Di¤erential Oligopoly Games where the Closed-Loop Memoryless and Open-Loop Equilibria Coincide

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February 14, 2001

Abstract

We illustrate two di¤erential oligopoly games using, respectively, the capital accumulation dynamics à la Nerlove-Arrow, and the capital accumulation dynamics à la Ramsey. We prove that these games bene…t from the property that closed-loop memoryless solutions degenerate into open-loop solutions, since the best reply of a generic …rm is independent of the rivals' state variables, which entails that the cross e¤ect from rivals' states to own controls disappears.

J.E.L. Classi…cation: D43, D92, L13

Keywords: di¤erential games, capital accumulation, open-loop equilibria, closed-loop equilibria

1 Introduction

The existing literature on di¤erential games applied to …rms' behaviour mainly concentrates on two kinds of solution concepts:[1] the-open loop and the closed-loop. In the former case, …rms precommit their decisions on the control variables to a path over time and the relevant equilibrium concept is the open-loop Nash equilibrium. In the latter, …rms do not precommit on any path and their strategies at any instant may depend on all the preceding history. In this situation, the information set used by …rms in setting their strategies at any given time is often simpli…ed to be only the current value of the capital stocks at that time. The relevant equilibrium concept, in this (sub-)case, is the closed-loop memoryless Nash equilibrium, which is strongly time consistent and therefore subgame perfect. When players (…rms) adopt the open-loop solution concept, they design the optimal plan at the initial time and then stick to it forever. The resulting open-loop Nash equilibrium is only weakly time consistent and therefore, in general, it is not subgame perfect. A re…nement of the closed-loop Nash equilibrium, which is known as the feedback Nash equilibrium, can also be adopted as the solution concept.[2] While in the closed-loop memoryless case the initial and current levels of all state variables are taken into account, in the feedback case only the current stocks of states are considered.[3]

The existing literature on di¤erential games devotes a considerable amount of attention to identifying classes of games where either the feedback or the closed-loop equilibria degenerate into open-loop equilibria. This interest is motivated by the following reason. Whenever an open-loop equilibrium is a degenerate closed-loop or feedback equilibrium, then the former is also subgame perfect; therefore one can rely on the open-loop equilibrium which, in general, is much easier to derive than feedback and closed-loop ones. Classes of games where this coincidence arises are illustrated in Clemhout and Wan (1974); Reinganum (1982); Mehlmann and Willing (1983); Dockner, Feichtinger and Jørgensen (1985); Fershtman (1987); Fershtman, Kamien and Muller (1992). For an overview, see Mehlmann (1988) and Fershtman, Kamien and Muller (1992).

[1] See Kamien and Schwartz (1981); Başar and Olsder (1982); Mehlmann (1988).

[2] For oligopoly models where …rms follow feedback rules, see Simaan and Takayama (1978), Fershtman and Kamien (1987, 1990), Dockner and Haug (1990), inter alia.

[3] For a clear exposition of the di¤erence among these equilibrium solutions see Başar and Olsder (1982, pp. 318-327, and chapter 6, in particular Proposition 6.1).

We model a dynamic capital accumulation game in a Cournot oligopoly. To this end, we will consider both the model of reversible investment à la Nerlove-Arrow (1962), i.e., capital accumulation through costly investment, and the model à la Ramsey (1928), i.e., a "corn-corn" growth model, where accumulation coincides with consumption postponement.

The main results are as follows. Both under the Nerlove-Arrow and the Ramsey capital accumulation dynamics, the open-loop Nash equilibrium coincides with the closed-loop memoryless equilibrium, and hence the former is subgame perfect. This depends upon two features common to both settings: (a) the dynamic behaviour of any firm's state variable does not depend on the rivals' control and state variables, which makes the kinematic equations concerning other firms redundant; and (b) for any firm, the first order conditions taken w.r.t. the control variables are independent of the rivals' state variables, which entails that the cross effect from rivals' states to own controls (which characterises the closed-loop information structure) disappears.

The remainder of the paper is structured as follows. The model is laid out in section 2. Section 3 examines the two capital accumulation games. Section 4 contains concluding remarks.

2 The general setup

The game is played over $t \in [0; \infty)$: Define the set of players as $P \equiv \{1; 2; 3; \ldots N\}$: Moreover, let $x_i(t)$ define the state variable for player i. Its dynamics can be described by the following:

$$\frac{dx_i(t)}{dt} \equiv \dot{x}_i(t) = f\left(x_i(t); \{u_i(t)\}_{i=1}^N\right) \tag{1}$$

where $\{u_i(t)\}_{i=1}^N$ is the vector of players' actions at time t; i.e., it is the vector of the values of control variables at time t: Note that we are focussing on the case where the dynamics of the state variable relevant for player i does not depend on the state variables relevant for different players.. The value of the state variables at $t = 0$ is assumed to be known: $\{x_i(0)\}_{i=1}^N = \{x_{0;i}\}_{i=1}^N$:

Each player has an objective function, defined as the discounted value of the flow of payoffs over time. The instantaneous payoff must depend upon the choices made by player i as well as its rivals, that is, $\pi_i(t) = \pi_i(x_i(t); X_{-i}(t); u_i(t); U_{-i}(t))$; where $X_{-i}(t)$ is the vector of the values of states of all other players, at time t; and $U_{-i}(t)$ summarises the actions of all other players at time t. Player i's objective is then

$$\max_{u_i(t)} J_i \equiv \int_0^T \pi_i(t)e^{-\rho t}dt \tag{2}$$

subject to the dynamic constraint represented by the behaviour of the state variables (1) for $i = 1, ..., N$. The factor $e^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant and common to all players. In order to solve his optimisation problem, each player a time path for his control, Under the closed-loop memoryless information structure, the Hamiltonian of firm i writes as follows:

$$\begin{aligned} H_i(x_i(t); u_i(t)) \equiv \; & e^{-\rho t}\Big[\pi_i(x_i(t); x_{-i}(t); u_i(t); u_{-i}(t)) + \\ & + \lambda_{ii}(t) \cdot f\left(x_i(t); \{u_i(t)\}_{i=1}^N\right) + \sum_{j \neq i} \lambda_{ij}(t) \cdot f\left(x_j(t); \{u_i(t)\}_{i=1}^N\right)\Big] ; \end{aligned} \tag{3}$$

where $\lambda_{ij}(t) = \mu_{ij}(t)e^{\rho t}$ is the costate variable (evaluated at time t) associated with state variable x_j. The first order conditions are:

$$\frac{\partial H_i(x_i(t); u_i(t))}{\partial u_i(t)} = 0 ; \tag{4}$$

and

$$-\frac{\partial H_i(x_i(t); u_i(t))}{\partial x_j(t)} - \sum_{j \neq i} \frac{\partial H_i(x_i(t); u_i(t))}{\partial u_j(t)} \frac{\partial u_j^*(t)}{\partial x_j(t)} = \frac{\partial \lambda_{ij}(t)}{\partial t} - \rho\lambda_{ij} \;\; \forall j = 1, 2, ..., N; \tag{5}$$

along with the initial condition $x_i(0) = x_0$ and the transversality condition, which sets the final value (at time T) of the state and/or co-state variables; in problems defined over an infinite time horizon, it is usual to set

$$\lim_{t \to \infty} \lambda_{ij}(t) \cdot x_j(t) = 0 \text{ for all } j . \tag{6}$$

The terms

$$\frac{\partial H_i(x_i(t); u_i(t))}{\partial u_j(t)} \frac{\partial u_j^*(t)}{\partial x_j(t)} \tag{7}$$

capture strategic interaction through the feedback from states to controls, which is by definition absent under the open-loop solution concept. In (5) and (7), $u_j^*(t)$ is the solution to the first order condition of firm j w.r.t. her control variable. Whenever the expression in (7) is zero for all j, then the closed-loop memoryless equilibrium collapses into the open-loop Nash equilibrium (see, e.g., Driskill and McCafferty, 1989, pp. 327-28). This can happen either because:

$$\frac{\partial H_i\,(x_i(t); u_i(t))}{\partial u_j\,(t)} = 0 \text{ for all } j\,; \tag{8}$$

which obtains if the Hamiltonian of player i is a function of his control variable but not of the rivals'; or because:

$$\frac{\partial u_j^*(t)}{\partial x_j\,(t)} = 0 \text{ for all } j\,; \tag{9}$$

which means that the first order condition of firm j with respect to her control variable does not contain the state x_j. Of course, it could also be that (8) and (9) hold simultaneously.

3 Two relevant examples

We consider two well known market models. In both models, the market exists over $t \in [0, \infty)$, and is served by N firms producing a homogeneous good. Let $q_i(t)$ define the quantity sold by firm i at time t. The marginal production cost is constant and equal to c for both firms. Firms compete à la Cournot, the demand function at time t being:

$$p(t) = A - BQ(t)\,;\; Q(t) \equiv \sum_{i=1}^{N} q_i(t)\,. \tag{10}$$

In order to produce, firms must accumulate capacity or physical capital $k_i(t)$ over time. The two models we consider in the present paper are characterised by two different kinematic equations for capital accumulation.

A] The Nerlove-Arrow (1962) setting, with the relevant dynamic equation being:

$$\frac{\partial k_i(t)}{\partial t} = I_i(t) - \delta k_i(t)\,; \tag{11}$$

where $I_i(t)$ is the investment carried out by firm i at time t, and δ is the constant depreciation rate. The instantaneous cost of investment is $C_i[I_i(t)] = b[I_i(t)]^2/2$, with $b > 0$. We also assume that firms operate with a decreasing returns technology $q_i(t) = f(k_i(t))$, with $f' \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f'' \equiv \partial^2 f(k_i(t))/\partial k_i(t)^2 < 0$. The demand function rewrites as:[4]

$$p(t) = A - B \sum_{i=1} f(k_i(t)). \tag{12}$$

Here, the control variable is the instantaneous investment $I_i(t)$, while the state variable is obviously $k_i(t)$.

B] The Ramsey (1928) setting, whit the following dynamic equation:

$$\frac{\partial k_i(t)}{\partial t} = f(k_i(t)) - q_i(t) - \delta k_i(t); \tag{13}$$

where $f(k_i(t)) = y_i(t)$ denotes the output produced by firm i at time t. As in setting [A], we assume $f' \equiv \partial f(k_i(t))/\partial k_i(t) > 0$ and $f'' \equiv \partial^2 f(k_i(t))/\partial k_i(t)^2 < 0$. In this case, capital accumulates as a result of intertemporal relocation of unsold output $y_i(t) - q_i(t)$.[5] This can be interpreted in two ways. The first consists in viewing this setup as a corn-corn model, where unsold output is reintroduced in the production process. The second consists in thinking of a two-sector economy where there exists an industry producing the capital input which can be traded against the final good at a price equal to one (for further discussion, see Cellini and Lambertini, 2000).

In this model, the control variable is $q_i(t)$, while the state variable remains $k_i(t)$. The demand function is (10).

[4] Notice that the assumption $q_i(t) = f(k_i(t))$ entails that firms always operate at full capacity. This, in turn, amounts to saying that this model encompasses the case of Bertrand behaviour under capacity constraints, as in Kreps and Scheinkman (1983), inter alia. The open-loop solution of the Nerlove-Arrow differential game in a duopoly model is in Fershtman and Muller (1984) and Reynolds (1987). The latter author also derives the feedback solution through Bellman's value function approach.

[5] In the Ramsey model, firms operate at full capacity in steady state, where any investment is just meant to make up for depreciation.

3.1 The Nerlove-Arrow model

When capital accumulates according to (11), the relevant Hamiltonian for firm i is:

$$H_i = e^{-\rho t}\left\{\left[A - Bf(k_i(t)) - B\sum_{i\neq j} f(k_j(t)) - c\right] f(k_i(t)) - \frac{b}{2}[I_i(t)]^2 + \right.$$
$$\left. + \lambda_{ii}(t)[I_i(t) - \delta k_i(t)] + \sum_{i\neq j}\lambda_{ij}(t)[I_j(t) - \delta k_j(t)]\right\} . \qquad (14)$$

Necessary conditions for the closed-loop memoryless equilibrium are:

$$\begin{array}{l}
\text{(i) } \dfrac{\partial H_i(t)}{\partial I_i(t)} = 0 \Rightarrow -bI_i(t) + \lambda_{ii}(t) = 0 \Rightarrow \lambda_{ii}(t) = bI_i(t) \\
\text{(ii) } -\dfrac{\partial H_i(t)}{\partial k_i(t)} - \dfrac{\partial H_i(t)}{\partial I_j(t)}\dfrac{\partial I_j^*(t)}{\partial k_i(t)} = \dfrac{\partial \lambda_{ii}(t)}{\partial t} - \rho\lambda_{ii}(t) \Rightarrow \\
\Rightarrow \dfrac{\partial \lambda_{ii}(t)}{\partial t} = (\rho + \delta)\lambda_{ii}(t) + f'(k_i(t))\left[2Bf(k_i(t)) + B\sum_{j\neq i} f(k_j(t)) - (A - c)\right] \\
\text{(iii) } -\dfrac{\partial H_i(t)}{\partial k_j(t)} - \dfrac{\partial H_i(t)}{\partial I_j(t)}\dfrac{\partial I_j^*(t)}{\partial k_j(t)} = \dfrac{\partial \lambda_{ij}(t)}{\partial t} - \rho\lambda_{ij}(t) ;
\end{array} \qquad (15)$$

with the transversality conditions:

$$\lim_{t\to\infty} \mu_{ij}(t)\cdot k_i(t) = 0 \text{ for all } i, j . \qquad (16)$$

Now observe that, on the basis of (15-i), we have:

$$\frac{\partial I_j^*(t)}{\partial k_i(t)} = 0 \text{ for all } i, j . \qquad (17)$$

Moreover, condition (15-iii), which yields $\partial\lambda_{ij}(t)/\partial t$, is redundant in that $\lambda_{ij}(t)$ does not appear in the first order conditions (15-i) and (15-ii). Therefore, the open-loop solution is indeed a degenerate closed-loop solution.[6]

Differentiating (15.i) w.r.t. time we obtain:

$$\frac{\partial I_i(t)}{\partial t} = \frac{1}{b}\cdot\frac{\partial \lambda_{ii}(t)}{\partial t} . \qquad (18)$$

[6] Note that, however, the open-loop solution does not coincide with the feedback solution, where each firm holds a larger capacity and sells more than in the open-loop equilibrium (see Reynolds, 1987).

Then, replace (15-i) into (15-ii), to get the following expression for the dynamics of the costate variable $\lambda_{ii}(t)$:

$$\frac{\partial \lambda_{ii}(t)}{\partial t} = b(\rho + \delta) I_i(t) + f'(k_i(t)) \left[2Bf(k_i(t)) + B \sum_{j \neq i} f(k_j(t)) - (A - c) \right]; \tag{19}$$

which can be plugged into (18), that rewrites as:

$$\frac{\partial I_i(t)}{\partial t} = (\rho + \delta) I_i(t) + \frac{f'(k_i(t))}{b} \left[2Bf(k_i(t)) + B \sum_{j \neq i} f(k_j(t)) - (A - c) \right]; \tag{20}$$

The discussion carried out so far establishes:

Proposition 1 Under the Nerlove-Arrow capital accumulation dynamics, the closed-loop memoryless equilibrium coincides with the open-loop equilibrium, which therefore is subgame perfect.

Invoking symmetry across firms and simplifying, we can rewrite (20):

$$\frac{\partial I(t)}{\partial t} = \frac{1}{b} \left\{ b(\rho + \delta) I(t) - f'(k(t)) \left[A - c - B(N+1) f(k(t)) \right] \right\}; \tag{21}$$

with the r.h.s.s being zero at:

$$I(t) = \frac{f'(k(t)) \left[A - c - B(N+1) f(k(t)) \right]}{b(\rho + \delta)}; \tag{22}$$

while $\partial k(t)/\partial t = 0$ at $k(t) = I(t)/\delta$. Of course, the explicit steady state solution requires a specific functional form for the production technology $f(k(t))$. The case where $f(k(t)) = k$ is treated in Fershtman and Muller (1984) and Calzolari and Lambertini (2000).

3.2 The Ramsey model

Under the dynamic constraint (13), the Hamiltonian of firm i is:

$$H_i = e^{-\rho t} \left\{ [A - Bq_i(t) - BQ_{-i}(t) - c] q_i(t) + \lambda_{ii}(t) [f(k_i(t)) - q_i(t) - \delta k_i(t)] + \sum_{j \neq i} \lambda_{ij}(t) [f(k_j(t)) - q_j(t) - \delta k_j(t)] \right\}; \tag{23}$$

where $Q_{-i}(t) = \sum_{j \neq i} q_j(t)$:

The first order condition concerning the control variable is:

$$\frac{\partial H_i(t)}{\partial q_i(t)} = A - 2Bq_i(t) - BQ_{-i}(t) - c - \lambda_{ii}(t) = 0 : \qquad (24)$$

Now examine at the co-state equation of firm i calculated for the state variable of firm i herself, for the closed-loop solution of the game:

$$-\frac{\partial H_i(t)}{\partial k_i(t)} - \frac{\partial H_i(t)}{\partial q_j(t)}\frac{\partial q_j^*(t)}{\partial k_i(t)} = \frac{\partial \lambda_{ii}(t)}{\partial t} - \rho\lambda_{ii}(t) \Rightarrow \qquad (25)$$
$$\frac{\partial \lambda_{ii}(t)}{\partial t} = \lambda_{ii}(t)\left[\rho + \delta - f(k_i(t))\right]$$

with

$$\frac{\partial q_j^*(t)}{\partial k_i(t)} = 0 \qquad (26)$$

as it emerges from the best reply function obtained from the analogous to (24):

$$q_j^*(t) = \frac{A - c - BQ_{-j}(t) - \lambda_{jj}(t)}{2B} ; \qquad (27)$$

Moreover, (27) also suffices to establish that the co-state equation:

$$-\frac{\partial H_i(t)}{\partial k_j(t)} - \frac{\partial H_i(t)}{\partial q_j(t)}\frac{\partial q_j^*(t)}{\partial k_j(t)} = \frac{\partial \lambda_{ij}(t)}{\partial t} - \rho\lambda_{ij}(t) \qquad (28)$$

is indeed redundant since $\mu_{ij}(t) = \lambda_{ij}(t)e^{-\rho t}$ does not appear in firm i's first order condition (24) on the control variable. This amounts to saying that, in the Ramsey game, the open-loop solution is a degenerate closed-loop solution because the best reply function of firm i does not contain the state variable pertaining to the same firm or any of her rivals. Therefore, we have proved the following analogous to Proposition 1:

Proposition 2 Under the Ramsey capital accumulation dynamics, the closed-loop memoryless equilibrium coincides with the open-loop equilibrium, which therefore is subgame perfect.

Then, using (13) and (27), it is possible to find the analytical steady state solutions:

$$q^{SS} = \frac{A - c}{B(N+1)} ; \quad f'(k) = \rho + \delta \qquad (29)$$

and to study their dynamic properties. This model can be easily extended to more general cases accounting for product di¤erentiation or non-linear market demand, under either Cournot or Bertrand competition (see Cellini and Lambertini, 1998, 2000).

The interest of the foregoing discussion lies in the possibility of using the open-loop information structure to generate subgame perfect equilibria. This feature makes it possible to easily extend the above models to investigate policy issues (Calzolari and Lambertini, 2000; Baldini and Lambertini, 2001), or the optimal behaviour of …rms in a wider strategy space, provided that such extensions do not compromise the properties outlined in Propositions 1 and 2.

4 Conclusions

It is well known that the explicit solution of di¤erential games with closed-loop or feedback information structures is, in general, cumbersome and often analytically impossible. For this reason, several contributions in the existing literature con…ne to the characterisation of open-loop equilibria, even if the limitations of the open-loop decision rules are well known.

In the foregoing analysis, we have shown that games of capital accumulation represent a …eld where the di¤erential game approach can be easily applied, as long as two dynamic rules of capital accumulation, widely adopted in the literature, exhibit the property that closed-loop no-memory solutions degenerate into open-loop solutions: speci…cally, the capital accumulation rule à la Nerlove-Arrow, and the capital accumulation rule à la Ramsey. This happens because the …rst order conditions of any given …rm does not contain the state variables pertaining to other …rms, even if strategic interdependence in the market phase is indeed present. Hence, these settings can be usefully applied along several directions using the open-loop approach, being the resulting equilibria subgame perfect. For instance, these two models can be used to characterise the e¤ects of pro…t taxation (Baldini and Lambertini, 2001), or tari¤s and quotas in a trade model (Calzolari and Lambertini, 2000). They can also be extended to investigate more complex situations where …rms invest in advertising or cost-reducing R&D.

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