u_{cm}}{u_c} \left[cov_t \left(\frac{dp_{i,t}^a}{p_{i,t}^a}, \frac{dM_t}{M_t} \right) - cov_t \left(\frac{dp_{i,t}^a}{p_{i,t}^a}, \frac{dP_t}{P_t} \right) \right] \tag{49}$$

From (49) we get that if $cov_t \left(\frac{dp_{i,t}^a}{p_{i,t}^a}, \frac{dP_t}{P_t} \right) > 0$, i.e. if asset i is positively correlated with the inflation rate, then its risk premium will be lower compared to other assets for which this is not true[9]. Therefore,

[9] Note that the key assumption in this case is given by $u_{cm} < 0$.

assets with such characteristics can be employed as tools to hedge against the inflation by risk-averse investors. Moreover, given that $u_{cc} < 0$, the risk premium is higher for assets positively correlated with real activities and with the growth rate of nominal money supply.

Proposition 2 *The real price of the equity share is given by:*

$$P^a_{e,t} = E_t \int_t^\infty e^{-\beta(s-t)} \frac{u_c(Y_s, m_s)}{u_c(Y_t, m_t)} Y_s ds \tag{50}$$

Proof. From equation (44) recall that:

$$\frac{\Delta p^a_{i,t+\Delta t}}{p^a_{i,t}} = \frac{P^a_{e,t+\Delta t} + Y_{t+\Delta t}\Delta t}{P^a_{e,t}} \tag{51}$$

Thus plugging (51) into (44), we get:

$$P^a_{e,t} = E_t \left\{ e^{-\beta\Delta t} \frac{u_c(Y_{t+\Delta t}, m_{t+\Delta t})}{u_c(Y_t, m_t)} \left(P^a_{e,t+\Delta t} + Y_{t+\Delta t}\Delta t \right) \right\} \tag{52}$$

iterate equation (52) to obtain:

$$P^a_{e,t} = E_t \sum_{j=1}^{\infty} e^{-\beta(j\Delta t)} \frac{u_c(Y_{t+j\Delta t}, m_{t+j\Delta t})}{u_c(Y_t, m_t)} Y_{t+j\Delta t}\Delta t \tag{53}$$

Thus, by taking the limit for $\Delta t \to 0$ in equation (53) we finally get the result given in (50). ■ See Appendix. ■

The stochastic process of the price level is given by the following results.

Proposition 3 *In the continuous time limit equilibrium, the commodity price level is given at time t by:*

$$\frac{1}{P_t} = E_t \int_t^\infty e^{-\beta(s-t)} \frac{u_m(Y_s, m_s)}{u_c(Y_t, m_t)} \frac{1}{P_s} ds \tag{54}$$

The expected inflation rate is given by:

$$\begin{aligned} \pi_t &\equiv \frac{1}{dt} E_t \left\{ \frac{dP_t}{P_t} \right\} = \\ &= i_t - R_t + var_t \left\{ \frac{dP_t}{P_t} \right\} - \frac{u_{cc} Y_t}{u_c} cov_t \left(\frac{dY_t}{Y_t}, \frac{dP_t}{P_t} \right) - \frac{u_{cm} m_t}{u_c} cov_t \left(\frac{dP_t}{P_t}, \frac{dm_t}{m_t} \right) \end{aligned} \tag{55}$$

Proof. see Appendix A. ■ See Appendix. ■.

From equation (54) we see that the price level equates the future discounted value of all the marginal benefits provided by holding one unit of dollar cash. This equation is one of the key elements to solve for the price level as function of the stochastic processes of money and output .

A more interesting case is probably offered by equation (55) which, after a simple rearrangement, shows that the Fischer equation does not hold when the second order terms are included. In fact, in this framework the riskiness on bonds given by $\left\{i_t - \pi_t + var_t\left\{\frac{dP_t}{P_t}\right\} - R_t\right\}$ becomes a function of the overall riskiness structure of the economy. Therefore, nominal risk free assets - like nominal bonds - become 'risky' assets, through the riskiness role of the price level. The stochastic process of the price level is a function of the overall riskiness structure of the economy, including the role of monetary and fiscal policy.

The definition of the real interest rate is in the following proposition:

Proposition 4 *The equilibrium real interest rate is given by:*

$$\begin{aligned} R_t &= \beta - \frac{u_{cc}Y_t}{u_c}\frac{1}{dt}E_t\left\{\frac{dY_t}{Y_t}\right\} - \frac{1}{2}\frac{u_{cc}Y_t^2}{u_c}var_t\left\{\frac{dY_t}{Y_t}\right\} - \frac{u_{cm}m_t}{u_c}\frac{1}{dt}E_t\left\{\frac{dm_t}{m_t}\right\} + \\ &\quad -\frac{1}{2}\frac{u_{cmm}m_t^2}{u_c}var_t\left\{\frac{dm_t}{m_t}\right\} - \frac{Y_t m_t u_{ccm}}{u_c}cov_t\left(\frac{dY_t}{Y_t}, \frac{dm_t}{m_t}\right) \end{aligned} \tag{56}$$

Proof. see appendix A. ■ See Appendix. ■

Proposition 5 *The equilibrium nominal interest rate is given by:*

$$i_t = \frac{u_m(Y_t, m_t)}{u_c(Y_t, m_t)} \tag{57}$$

Proof. see appendix A. ■ See Appendix A.

Few additional comments are in order. If we look at equation (56) we observe that the stochastic process of money supply affects the real interest rate, if the utility function is non-separable in money and output. However, if the utility function is logarithmic in both money and output - like f.e. in Stulz (1986) - (this is equivalent to say that $u_{cm} = u_{ccm} = u_{cmm} = 0$) there is no way by which monetary policy - and in the present particular case also fiscal policy - can affect real interest rate.

The right hand side of equation (57) is the marginal rate of substitution between consumption and real cash holdings and it can be viewed as the marginal benefit of holding one additional unit of cash balance. In equilibrium, this cost must be equal to the benefit.

Let us examine the following results on the term structure.

Proposition 6 *(a) The nominal term structure equation is defined as:*

$$\frac{N(t,\tau)}{P_t} = e^{-\beta\tau}E_t\left[\frac{u_c(Y_{t+\tau}, m_{t+\tau})}{u_c(Y_t, m_t)}\frac{1}{P_{t+\tau}}\right] \tag{58}$$

where $N(t,\tau)$ is the time t nominal price of a discount bond paying a dollar in τ periods.
(b) The real term structure is:

$$b(t,\tau) = e^{-\beta\tau}E_t\left[\frac{u_c(Y_{t+\tau}, m_{t+\tau})}{u_c(Y_t, m_t)}\right] \tag{59}$$

where $b(t,\tau)$ is the time t nominal price of a discount bond paying a unit of consumption good in τ periods.

Proof. The result follows directly from the Euler equation (41) ■

Equations (58) and (59) will be particularly useful in the following examples to establish the behavior of the term structure.

5 A specific example

In what follows we will provide a specific model from which we will derive the stochastic processes for the price level (CPI) and the other various elements of the model. The stochastic processes specified here describe a more realistic economic setting. The setup is closed to what has already been proposed in the literature by Bakshi and Chen (1996) and Balduzzi (2000). The novelty of the following approach lies in the approach taken in the solution method. Differently from the other approaches undertaken in the literature I explicitly consider a solution for the term structure of interest rates which is directly derived from the assumptions here considered by following the method described in Cox, Ingersoll and Ross (1985), instead of imposing a structure to the movement law of the nominal interest rate and later verifying the model after this imposition. Basically, the methodology followed here builds on an arbitrage-free model.

In order to simplify algebra, let us assume the following strongly separable utility function given by:

$$u\left(C_t, \frac{M_t}{P_t}\right) = \phi \log C_t + (1-\phi)\log m_t \tag{60}$$

where $m_t = M_t/P_t$.

Assume that the stochastic process leading output is given by:

$$dY_t = (\mu_Y + \eta_Y x_t)\, dt + \sigma_Y \sqrt{x_t} dW_{x,t} \tag{61}$$

where the process for the technology factor x_t is given by:

$$dx_t = a\,(b - x_t)\, dt + \sigma_x \sqrt{x_t} dW_{x,t} \tag{62}$$

where $(W_{x,t})_t$ is a unidimensional $\mathbb{Q}$-Brownian motion, μ_Y, η_Y, σ_Y, a, b, and σ_x are fixed real numbers.

Let us assume the following two stochastic processes for the two monetary aggregates H_t and F_t:

$$d\ln H_t = \mu_H^* dt + d\ln(q_t) \tag{63}$$

$$d\ln F_t = \overline{\mu}_F dt + d\ln(q_t) \tag{64}$$

where q_t is the detrended money supply process. Basically from (63)-(64) we see that the stochastic process for the two-types of money supply has two components: a drift term and a stochastic component q_t given by the detrended component. In particular, μ_H^* is assumed to be constant and positive, while $\overline{\mu}_F$ is the drift term depending upon the fiscal policy parameters as discussed in a previous section and is still defined by (**??**) Following Bakshi and Chen [**?**], q_t is assumed to follow the stochastic process:

$$\frac{dq_t}{q_t} = k_q\left(\mu_q - q_t\right) dt + \sigma_q \sqrt{q_t} dW_{i,t}, \quad i = H, F \tag{65}$$

where $(W_{i,t})_t$ is a unidimensional $\mathbb{Q}$-Brownian motion independent upon $(W_{x,t})_t$. Therefore, by using the definition of money supply given in (12) and the above assumptions, we get that the stochastic process leading money supply is given by:

$$\frac{dM_t}{M_t} = \mu_{M,t}dt + \sigma_q\sqrt{q_t}dW_{M,t} \tag{66}$$

where $(W_{M,t})_t$ is a unidimensional $\mathbb{Q}$-Brownian motion independent upon $(W_{x,t})_t$ and $(W_{i,t})_t$ and where:

$$\mu_M = \mu_M^* + 2k_q\left(\mu_q - q_t\right) \tag{67}$$

with:

$$\mu_M^* = \mu_H^* + \overline{\mu}_F \tag{68}$$

and $d\Omega_{M,t} = d\Omega_{H,t} + d\Omega_{F,t}$. Following the same steps we did in the section on fiscal policy[10], we have that $\overline{\mu}_F$ is given by:

$$\overline{\mu}_F = \sigma_{pH} + \frac{\left(\mu_i - \phi_1 + \psi\sigma_q^2\right)}{1+\psi} \tag{69}$$

where σ_{pH} the correlation between the price level and the monetary aggregate H.

Given our set of assumptions, above considered, we can now derive the main results related with the commodity price level and the inflation process, as reported and discussed in the following Theorem:

Proposition 7 *Given the utility function of the representative agent as described by equation* (60)*, then the equilibrium price level is given by:*

$$P_t^c = \frac{\phi}{1-\phi}\frac{q_t^2\left(\beta+\mu_M^*\right)\left(\beta+\mu_M^*+2k_q\mu_q\right)}{\left(\beta+\mu_M^*\right)+\left(k_q+3\sigma_q^2\right)q_t 2k_q\mu_q}\frac{M_t}{Y_t} \tag{70}$$

The stochastic process of the CPI is given by:

$$\frac{dP_t^c}{P_t^c} = \pi_t dt + \sigma_q\sqrt{q_t}\left[1+\frac{\left(\Delta_q\Psi - \Delta\Psi_q\right)}{\Delta\Psi}q_t\right]dW_{M,t} - \sigma_y\sqrt{x_t}dW_{x,t} \tag{71}$$

where the inflation rate is given by:

$$\begin{aligned}\pi_t &= \mu_M^* - \mu_y + \left(\sigma_y^2 - \eta_y\right)x_t + \frac{\left(\Delta_q\Psi - \Delta\Psi_q\right)}{\Delta\Psi}q_t\left(k_q\left(\mu_q - q_t\right)+\frac{\sigma_q^2 q_t}{2}\right)+ \\ &\quad + \frac{\left[2\left(\Delta_{qq}\Psi - \Delta\Psi_{qq}\right) - \Delta_q\Psi + \Delta\Psi_q\right]\sigma_q^2 q_t^{3/2}}{2\Psi^2\Delta}\end{aligned} \tag{72}$$

with:

$$\Delta(q) = \frac{\phi}{1-\phi}\left[q_t^2\left(\beta+\mu_M^*\right)\left(\beta+\mu_M^*+2k_q\mu_q\right)\right] \tag{73}$$

$$\Psi(q) = \left(\beta+\mu_M^*\right)+\left(k_q+3\sigma_q^2\right)q_t 2k_q\mu_q \tag{74}$$

and $\Delta_q = \frac{\partial\Delta(q)}{\partial q}$; $\Psi_q = \frac{\partial\Psi(q)}{\partial q}$; $\Delta_{qq} = \frac{\partial\Delta(q)}{\partial q}$.

[10] Recall that in order to derive the expression of μ_F (nominal). We need to consider the government budget constraint written in real terms, and exploting the definition of F and H weitten in real terms.

Proof. See Appendix A. ■

The price level (equation (70)) and the inflation rate given in equation (72) are function of both fiscal and monetary parameters in a very complicated way. In the section dedicated to the simulation analysis I will provide an intuitive explanation of the pattern behavior of such variables.

In the following pages I will describe all the necessary steps to solve out the present model.

6 The real spot interest rate

In this section I am going to derive the dynamic of real spot interest rate driven by the exogenous dynamic of technology. Since the technology is the solution of the EDS given by (62), x_t is Markov and satisfy the necessary technical conditions to apply the Representation Theorem of Feyman-Kač[11]. We can now follow the PDE approach to derive the dynamic of real spot rate as a function of time and technology.

First, let us consider how to derive the real spot rate dynamics.

Proposition 8 *Given the dynamic stochastic process for the technology as in* (62) , *the real spot rate* R_t *is a function* $\phi(t, x_t)$ *which is the unique solution of the Kolmogorov PDE:*

$$\begin{cases} \frac{1}{2}\sigma_x^2 x_t \frac{\partial^2 \phi(t,x_t)}{\partial x_t^2} + a(b - x_t)\frac{\partial \phi(t,x_t)}{\partial x_t} + \frac{\partial \phi(t,x_t)}{\partial t} - x_t \phi(t, x_t) = 0 \\ \phi(T, x_t) = \frac{2ab}{\gamma+a} \end{cases}$$

where the final condition is the long time spot rate determined as in Cox, Ingersoll and Ross (1985). So we have:

$$R_t = A(\theta) e^{C(\theta)x_t} \tag{75}$$

where $\theta = T - t$ *and*

$$C(\theta) = \frac{\sigma_x^2 (2 + a)(1 - e^{\gamma\theta}\gamma)}{2a[\sigma_x^2 - a + e^{\gamma\theta}(a - \sigma_x^2) - (1 + e^{\gamma\theta})\gamma]} \tag{76}$$

$$A(\theta) = \frac{1}{\gamma + a}\left\{2^{\chi} abe^{\nu(\theta)}\gamma^{\zeta}\left[a + \gamma - \sigma_x^2 + e^{\gamma\theta}(\gamma - a + \sigma_x^2)\right]^{-\zeta}\right\} \tag{77}$$

and where

$$\begin{aligned} \chi &= \frac{a^2 - \gamma^2 - 2\sigma_x^2(a + b + ab) + \sigma^4(1 + b)}{(a - \sigma_x^2)^2 - \gamma^2} \\ \nu(\theta) &= \frac{b\theta\sigma_x^2(2 + a - \gamma)}{2(a + \gamma - \sigma_x^2)} \\ \zeta &= \frac{b\sigma_x^2(\sigma_x^2 - 2a - 2)}{(a - \sigma^2)^2 - \gamma^2} \\ \gamma &= \sqrt{a^2 + 2\sigma_x^2} \end{aligned}$$

[11] In particular the drift and volatility term in (62) must be Lipschitz and bounded on $\Re$.

Proof. See Appendix A. ■

Now we can derive the dynamic of the real spot interest rate in the following Lemma.

Proposition 9 *The dynamic of real spot interest rate is:*

$$dR_t = a^*(\theta)[b^*(\theta) - R_t]\,dt + C(\theta)\,R_t\sigma_x\sqrt{x_t}dW_{x,t} \tag{78}$$

where $(W_{x,t})_t$ *is a unidimensional* $\mathbb{Q}-$*Brownian motion, and where:*

$$\begin{aligned} a^*(\theta) &= -\left[a(b-x_t)\,C(\theta) + C'(\theta)\,x_t + \frac{1}{2}\sigma_x^2 x_t C^2(\theta)\right] \\ b^*(\theta) &= \frac{A'(\theta)\,e^{C(\theta)x_t}}{a^*(\theta)} \end{aligned}$$

Proof. See Appendix A. ■

Few comments are in order. One might naturally ask whether the real interest rate here derived is an equilibrium real rate, given the solution method here adopted not orthodox to the traditional equilibrium term structure theory. In truth, the real rate derived in the above theorem is effectively an equilibrium real rate, even if the solution method adopted is very much close to the arbitrage-free models. In fact, according to Cox, Ingersoll and Ross (1985), the choice made for the functional form of $\phi(t, x_t)$ is the only possible choice which makes the solution here derived exactly equal to the equilibrium solution.

Now we have all the elements to derive the term structure of interest rates, in real terms.

7 Zero coupon bonds and term structure of real interest rates

In this section we derive the price of zero coupon bonds as a function of time, technology and real spot interest rate whose dynamics was determined previously. We follow again a PDE approach because x_t and r_t both follow Markov processes and satisfy the necessary technical conditions to apply the Representation Theorem of Feyman-Kač[12]. The solution is not explicit but it is necessary to follow a numerical method. The approach taken here can be defined as a 'two step' procedure, because I first derive the zero coupon bonds price and only afterward we find the term structure of real interest rates. This approach is slightly different from what has been proposed by Bakshi and Chen (1996). Here the stochastic process of both real and nominal interest rate is directly derived from the assumptions made on the stochastic processes for the technology and policy processes. Instead Bakshi and Chen (1996) impose the stochastic process of the nominal interest rate in less orthodox way. In some sense, the structure they use considers some ad hoc assumptions, that I removed here.

Proposition 10 *Given the dynamic for technology and real spot rate, as in* (62) *and* (107)*, respectively, then the zero coupon bond* $B(t,T)$ *is a function* $\vartheta(t, x_t, R_t)$ *that is the unique solution of the following*

[12] In particular, the drift and volatility term in (107) must be Lipschitz and bounded on $\Re$.

Kolmogorov PDE:

$$\begin{cases} \frac{1}{2}\sigma_x^2 x_t \frac{\partial^2\vartheta(t,x_t,R_t)}{\partial x_t^2} + a\left[b - x_t\right] \frac{\partial\vartheta(t,x_t,R_t)}{\partial x_t} + \frac{1}{2}C^2\left(\theta\right)C^{*2}\left(\theta\right)R_t^2\sigma_x^2 x_t \frac{\partial^2\vartheta(t,x_t,R_t)}{\partial R_t^2} + \\ +a^*\left(\theta\right)\left[b^*\left(\theta\right) - R_t\right] \frac{\partial\vartheta(t,x_t,R_t)}{\partial R_t} + R_t C\left(\theta\right)\sigma_x^2 x_t \frac{\partial^2\vartheta(t,x_t,R_t)}{\partial r_t \partial x_t} + \frac{\partial\vartheta(t,x_t,R_t)}{\partial t} + \\ -R_t\vartheta\left(t,x_t,R_t\right) = 0 \\ \vartheta\left(T,x_t,R_t\right) = 1 \end{cases} \tag{79}$$

Proof. See Appendix A. ■

It is now straightforward to derive the dynamic for the real term structure.

Lemma 11 *The term structure of real interest rates is:*

$$r\left(t,T\right) = -\frac{1}{\theta}\left[\ln A^*\left(\theta\right) - R_t C^*\left(\theta\right)\right] \tag{80}$$

where $A^*\left(\theta\right)$ and $C^*\left(\theta\right)$ are the solutions of system (110).

Proof. The term structure of real interest rates is immediately derived by the relation between the price of zero coupon bond and the continuous real interest rate:

$$B\left(t,T\right) = e^{-\theta r(t,T)}$$

■

With this machinery at hand we can now derive the nominal term structure of interest rates.

8 The Nominal term structure of interest rates

In what follows we will derive the equilibrium nominal spot interest rate and the analytical expression of the nominal zero coupon bond and the nominal term structure of interest rates (in implicit form). To start with, consider together the expression for the real interest rate and the expression for the price level given, respectively, by (75), and (95).

Lemma 12 *The nominal spot interest rate is given by:*

$$i_t = A\left(\theta\right)e^{C(\theta)x_t}\left[\left(\frac{\phi}{1-\phi}\right)\frac{q_t^2\left(\beta+\mu_M^*\right)\left(\beta+\mu_M^*+2k_q\mu_q\right)}{\left(\beta+\mu_M^*\right)+\left(k_q+3\sigma_q^2\right)q_t 2k_q\mu_q}\frac{M_t}{Y_t}\right] \tag{81}$$

where $A\left(\theta\right)$, $C\left(\theta\right)$ are given by (76) *and* (77)*, respectively.*

Proof. To get (81) it is enough to multiply (75) by (70). ■

Lemma 13 *The nominal zero coupon bond is given by*

$$N\left(t,T\right) = B\left(t,T\right)\left[\left(\frac{\phi}{1-\phi}\right)\frac{q_t^2\left(\beta+\mu_M^*\right)\left(\beta+\mu_M^*+2k_q\mu_q\right)}{\left(\beta+\mu_M^*\right)+\left(k_q+3\sigma_q^2\right)q_t 2k_q\mu_q}\frac{M_t}{Y_t}\right] \tag{82}$$

where $B\left(t,T\right)$ is the solution of (108).

Proof. To get (82) it is enough to multiply the solution of (79) by (70). ■

Lemma 14 *The nominal term structure of interest rate is given by*

$$I(t,T) = r(t,T)\left[\left(\frac{\phi}{1-\phi}\right)\frac{q_t^2(\beta+\mu_M^*)\left(\beta+\mu_M^*+2k_q\mu_q\right)}{(\beta+\mu_M^*)+\left(k_q+3\sigma_q^2\right)q_t 2k_q\mu_q}\frac{M_t}{Y_t}\right] \tag{83}$$

where $R(t,T)$ *is given by* (80)

Proof. To get (83) it is enough to multiply (80) by (70). ■

By looking at the expressions for (81), (82) and (83) we observe that the nominal variables are bigger than the corresponding real variable if (70) is bigger than one. According to that, we can easily find a condition on ϕ_1 (contained in μ_M^*) which verifies this assertion. Therefore, the condition on μ_M^* is

$$\mu_M^* < \mu_M^{*a} \quad \text{and} \quad \mu_M^* > \mu_M^{*b} \tag{84}$$

where μ_M^{*a}, μ_M^{*b} are given by:

$$\mu_M^{*a,b} = \frac{-\Gamma \pm \sqrt{\Gamma^2 + 4M_t\phi q_t^2\Lambda}}{2M_t\phi q_t^2}$$

where:

$$\begin{aligned}\Gamma &= 2M_t\phi q_t^2\left(\beta+k_q\mu_q\right) - Y_t(1-\phi)\\ \Lambda &= -Y_t(1-\phi)\beta - 2Y_tk_q\mu_q q_t\left(k_q+3\sigma_q^2\right)(1-\phi) + M_t\phi q_t^2\beta\left(\beta+2k_q\mu_q\right)\end{aligned}$$

These equations establish a bound on the drift of the stochastic process leading the money growth rate. By virtue of the definition of μ_M^* given in (68) we get that the bound for ϕ_1 is given by:

$$\phi_1^b < \phi_1 < \phi_1^a$$

where:

$$\phi_1^{a,b} = \left(\mu_i + \sigma_q^2\right) - \frac{(1+\psi)}{\psi}\left(\mu_M^{*a,b} - \mu_H^*\right) \tag{85}$$

with μ_H^* defined as in (68).

9 Simulation Results

The results outlined in these pages are not easily interpretable without the help of a set of simulations. In what follows, I will construct a set of simulation results that will highlights the main properties of the present model. To do so, we need to specify some parameters. For the choice of the core parameters I follow the work by Balduzzi (2000) who presented an estimated version of the stochastic processes, through the methods of moments estimation method. The parameter chosen are collected in the following table:

β	ϕ	k_q	μ_q	a	b	μ_Y	η_Y	σ_Y	σ_x	σ_q
0.998	0.01	0.56	0.11	0.86	0.06	1.3	0.7	0.06	0.03	0.02

Table 1

The parameter β is set equal to 0.998, as in the traditional Real Business Cycle literature. Also the preference parameter ϕ is chosen in accord to Balduzzi (2000), who derived it according to an estimate of the key equations of the model through methods of moments. Moreover, all the other parameters are taken from the very few empirical studies existing in the literature on this subject. This is done especially for comparisons maters. However, it should be remembered that the main goal of the present paper is to show the joint role of monetary and especially fiscal policy parameters in the determination of the term structure of interest rates. The search for the model which best could fit the data is still ongoing. A final remarks before the discussion of results regarding the treatment of the stochastic processes for money supply, output growth, technology and the stochastic process describing the stochastic trend in money supply. To run simulations I integrated all these diffusion processes[13] and I initialized them by setting an initial condition equal to one, i.e. $x_0 = Y_0 = M_0 = q_0 = 1$.

The first step is to consider the simulation analysis for both the price level and the inflation rate for different values of both the fiscal policy parameter ϕ_1 and the monetary parameter μ_H. In figures 1-2 I reported the pictures to detect the effect of changes in these two parameters. It is worth to recall that the simulations reported in Figures 1 and 2 are conducted within an a-temporal context, without any reference to time elapsing. This is done in order to show the impact effect on both the price level and the inflation rate induced by changes in parameters ϕ_1 and μ_H. In Figure 1 we observe that a progressive increase in ϕ_1 creates a proportional reduction of the level of the price index and, at the same time of the inflation rate. This means that a tighter fiscal policy has a deflationary impact. These pictures confirms the intuition provided in a simpler model in the previous sections. These effects are interpretable along the lines of the Fiscal Theory of the Price Level and , in particular, under the new Wiksellian monetary policy framework highlighted by Woodford (2000).

In Figure 2, I show the effects on the same variables after changing the monetary policy parameter. In this case, we observe an almost reversed effect: In the simulation here considered I have set the fiscal policy parameter equal to 0.55. This is a value coherent with the bounds here assumed and with the literature on FTPL. It should be noted that the monetary policy parameter of interest here is given by the drift term of the stochastic process of the money supply growth rate which goes into the public's hands, i.e. the monetary base held by the public. The inflationary effect after and increase of money supply is normal and expected, according to more than one hundred monetary history.

Probably the more interesting set of simulations is represented by Figures 3-5, where I reported the effects of changes of the fiscal policy parameter for the pattern behavior of the nominal and real spot rate, the zero coupon bond (nominal and real) and the term structure of interest rates (nominal and real), respectively. All the pictures here represented show dynamical effects of the various policies. In fact, the simulations are run over a set of 10 periods. Each simulation step is 0.01, in the interval [0, 10]. All the curve showed in this set of picture refers to particular values for ϕ_1.

[13] The analytical expressions for the itegrated processes are reported in Appendix B.

In Figure 3, I have considered different plots, according to different values for ϕ_1. The picture shows that if fiscal policy becomes tighter (ϕ_1 raises) the spot rate curve shifts down. However, even if for very small values of ϕ_1 (implied by a very high taxation policy) the curve appears to loose its sensitiveness to this parameter.

The starred curve represents the real term structure. We observe that the starred curve lies below all the nominal curves. This is because the nominal curve incorporate the inflationary effect (expected and/or actual). Since the choice of the parameter is done punctual and kept constant over time horizon chosen for the simulation period, it might happen that the curve of the real term structure overcome the nominal term structure, showing an increase of the inflation rate. This effect would disappear if fiscal authority will keep on raising the coefficient on the fiscal policy reaction function. Note that the type of pathology is evident especially for very long periods far away the simulation staring point.

Let us look now at Figure 4, where, in order to simulate the model and have a closed form solution useful for the simulation analysis, I have made a simplifying assumption by supposing that the coefficients of equations (110) are time independent. We are aware of the limitation of this choice but this is motivated by the need to present some results in an intuitive way which can be readable according to an handy economic sense. In any case, in our view this represents a remarkable step ahead in the current literature, where all the results about the term structure are almost imposed and not explicitly derived as considered here. The parameters chosen for the simulation are exactly the same of those considered for simulating the nominal and real spot rate curve.

The picture shows that the position in the plane of the term structure strictly depends upon the value assumed for the tax rate ϕ_1. In particular, it is evident that if the fiscal pressure decreases (i.e. if ϕ_1 decreases), the curve of the nominal term structure zero coupon bond shifts down, even if for very small values of ϕ_1 (implied by a very high taxation policy) the curve appears to loose its sensitiveness to this parameter. The behavior of the curve is coherent with both the results from Fiscal Theory and the empirical evidence. In fact, if the tax rate get reduced, this means, that the price asked for newly issued debt must be reduced in order to convince new subscribers to buy additional debt. In fact, newly issued debt, without a backing of more taxes implies an inflationary risk for the future (causing a reduction of the nominal value of debt right by now). This will even more evident when we will discuss the Term Structure for both Nominal and Real Zero Coupon Bonds in the next picture.

With respect to the punctual choice for ϕ_1 can be repeated here the considerations of the same sort done before for the nominal and real spot rate curve.

Finally, consider the simulation for the term structure of nominal and real rates reported in Figure 5. The picture shows that the position in the plane of the term structure strictly depends upon the value assumed for ϕ_1. In particular, it is evident that if the fiscal parameter ϕ_1 tax rate decreases, the curve of the nominal term structure shifts up, even if for very high values of ϕ_1 the curve appears to loose its sensitiveness to this parameter. The behavior of the curve is coherent with both the results from Fiscal Theory and the empirical evidence. In fact, if the tax rate raises, this means that it will be possible to reduce the number of issue of new public debt in order to finance the current position of the government. This will call for a lower interest rate, and for a bigger government credibility. This shows not only monetary policy is important for the term structure, but also fiscal policy is crucial in order to determine the position of the curve. Empirically, this has been verified as true in the experience of Italy after 1996-1997 episode of fiscal retrenchment which shifted down the position of the nominal term structure after a crucial fiscal consolidation, necessary to get into the EU.

Here too the starred curve represents the real term structure and we can replicate here the same kind of considerations done for the punctual choice of ϕ_1. We observe that even in this case the starred curve lies below all the nominal curves, in accord with economic theory. From the economic point of view, the case where the nominal curve stays above the real, can be thought as the result of a very tight fiscal policy which produces a deflationary equilibrium, such that the nominal term structure is below (or at) the real term curve. Such deflationary equilibria are not impossible within the Fiscal Theory framework, as discussed by Sims (1994): this can be interpreted also by thinking about a contraction of expenses (and of aggregate demand) determined by a very high level of fiscal pressure.

10 Concluding Remarks

In this paper I have considered a simple intertemporal model for the determination of the nominal and real term structure where fiscal and monetary policies play a joint key role in the determination of the position of the term structure and the zero coupon bond curve in the plane. This shows to be an important development in a literature where only monetary and technological factors were seen as unique determinants of the term structure of interest rates. The novelty of the present paper is represented by a different way employed to determine the term structure of interest rates in the monetary models, relying on an arbitrage-free model, rather than relying on equilibrium arguments.

The results show that the introduction of both monetary and fiscal policy rules adds an important element in the determination of the position in the plane of the term structure of interest rates. In particular, it is shown that as showed by important contributions in the Fiscal Theory of the Price Level (FTPL), inflation is not necessarily a monetary phenomena. An expected fiscal retrenchment shifts downwards the entire term structure of nominal interest rates and the nominal spot rate curve. The effect is entirely due to the reaction of expected inflation: if government adopts a tighter fiscal policy, bonds become more 'secure' in the sense, that an expected higher level of taxes will make the government able to pay back taxes. Thus, bond prices will increase, while their return will decrease. The nominal terms structure equation depends upon the technological parameters. However, the real curve does not seem to be affected by the same set of parameters affecting the nominal curve. From this point of view, this model presents similar conclusion with respect to Bakshi and Chen (1996). The approach taken here, is however, general enough which will allow to design a more complex set of stochastic processes able to create a set of interrelations between nominal and real term structure of interest rates which has not been explored in the literature.

The results here presented generalize two distinct flow of literature: the FTPL and the term structure models in a promising way. Further generalization could go, for example, towards a more careful modelling of the monetary policy function.

The pattern of the nominal term structure here considered captures quite well the dynamic of the the term structure pattern of many countries after having experienced a fiscal retrenchment, as discussed by Corsetti and Pesenti (1999), in the case of European Countries.

Appendix A

Proof of Proposition 1.

Subtract equation (41) from equation (44). Then manipulate the resulting expression and use the definition of the stochastic process for $p_{i,t}^{a}$ given in (5) to get:

$$e^{-\beta\Delta t}E_t\left\{\frac{u_c\left(Y_{t+\Delta t},m_{t+\Delta t}\right)}{u_c\left(Y_t,m_t\right)}\left[\left(\mu_{i,t}^{a}-R_t\right)\Delta t+\sigma_{i,t}^{a}\Omega_{i,t}^{a}\sqrt{\Delta t}\right]\right\}=0 \tag{86}$$

Take now a Taylor expansion of equation (86) around the steady state, we obtain:

$$\begin{aligned}e^{-\beta\Delta t}E_t\Bigg\{&\left[\left(\mu_{i,t}^{a}-R_t\right)\Delta t+\sigma_{i,t}^{z}\Omega_{i,t}^{z}\sqrt{\Delta t}\right]\cdot\left[1+\frac{u_c\left(Y_{t+\Delta t},m_{t+\Delta t}\right)Y_t}{u_c\left(Y_t,m_t\right)}\frac{\Delta Y_t}{Y_t}+\right.\\&\left.+\frac{u_{cm}\left(Y_{t+\Delta t},m_{t+\Delta t}\right)m_t}{u_c\left(Y_t,m_t\right)}\frac{\Delta m_t}{m_t}\right]\Bigg\}\frac{1}{\Delta t}=0\end{aligned} \tag{87}$$

letting $\Delta t\to 0$ in (87) and applying the Ito's multiplication rule, we easily derive (47). ■

Proof of Proposition 2

From Equation (44) recall that:

$$\frac{\Delta p_{i,t+\Delta t}^{a}}{p_{i,t}^{a}}=\frac{P_{e,t+\Delta t}^{a}+Y_{t+\Delta t}\Delta t}{P_{e,t}^{a}} \tag{88}$$

Thus, plugging (88) into (44) we get:

$$P_{e,t}^{a}=E_t\left\{e^{-\beta\Delta t}\frac{u_c\left(Y_{t+\Delta t},m_{t+\Delta t}\right)}{u_c\left(Y_t,m_t\right)}\left(P_{e,t+\Delta t}^{a}+Y_{t+\Delta t}\Delta t\right)\right\} \tag{89}$$

iterate equation (89) to get:

$$P_{e,t}^{a}=E_t\sum_{j=1}^{\infty}e^{-\beta(j\Delta t)}\left[\frac{u_c\left(Y_{t+j\Delta t},m_{t+j\Delta t}\right)}{u_c\left(Y_t,m_t\right)}Y_{t+j\Delta t}\Delta t\right] \tag{90}$$

Thus, by taking the limit for $\Delta t\to 0$ in equation (90) we finally obtain the result in (50). ■

Proof of Proposition 3.

Rewrite the first order condition (42) as follows:

$$\frac{1}{P_t}=e^{-\beta\Delta t}E_t\left\{\left[\frac{u_c\left(Y_{t+\Delta t},m_{t+\Delta t}\right)}{u_c\left(Y_t,m_t\right)}\frac{1}{P_{t+\Delta t}}+\frac{u_m\left(Y_{t+\Delta t},m_{t+\Delta t}\right)}{u_c\left(Y_t,m_t\right)}\right]\frac{\Delta t}{P_{t+\Delta t}}\right\} \tag{91}$$

then, iterate equation (91) to get:

$$\frac{1}{P_t}=E_t\left\{\sum_{j=1}^{\infty}e^{-\beta(j\Delta t)}\frac{u_m\left(Y_{t+j\Delta t},m_{t+j\Delta t}\right)}{u_c\left(Y_t,m_t\right)}\frac{\Delta t}{P_{t+j\Delta t}}\right\} \tag{92}$$

Taking the limit of the equation (92) we get the result under (54).

To obtain the expression for the inflation rate, divide the two first order conditions (41) and (43), to get:

$$E_t \left\{ \frac{u_c(Y_{t+\Delta t}, m_{t+\Delta t})}{u_c(Y_t, m_t)} (1 + R_t \Delta t) \right\} = E_t \left[u_c(Y_{t+\Delta t}, m_{t+\Delta t})(1 + i_t \Delta t) \frac{P_t}{P_{t+\Delta t}} \right] \tag{93}$$

Expand equation (93), in Taylor's series to get, after rearrangement:

$$(i_t - r_t)\Delta t = \left[1 + \frac{u_{cc} Y_t}{u_c} \left(\frac{\Delta Y_t}{Y_t} \right) + \frac{u_{cm}}{u_c} m_t \left(\frac{\Delta m_t}{m_t} \right) \right] \left[\frac{\Delta P_t}{P_t} - \left(\frac{\Delta P_t}{P_t} \right)^2 \right] + o(\Delta t)^{3/2} \tag{94}$$

Thus, by taking the limit of equation (94), for $\Delta t \to 0$ we have:

$$i_t - r_t = \frac{1}{dt} E_t \left\{ \frac{dP_t}{P_t} \right\} - var_t \left\{ \frac{dP_t}{P_t} \right\} + \frac{u_{cc} Y_t}{u_c} cov_t \left(\frac{dY_t}{Y_t}, \frac{dP_t}{P_t^C} \right) + \frac{u_{cm} m_t}{u_c} cov_t \left(\frac{dP_t}{P_t}, \frac{dm_t}{m_t} \right)$$

Thus, by using $\pi_t = \frac{1}{dt} E_t \left\{ \frac{dP_t}{P_t} \right\}$, and rearranging we easily get the equation (55). ■

Proof of Proposition 4.

Consider the First Order Condition in equation (41) and take a Taylor expansion around the equilibrium, to get:

$$u_c(Y_t, m_t)(1 + R_t \Delta t) = E_t \left\{ \left[u_c(Y_t, m_t) + u_{cc}(Y_t, m_t) \Delta Y_t + u_{cm}(Y_t, m_t) \Delta m_t + \frac{u_{ccc}(Y_t, m_t)}{2} (\Delta Y_t)^2 + \right.\right.$$
$$\left.\left. + \frac{u_{ccm}(Y_t, m_t)}{2} \Delta Y_t \Delta m_t + \frac{u_{cmm}(Y_t, m_t)}{2} (\Delta m_t)^2 \right] (1 + R_t \Delta t) \right\} + o(\Delta t)^{3/2} \tag{95}$$

Thus, simplifying, we get:

$$R_t = \beta - \frac{1}{\Delta t} E_t \left\{ \frac{u_{cc}(Y_t, m_t) Y_t}{u_c(Y_t, m_t)} \left(\frac{\Delta Y_t}{Y_t} \right) + \frac{u_{cm} m_t}{u_c} \left(\frac{\Delta m_t}{m_t} \right) + \frac{u_{ccc} Y_t^2}{2u_c} \left(\frac{\Delta Y_t}{Y_t} \right)^2 + \right.$$
$$\left. + \frac{u_{ccm} Y_t m_t}{2u_c} \left(\frac{\Delta Y_t}{Y_t} \right) \left(\frac{\Delta m_t}{m_t} \right) + \frac{u_{cmm} Y_t m_t^2}{2u_c} \left(\frac{\Delta m_t}{m_t} \right)^2 \right\} + o(\Delta t)^{3/2} \tag{96}$$

Recall that:

$$\frac{1}{dt} E_t \left\{ \left(\frac{dY_t}{Y_t} \right)^2 \right\} = var_t \left(\frac{dY_t}{Y_t} \right);$$
$$\frac{1}{dt} E_t \left\{ \left(\frac{dm_t}{m_t} \right)^2 \right\} = var_t \left(\frac{dm_t}{m_t} \right);$$

If we now take the limit of equation (96) for $\Delta t \to 0$ and apply the Ito's Lemma, we finally get (56). ■

Proof of Proposition 5.

Subtract First Order Condition (43) from (42) to get:

$$E_t \left[\frac{u_m(Y_{t+\Delta t}, m_{t+\Delta t})}{u_c(Y_t, m_t)} \frac{P_t}{P_{t+\Delta t}} \right] = E_t \left[\frac{u_c(Y_{t+\Delta t}, m_{t+\Delta t})}{u_c(Y_t, m_t)} \frac{P_t}{P_{t+\Delta t}} i_t \right] \tag{97}$$

Consider now a Taylor expansion of the price ratio, to get:

$$\frac{P_t}{P_{t+\Delta t}} = 1 - \frac{\Delta P_t}{P_t} + \left(\frac{\Delta P_t}{P_t}\right)^2 + o\left(\Delta t\right)^{3/2} \tag{98}$$

Expand also in Taylor series the LHS of equation (97) so that:

$$\begin{aligned} E_t\left[\frac{u_c\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)}\frac{P_t}{P_{t+\Delta t}} i_t\right] &= E_t\left\{\underbrace{\left[1 + \frac{u_{cc}Y_t}{u_c}\left(\frac{\Delta Y_t}{Y_t}\right) + \frac{u_{cm}m_t}{u_c}\left(\frac{\Delta m_t}{m_t}\right)\right]}\left(1 - \frac{\Delta P_t}{P_t}\right)\right. \\ &\quad \left. + o\left(\Delta t\right)^{3/2}\right\} i_t \end{aligned} \tag{99}$$

Note that the term underbraced in the first line of equation (99) tends to 1, as $\Delta t \to 0$, after having applied the Ito's multiplication rule. Moreover, we have that:

$$\lim_{\Delta t \to 0}\left\{\frac{u_m\left(Y_{t+\Delta t}, m_{t+\Delta t}\right)}{u_c\left(Y_t, m_t\right)}\frac{P_t}{P_{t+\Delta t}}\right\} = \frac{u_m\left(Y_t, m_t\right)}{u_c\left(Y_t, m_t\right)} \tag{100}$$

which proves the proposition. ■

Proof of Lemma 7

(i) To prove result (**??**) it is enough to apply the result in Proposition 1, by substituting out the derivative of the utility function into their respective expressions.

(ii) To get real rate in (**??**) we just need to apply the results from Proposition 2, after having plugged the expression for the derivative into equation (56).

(iii) To prove result (**??**) recall that from Proposition 2 and equation (50) we have a formula directly usable for the present case. Taking the appropriate derivatives of the utility function (**??**) in equation (50) we get:

$$\begin{aligned} p^a_{e,t} &= E_t \int_t^\infty e^{-\beta(s-t)} \frac{C_s^\eta m_s^{1-\eta}}{C_t^{\eta-1} m_t^{1-\eta}} ds = \end{aligned} \tag{101}$$

$$\begin{aligned} &= \frac{1}{Y_t^{\eta-1} m_t^{1-\eta}} E_t \int_t^\infty e^{-\beta(s-t)} Y_s^\eta m_s^{1-\eta} ds \end{aligned} \tag{102}$$

where I also inserted the equilibrium condition $C_t = Y_t$. After taking the integral of (102) we get exactly the result given in equation (**??**).

(iv) The dynamics of $p^a_{e,t}$ can be obtained by simply applying the Ito's Lemma to equation (**??**).

(v) To obtain the price level in (**??**) recall that from Proposition 3 a general definition of the price level is given by equation (54). In our case, this implies that:

$$\frac{1}{P_t} = \left(\frac{1-\eta}{\eta}\right) \frac{1}{Y_t^{\eta(1-\rho)-1} m_t^{(1-\eta)(1-\rho)}} E_t \int_t^\infty e^{-\beta(s-t)} Y_s^{\eta(1-\rho)} m_s^{(1-\eta)(1-\rho)} \frac{1}{P_s} ds \tag{103}$$

which can be rewritten as:

$$\frac{1}{P_t Y_t} = \left(\frac{1-\eta}{\eta}\right) \frac{1}{Y_t^{\eta(1-\rho)} m_t^{(1-\eta)(1-\rho)}} \int_t^\infty e^{-\beta(s-t)} Y_s^{\eta(1-\rho)} m_s^{(1-\eta)(1-\rho)} E_t\left[\frac{1}{M(s)}\right] ds \tag{104}$$

To calculate the expectational term $E_t\left[\frac{1}{M(s)}\right]$ in (104) we need to integrate the stochastic process for $M(t)$ given by equation (**??**) so:

$$M(s) = M(t)\exp\left\{-\left[\left(\mu_M - \frac{\sigma_M^2}{2}\right)(s-t) - \sigma_M\int_t^s d\omega_{M,t}\right]\right\} \tag{105}$$

where $d\omega_{M,t} = \Omega_{M,t}\sqrt{dt}$. So that:

$$E_t\left[\frac{1}{M(s)} \mid M(t)\right] = \frac{1}{M(t)}\exp\left\{-\left[\left(\mu_M - \sigma_M^2\right)(s-t)\right]\right\} \tag{106}$$

Plugging equation (106) into (104) and integrate the resulting equation, we get exactly the result given by (**??**).

(vi) To get the dynamics of the price level (**??**) it is enough to apply the Ito's Lemma to equation (**??**).

(vii) The inflation rate (**??**) is the drift term of the stochastic process found in (vi). ■

Proof of Proposition 9

Let us start by showing how to get (70). From (66) we have that:

$$\ln M_t = \mu_M^* + 2\ln q_t$$

so that:

$$M_t = e^{\mu_M^*} q_t^2$$

inverting it:

$$\frac{1}{M_t} = e^{-\mu_M^*} q_t^{-2}$$

Define $G(q) = \frac{1}{q_t^2}$. Thus by using Ito's Lemma, we get:

$$d\left[\frac{e^{2k_q\mu_q}}{q_t^2}\right] = \frac{2}{q_t}\left(k_q + 3\sigma_q^2\right)e^{2k_q\mu_q t}dt - \frac{2\sigma_q}{q_t\sqrt{q_t}}dW_{M,t}$$

Thus, let us consider the expected value, as follows:

$$\begin{aligned} E_t\left[\frac{1}{q_s^2}\right] &= E_t\left[\frac{1}{q_s^2} \mid q_t\right] = \\ &= E_t\left[e^{-2k_q\mu_q s}\left\{\int_t^s d\left[\frac{e^{2k_q\mu_q z}}{q_z^2}\right] + \frac{e^{2k_q\mu_q t}}{q_t^2}\right\} \mid q_t\right] = \\ &= \frac{e^{-2k_q\mu_q(s-t)}}{q_t} + \frac{\left(k_q + 3\sigma_q^2\right)}{q_t k_q \mu_q}\left(1 - e^{-2k_q\mu_q(s-t)}\right) \end{aligned}$$

Finally, from the FOC of the problem of the representative agent, we get:

$$\frac{1}{P_t^C Y_t} = \frac{1-\phi}{\phi}\int_t^\infty E_t\left[\frac{1}{q_s^2}\right]e^{-\beta(s-t)-\mu_M^*}ds$$

which, after solving for the integral, we get exactly the result under (71).

To get the expression of the inflation rate, it is enough to apply Ito's lemma to (71) by setting $V(M_t, Y_t, q_t) = \frac{\Delta(q_t)}{\Psi(q_t)} \frac{M_t}{Y_t}$, so that:

$$dP_t^c = G_M dM_t + G_Y dY_t + G_q dq_t + \frac{1}{2}\left[G_{YY}(dY_t)^2 + G_{qq}(dq_t)^2 + G_{Mq}(dM_t)(dq_t)\right]$$

thus, performing the calculations by taking into account (73), (74) and the definitions of the stochastic processes for Y_t, q_t and M_t (61), (65), (66) we get exactly the result given by (72).■

Proof of Proposition 10

If $\phi(t, x_t) = R_t$ like in equation (75) so the Kolmogorov PDE is:

$$\begin{cases} \frac{1}{2}\sigma^2 x_t C^2(\theta) R_t + a(b - x_t) C(\theta) R_t + A'(\theta) e^{C(\theta)x_t} + \\ +C'(\theta) x_t R_t - x_t R_t = 0 \\ \phi(T, x_t) = \frac{2ab}{\gamma + a} \end{cases}$$

where $A'(\theta)$ and $C'(\theta)$ represent the derivative with respect to time of functions $A(\theta)$ and $C(\theta)$ given by equations (76) and (77). Given that the Kolmogorov PDE is verified for all t and x_t, we can separate into two equations, one dependent and one independent upon x_t. Now our task substantiates in solving the following DE system:

$$\begin{cases} \frac{1}{2}\sigma_x^2 C^2(\theta) - aC(\theta) + C'(\theta) - 1 = 0 & C(0) = 1 \\ A'(\theta) + abA(\theta) C(\theta) = 0 & A(0) = \frac{2ab}{\gamma + a} \end{cases}$$

where the first DE is a Riccati type equation whose solution is (76). The second equation, instead, is a regular first order DE whose solution is given by (77).■

Proof of Proposition 11

By applying Ito's Lemma to (75), the dynamic of $R_t = \phi(t, x_t)$ is:

$$\begin{aligned} dR_t &= C(\theta) R_t dx_t + \frac{\partial R_t}{\partial t} dt + \frac{1}{2}\sigma_x^2 x_t C^2(\theta) R_t dt \\ &= R_t \left[a(b - x_t) C(\theta) + C'(\theta) x_t + \frac{1}{2}\sigma_x^2 x_t C^2(\theta)\right] dt + \\ &\quad +A'(\theta) e^{C(\theta)x_t} dt + C(\theta) R_t \sigma_x \sqrt{x_t} dW_{t,x} \end{aligned} \tag{107}$$

Finally, we may write (107) as a mean reverting process in square root with coefficients time dependent as in (78).■

Proof of Proposition 12

If $\vartheta(t, x_t, R_t) = B(t, T)$ like in the following equation (109) so the Kolmogorov PDE must be:

$$\begin{cases} -\frac{1}{2}\sigma_x^2 x_t C^2(\theta) R_t C^{*2}(\theta) B(t,T)\left[1 - C^{*2}(\theta) R_t\right] + \\ -a[b - x_t] C^*(\theta) B(t,T) R_t C(\theta) + \frac{1}{2}\sigma_x^2 x_t C^2(\theta) R_t^2 C^{*2}(\theta) B(t,T) + \\ -a^*(\theta)[b^*(\theta) - R_t] C^*(\theta) B(t,T) + \sigma_x^2 x_t C^2(\theta) R_t^2 C^{*2}(\theta) B(t,T) + \\ +A^{*'}(\theta) e^{C^*(\theta)R_t} - C^{*'}(\theta) R_t B(t,T) - R_t B(t,T) = 0 \\ \vartheta(T, x_t, R_t) = 1 \end{cases} \tag{108}$$

where we define

$$B(t,T) = A^*(\theta)\, e^{-C^*(\theta) R_t} \tag{109}$$

The Kolmogorov PDE (108) is verified for all t, x_t and r_t; we can separate Kolmogorov PDE into two equation, one dependent and one independent upon x_t and r_t. Now the problem is to solve the following DE system:

$$\begin{cases} \frac{1}{2}\sigma^{*2}(\theta)\, C^{*2}(\theta) - \Psi(t)\, C^*(\theta) - C^{*'}(\theta) - 1 = 0 & C^*(0) = 0 \\ A^{*'}(\theta) - a^*(\theta)\, b^*(\theta)\, A^*(\theta)\, C^*(\theta) = 0 & A^*(0) = 1 \end{cases} \tag{110}$$

where $\sigma^*(t)$ and $\Psi(t)$ are

$$\sigma^*(\theta) = 2\sigma_x C(\theta)\sqrt{R_t x_t} \tag{111}$$

$$\Psi(\theta) = \frac{1}{2}\sigma_x^2 x_t C^2(\theta) + abC(\theta) - aC(\theta)\, x_t - a^*(\theta) \tag{112}$$

The first DE is a Riccati equation with coefficients time dependent and the second is a first order DE with coefficients time dependent. We can find solutions only by numerical methods.■

Appendix B

Analytical expressions for the integrated stochastic processes.

$$x_t = x_0 e^{-a\theta} + \frac{b}{a}\left(1 - e^{-a\theta}\right)$$

$$q_t = q_0 e^{-k_q\theta} + \frac{\mu_q}{k_q}\left(1 - e^{-k_q\theta}\right)$$

$$Y_t = Y_0 + \left(\mu_y + \eta_y x_t\right) e^{\theta R_t}$$

$$M_t = M_0 + \mu_M e^{\theta R_t}$$

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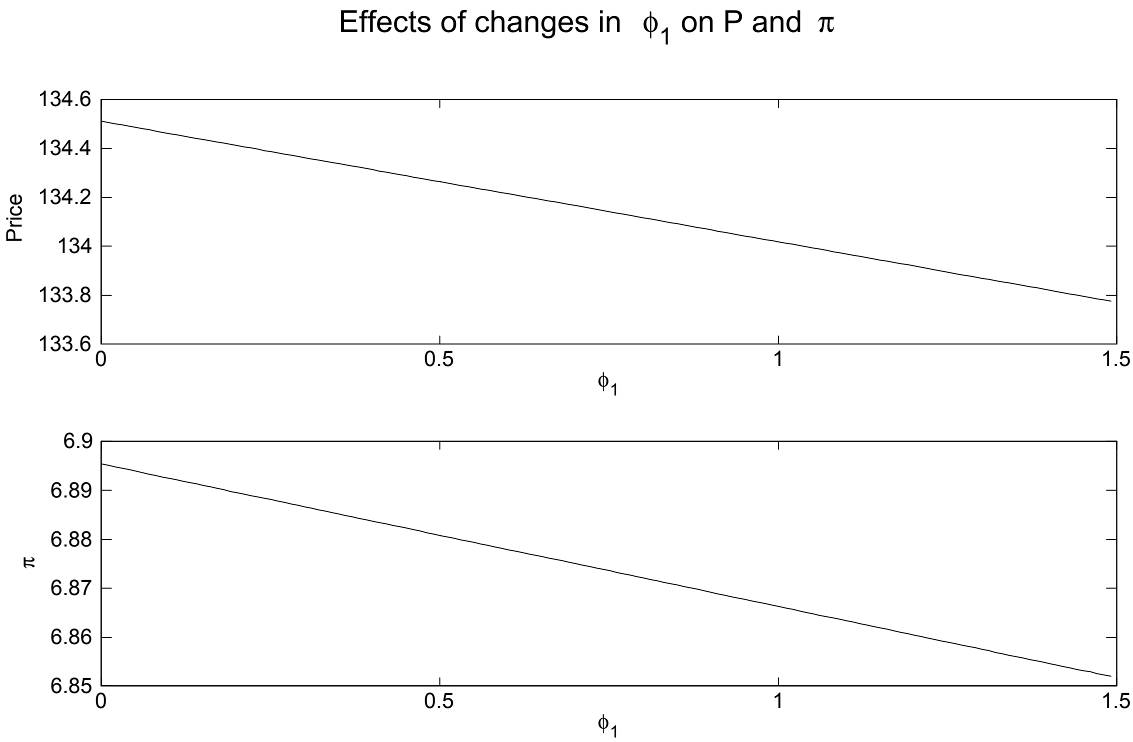
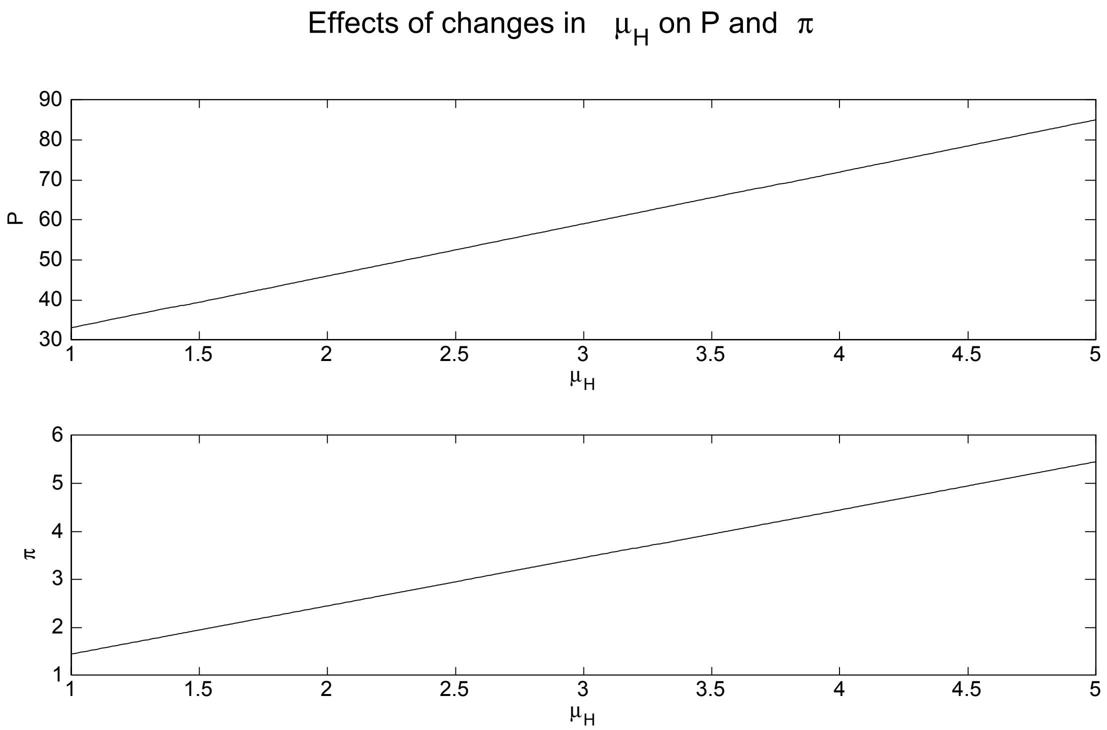

Effects of changes in ϕ_1 on P and π
134.6
134.4
134.2
134
133.8
133.6
Price
0
0.5
1
1.5
ϕ_1
6.9
6.89
6.88
6.87
6.86
6.85
π
0
0.5
1
1.5
ϕ_1

Figure 1:



Effects of changes in μH on P and π
90
80
70
60
50
40
30
P
1
1.5
2
2.5
3
3.5
4
4.5
5
μH
6
5
4
3
2
1
π
1
1.5
2
2.5
3
3.5
4
4.5
5
μH

Figure 2:

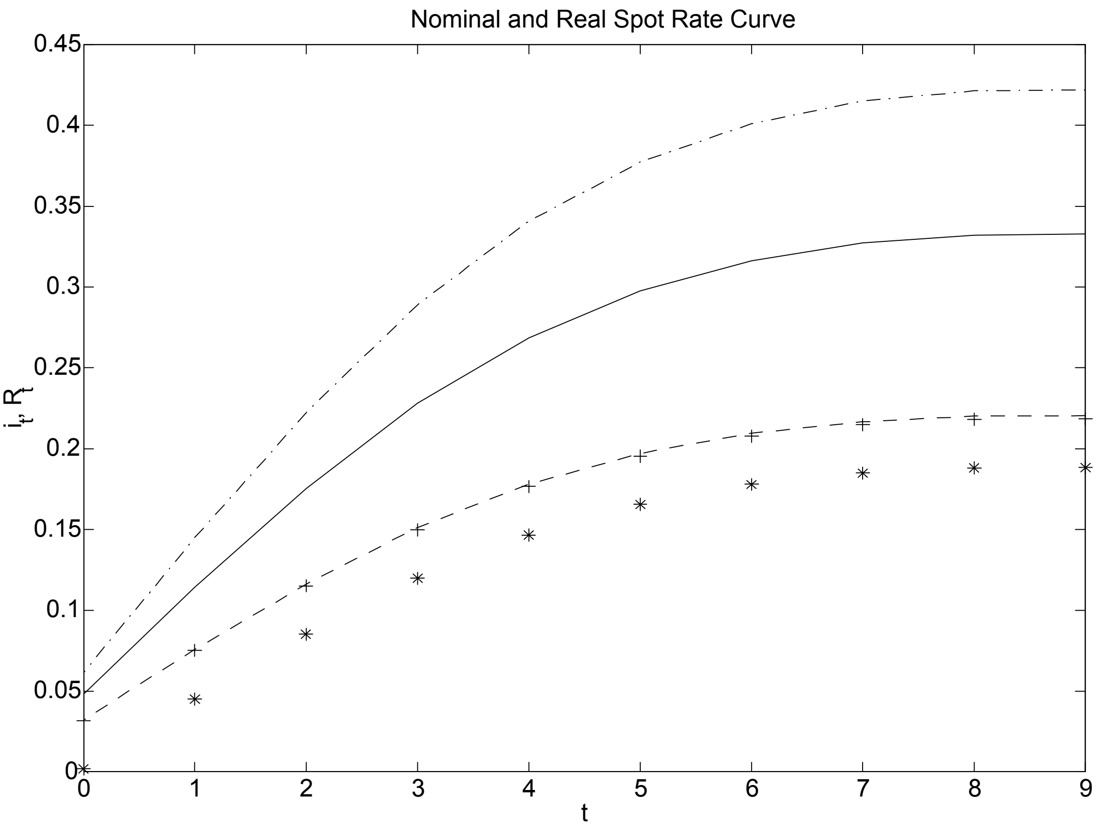
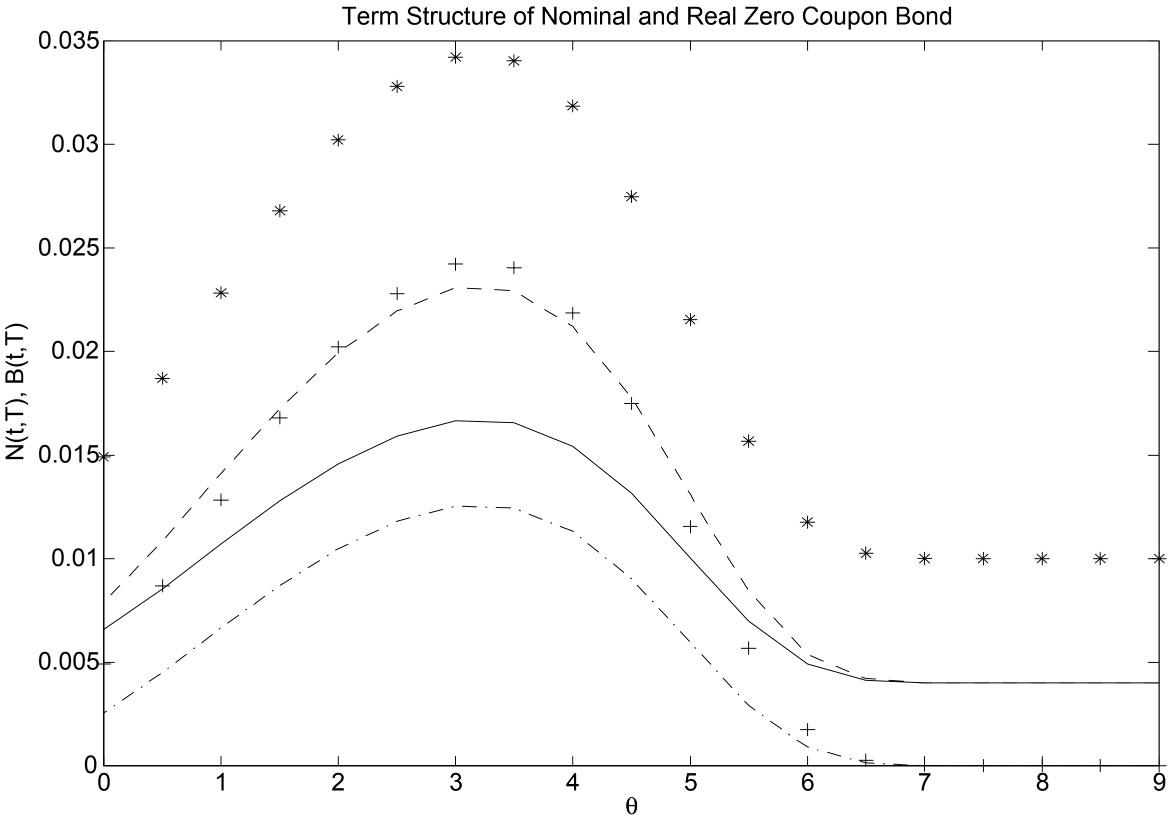

Nominal and Real Spot Rate Curve
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
i_t, R_t
0
1
2
3
4
5
6
7
8
9
t

Figure 3:



Term Structure of Nominal and Real Zero Coupon Bond
N(t,T), B(t,T)
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
1
2
3
4
5
6
7
8
9
θ

Figure 4:

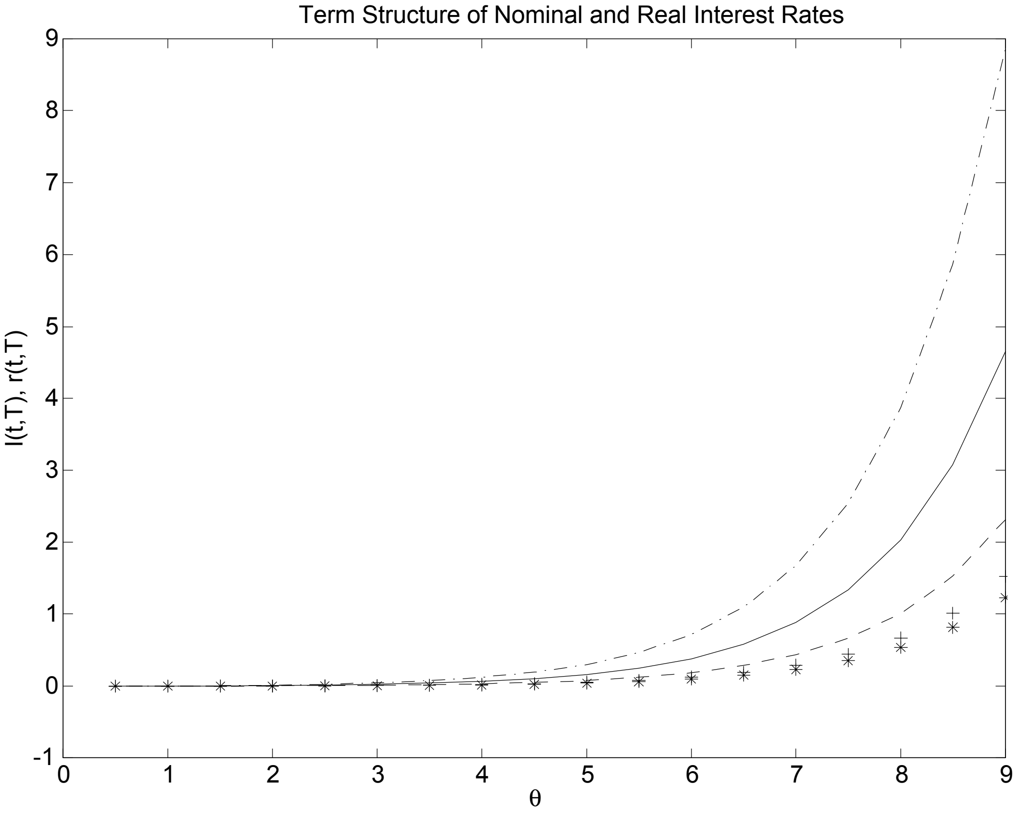

Term Structure of Nominal and Real Interest Rates
I(t,T), r(t,T)
θ
9
8
7
6
5
4
3
2
1
0
-1
0
1
2
3
4
5
6
7
8
9

Figure 5: