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Locally- but not Globally-identified SVARs*

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Abstract

This paper analyzes Structural Vector Autoregressions (SVARs) where identification of structural parameters holds locally but not globally. In this case there exists a set of isolated structural parameter points that are observationally equivalent under the imposed restrictions. Although the data do not inform us which observationally equivalent point should be selected, the common frequentist practice is to obtain one as a maximum likelihood estimate and perform impulse response analysis accordingly. For Bayesians, the lack of global identification translates to non-vanishing sensitivity of the posterior to the prior, and the multi-modal likelihood gives rise to computational challenges as posterior sampling algorithms can fail to explore all the modes. This paper overcomes these challenges by proposing novel estimation and inference procedures. We characterize a class of identifying restrictions that deliver local but non-global identification, and the resulting number of observationally equivalent parameter values. We propose algorithms to exhaustively compute all admissible structural parameters given reduced-form parameters and utilize them to sample from the multi-modal posterior. In addition, viewing the set of observationally equivalent parameter points as the identified set, we develop Bayesian and frequentist procedures for inference on the corresponding set of impulse responses. An empirical example illustrates our proposal.

Keywords: local identification, Bayesian inference, Markov Chain Monte Carlo, robust Bayesian inference, frequentist inference, multi-modal posterior
JEL codes: C01,C13,C30,C51.

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Non-Technical Summary

Structural Vector Autoregressions (SVARs) represent a standard tool for macroeconomic policy analysis. Various types of identifying assumptions have been proposed, including equality and sign restrictions, and analytical investigation of whether they point- or set-identify the objects of interest is an active area of research. The seminal work of Rubio-Ramirez et al. (2010) shows a necessary and sufficient condition for zero restrictions to achieve global identification. This class of zero restrictions, however, does not exhaust the universe of zero and non-zero equality restrictions that are relevant in practice. Questions regarding identification, estimation, and inference when identification is not global remain largely open.

This paper analyzes Structural Vector Autoregressions where identification of structural parameters holds locally but not globally. In this case there exists a set of isolated structural parameter points that are observationally equivalent under the imposed restrictions. In this respect, the main issue is that the data do not inform us which observationally equivalent point should be selected. The common frequentist practice is to obtain one as a maximum likelihood estimate and perform impulse response analysis accordingly. All the other observationally equivalent parameter points are completely ignored. For Bayesians, instead, the lack of global identification translates to non-vanishing sensitivity of the posterior to the prior, and the multi-modal likelihood gives rise to computational challenges as posterior sampling algorithms can fail to explore all the modes. This paper overcomes these challenges by proposing novel estimation and inference procedures.

The first part of the paper is mainly dedicated to the identification issue. We characterize a class of identifying restrictions that deliver local but non-global identification, and the resulting number of observationally equivalent parameter values. Moreover, we provide a geometric interpretation of the local identification case and, through a simple example, we show to what extent this phenomenon can lead to misleading results. The second part of the paper, instead, is dedicated to the estimation and inference of locally- but not globally-identified SVARs. We propose algorithms to exhaustively compute all admissible structural parameters given reduced-form parameters and utilize them to sample from the multi-modal posterior. In addition, viewing the set of observationally equivalent parameter points as the identified set, we develop Bayesian and frequentist procedures for inference on the corresponding set of impulse responses. These approaches for conducting inference are general enough to be applied to other locally-identified econometric models, like proxy or non-Gaussian SVARs, or SVARs identified through heteroskedasticity. Finally, an empirical example illustrates our proposal.

I Introduction

Macroeconomic policy analysis makes extensive use of impulse response analysis based on Structural Vector Autoregressions (SVARs). Various types of identifying assumptions have been proposed, including equality and sign restrictions, and analytical investigation of whether they point- or set-identify the objects of interest is an active area of research. The seminal work of Rubio-Ramirez et al. (2010) (henceforth RWZ) shows a necessary and sufficient condition for zero restrictions to achieve global identification. This class of zero restrictions, however, does not exhaust the universe of zero and non-zero equality restrictions that are relevant in practice. Questions regarding identification, estimation, and inference when identification is not global remain largely open.

This paper focuses on a class of SVARs where the imposed identifying restrictions guarantee local identification but do not attain global identification. The set of observationally equivalent structural parameters then consists of multiple isolated points, which implies that the likelihood can have multiple peaks of the same height. Such locally- but non-globally identified SVARs appear in various settings of practical relevance. Examples include non-zero restrictions which set the structural parameters to calibrated values, non-recursive zero restrictions, equality restrictions across shocks and/or equations, and heteroskedastic SVARs with across-regime restrictions on the structural coefficients. Although the data do not inform us which observationally equivalent point should be selected, the common frequentist practice is to obtain one as a maximum likelihood estimator and perform impulse response analysis as if it were the only maximizer. This approach is problematic as different maximizers of the likelihood may imply very different impulse responses. In our view, this practice is prevalent due to the lack of an efficient algorithm that can uncover all the local maxima. Standard Bayesian analysis also faces challenges when the likelihood has multiple modes. First, the lack of global identification leads the posterior to remain sensitive to the choice of prior even asymptotically. Second, posterior sampling algorithms may fail to explore all the modes, resulting in an inaccurate approximation of the posterior.

This paper proposes methods for estimation and inference that overcome these challenges. We first characterize a class of equality and sign restrictions that delivers local but non-global identification. Second, we show a necessary and sufficient condition for local identification that can be easily checked under a general class of equality constraints imposed on the structural parameters or functions of them. Third, we investigate how many observationally equivalent parameter values exist under such identifying restrictions, and propose algorithms to exhaustively compute them given reduced-form parameter values. Specifically, we exploit the orthogonal matrix parametrization of Uhlig (2005) and Rubio-Ramirez et al. (2010) and pin down the observationally equivalent parameter points (conditional on the reduced-form parameters) by sequentially exhausting the admissible orthogonal vectors satisfying the imposed restrictions, or in some cases solving a system of polynomial equations. We provide an intuitive geometric exposition that illustrates the mechanism driving the lack of global identification and the number of observationally equivalent parameter values. As a byproduct, we also characterize the set of reduced-form parameter values that yield no admissible structural parameters (i.e, an empty identified set) despite the condition for local identification being met.

Our proposal for computing the identified set contributes to standard Bayesian inference by simplifying and stabilising sampling from the multi-modal posterior. The way we obtain a draw of the structural parameters or an impulse response from the posterior incorporates the following three steps. The first step is to obtain a draw of the reduced-form parameters either by directly sampling it or transforming a draw of the structural parameters into the reduced-form parameters. In the second step, given the draw of the reduced-form parameters, we compute all the observationally equivalent orthogonal matrices using our proposed algorithm. In the third step, we draw one of the observationally equivalent orthogonal matrices according to the probability weights implied by the prior distribution. Combining the draws of the reduced-form parameters and orthogonal matrix provides a draw of the structural parameters and impulse responses. Thus-constructed new draw stochastically moves across the modes supported by the prior. Hence, merging these extra steps into Gibbs or Metropolis-Hasting algorithm helps us explore all the posterior modes of the structural parameters.

Bayesian inference for non-identified parameters requires specifying a prior over the observationally equivalent parameter values and its influence to posterior remains even asymptotically. This phenomenon persists to the current case of only locally-identified structural parameters. To deal with the case where the user cannot form the prior or wants to draw prior-free frequentist inference, we propose projection-based frequentist inference procedures for the impulse responses. Viewing the set of observationally equivalent parameter points as the identified set (a set-valued map from the reduced-form parameters to the set of observationally equivalent structural parameters), we extend the approach of Norets and Tang (2014), Kline and Tamer (2016), and Giacomini and Kitagawa (2021), designed primarily for models with interval identified sets, to cases where the identified set consists of a finite number of points. Specifically, we consider projecting the posterior credible region for the reduced-form parameters to the impulse responses through the discrete identified set mapping. This approach obtains asymptotically frequentist valid confidence intervals in the presence of local identification. A complication unique to the current case of discrete identified set is how to label these observationally equivalent parameter points in a manner consistent over different values of the reduced-form parameters. We propose two different ways to do so. As shown in Giacomini and Kitagawa (2021), our frequentist procedure can be interpreted as (multiple prior) robust Bayesian procedures performing global sensitivity analysis with respect to a certain class of priors.

To illustrate our proposal, we apply the method to a locally identified New-Keynesian monetary policy SVAR. We show that, when a single element is selected from the identified set, the choice of the element can lead to significantly different and arguably contradictory results. We perform Bayesian inference with the prior that equally weights these observationally equivalent contradictory impulse responses, and show that the posterior distribution, approximated by our sampling procedure, well captures these contradictory impulse responses by its multimodality. Our proposals of asymptotically valid frequentist inference also explore all the admissible impulse responses and provide their summary by interval estimates.

I.1 Related literature

The theory of identification for linear simultaneous equation models has a long history in econometrics. See Dhrymes (1978), Fisher (1966) and Hausman (1983), among others. Rothenberg (1971) analyses identification in parametric models. Building on this, Giannini (1992) proposes a criterion for verifying local identification for SVAR models. This criterion takes the form of rank conditions for the Hessian matrix of the average likelihood. It is much weaker than the necessary and sufficient condition for global identification shown in Rubio-Ramirez et al. (2010). The focus of the present paper is the class of identifying restrictions that satisfies the former but not the latter. Once local identification is guaranteed, Giannini (1992) proposes estimating the parameters of the SVAR by numerically maximizing the likelihood. This approach is also recommended by the textbooks Amisano and Giannini (1997), Hamilton (1994), Lütkepohl (2006), and Kilian and Lütkepohl (2017). For the locally identified models considered in this paper, however, the maximum likelihood estimate is not necessarily unique, and a typical numerical maximization routine will select only one point in a non-systematic manner (e.g. depending on a choice of initial value). Sims and Zha (1999) and Hamilton et al. (2007) include discussions of the existence of multiple likelihood peaks due to local identification.

Following Uhlig (2005), Rubio-Ramirez et al. (2010), Arias et al. (2018) and Granziera et al. (2018), we parameterize an SVAR by its reduced-form VAR-parameters and the orthogonal matrix relating its reduced-form error covariance matrix and structural parameters. Fixing the reduced-form parameters, finding all the observationally equivalent structural parameters reduces to finding all the admissible orthogonal matrices that satisfy the imposed identifying restrictions. Compared to expressing the non-linear equation system by the reduced form and structural parameters, this formulation is advantageous in terms of geometric interpretability and analytical tractability. In addition, it simplifies not only assessing local identification (e.g., Magnus and Neudecker, 2007), but also obtaining all the solutions given the reduced-form parameters.

Our paper is related to the growing literature on SVARs that are set-identified through sign and zero restrictions (Faust 1998; Canova and de Nicoló 2002; Uhlig 2005; Mountford and Uhlig 2009, Arias et al. 2018, Gafarov et al. 2018, Giacomini and Kitagawa 2021, Granziera et al. 2018, among others). The identified set of impulse responses in this class of models is a set with a positive measure if nonempty, whereas the identified set here consists of a finite number of isolated points, each corresponding to a solution of a non-linear system of equations. This difference in the topological features of the identified set distinguishes our inferential procedure from these works.

Our proposals of asymptotically frequentist-valid inference build on Bayesian approach to inference on identified set as considered in Chen et al. (2018), Giacomini and Kitagawa (2021), Kline and Tamer (2016), Liao and Simoni (2019), Moon and Schorfheide (2012), and Norets and Tang (2014). To our knowledge, however, none of these proposals have been applied to the case where the identified set consists of isolated points. As discussed in Giacomini and Kitagawa (2021), our approach for drawing frequentist inference has a robust Bayes interpretation, where ambiguity within the identified set is introduced through a set of unrevisable priors. In this sense, it can be appealing to Bayesians who cannot form a credible prior for the structural parameters or want to perform global sensitivity analysis. Depending on the application, the class of priors considered in Giacomini and Kitagawa (2021) could be too large. In such cases, refining the set of priors would be sensible. This can be done, for instance, by applying the approaches considered in Giacomini et al. (2020) and Giacomini et al. (2019), although we do not present them in this paper.

The results and proposals of this paper, from identification to estimation and inference, can also contribute to the literature that bridges Dynamic Stochastic General Equilibrium (DSGE) and VAR models. The solution of a linearized DSGE model can be summarized by a state-space representation that implies, under appropriate invertibility conditions, an (infinite order) SVAR subject to specific identifying restrictions (see, Christiano et al. 2006, Fernandez-Villaverde et al. 2007, and Ravenna, 2007 for example). As stressed by Canova (2005, chapter 4) among others, popular identification schemes that lead to global identification, such as the Cholesky decomposition, cannot be justified in a large class of DSGE models. Hence, if the mapping between the DSGE and the SVAR is unique as in Christiano et al. (2006, Proposition 1), DSGE-based identifying restrictions can result in local (but not global) identification. This is due to the non-recursive nature of the identification scheme, and the possible multiplicity of solutions characterizing the DSGE model. See Iskrev (2010), Komunjer and Ng (2011) and Qu and Tkachenko (2012) for DSGE models, and Al-Sadoon and Zwiernik (2019) for local identification in linear rational expectation models. This paper is related to the DSGE literature for two reasons: firstly, DSGE models may imply non-recursive identifying restrictions in an SVAR, resulting thus in local identification of the SVAR, and, secondly, our estimation and inference methods to handle local identification can also be extended to locally identified (linearised) DSGE models.

Although identification analysis of this paper mainly focuses on Gaussian SVARs, the inferential proposals we propose can also contribute to the growing literature on identification using non-Gaussianity and/or heteroskedasticity. Concerning the former, Lanne et al. (2017), Lanne and Luoto (2019), Gourieroux et al. (2017) exploit higher moments and Independent Component Analysis (ICA) to point identify all or a subset of the structural shocks of SVARs. For the latter, Sentana and Fiorentini (2001), Rigobon (2003), Lanne and Lütkepohl (2008), Brunnermeier et al. (2021), Lewis (2021), among others, propose to use the heteroskedasticity in the data to reach point identification for factor models, simultaneous equations, and SVARs. Without a enough number of restrictions that can pin down the labeling of the structural shocks, the identification under these approaches are inherently local but not global due to the observationally equivalent representations through permutations of the structural shocks or structural equations. As pursued in Drautzburg and Wright (2021), adding sign restrications motivated via economic models to non-Gaussian SVARs help reduce the number of admissible impulse responses, while it is not known if the sign restrictions can guarantee global identification in non-Gaussian SVARs. Our inferential proposals can be implemented in these models if we can compute the set of admissible structural parameters given the reduced-form parameters (e.g., moments of the data).

The remainder of the paper is organized as follows. Section II introduces notation and a general analytical framework for SVARs whose identifying restrictions take the form of equality and sign restrictions. It also presents a new necessary and sufficient condition for local identification in SVARs. Section III discusses a battery of examples of locally- but not globally-identified SVARs. Section IV presents algorithms for computing observationally equivalent parameter values, and Section V proposes inference methods that accommodate frequentist, Bayesian, and robust Bayesian perspectives. Section VI presents an empirical example and Section VII concludes. Further results on local identification are reported in Appendices A and B, and the proofs omitted from the main text are presented in Appendix C.

II Econometric framework

Let y_t be a $n \times 1$ vector of variables observed over $t = 1 \ldots T$. The SVAR model is specified as

$$A_0 y_t = a + \sum_{j=1}^{p} A_j y_{t-j} + \varepsilon_t \tag{1}$$

where ε_t is a $n \times 1$ multivariate normal white noise process with null expected value and covariance matrix equal to the identity matrix I_n. The quantities $A_0,\, A_1, \ldots,\, A_p$ are $n \times n$ matrices of parameters, and a is a $n \times 1$ vector of constant terms. The set of structural parameters is denoted by $A = (A_0, A_+) \in \mathcal{A} \subset \mathbb{R}^{(n+m)n}$, with $m \equiv np + 1$ and $A_+ \equiv (a,\, A_1, \ldots,\, A_p)$ being a $n \times m$ matrix. We also assume that the initial conditions $y_1,\, \ldots,\, y_p$ are given.

The reduced-form representation of the SVAR, obtained by pre-multiplying by the inverse of A_0, is the standard VAR model

$$y_t = b + \sum_{j=1}^{p} B_j y_{t-j} + u_t \tag{2}$$

where $B_j = A_0^{-1} A_j$, $j = 1, \ldots, p$, $b = A_0^{-1} a$, $u_t = A_0^{-1} \varepsilon_t$ and $E(u_t\, u_t') \equiv \Sigma = A_0^{-1} A_0^{-1\prime}$. The set of reduced-form parameters is $\phi = (B,\, \Sigma) \in \Phi \subset \mathbb{R}^{n+n^2 p} \times \Omega$, where $B = (b,\, B_1, \ldots,\, B_p)$ and Ω is the space of positive semi-definite matrices.

Assuming further that the VAR Eq. (2) is invertible, it has the VMA(∞) representation:

$$y_t = c + \sum_{j=0}^{\infty} C_j(B) u_{t-j} = c + \sum_{j=0}^{\infty} C_j(B) A_0^{-1} \varepsilon_{t-j}$$

where $C_j(B)$ is the j-th coefficient matrix of the inverted lag polynomial $\left(I_n - \sum_{j=1}^{p} B_j L^j\right)^{-1}$. We define the impulse response matrix at horizon h (IR^h) and the long-run cumulative impulse response matrix (CIR^∞) to be

$$IR^h = C_h(B) A_0^{-1}, \tag{3}$$

$$CIR^\infty = \sum_{j=0}^{\infty} IR^h = \left(\sum_{j=0}^{\infty} C_j(B)\right) A_0^{-1}, \tag{4}$$

In what follows throughout, we denote the Cholesky decomposition of Σ by $\Sigma = \Sigma_{tr} \Sigma_{tr}'$, where Σ_{tr} is the unique lower-triangular Cholesky factor with non-negative diagonal elements. The column vectors of Σ_{tr}^{-1} and Σ_{tr}' are denoted by $\Sigma_{tr}^{-1} \equiv (\tilde{\sigma}_1,\, \tilde{\sigma}_2, \ldots,\, \tilde{\sigma}_n)$ and $\Sigma_{tr}' \equiv (\sigma_1,\, \sigma_2, \ldots,\, \sigma_n)$. The i-th entry of $\tilde{\sigma}_j$ and σ_j are denoted by $\tilde{\sigma}_{j,i}$ and $\sigma_{j,i}$, respectively.

II.1 Identification of SVAR models

Identification analysis of SVAR models concerns solving $\Sigma = A_0^{-1} A_0^{-1\prime}$ to decompose the reduced form error variance-covariance matrix Σ into the matrix of structural coefficients A_0. Following Uhlig (2005), any structural matrix A_0 defined by a rotation of the Cholesky factor $A_0 = Q' \Sigma_{tr}^{-1}$

admits the decomposition $\Sigma = A_0^{-1}A_0^{-1\prime}$ and, given the reduced-form parameters ϕ, the set of admissible A_0 matrices can be represented by $\mathcal{A}_0(\phi) \equiv \{A_0 = Q'\Sigma_{tr}^{-1} : Q \in \mathcal{O}(n)\}$, where $\mathcal{O}(n)$ is the set of $n \times n$ orthogonal matrices. Let R be generic notation denoting identifying restrictions. The identifying restrictions constrain the admissible values of A to a subset of $\mathcal{A}$. We denote this subset by $\mathcal{A}_R$, and its projection for A_0 by $\mathcal{A}_{R,0}$. Accordingly, let $\Phi_R \subset \Phi$ be the space of reduced-form parameters formed by projecting $A \in \mathcal{A}_R$, and let

$$\mathcal{A}_{R,0}(\phi) \equiv \mathcal{A}_0(\phi) \cap \mathcal{A}_{R,0}, \tag{5}$$

which is nonempty for $\phi \in \Phi_R$.

We define global and local identification for an SVAR as follows.

Definition 1 (Global identification)**.** An SVAR model is globally identified under identifying restrictions R if for almost every $A \in \mathcal{A}_R$ there is no other observationally equivalent A in $\mathcal{A}_R$.

Definition 2 (Local identification)**.** An SVAR model is locally identified under identifying restrictions R if for almost every $A \in \mathcal{A}_R$, there exists an open neighborhood G such that $G \cap \mathcal{A}_R$ contains no other observationally equivalent A.

Some remarks on these two notions of identification are in order. An equivalent definition of global identification would be that, for almost every $\phi \in \Phi_R$, there exists a unique corresponding structural parameter point. In other words, $\mathcal{A}_{R,0}(\phi)$ is singleton-valued at almost every $\phi \in \Phi_R$. In addition, the case where $\Phi_R = \Phi$, i.e. the imposed identifying assumptions are not observationally restrictive, is what RWZ refer to as *exact identification*. In contrast, the definition of local identification says that, if there are multiple observationally equivalent structural parameter points, they must be far apart. This implies that for almost every $\phi \in \Phi_R$, if $\mathcal{A}_{R,0}(\phi)$ is not singleton, it consists of isolated points. In Proposition 2 below, we characterize a class of locally identified SVARs. For this class of SVARs, the space of reduced-form parameters Φ can be partitioned into three subsets. The first, of positive measure, contains parameters for which the model is locally- but not globally-identified; the second, of positive measure, on which there is no structural parameter satisfying the identifying assumption (i.e., $\mathcal{A}_{R,0}(\phi)$ is empty); and the third, of measure zero, on which the model is globally identified. This feature of locally identified SVARs stands in contrast to exactly identified SVARs and globally and over-identified SVARs, where the mapping from the reduced-form parameter space Φ to structural parameters that satisfy the identifying restrictions is guaranteed to be either singleton-valued or empty at almost every $\phi \in \Phi$.

II.2 Normalization, sign, zero and non-zero identifying restrictions

This section introduces the types of identifying restriction considered in this paper. We begin with sign normalization restrictions, and then move to zero, non-zero, and sign restrictions.

Sign Normalization restrictions

Following Waggoner and Zha (2003) and Hamilton et al. (2007), and in line with RWZ and Giacomini and Kitagawa (2021), we impose sign normalization restrictions on the structural shocks. Specifically, we restrict the diagonal elements of A_0 to be non-negative.

$$\text{diag}\,(Q'\Sigma_{tr}^{-1}) \geq 0. \tag{6}$$

Under these assumptions, a unit positive change in a structural shock can be interpreted as a one standard-deviation *ceteris paribus* positive shock in the corresponding endogenous variable.

Zero and non-zero equality restrictions

While sign normalization restrictions on the diagonal elements of A_0 restrict the set of admissible structural matrices, they are not enough to obtain point identification. The standard approach in the literature is to impose equality restrictions either on the structural parameters or particular linear and non-linear functions of them.[1]

Following RWZ, we represent identifying restrictions as restrictions on the reduced-form parameters ϕ and the column vectors $(q_1, q_2, \ldots, q_n)$ of the orthogonal matrix Q.

[1]Other proposals of identification strategies include the use of external instruments as in Mertens and Ravn (2013) and Stock and Watson (2018), heteroskedasticity of the structural shocks as in Rigobon (2003), Bacchiocchi and Fanelli (2015) and Bacchiocchi (2017), and the presence of non-normality as in Lanne and Lütkepohl (2010) and Lanne et al. (2017).

$$\begin{aligned}
((i,j)\text{-th element of } A_0^{-1}) = c &\iff (e_i' \Sigma_{tr}) q_j = c, && (7)\\
((i,j)\text{-th element of } A_0) = c &\iff (\Sigma_{tr}^{-1} e_j)' q_i = c, && (8)\\
((i,j)\text{-th element of } A_l) = c &\iff (\Sigma_{tr}^{-1} B_l e_j)' q_i = c, && (9)\\
((i,j)\text{-th element of } CIR^{\infty}) = c &\iff \left[e_i' \sum_{h=0}^{\infty} C_h(B) \Sigma_{tr} \right] q_j = c, && (10)\\
\text{(linear restriction between } (i,j)\text{-th} & \\
\text{and } (h,k)\text{-th elements of } A_0^{-1}) &\iff (e_i' \Sigma_{tr}) q_j - d(e_h' \Sigma_{tr}) q_k = c, && (11)\\
\text{(linear restriction between } (i,j)\text{-th} & \\
\text{and } (h,k)\text{-th elements of } A_0) &\iff (\Sigma_{tr}^{-1} e_j)' q_i - d(\Sigma_{tr}^{-1} e_k)' q_h = c, && (12)
\end{aligned}$$

where e_i is the i-th column of the identity matrix I_n, and c and d are known non-zero scalars. Eq. (7) and Eq. (8) cover short-run identifying restrictions including the causal ordering restrictions of Sims (1980) and Bernanke (1986). Eq. (9) corresponds to restrictions that exclude some of the right-hand side variables in the structural equations. Eq. (10) corresponds to long-run identifying restriction as considered in Blanchard and Quah (1989). These first four equality restrictions, when $c = 0$, were considered in RWZ and Giacomini and Kitagawa (2021), but the remaining two were not. The additional restrictions we allow are non-zero equality restrictions (i.e., when $c \neq 0$ in Eq.s (7)-(10)), and cross-equation restrictions on the structural parameters and impulse responses. As we clarify in Section III, these last types of restriction drive a departure from global identification to local identification.[2]

We represent these equality restrictions by

$$\begin{aligned}
\mathbf{F}(\phi, Q) &\equiv \begin{pmatrix} F_{11}(\phi) & F_{12}(\phi) & \cdots & F_{1n}(\phi) \\ F_{21}(\phi) & F_{22}(\phi) & \cdots & F_{2n}(\phi) \\ \vdots & \vdots & \ddots & \vdots \\ F_{n1}(\phi) & F_{n2}(\phi) & \cdots & F_{nn}(\phi) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} - \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0} \\
&\equiv \mathbf{F}(\phi) \operatorname{vec} Q - \mathbf{c} = \mathbf{0} \qquad (13)
\end{aligned}$$

where $F_{ij}(\phi)$, $1 \leq i, j \leq n$, is a matrix of dimension $f_i \times n$, which depends only on the reduced-form parameters $\phi = (B,\ \Sigma)$. The dimension of $\mathbf{F}(\phi)$ is $f \times n^2$, where $f = f_1 + \cdots + f_n$ denotes

[2] Starting from Eq. (11) it is straightforward to extend the restrictions to CIR^{∞} or responses at any horizon. Similarly, starting from Eq. (12), we can also restrict the elements $A_1, \ldots, A_p$ across two or more equations. The general form to be given in Eq. (13) can accommodate these two extensions.

the total number of restrictions imposed. We allow $f_i = 0$ for some i, in which case the i-th block row in $\mathbf{F}(\phi)$ is null. Finally, $\text{vec}\, Q \equiv (q_1', \ldots q_n')'$ is the vectorization of Q, and $\mathbf{c} \equiv (c_1' \ldots, c_n')'$ is a vector of known constants with length f, where each c_i is a $f_i \times 1$ vector.

If $F_{ij}(\phi) = 0$ for all $i \neq j$, there are no cross equation restrictions or restrictions across the effects of the shocks. If $c_i = 0$ for all i, then only zero restrictions are imposed. This representation of the identifying restrictions is in line with Lütkepohl (2006) and Bacchiocchi and Lucchetti (2018), both of which allow non-homogeneous and across-shock restrictions. We provide the following formal definitions.

Definition 3 (Recursive restrictions)**.** The restrictions are said to be *recursive* if $F_{ij}(\phi) = 0$ for $j > i$, and $f_i = n - i$, for $i = 1, \ldots, n$.

Definition 4 (Homogeneous and non-homogeneous restrictions)**.** The restrictions are said to be homogeneous if $\mathbf{c} = \mathbf{0}$ and non-homogeneous if $\mathbf{c} \neq \mathbf{0}$.

Defined in this way, recursive restrictions pin down a unique ordering of the variables with Eq. (13) becoming a lower-triangular block matrix. Otherwise, our framework allows for the ordering of variables to be non-unique. Even with the order of variables fixed, if the restrictions include across-shock restrictions, then Eq. (13) allows for a multiple block-matrix representation. The general identification results of this section are valid independent of how the variables are ordered or how the imposed restrictions are represented within Eq. (13), unless the recursive structure is assumed explicitly.

Sign restrictions

In addition to equality restrictions, sign restrictions can be imposed on impulse responses or structural parameters. These sign restrictions can be seen as additional constraints on the columns of the Q matrix. Suppose we impose $s_{h,i} \leq n$ number of sign restrictions on the impulse responses to i-th shock at h-th horizon. They can be expressed as

$$S_{h,i}(\phi) q_i \ \geq\ \mathbf{0}, \tag{14}$$

where $S_{h,i} \equiv D_{h,i}\, C_h(B) \Sigma_{tr}$ is a $s_{h,i} \times n$ matrix, $D_{h,i}$ is the $s_{h,i} \times n$ signed selection matrix, which indicates by 1 (-1) the impulse responses whose signs are retricted to being positive (negative), and $C_h(B)$ is from the definition of an impulse response Eq. (3). The inequality in Eq. (14) is component-wise. Sign restrictions on structural parameters are linear inequality constraints on the columns of the matrix Q, so can also be accommodated. Stacking all the $S_{h,i}$ matrices involving sign restrictions on q_i at different horizons into a matrix S_i, we have

$$S_i(\phi)q_i \geq \mathbf{0}. \tag{15}$$

We represent the set of all sign restrictions by

$$\mathbf{S}(\phi, Q) \geq \mathbf{0}. \tag{16}$$

Admissible structural parameters and identified set

Given identifying restrictions of the form introduced above, we hereafter let R be the collection of restrictions $\{\mathbf{F}(\phi, Q) = 0, \mathbf{S}(\phi, Q) \geq 0, \text{diag}(Q'\Sigma_{tr}^{-1}) \geq 0\}$, or $R = (F, S)$ for short. We call $A = (A_0, A_+)$ *admissible* if it satisfies R. The set of all these admissible structural parameters can be represented by

$$\mathcal{A}_R(\phi) \equiv \{(A_0, A_+) = (Q'\Sigma_{tr}^{-1}, Q'\Sigma_{tr}^{-1}B) \,:\, Q \in \mathcal{O}(n), \mathbf{F}(\phi, Q) = \mathbf{0}, \mathbf{S}(\phi, Q) \geq \mathbf{0}, \ \text{diag}\,(Q'\Sigma_{tr}^{-1}) \geq 0\}.$$

The projection of $\mathcal{A}_R(\phi)$ for A_0 gives $\mathcal{A}_{R,0}(\phi)$ as defined in Eq. (5). The identified set for Q is defined as the set of admissible orthogonal matrices given the reduced-form parameters:

$$\mathcal{Q}_R(\phi) \equiv \{Q \in \mathcal{O}(n) \,:\, \mathbf{F}(\phi, Q) = \mathbf{0}, \ \mathbf{S}(\phi, Q) \geq \mathbf{0}, \ \ \text{diag}\,(Q'\Sigma_{tr}^{-1}) \geq 0\}.$$

The objects of interest may also include transformations of structural parameters such as impulse response functions. We denote a scalar parameter of interest by $\eta = \eta(\phi, Q)$ and define its identified set as

$$IS_\eta(\phi) \equiv \{\eta(\phi, Q) \,:\, Q \in \mathcal{Q}_R(\phi)\},$$

When $\eta(\phi, Q)$ is a restriction on an impulse response

$$\eta(\phi, Q) = IR_{ij}^h = e_i'C_h(B)\Sigma_{tr}Qe_j \equiv c_{ih}'(\phi)\, q_j,$$

where IR_{ij}^h is the (i,j)-th element of IR^h and $c_{ih}'(\phi)$ is the i-th row of $C_h(B)\Sigma_{tr}$.

When A is globally identified, $IS_\eta(\phi)$ is a singleton for almost every $\phi \in \Phi_R$. If A is only locally identified, $IS_\eta(\phi)$ can be a set of multiple isolated points generated by observationally equivalent structural parameters. Local identification can be certainly viewed as a special case of set identification, although it is not covered by standard set identification analysis where the identified set is typically an interval or a set with positive Lebesgue measure.

II.3 Conditions for local identification

This section presents conditions for global and local identification when the identifying restrictions are equality restrictions in the form Eq. (13). In the case of local identification, we present an analytical characterization of the number of observationally equivalent structural parameter values.

We begin with the well known condition for global identification developed in Theorem 7 of RWZ and recently extended in Bacchiocchi and Kitagawa (2021).[3] This condition for global identification acts as a reference point in our discussion of local identification.

Proposition 1 (Necessary and sufficient condition for global identification, RWZ and Bacchiocchi and Kitagawa (2021))**.** *Consider an SVAR with identifying restrictions of the form Eq. (7) - Eq. (12) collected in* $\boldsymbol{F}(\phi, Q)$. *Assume* $F_{ij}(\phi) = 0$ *for* $i \neq j$, *and* $\mathbf{c} = \mathbf{0}$. *The SVAR is globally identified at* $A = (A_0, A_+) \in \mathcal{A}_R$ *if and only if the following conditions hold at* ϕ *implied by A:*

1. *It holds*

$$\operatorname{rank}\big(F_{11}(\phi)', \tilde{\sigma}_1\big) = n. \tag{17}$$

2. *Let* q_1 *be a unit length vector satisfying* $F_{11}(\phi)\, q_1 = 0$ *and the sign normalization restriction, which is unique under Eq. (17). For* $i = 2, \ldots, n$

$$\operatorname{rank}\big(F_{ii}(\phi)', q_1, \ldots, q_{i-1}, \tilde{\sigma}_i\big) = n, \tag{18}$$

   hold, where the orthonormal vectors $q_2, \ldots, q_n$ *solve*

$$\big(F_{ii}(\phi)', q_1, \ldots, q_{i-1}\big)' q_i = \mathbf{0} \tag{19}$$

   sequentially, and satisfy the sign normalization restrictions.

This proposition characterizes a boundary separating cases where an SVAR is globally identified and cases where it is not guaranteed to be globally identified. In what follows, we consider departures from this proposition's conditions for global identification, and show implications for local identification and the failure of global identification. In particular, we allow $F_{ij}(\phi)$ to be nonzero for some $i \neq j$ and/or $\mathbf{c} \neq \mathbf{0}$ by including restrictions of the form Eq. (11)- Eq. (12). With this expanded set of identifying restrictions, Proposition 2 derives a rank condition that is

[3] Theorem 7 of Rubio-Ramirez et al. (2010) claims that under a set of regularity conditions, the exact identification of an SVAR holds if and only if $f_i = n - i$ for all $i = 1, \ldots, n$. Bacchiocchi and Kitagawa (2021) show that relaxing one of their regularity conditions, the condition of $f_i = n - i$ for all $i = 1, \ldots, n$ is no longer sufficient and it needs to be augmented by rank conditions, which, in the current setting, is equivalent to Eq. (17) and Eq. (18). See Bacchiocchi and Kitagawa (2021) for further detail.

necessary and sufficient for local identification. Lütkepohl (2006) and Bacchiocchi and Lucchetti (2018) provide similar conditions for local identification in a setting that is less general in terms of the kind of restrictions that can be imposed. Their rank condition is expressed in terms of the structural parameter matrices A, while our Proposition 2 presents the rank condition in terms of the coefficient matrix of the equality restrictions $\boldsymbol{F}(\phi)$ and the orthogonal matrix Q. We define $Chol(\cdot)$ to be the Cholesky factor of $(\cdot)$ and $g : \mathbb{R}^{(n+m)n} \to \mathbb{R}^{n+n^2p} \times \Omega \times \mathcal{O}(n)$, to be the function mapping structural to reduced-form parameters and the admissible orthogonal matrix.

Proposition 2 (Rank condition - necessary and sufficient condition for local identification)**.** *Consider an SVAR with equality restrictions of the form Eq. (7) - Eq. (12) collected in* $\boldsymbol{F}(\phi, Q)$. *Let* $\tilde{D}_n$ *be the* $n^2 \times n(n-1)/2$ *full-column rank matrix such that for any* $n(n-1)/2$*-dimensional vector* v, $\tilde{D}_n v \equiv vec\,(H)$ *holds, where* H *is an* $n \times n$ *skew-symmetric matrix satisfying* $H = -H'$ *(see Appendix D for the specific construction of* $\tilde{D}_n$ *for* $n = 2, 3, 4$*).*
(i) The SVAR is locally identified at $A = (A_0, A_+) \in \mathcal{A}_R$ *if and only if*

$$\text{rank}\left[\boldsymbol{F}(\phi)\big(I_n \otimes Q\big)\tilde{D}_n\right] = n(n-1)/2 \tag{20}$$

holds, where the reduced-form parameters $\phi = (B, \Sigma) \in \Phi$ *and the orthogonal matrix* $Q \in \mathcal{O}(n)$ *are such that* $(B, \Sigma, Q) = g(A_0, A_+) = (A_0^{-1}A_+, A_0^{-1}A_0^{-1\prime}, Chol(A_0^{-1}A_0^{-1\prime})'A_0')$. *Hence, a necessary condition for the rank condition Eq. (20) is* $f = \sum_{i=1}^{n} f_i \geq n(n-1)/2$.
(ii) Let $\mathcal{K}$ *be the set of structural parameters in* $\mathcal{A}_R$ *satisfying the rank condition of Eq. (20),*

$$\mathcal{K} \equiv \left\{A \in \mathcal{A}_R : \text{rank}\left[\boldsymbol{F}(\phi)\big(I_n \otimes Q\big)\tilde{D}_n\right] = n(n-1)/2\right\}.$$

Either $\mathcal{K}$ *is empty or the complement of* $\mathcal{K}$ *in* $\mathcal{A}_R$ *is of measure zero.*

Proof. See Appendix C. □

Statement (i) of this proposition provides a necessary and sufficient condition for local identification at a given $A \in \mathcal{A}_R$ in the form of a rank condition for a matrix that is a function of A, i.e., (ϕ, Q) is a function of A. Eq. (20) as stated is of limited practical use since the true A is generally unknown, which means that verifying Eq. (20) is infeasible.

Statement (ii) of this proposition makes the rank condition Eq. (20) useful by showing that it holds either nowhere or almost everywhere in the parameter space $\mathcal{A}_R$. This means that, similar to the proposals following Theorem 3 in RWZ and Theorem 1 in Bacchiocchi and Lucchetti (2018), one can assess local identification by randomly generating structural parameters $A \in \mathcal{A}_R$ and checking whether the rank condition holds or not. Specifically, we can consider drawing reduced-form parameters $\phi \in \Phi_R$ from its prior or posterior and solving a constrained non-linear optimization problem of the form[4]

$$\begin{aligned} &\arg \min_{Q \in \mathbb{R}^{n^2}} \left(\mathbf{F}(\phi)\mathrm{vec}\, Q - \mathbf{c}\right)' \left(\mathbf{F}(\phi)\mathrm{vec}\, Q - \mathbf{c}\right) \\ &\text{s.t.} \quad \mathrm{diag}\ (Q' \Sigma_{tr}^{-1}) \geq 0, \ \ \mathbf{S}(\phi, Q) \geq \mathbf{0} \text{ and } Q'Q = I_n. \end{aligned} \tag{21}$$

If the value of the optimization is zero, then the obtained Q is an admissible orthogonal matrix at the given ϕ. If such an admissible Q satisfies the rank condition in Eq. (20), then the SVAR is locally identified at (ϕ, Q). If the rank condition is not met, the SVAR is not locally identified at (ϕ, Q). Proposition 2 (ii) says that only one of the two possibilities occurs with positive measure, while the other has zero measure. Hence, by checking the rank condition at a few parameter values drawn from a probability distribution supporting $\mathcal{A}_R$ or Φ_R, we can learn whether the rank condition holds nowhere or almost everywhere on the space of structural parameters. Confirming the latter can be seen as a strong support for local identification holding at the true A, unless the true structural parameter value is believed to belong to the null set in the parameter space.

In many empirical applications the interest is only on one single shock, or at most on a small subset of them. The condition in Proposition 2 could be, thus, too stringent. The next result allows to check whether a subset of shocks is locally identified. First of all, without loss of generality, let the shock be ordered such that $f_1 \leq f_2 \leq \ldots \leq f_n$. Conditional on this ordering, let the shocks be partitioned into two groups such that: a) the shocks of interest belong to the first group of s shocks, and b) the shocks in the former group do not present restrictions with those of the latter. These features allow to write the restrictions as

$$\begin{pmatrix} \mathbf{F}_{11}(\phi) & 0 \\ 0 & \mathbf{F}_{22}(\phi) \end{pmatrix} \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix} - \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} = \mathbf{0} \tag{22}$$

where the orthogonal matrix Q has been partitioned as $Q = \left[\, Q_1 \mid Q_2 \,\right]$, with $\mathbf{q}_1 = \mathrm{vec}\ Q_1$ and $\mathbf{q}_2 = \mathrm{vec}\ Q_2$.

Corollary 1 (Rank condition - necessary and sufficient condition for local identification of a subset of shocks)**.** *Consider an SVAR with equality restrictions of the form Eq. (7) - Eq. (12) collected as in Eq. (22), where the shocks are ordered such that* $f_1 \leq f_2 \leq \ldots \leq f_n$*. A necessary and sufficient condition for the first* s *shocks to be locally identified at the parameter*

[4]This minimization problem is constrained by the orthogonality constraints $Q'Q = I_n$, which is known as Stiefel manifold following Stiefel (1935-1936). Edelman et al. (1998) develop algorithms for optimization in the Stiefel manifold, while Boumal et al. (2014) propose a Matlab toolbox for optimization on manifolds including the Stiefel one. A Matlab code for this optimization is available from the authors upon request.

point $A = (A_0, A_+) \in \mathcal{A}_R$ *is that*

$$\text{rank}\begin{pmatrix} \boldsymbol{F}_{11}(\phi) \\ N_{ns}\left(I_n \otimes Q_1'\right) \end{pmatrix} = ns \tag{23}$$

where the completely known matrix $N_{ns} \equiv 1/2(I_{ns} + K_{ns})$*, with* k_{ns} *being the commutation matrix.*[5]

Proof. See Appendix C. □

The implementation of the rank condition in Corollary 1 can be easily performed by a slight modification of the procedure used for the general condition in Proposition 2. We first derive an admissible matrix Q_1 by solving a constrained non-linear optimization problem as in Eq. (21), where Q_1 substitutes the entire Q. The second step, thus, consists in checking the rank condition in Eq. (23) using the obtained Q_1 matrix.

Allowing only recursive identifying restrictions, the next proposition provides a simple necessary and sufficient condition for the rank condition of Proposition 2 (i). It extends, to local identification, the condition for global identification presented in Proposition 1.

Proposition 3 (Necessary and sufficient condition for local identification in recursive SVARs)**.** *Consider an SVAR with recursive identifying restrictions of the form Eq. (13). Let* $\tilde{F}_{ii}(\phi) = F_{11}(\phi)$ *for* $i = 1$*, and*

$$\tilde{F}_{ii}(\phi) = \left(F_{ii}'(\phi), q_1, \ldots, q_{i-1}\right)' \tag{24}$$

for $i = 2, \ldots, n$*, where* $q_1, \ldots, q_i$ *are the first* i *column vectors of* $Q \in \mathcal{O}(n)$ *satisfying the equality restrictions* $\boldsymbol{F}(\phi)vec\,Q - \mathbf{c} = \mathbf{0}$ *given* $\phi \in \Phi_R$*. The rank condition of Eq. (20) holds at* (ϕ, Q) *if and only if* $\text{rank}\big(\tilde{F}_{ii}(\phi)\big) = n - 1$ *holds for all* $i = 1, \ldots, n$*.*

Proof. See Appendix C. □

Since the rank condition of Proposition 2 (i) is necessary and sufficient for local identification, the condition shown in Proposition 3 is also necessary and sufficient for local identification for SVARs under recursive identifying restrictions. Moreover, the claim of Proposition 2 (ii) carries over to the setting of Proposition 3, so knowing that the condition shown in Proposition 3 holds at a few $\phi \in \Phi_R$ drawn from its prior or posterior allows us to conclude local identification holds almost everywhere in the parameter space. The condition in Proposition 3 exploits sequential determination of q_i, $i = 1, \ldots, n$, given ϕ, so checking it does not require nonlinear optimization for Q.

[5] A commutation matrix K is defined such that, given a generic matrix A, then $K\,\text{vec}\,A = \text{vec}\,A'$. See Magnus and Neudecker (2007) for some properties of the commutation matrix.

The proof of Proposition 3 leads to the following corollary showing a necessary and sufficient condition for the local identification of impulse responses to a particular shock.

Corollary 2 (Sufficient condition for local identification of the j-th shock)**.** *Under the assumptions of Proposition 3, the impulse responses for the j-th structural shock, $1 \leq j \leq n$, are locally identified at the parameter point $A = (A_0, A_+) \in \mathcal{A}_R$ if and only if* $\text{rank}(\tilde{F}_{ii}(\phi)) = n - 1$ *holds for all $i = 1, \ldots, j$.*

II.4 The number of observationally equivalent parameter points

The results presented so far are silent about how many observationally equivalent structural parameter points there are. As the next proposition shows, our constructive identification argument through the orthogonal matrix Q allows us to characterize the number of observationally equivalent parameter points.

Proposition 4 (Number of locally identified points)**.** *Consider an SVAR with equality restrictions of the form Eq. (7)-Eq. (12) collected in $\boldsymbol{F}(\phi, Q) = \mathbf{0}$. Given $\phi \in \Phi$ and provided that the rank condition in Eq. (20) is met, the number of admissible Q matrices (Q matrices solving $\boldsymbol{F}(\phi, Q) = \mathbf{0}$) is zero or finite. In particular, if the equality identifying restrictions are recursive, the number of admissible Q matrices is at most 2^n. If the equality identifying restrictions are non-recursive, the number of admissible Q matrices is at most $2^{n(n+1)/2}$.*

Proof. See Appendix C. □

The proposition provides an upper bound for the number of locally identified observationally equivalent parameter points. It corresponds to the maximal number of modes that the likelihood of the structural parameters can have. The maximum number of observationally equivalent structural parameters is considerably lower when the SVAR is identified through recursive equality restrictions rather than non-recursive restrictions. The intuition for this result is that, if the identification of the columns of Q can be performed recursively, the equations concerning the orthogonality conditions among the columns of Q are linear, rather than quadratic.

In comparison to the exact (global) identification case of RWZ and Proposition 1, Proposition 4 highlights that non-homogenous restrictions ($\mathbf{c} \neq \mathbf{0}$) lead to the possibility that, given $\phi \in \Phi$, (i) an admissible Q does not exist, or (ii) the admissible Q is no longer unique. Adding sign restrictions to the sign normalization restrictions can reduce the number of admissible Q's, but cannot generally guarantee uniqueness of the admissible Q's. Section II.5 below illustrates the transition from exact global identification to local identification through a simple example.

The following corollary, instead, focuses on the identification of a subset of all the structural shocks and derives the maximum number of solutions for this specific case. We consider, thus, the same situation as in Corollary 1 and assume that the rank condition therein is met.

Corollary 3 (Number of locally identified points for a subset of shocks)**.** *Consider an SVAR with equality restrictions of the form Eq. (7)-Eq. (12) collected as in Eq. (22). Given $\phi \in \Phi$ and provided that the rank condition in Eq. (23) is met, the number of admissible Q_1 matrices is zero or finite. In particular, the number of admissible Q_1 matrices is at most $2^{s(s+1)/2}$.*

Proof. The result is a by-product of Proposition 4 and can be easily derived from its proof. □

II.5 The geometry of identification

We present an intuitive geometric exposition for why the introduction of nonhomogeneous restrictions Eq. (11) and/or across-shock restrictions Eq. (12) can lead to local identification. This exposition also provides intuition for the number of local identified parameter points shown in Proposition 4. Appendix A provides the algebraic analysis behind our geometric discussion.

To make exposition as simple as possible, consider a bivariate VAR with a single non-homogeneous identifying restriction imposed on the structural parameters:

$$(A_0)^{-1}_{[1,1]} = c \iff (e_1' \Sigma_{tr})\, q_1 = c \tag{25}$$

where $c > 0$ is a known (positive) scalar and e_1 is the first column of I_2. Denoting the first column of $\Sigma_{tr}' = \begin{pmatrix} \sigma_{1,1} & \sigma_{2,1} \\ 0 & \sigma_{2,2} \end{pmatrix}$ by $\sigma_1 = (\sigma_{1,1}, 0)'$, this identifying restriction can be written as $\sigma_1'\, q_1 = c$.

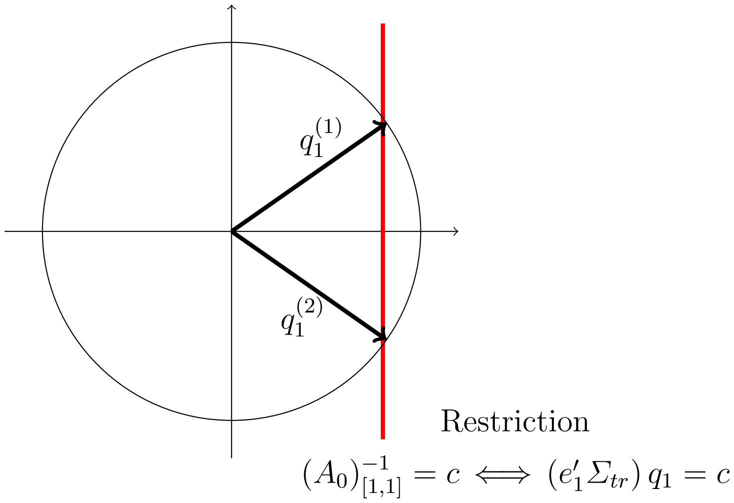
Hence, given ϕ, q_1 must satisfy the two equations,

$$\begin{cases} \sigma_1'\, q_1 & = & c \\ q_1'\, q_1 & = & 1. \end{cases}$$

Figure 1 depicts these two constraints. Letting the x-axis correspond to the vector σ_1, the set of q_1 vectors satisfying the first constraint is a vertical line whose location is determined by σ_1 and c. The second constraint imposes that q_1 lies on the unit circle. Points at the intersection of the vertical line and the unit circle, if any exist, are solutions to this system of equations.

When the imposed restriction is a zero restriction ($c = 0$), the vertical line passes through the origin and intersects the circle at two points. The two solutions for q_1, $q_1^{(1)}$ and $q_1^{(2)}$ are symmetric across the origin, and the sign normalization restriction Eq. (6) is guaranteed to rule one of them out (see Appendix A for details). Thus, the first column of Q is globally identified.

Figure 1: Identification of q_1 in the bivariate SVAR with non-zero restriction.



Notes: The vertical red line represents the non-zero restriction $(A_0)^{-1}_{[1,1]} = c$. The two black arrows represent the identified vectors $q_1^{(1)}$ and $q_1^{(2)}$.

The vertical line in Figure 1 corresponds to a non-zero restriction ($c > 0$). If the vertical line is perfectly tangent to the unit circle, we continue to have global identification. Otherwise, there are two distinct solutions for q_1, as shown in Figure 1. Compared to the case where $c = 0$, a crucial difference is that there are some values of ϕ and c where the sign normalization restriction cannot rule out one solution. In this case, they are both admissible and the first column of Q is locally- but not globally-identified.[6]

The second column of Q, i.e. the unit-length vector q_2, can be pinned down through its orthogonality with q_1

$$\begin{cases} q_2' q_1 &= 0 \\ q_2' q_2 &= 1. \end{cases} \tag{26}$$

If q_1 is only locally identified with two admissible vectors $q_1^{(1)}$ and $q_1^{(2)}$, Eq. (26) needs to be solved given both. Solving the system when $q_1 = q_1^{(1)}$ provides two solutions for q_2 that are depicted in the left panel of Figure 2. As the two solutions mirror each other across the origin, only one will satisfy the sign normalization restriction for the second shock. A similar picture is obtained when $q_1 = q_1^{(2)}$ (the right panel of Figure 2), and here too one of the solutions for q_2 can be ruled out by the sign normalization restriction.

[6]For $\phi \notin \Phi_F$, the vertical line does not intersect the unit circle, and no real solution for q_1 exists. If $c \neq 0$, the identifying restriction becomes observationally restrictive, and the identifying restriction can be refuted by the reduced-form models.

To summarize, an equality restriction with $c > 0$ leads to local but non-global identification for q_1, and there are then two admissible Q matrices, $Q_1 = [q_1^{(1)}, q_2^{(1)}]$ and $Q_2 = [q_1^{(2)}, q_2^{(2)}]$ given ϕ. This implies that both $A_0 = Q_1' \Sigma_{tr}^{-1}$ and $A_0 = Q_2' \Sigma_{tr}^{-1}$ are admissible. In this example, we obtain two observationally equivalent Q matrices, which is consistent with the upper bound on the number of observationally equivalent Q matrices in Proposition 4.

For a specific numerical illustration, let the bivariate VAR be characterized by constants that are zero and a single lag with reduced-form parameters

$$B_1 = \begin{pmatrix} 0.8 & -0.2 \\ 0.1 & 0.6 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 0.49 & -0.14 \\ -0.14 & 0.13 \end{pmatrix}, \qquad \Sigma_{tr} = \begin{pmatrix} 0.7 & 0 \\ -0.2 & 0.3 \end{pmatrix},$$

and consider imposing restriction $(A_0)^{-1}_{[1,1]} = 0.5 \iff (e_1' \Sigma_{tr})\, q_1 = 0.5$. Following Eq. (55) and Eq. (60) - Eq. (61) in Appendix A, we calculate the two admissible matrices Q_1 and Q_2

$$Q_1 = \begin{pmatrix} 0.714 & -0.700 \\ 0.700 & 0.714 \end{pmatrix} \qquad \text{and} \qquad Q_2 = \begin{pmatrix} 0.714 & 0.700 \\ -0.700 & 0.714 \end{pmatrix}$$

with associated admissible A_0 matrices

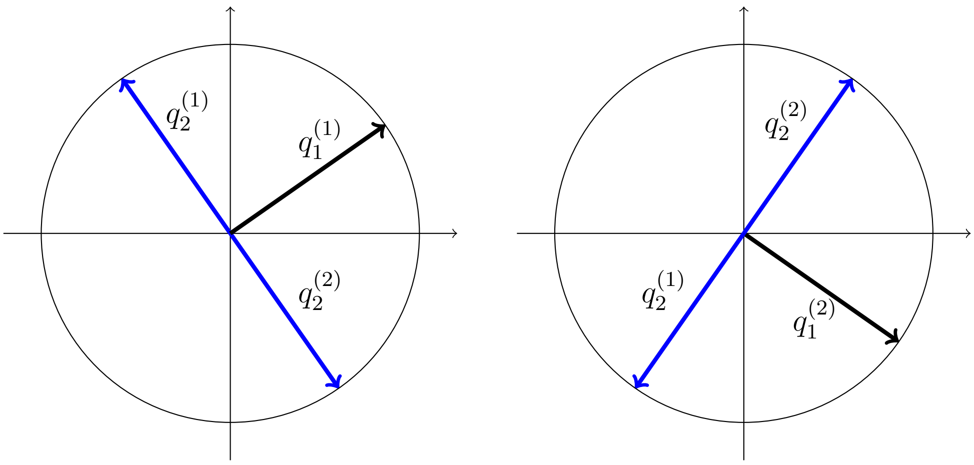
$$A_0^{(1)} = \begin{pmatrix} 1.687 & 2.333 \\ -0.320 & 2.381 \end{pmatrix} \qquad \text{and} \qquad A_0^{(2)} = \begin{pmatrix} 0.354 & -2.333 \\ 1.680 & 2.381 \end{pmatrix}.$$

Based on these structural parameter values, Figure 3 shows the impulse response of $y_t = (y_{1t}, y_{2t})'$ to the structural shocks ε_{1t} and ε_{2t}. Despite the simplicity of this example, it clearly illustrates the extent to which conclusions depend on the choice of observationally equivalent Q matrices.

III Locally identified SVARs: some examples

Hamilton et al. (2007) discuss local identification as a normalization problem. As shown in the previous section, in the presence of non-homogeneous equality restrictions Eq. (11) and/or across-shock restrictions Eq. (12), proper sign normalization restrictions are not enough to resolve the issue of local identification in SVARs. The examples below illustrate that this issue is of practical relevance.

Figure 2: Identification of q_2 in the bivariate SVAR with non-zero restriction.



Case 1: $q_1^{(1)}$ Case 2: $q_1^{(2)}$

Notes: The left panel shows the identification of the $q_2^{(1)}$ and $q_2^{(2)}$ vectors (in blue), conditional on the identified $q_1^{(1)}$ (in black). Similarly, the right panel shows the identification of the $q_2^{(1)}$ and $q_2^{(2)}$ vectors (in blue), conditional on the identified $q_1^{(2)}$ (in black).

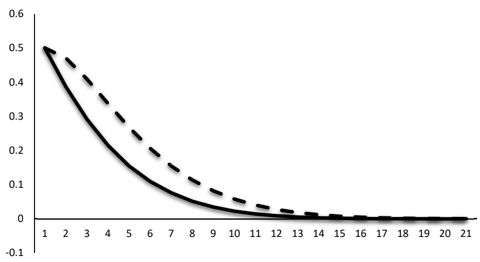
III.1 Calibrated identifying restrictions and restrictions across shocks or across equations

One strategy employed in the literature is to calibrate some parameters instead of estimating them. Calibration can be viewed as imposing non-homogeneous restrictions and, as we have shown in the simple example in Section II.5, this can lead to local identification. For example, Blanchard and Perotti (2002) and Blanchard and Watson (1986) impose nonzero values for some structural parameters in A_0, based on external information. Abraham and Haltiwanger (1995), Davis and Kilian (2011) and Kilian (2010), instead of imposing fixed values, explore a grid of possible values for some structural parameters in order to provide robustness checks for their main model specification.

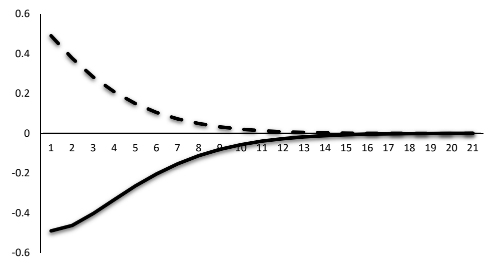
Cross-equation restrictions have been investigated in the classical literature of simultaneous equation systems (Fisher, 1966, and Kelly, 1975).[7] Among others, examples in this direction

[7] In particular, Kelly (1975) presents cases in which economic theory might suggest imposing such restrictions. However, constraining parameters across equations is conditional on the kind of normalization considered. In simultaneous equation systems, normalization rules were generally based on imposing a unit coefficient for the variable playing the role of endogenous variable in that specific equation. In the parametrization proposed by RWZ for SVAR models the normalization rule instead consists of imposing unit variance on the uncorrelated structural shock. In this case, imposing restrictions on elasticities across equations would involve non-linear restrictions on the estimated coefficients. In fact, to obtain the elasticities, we need to normalize the coefficient

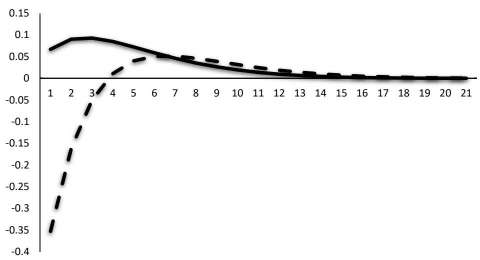
Figure 3: Impulse response functions related to the locally identified SVAR discussed in Section II.5.



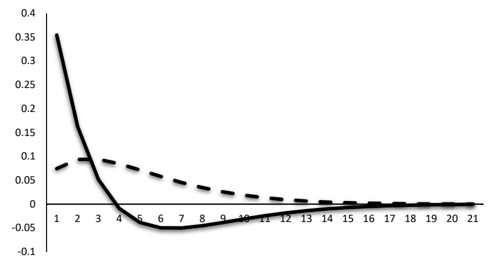
(a) response of y_{1t} to ε_{1t}



(b) response of y_{1t} to ε_{2t}



(c) response of y_{2t} to ε_{1t}



(d) response of y_{2t} to ε_{2t}

Notes: The bivariate SVAR is characterized by a non-zero restriction. In solid lines we report the IRFs obtained through $Q_1 = (q_1^{(1)}, q_2^{(1)})$ while in dashed lines those obtained through $Q_2 = (q_1^{(2)}, q_2^{(2)})$.

can be found in Deaton and Muellbauer (1980) and related papers on demand systems, where cross-equation restrictions are imposed in order to test for the Slutsky symmetry assumptions. See Kilian (2013) for a simple example on restrictions across equations in a bivariate SVAR. Similar situations arise in SVARs when restrictions are imposed on impulse responses to different structural shocks. In both approaches, such kinds of constraints involve restrictions across the columns of the orthogonal matrix Q, as those reported in Eqs. (11)-(12). As for calibrated parameters, this identification strategy can lead to local identification.

for the *endogenous* variable in each equation. See Hamilton et al. (2007) and Waggoner and Zha (2003) for specific details on the normalization issue in SVAR models.

III.2 Non-recursive SVAR models

RWZ provide an example of a locally- but not globally-identified SVAR where the sufficient condition for local identification of Proposition 3 is not met. This example involves non-recursive causal ordering restrictions and has practical importance, as we illustrate below.

Cochrane (2006) considers the following New-Keynesian model for inflation π_t, output gap x_t, and the nominal interest rate i_t:

$$\begin{aligned} \pi_t &= \beta E_t\, \pi_{t+1} \;+\; \kappa x_t \;+\; u_t^s \\ x_t &= E_t\, x_{t+1} \;-\; \tau(i_t - E_t\, \pi_{t+1}) \;+\; u_t^d \\ i_t &= \phi_\pi\, \pi_t \;+\; u_t^{mp} \end{aligned} \tag{27}$$

with u_t^s, t_t^d and u_t^{mp} being, respectively, the independent supply, demand, and monetary policy shocks with variances σ_s^2, σ_d^2 and σ_{mp}^2. Fukac et al. (2007) show that this model can be written as an SVAR of the form

$$A_0\, y_t \;=\; \varepsilon_t$$

where $y_t = (\pi_t,\, x_t,\, i_t)'$ is the vector of observable variables, $\varepsilon_t = (\varepsilon_t^s,\, \varepsilon_t^d,\, \varepsilon_t^{mp})$ collects the unit-variance uncorrelated structural shocks and

$$A_0 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}.$$

Note that there is a well-defined mapping between the parameters in A_0 and those in the DSGE representation Eq. (27).[8]

This model includes the following restrictions:

$$\begin{aligned} a_{13} = 0 &\iff (\Sigma_{tr}^{-1} e_3)' q_1 = 0 \\ a_{21} = 0 &\iff (\Sigma_{tr}^{-1} e_1)' q_2 = 0 \\ a_{32} = 0 &\iff (\Sigma_{tr}^{-1} e_2)' q_3 = 0, \end{aligned} \tag{28}$$

with $f_1 = f_2 = f_3 = 1$. The sufficient condition for local identification in Proposition 3 is clearly not satisfied, but it can be shown that the rank condition of Proposition 2 is satisfied. Specifically, Rubio-Ramirez et al. (2008) show the existence of two orthogonal matrices Q_1 and Q_2 transforming the reduced-form parameters into admissible structural parameters. The em-

[8] As pointed out by Canova (2005), zero restrictions implied by DSGE models do not match the recursive identification schemes common in SVAR analyses.

pirical application shown in Section VI performs estimation and inference for this model. Other examples of non-recursive SVARs, among others, are Sims (1986), Bernanke (1986), Blanchard and Watson (1986) and Sims and Zha (2006).

III.3 SVAR with breaks

Bacchiocchi and Fanelli (2015) consider SVARs with breaks in the structural error variances and regime-dependent structural coefficients. They consider identifying assumptions that restrict some structural parameters to being invariant across the regimes.

Suppose that the two regimes are characterized by two different reduced-form error covariance matrices Σ_1 and Σ_2, which are related to the regime-dependent structural parameters through

$$\Sigma_1 = A_{01}^{-1} A_{01}^{-1\prime} \qquad \text{and} \qquad \Sigma_2 = A_{02}^{-1} A_{02}^{-1\prime}, \tag{29}$$

where A_{01} and A_{02} are the matrices of regime-specific structural parameters. Let Q_1 and Q_2 be the regime specific orthogonal matrices mapping the reduced-form error variances to the structural coefficients,

$$A_{01} = Q_1' \, \Sigma_{1,tr}^{-1} \qquad \text{and} \qquad A_{02} = Q_2' \, \Sigma_{2,tr}^{-1} \tag{30}$$

with $Q_i = [q_{1(i)}, \ldots, q_{n(i)}]$, $\forall \, i, \in \{1, 2\}$. We denote the j-th column vector of $\Sigma_{i,tr}'$ by $\sigma_{j(i)}$ for $j = 1, 2$ and $i = 1, 2$.

For simplicity, consider a bivariate SVAR with two regimes. Impose the following identifying restrictions:

$$\begin{array}{lcl} (A_{01})^{-1}_{[1,2]} = 0 & \iff & (e_1' \, \Sigma_{1,tr}) \; q_{2(1)} = 0 \\ (A_{01})^{-1}_{[2,1]} = (A_{02})^{-1}_{[2,1]} & \iff & (e_2' \, \Sigma_{1,tr}) \; q_{1(1)} = (e_2' \, \Sigma_{2,tr}) \; q_{1(2)}. \end{array} \tag{31}$$

The first zero restriction combined with the sign normalization pins down the orthogonal matrix Q_1 in the first regime. The second restriction in Eq. (31) gives rise to the following system of equations:

$$\begin{cases} \sigma_{1(2)}' \, q_{1(2)} & = & c \\ q_{1(2)}' q_{1(2)} & = & 1 \end{cases} \tag{32}$$

where $c = \sigma_{2(1)}' q_{1(1)}$, which is a known constant once the Q_1 for the first regime is identified. Hence, the problem of identification for structural parameters in the second regime is reduced to the example discussed in Section II.5, in which local identification holds with two distinct solutions.

III.4 Heteroskedastic SVAR

Rigobon (2003) proposes a specification of systems of simultaneous equations where the structural shocks present different volatility regimes. Lanne and Lütkepohl (2008) extends this approach to SVAR models. In synthesis, exploiting the presence of two or more volatility regimes makes the SVAR model already identified without the need for any kind of restrictions. However, Rigobon (2003) recognized that the identification of the model is obtained up to a permutation of the equations or structural shocks. Bacchiocchi et al. (2022), among others, formalize this statement by transforming the identification issue into an eigenvalue problem.

Suppose the data show, at time T_B, a break in the covariance matrix, say Σ_1 and Σ_2. This literature assumes the two different covariance matrices to be characterized by different variances in the structural shock, i.e. the identity matrix in the first regime and a diagonal matrix Λ, with strictly positive elements, in the second regime. This assumption leads to

$$E(u_t u_t') = \begin{cases} \Sigma_1 = A_0^{-1} A_0^{-1\prime} & \text{if} \quad 1 \leq t \leq T_B \\ \Sigma_2 = A_0^{-1} \Lambda A_0^{-1\prime} & \text{if} \quad T_B < t \leq T. \end{cases} \tag{33}$$

Let $\Sigma_{1,tr}$ be the lower triangular Cholesky decomposition of Σ_1 and, as before, let $Q \in \mathcal{O}(n)$. Bacchiocchi et al. (2022) show that the identification issue reduces to

$$\begin{aligned} &A_0^{-1} = \Sigma_{1,tr} Q \\ &\Sigma_{1,tr}^{-1} \Sigma_2 \Sigma_{1,tr}^{-1\prime} = Q \Lambda Q' \end{aligned} \tag{34}$$

that is an eigen-decomposition problem, where Λ collects the eigenvalues and the columns of Q are the related eigenvectors. Now, let P be a permutation matrix. It becomes easy to see that $\tilde{Q} = QP$ and $\tilde{\Lambda} = P \Lambda P'$ are admissible solutions, too, but with different ordering of the columns of Q and the elements in Λ. Although the heteroskedastic SVAR doesn't require any kind of restrictions, it can be only locally-identified. As we will see, the strategies for doing estimation and inference in locally-identified SVARs that we propose in the next sections can be extremely useful for heteroskedastic SVARs too, that are receiving increasing attention in the recent years.[9]

III.5 Proxy-SVAR

A set of identifying restrictions similar to the non-recursive zero restrictions discussed above can appear when the identification strategy exploits proxy variables for the structural shocks.

Consider again a three-variable SVAR. Instead of imposing zero restrictions directly on any

[9] See, among many others, the recent contributions by Sims (2020), Brunnermeier et al. (2021), Lewis (2021,2022) and all the references in Kilian and Lütkepohl (2017, Chapter 14).

element of A_0, we consider observable variables that proxy some of the underlying structural shocks. The idea of using proxy variables to identify the structural impulse responses has been considered in Stock and Watson (2012) and Mertens and Ravn (2013), amongst others. We restrict our analysis to SVARs and focus on identification of the full system of SVARs rather than subset identification of the impulse responses.[10] To be specific, consider introducing the external variables $m_t = (m_{1t}, m_{2t}, m_{3t})'$, each of which acts as a proxy for some contemporaneous structural shocks. Following Angelini and Fanelli (2019), Arias et al. (2021), and Giacomini et al. (2022), we augment m_t into the original SVAR,

$$\begin{pmatrix} A_0 & O \\ \mathit{\Gamma}_1 & \mathit{\Gamma}_2 \end{pmatrix} \begin{pmatrix} y_t \\ m_t \end{pmatrix} = \begin{pmatrix} \epsilon_t \\ \nu_t \end{pmatrix}, \quad (\epsilon_t, \nu_t)' \sim \mathcal{N}(0, I_{6\times 6}), \tag{35}$$

where O is 3×3 matrix of zeros, $\mathit{\Gamma}_1$ and $\mathit{\Gamma}_2$ are 3×3 coefficient matrices in the augmented equations, and the shocks ν_t in the second block component of the augmented system are interpreted as measurement errors in the proxy variables. Inverting Eq. (35) leads to

$$m_t = -\mathit{\Gamma}_2^{-1}\mathit{\Gamma}_1 A_0^{-1}\epsilon_t + \mathit{\Gamma}_2^{-1}\nu_t \tag{36}$$

In the Proxy-SVAR approach, the identifying restrictions are zero restrictions on the covariance matrix of m_t and ϵ_t. Consider imposing the following restrictions:

$$E(m_t\epsilon_t') = \begin{pmatrix} 0 & \rho_{12} & \rho_{13} \\ \rho_{21} & 0 & \rho_{23} \\ \rho_{31} & \rho_{32} & 0 \end{pmatrix} \tag{37}$$

where ρ_{ij} is the (unconstrained) covariance of m_{it} and ϵ_{jt}. The zero-covariance restrictions represented in (37) imply that variable m_{it}, $i = 1, 2, 3$, proxies a combination of the structural shocks *excluding* ϵ_{it}. Combining eq. (36) with eq. (37) and substituting $A_0^{-1} = \mathit{\Sigma}_{tr}Q$, $Q = [q_1, q_2, q_3]$, the exogeneity restrictions of eq. (37) can be expressed as

$$\begin{aligned} (e_1'\mathit{\Gamma}_2^{-1}\mathit{\Gamma}_1\mathit{\Sigma}_{tr})q_1 &= 0, \\ (e_2'\mathit{\Gamma}_2^{-1}\mathit{\Gamma}_1\mathit{\Sigma}_{tr})q_2 &= 0, \\ (e_3'\mathit{\Gamma}_2^{-1}\mathit{\Gamma}_1\mathit{\Sigma}_{tr})q_3 &= 0. \end{aligned} \tag{38}$$

Since $\mathit{\Gamma}_2^{-1}\mathit{\Gamma}_1$ can be identified by the covariance matrix of the reduced-form VAR errors in the

[10]The proxy-variable identification strategy has been shown to be useful for non-invertible structural MA models. See Stock and Watson (2018) and Plagborg-Møller and Wolf (2022).

augmented system Eq. (35), the zero restrictions of Eq. (38) have the same form as Eq. (28). Hence, Proxy-SVAR identification under the exogeneity restrictions Eq. (37) delivers local but non-global identification of the A_0 matrix.

IV Computing identified sets of locally identified SVARs

A common approach to estimating SVAR structural parameters is constrained maximum likelihood (Amisano and Giannini, 1997), with the maximization performed numerically given some initial values. The standard gradient-based algorithm stops once it reaches a local maximum, and does not check for the existence of other observationally equivalent parameter values. Hence, the conventional maximum likelihood procedure applied to an SVAR that is locally but not globally identified will select one of the observationally equivalent structural parameters in a nonsystematic way, limiting the credibility of the resulting estimates and inference.

This section proposes computational methods that produce estimates of all the observationally equivalent A matrices given the identifying restrictions. Our approach is first to obtain $\hat{\phi} = (\hat{B}, \hat{\Sigma})$, an estimate of the reduced-form parameters ϕ, and then compute the identified set for A_0 given $\hat{\phi}$, $\mathcal{A}_0(\hat{\phi}|F, S)$, by solving a system of equations for the Q matrix given $\hat{\phi}$. For estimators of ϕ, we consider (i) the unconstrained reduced-form VAR estimator for ϕ denoted by $\hat{\phi}_u$ and (ii) the estimator for ϕ induced by a constrained maximum likelihood estimate of A under the identifying restrictions (i.e., one of the locally identified structural parameter points maximizing the likelihood), denoted by $\hat{\phi}_r$. In the Bayesian inference methods considered in Section V, we view $\hat{\phi}$ as a draw from the posterior of ϕ.

In what follows, we propose two procedures to compute $\mathcal{A}_0(\hat{\phi}|F, S)$. The first procedure is general and invokes a non-linear solver. The second procedure is more constructive and involves only elementary calculus, but the allowed type of identifying restrictions is more limited. Both algorithms deal with just identified SVARs, and we presume the rank condition in Proposition 2 or the sufficient condition in Proposition 3 are ensured or have been checked empirically prior to implementation.

IV.1 A general computation procedure for locally identified SVARs

Given $\hat{\phi}$, this method computes the orthogonal matrices subject to the identifying restrictions by solving a non-linear system of equations.[11] If the model is locally identified, then it yields at most $2^{n(n+1)/2}$ solutions for Q. Some of these will be discarded by normalization and sign restrictions.

[11] Kociecki and Kolasa (2018) similarly check global identification of DSGE models by examining the solutions of a non-linear system of equations.

The remaining solutions for Q are then used to span the identified set for A_0, $\mathcal{A}_0(\hat{\phi}|F,S)$, and its projection leads to the identified set of an impulse response $IS_\eta(\hat{\phi})$. All these steps are stated formally in the next algorithm.

Algorithm 1. *Consider a SVAR with equality restrictions Eq. (13) and sign restrictions Eq. (16), and assume $f = n(n-1)/2$ equality restrictions are imposed. Let $\hat{\phi}$ be a given estimator for ϕ such as $\hat{\phi}_u$ or $\hat{\phi}_r$.*

1. *Solve the system of equations for Q:*

$$\begin{cases} \boldsymbol{F}(\hat{\phi})vec\,Q - \mathbf{c} &= \boldsymbol{0} \\ Q'\,Q &= I_n; \end{cases} \tag{39}$$

2. *If the set of real solutions for Q is non-empty (which is guaranteed if $\hat{\phi} = \hat{\phi}_r$), then retain only those satisfying the normalization and sign restrictions to obtain $\mathcal{Q}_R(\hat{\phi})$. $\mathcal{A}_0(\hat{\phi}|F,S)$ is constructed accordingly by $\{A_0 = Q'\hat{\Sigma}_{tr}^{-1} : Q \in \mathcal{Q}_R(\hat{\phi})\}$.*
3. *When $\hat{\phi} = \hat{\phi}_u$, it is possible that no real solution for Q exists in Step 1. If so, we return $\mathcal{Q}(\hat{\phi}|F,S) = \emptyset$, i.e., $\hat{\phi}$ is not compatible with the imposed identifying restrictions.*

The crucial step in this algorithm is obtaining all the solutions to the equation system (39). This is a system of polynomial equations consisting of linear and quadratic equations.[12] Closed-form solutions do not seem available, but numerical algorithms to compute all the roots of the polynomial equations are. Matlab, for example, has the function `vpasolve`, an algorithm to find all the solutions of a system of non-linear equations.[13] According to the Matlab documentation,[14] `vpasolve` returns the complete set of solutions in the case of polynomial equations. The strength of this algorithm is its generality, but it is a black-box function.[15]

When non-homogeneous restrictions or cross-shock restrictions are imposed, the model becomes observationally restrictive. Hence, when $\hat{\phi}$ is obtained from the unconstrained reduced-form VAR estimator $\hat{\phi}_u$, if $\hat{\phi}_u$ happens to be outside of Φ_R, then Step 3 of Algorithm 1 becomes relevant. When the algorithm returns $\mathcal{Q}_R(\hat{\phi}) = \emptyset$, the maximum likelihood reduced-form

[12]Sturmfels (2002) provides a good overview of systems of polynomial equations with potential applications in statistics and economics. As we saw in Section II.3, this system can be also seen as a minimization problem of the quadratic objective function subject to the orthogonality constraints $Q'\,Q = I_n$. Noting that the orthogonality constraints generate the Stiefel manifold, we can consider applying algorithms for optimization on the Stiefel manifold. See Edelman et al. (1998) and Boumal et al. (2014).

[13]An alternative approach could be to solve the system analytically through the Matlab fuction `solve`, and then approximate the roots numerically using the function `vpa`. For all cases investigated in our empirical analyses, the two strategies lead to the same set of results.

[14]https://uk.mathworks.com/help/symbolic/vpasolve.html

[15]Matlab solvers are not open source, and we fail to uncover the precise numerical algorithm `vpasolve` uses to find roots of nonlinear equation systems.

model suggests that some of the imposed identifying restrictions are misspecified. One can hence consider relaxing some of the imposed sign restrictions, or modify the value of $\mathbf{c} \neq 0$ if non-homogeneous restrictions are present. Alternatively, if we want to maintain the imposed restrictions, we can employ the constrained reduced-form estimate $\hat{\phi} = \hat{\phi}_r$ instead, so that $\mathcal{Q}_R(\hat{\phi})$ is guaranteed to be nonempty.

IV.2 Computational procedure for locally identified SVARs with recursive non-homogeneous restrictions

If the identifying restrictions imposed allow the sequential determination of the column vectors of Q as exploited in the identification arguments in the previous sections, we can modify Algorithm 1. In this section, we consider recursive SVARs with non-homogeneous and cross-shock restrictions, as covered in Proposition 3.

Let $Q_{1:i}$, $1 \leq i \leq n$, be a $n \times i$ matrix whose column vectors are orthonormal (i.e. it consists of the first i column vectors of Q). Given ϕ, define $\tilde{F}_{11}(\phi) = F_{11}(\phi)$ and the following matrices sequentially for $i = 2, \ldots, n$,

$$\tilde{F}_{ii}(\phi) = \begin{pmatrix} F_{ii}(\phi) \\ Q_{1:(i-1)}(\phi)' \end{pmatrix}, \tag{40}$$

where $Q_{1:(i-1)}(\phi)$ satisfies the identifying restrictions for the first $(i-1)$ orthogonal vectors, i.e., $(F_{j1}(\phi), \ldots F_{jj}(\phi))\mathrm{vec}Q_{1:(i-1)}(\phi) = c_j$ holds for $j = 1, \ldots, (i-1)$. For $i = 1, \ldots, n$, we define a $(n-1) \times 1$ vector,

$$\tilde{c}_i(\phi) = \begin{pmatrix} c_i - \big(F_{i1}(\phi), \ldots, F_{i(i-1)}(\phi)\big)\,\mathrm{vec}Q_{1:(i-1)}(\phi) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{41}$$

Then, for $i = 1, \ldots, n$, define

$$d_i(\phi) = \tilde{F}_{ii}(\phi)'\Big(\tilde{F}_{ii}(\phi)\tilde{F}_{ii}(\phi)'\Big)^{-1}\tilde{c}_i(\phi), \tag{42}$$

$$B_i(\phi) = \Big(I_n - \tilde{F}_{ii}(\phi)'\Big(\tilde{F}_{ii}(\phi)\tilde{F}_{ii}(\phi)'\Big)^{-1}\tilde{F}_{ii}(\phi)\Big), \tag{43}$$

and let $\alpha_i(\phi)$ be a $n \times 1$ basis vector of the linear space spanned by the vectors in $B_i(\phi)$. Note that $B_i(\phi)$ is the $n \times n$ matrix projecting onto the linear space orthogonal to the row vectors of

$\tilde{F}_{ii}(\phi)$. Hence, given the rank of $\tilde{F}_{ii}(\phi)$ is $n-1$, $B_i(\phi)$ has a rank of 1, so $\alpha_i(\phi)$ is unique up to sign, and $\tilde{F}_{ii}(\phi)\alpha_i(\phi) = 0$ holds.

Consider the $n \times 1$ vector, $x = d_i(\phi) + z\alpha_i(\phi)$, $z \in \mathbb{R}$. Due to the way $d_i(\phi)$ and $\alpha_i(\phi)$ are constructed, $\tilde{F}_{ii}(\phi)x = \tilde{c}_i$ holds. That is, by choosing z so that x is a unit-length vector, we can obtain q_i vectors satisfying $\tilde{F}_{ii}(\phi)q_i = \tilde{c}_i$. Solving for x is simple as it requires only finding the roots of a quadratic equation (see Eq. (44) and Eq. (45) in Algorithm 2 below). Given ϕ, we repeat this process for every $i = 1, \ldots, n$ to determine the q_i vectors sequentially, and compute all the Q matrices satisfying the equality restrictions $\mathbf{F}(\phi, Q) = \mathbf{0}$. $\mathcal{A}_0(\phi|F, S)$ and $IS_\eta(\phi)$ can then be obtained by retaining the Q that satisfy the normalization and sign restrictions. We summarize this computational procedure in the next algorithm.

Algorithm 2. *Consider a SVAR satisfying the normalization restrictions Eq. (6), the equality restrictions Eq. (13), and the sign restrictions Eq. (16), where the imposed equality restrictions satisfy the sufficient condition for local identification given in Proposition 3. Let $\hat{\phi}$ be a given estimator for ϕ such as $\hat{\phi}_u$ or $\hat{\phi}_r$. In the description of the algorithm below, we omit the argument $\hat{\phi}$ as far as it does not give rise confusion.*

Let $\boldsymbol{b} = (b_1, \ldots, b_n) \in \{0,1\}^n$ be a bit vector which will be used to index each of the at most 2^n possible solutions for the Q matrices. Beginning with $\boldsymbol{B} = \{0,1\}^n$, we will map each $\boldsymbol{b} \in \boldsymbol{B}$ to a possible solution of Q, check if it is feasible or not, and refine $\boldsymbol{B}$ accordingly. The following algorithm describes this process in detail:

1. *Solve for $z \in \mathbb{R}$ in*
$$d_1'd_1 + 2d_1'\alpha_1 z + \alpha_1'\alpha_1 z^2 = 1, \tag{44}$$
and denote the two solutions by $z_1^{b_1}$, $b_1 \in \{0,1\}$.
   - (a) *If they are real, then define $q_1^{b_1} = d_1 + \alpha_1 z_1^{b_1}$, $b_1 \in \{0,1\}$. Let $\boldsymbol{B}_1 \subset \{0,1\}$ be the set of b_1 such that $q_1^{b_1}$ satisfies the sign normalization and sign restrictions for q_1. If $\boldsymbol{B}_1$ is empty (i.e., no $q_1^{b_1}$ satisfies the sign normalization and sign restrictions for q_1), then stop and conclude $\mathcal{Q}_R(\hat{\phi}) = \emptyset$*
   - (b) *If the roots of Eq. (44) are not real, then stop and return $\mathcal{Q}_R(\hat{\phi}) = \emptyset$.*
2. *This step iterates sequentially for $i = 2, \ldots, n$, given $\boldsymbol{B}_{i-1} \subset \{0,1\}^{i-1}$.*
   - (a) *For each $(b_1, \ldots, b_{i-1}) \in \boldsymbol{B}_{i-1}$, construct $\boldsymbol{B}_i(b_1 b_2 \cdots b_{i-1}) \subset \{0,1\}^i$ by performing the following subroutines:*
      - i. *Construct $\tilde{F}_{ii}$ from Eq. (40) by setting $Q_{1:i-1} = [q_1^{b_1}, q_2^{b_1 b_2}, \ldots, q_{i-1}^{b_1 \cdots b_{i-1}}]$, and obtain*

d_i and α_i accordingly. Then,solve for $z \in \mathbb{R}$ in

$$d_i' d_i + 2 d_i' \alpha_i z + \alpha_i' \alpha_i z^2 = 1, \tag{45}$$

and denote the two solutions by $z_i^{b_1 b_2 \cdots b_i}$, $b_i \in \{0,1\}$.

ii. *If they are real, define $q_1^{b_1 b_2 \cdots b_i} = d_i + \alpha_i z_1^{b_1 b_2 \cdots b_i}$, $b_i \in \{0,1\}$. Let $\boldsymbol{B}_i(b_1 b_2 \cdots b_i)$ be the set of $(b_1, b_2, \ldots, b_i) \in \{0,1\}^i$ such that $q_i^{b_1 b_2, \cdots b_i}$ satisfies the sign normalization and sign restrictions for the i-th column vector of Q. This can be empty if no $q_i^{b_1 b_2, \cdots b_i}$ satisfies them.*

iii. *If the roots of Eq. (45) are not real, return $\boldsymbol{B}_i(b_1 b_2 \cdots b_{i-1}) = \emptyset$.*

(b) *Construct $\boldsymbol{B}_i = \bigcup_{(b_1, \ldots, b_{i-1}) \in \boldsymbol{B}_{i-1}} \boldsymbol{B}_i(b_1 \cdots b_{i-1})$. If $\boldsymbol{B}_i \neq \emptyset$, go back to the beginning of Step 2.*

(c) *If $\boldsymbol{B}_i = \emptyset$, then stop and return $\mathcal{Q}_R(\hat{\phi}) = \emptyset$.*

3. *We obtain*

$$\mathcal{Q}_R(\hat{\phi}) = \left\{ \left(q_1^{b_1}, q_2^{b_1 b_2}, \ldots, q_n^{b_1 b_2 \cdots b_n} \right) : \boldsymbol{b} \in \boldsymbol{B}_n \right\}.$$

Algorithm 2 computes the set of all admissible $Q \in \mathcal{Q}_R(\hat{\phi})$. In the description of the algorithm, they are indexed by the bit vectors $\mathbf{b} \in \mathbf{B}_n$. The algorithm is constructive and guaranteed to compute all the admissible Q matrices. Projecting this set of admissible matrices onto the impulse response of interest, we obtain a plug-in estimate of the identified set $IS_\eta(\hat{\phi})$.

Algorithm 2 is more constructive than Algorithm 1, but it restricts the set of equality restrictions to be recursive. Algorithm 2 can be extended to a class of models involving non-recursive identifying restrictions (e.g., examples in Sections III.2 and III.5) by incorporating steps that solve a certain system of quadratic equations. Such an algorithm is rather involved to present, so we do not include it in this paper. Algortihm 1 can be certainly applied to a general class of models with nonrecursive identifying restrictions.

V Inference for locally identified SVARs

V.1 Bayesian inference

Standard Bayesian inference specifies a prior distribution for either the structural parameters A (e.g., Baumeister and Hamilton, 2015), or the reduced-form parameters and rotation matrix (ϕ, Q) as a reparametrization of A (e.g., Uhlig, 2005). When identification is local, the likelihood for the joint parameter vector A can have multiple modes, which means that the posterior for the structural parameters and impulse responses may also have multiple modes. This leads to computational challenges as commonly used Markov Chain Monte Carlo (MCMC) methods can fail to adequately explore the posterior when it is multi-modal. For instance, in the standard Metropolis-Hastings algorithm, the presence of multiple modes complicates the choice of proposal distribution. If the proposal distribution in the Metropolis-Hastings algorithm does not support some modes well, a lack of irreducibility of the Markov chain can lead it to fail to converge to the posterior. Similarly, in the standard Gibbs sampler, the presence of multiple modes in the posterior for A leads its support to be almost disconnected, which can then lead a break down of irreducibility and the Gibbs sampler to fail to converge (see Example 10.7 in Robert and Casella, 2004). By combining our constructive algorithms (either Algorithm 1 or Algorithm 2) for computing $IS_\eta(\phi)$ with the posterior sampling algorithm for ϕ, we can overcome such computational challenges.

We consider approximating the posterior for a scalar impulse response $\eta(\phi)$. Assume that the reduced-form parameters yield nonempty $\mathcal{Q}_R(\phi)$. Let $IS_\eta(\phi)$ consist of $M(\phi) \geq 1$ distinct points,

$$IS_\eta(\phi) = \left\{\eta_1(\phi), \eta_2(\phi), \ldots, \eta_{M(\phi)}(\phi)\right\}, \tag{46}$$

where we index the observationally equivalent impulse responses to satisfy $\eta_1(\phi) < \eta_2(\phi) < \cdots < \eta_{M(\phi)}(\phi)$.

We follow the "agnostic" Bayesian approach of Uhlig (2005). The posterior for η is induced by the posterior for ϕ, $\pi_{\phi|Y}$, which is supported on $\Phi_R \equiv \{\phi : \mathcal{Q}_R(\phi) \neq \emptyset\}$, and Q has a uniform prior supported only on the admissible set of rotation matrices $\mathcal{Q}_R(\phi)$ given $\phi \in \Phi_R$. Local identification with the $M(\phi)$-point identified set as in (46) can be obtained by projecting the $M(\phi)$ admissible rotation matrices into the space of impulse responses if each of them leads to distinct values of impulse response. Hence, the uniform weights assigned over these rotation matrices imply that equal weights are assigned to the points in $IS_\eta(\phi)$. As a result, for $G \subset \mathbb{R}$, the posterior for η can be expressed as

$$\pi_{\eta|Y}(\eta \in G) \propto E_{\phi|Y}\left[\sum_{m=1}^{M(\phi)} 1\left\{\eta_m(\phi) \in G\right\}\right]. \tag{47}$$

Since the reduced-form VAR likelihood is unimodal and concentrated around the maximum likelihood estimate, MCMC algorithms will perform well when sampling from $\pi_{\phi|Y}$. Hence, the posterior (47) can be approximated by combining a posterior sampler for ϕ with the algorithm for computing $\left\{\eta_m(\phi) : m = 1, \ldots, M(\phi)\right\}$.

V.2 Frequentist-valid inference

Bayesian inference as considered above can be sensitive to the choice of prior even in large samples due to the lack of global identification. The standard Bayesian procedure (assuming a unique prior for the structural parameters) specifies an allocation of the prior belief over observationally equivalent impulse responses, $IS_\eta(\phi)$, conditional on ϕ. This conditional belief given ϕ is not updated by the data and, as a result, the shape and heights of the posterior around the modes remain sensitive to its specification. In this section, we propose an asymptotically valid frequentist inference procedure for the impulse response identified set that can draw inferential statements which are robust to the choice of prior weights over the set of locally identified parameter values.

Our approach is to project asymptotically valid frequentist confidence sets for the reduced-form parameters ϕ through the identified set mapping $IS_\eta(\phi)$. In standard set-identified models where the identified set is a connected interval with positive width, the projection approach to constructing the confidence set has appeared in the 2011 working paper version of Moon and Schorfheide (2012), Norets and Tang (2014), Kline and Tamer (2016), among others. This approach generally yields asymptotically valid (but conservative) confidence sets even when the identified set consists of discrete points. However, a challenge unique to the discrete identified set case is the computation of projection confidence sets for the impulse responses based on a finite number of grid points or draws of ϕ from their confidence set. In what follows, we propose methods to tackle this computational challenge.

Let $CS_{\phi,\alpha}$ be an asymptotically valid confidence set for ϕ with coverage probability $\alpha \in (0,1)$. If the maximum likelihood estimator $\hat{\phi}$ is $\sqrt{T}$-asymptotically normal, the likelihood contour set $CS_{\phi,\alpha}$ is determined by the α-th quantiles of the χ^2 distribution with the degree of freedom $dim(B)+n(n-1)/2$. If the posterior for ϕ satisfies the Bernstein-von Mises property, that is the posterior for $\sqrt{T}(\phi-\hat{\phi})$ asymptotically coincides with the sampling distribution of the maximum likelihood estimator, the Bayesian highest density posterior region with credibility α can be used for $CS_{\phi,\alpha}$. The MCMC confidence set procedure developed by Chen et al. (2018) can then be used to obtain draws of ϕ from the highest density posterior region with credibility α. We follow this procedure in our empirical application below. The inference procedures below allows for any $CS_{\phi,\alpha}$ with asymptotically valid coverage, and takes draws or grids of ϕ from $CS_{\phi,\alpha}$ as given.

The projection confidence set is defined as

$$CS^p_{\eta,\alpha} = \bigcup_{\phi \in CS_{\phi,\alpha}} IS_\eta(\phi). \tag{48}$$

We assume that $CS_{\phi,\alpha}$ is an asymptotically valid confidence set for ϕ in the sense that

$$\lim_{T\to\infty} p_{Y^T|\phi_0}(\phi_0 \in CS_{\phi,\alpha}) = \alpha,$$

where $p_{Y^T|\phi_0}$ is the sampling distribution of the data with sample size T and ϕ_0 is the true value of ϕ. Since $\{\phi_0 \in CS_{\phi,\alpha}\}$ implies $\{IS_\eta(\phi_0) \subset CS^p_{\eta,\alpha}\}$, $CS^p_{\eta,\alpha}$ (and any set including $CS^p_{\eta,\alpha}$) is an asymptotically-valid but potentially conservative confidence set for $IS_\eta(\phi_0)$,

$$\lim_{T\to\infty} p_{Y^T|\phi_0}(IS_\eta(\phi_0) \subset CS^p_{\eta,\alpha}) \geq \alpha.$$

Let $\{\phi_k : k = 1,\ldots,K\}$ be a finite number of Monte Carlo draws or grid points from $CS_{\phi,\alpha}$. A sample analogue of the projection confidence set, $\bigcup_{k=1,\ldots,K} IS_\eta(\phi_k)$, is less useful in approximating $CS^p_{\eta,\alpha}$, because each $IS_\eta(\phi_k)$ is a discrete set, whereas the underlying $CS^p_{\eta,\alpha}$ we want to approximate can be a union of disconnected intervals with positive widths. In addition, it is difficult to judge how many disconnected intervals $IS^p_{\eta,\alpha}$ has and where the possible gaps lie within $CS^p_{\eta,\alpha}$ from a finite number of draws of $IS_\eta(\phi_k)$, $k = 1,\ldots,K$. Reporting the convex hull of $\bigcup_{k=1,\ldots,K} IS_\eta(\phi_k)$ is simple, but it can lead to a connected confidence set that obscures the discrete feature of the identified set.

In what follows, we propose two different approaches for computing the projection confidence set for an impulse response given a set of Monte Carlo draws for ϕ. We refer to the first as *switching-label projection confidence sets*. It allows the labels indexing observationally equivalent impulse responses to vary across the horizons, and produces confidence sets that can capture multi-modality of the posterior distribution or the integrated likelihood for each impulse response at each horizon. We refer to the second approach as *fixed-label projection confidence set*. It maintains unique labels for observationally equivalent structural parameters across the impulse responses and over horizons, i.e., the labels for observationally equivalent structural parameters are defined in terms of the modes of the posterior for Q. This approach may produce confidence sets that are wider than the switching-label projection confidence set, but it can better capture and visualize dependence of the impulse responses over the horizons.

V.2.1 Switching-label projection confidence sets

The switching-label approach draws inference for each impulse response at each horizon one-by-one. We hence set $\eta(\phi)$ to a particular scalar impulse response.

Maintaining the notation of the previous subsection, let $IS_\eta(\phi_k) = \{\eta_1(\phi_k),\ldots,\eta_{M(\phi_k)}(\phi_k)\}$, where $M(\phi_k)$ is the number of distinct points in the identified set at $\phi = \phi_k$. We label these points in increasing order, $\eta_1(\phi_k) < \cdots < \eta_{M(\phi_k)}(\phi_k)$. Let $\bar{M} = \max_k M(\phi_k)$ be the largest cardinality of $IS_\eta(\phi_k)$ among the draws of ϕ_k, $k = 1, \ldots, K$. $\bar{M}$ indicates the largest possible number of disconnected intervals of $CS^p_{\eta,\alpha}$. We view these intervals as clusters, each of which is indexed by $\tilde{m} \in \{1, \ldots, \bar{M}\}$. Let $\tilde{K} = |\{\phi_k : M(\phi_k) = \bar{M}\}|$ be the number of ϕ draws that has the maximal number of observationally equivalent impulse responses and define estimates of the cluster-specific mean and variance by

$$\mu_{\tilde{m}} = \frac{1}{\tilde{K}} \sum_{\phi_k : M(\phi_k) = \bar{M}} \eta_{\tilde{m}}(\phi_k),$$
$$\sigma^2_{\tilde{m}} = \frac{1}{\tilde{K} - 1} \sum_{\phi_k : M(\phi_k) = \bar{M}} \big(\eta_{\tilde{m}}(\phi_k) - \mu_{\tilde{m}}\big)^2, \tag{49}$$

for each $\tilde{m} = 1, \ldots, \bar{M}$.

For each ϕ_k, $k = 1, \ldots, K$, we augment a binary vector of length $\bar{M}$, $D(\phi_k) = \big(D_{\tilde{m}}(\phi_k) \in \{0, 1\} : \tilde{m} = 1, \ldots, \bar{M}\big)$, which indicates whether or not any one point of $IS_\eta(\phi_k)$ can be associated with $\tilde{m}$-th cluster. The true $D(\phi_k)$ is not observed, so must be imputed by, for instance, maximizing the Gaussian log-likelihood criterion in the following manner. Let ρ_{ϕ_k} be an increasing injective map from $\{1, \ldots, M(\phi_k)\}$ to $\{1, \ldots, \bar{M}\}$, characterizing which cluster each $\eta_m(\phi_k)$, $m = 1, \ldots, M(\phi_k)$, belongs to. Define

$$\hat{\rho}_{\phi_k} \in \arg\min_{\rho_{\phi_k}} \sum_{m=1}^{M(\phi_k)} \frac{\Big(\eta_m(\phi_k) - \mu_{\rho_{\phi_k}(m)}\Big)^2}{\sigma^2_{\rho_{\phi_k}(m)}}, \tag{50}$$

which minimizes the sum of variance-weighted squared distances to the cluster-specific means. We then construct $D(\phi_k) = (D_{\tilde{m}}(\phi_k) : \tilde{m} = 1, \ldots, \bar{M}) \in \{0, 1\}^{\bar{M}}$ from the indicators for whether $\hat{\rho}_{\phi_k}$ maps any $m \in \{1, \ldots, M(\phi_k)\}$ to $\tilde{m}$, i.e., $D_{\tilde{m}}(\phi_k) = 1\{\exists\ m\ \ s.t.\ \ \rho_{\phi_k}(m) = \tilde{m}\}$. We then construct an interval for each cluster $\tilde{m} \in \{1, \ldots, \bar{M}\}$ by

$$C_{\tilde{m}} = \left[\min_{\phi_k : D_{\tilde{m}}(\phi_k) = 1} \eta_{\hat{\rho}^{-1}_{\phi_k}(\tilde{m})}(\phi_k), \max_{\phi_k : D_{\tilde{m}}(\phi_k) = 1} \eta_{\hat{\rho}^{-1}_{\phi_k}(\tilde{m})}(\phi_k) \right]. \tag{51}$$

An approximation of the projection confidence set is then formed by taking the union of $C_{\tilde{m}}$:

$$\widehat{CS}^p_{\eta,\alpha} \equiv \bigcup_{\tilde{m}=1}^{\bar{M}} C_{\tilde{m}}. \tag{52}$$

Note $\widehat{CS}^p_{\eta,\alpha}$ obtained in this way includes all the $IS_\eta(\phi_k)$, $k = 1, \ldots, K$, and at the same time, can yield a collection of disconnected intervals. Moreover, if the maximum likelihood estimator for ϕ is consistent for ϕ_0, $IS_\eta(\phi)$ is a continuous correspondence at ϕ_0 and $M(\phi)$ is constant in an open neighborhood of ϕ_0, it can be shown that $\widehat{CS}^p_{\eta,\alpha}$ converges to $IS_\eta(\phi_0)$ in the Hausdorff metric. Hence, $\widehat{CS}^p_{\eta,\alpha}$ can consistently uncover the true identified set consisting of potentially multiple points.

We construct $\widehat{CS}^p_{\eta,\alpha}$ separately for each impulse response at each horizon. Hence, the labeling of the clusters $\tilde{m} = 1, \dots, \bar{M}$ defined for one impulse response does not correspond to the labeling of the clusters defined for other impulse responses or horizons. For example, a particular impulse response function labeled as $\tilde{m} = 1$ in one horizon can be labeled as $\tilde{m} = 2$ in another horizon. We expect that switching-label projection confidence sets can visualize well the multi-modality of the marginal posterior for each impulse response.

V.2.2 Fixed-label projection confidence sets

In contrast to switching-label projection confidence sets, fixed-label projection confidence sets maintain fixed-labeling across impulse responses and over time horizons. For example, an impulse response function labeled as $\tilde{m} = 1$ at one horizon is labeled as $\tilde{m} = 1$ at other horizons.

To implement this procedure, we need to anchor the labels to a particular impulse response, say, the impulse response of i^*-th variable to j^*-th structural shock at a particular horizon $h = h^*$, denoted hereafter by $\eta^*(\phi, q_{j^*}) \equiv e'_{i^*} C_{h^*}(\phi) q_{j^*}$. Given a Monte Carlo draw of the reduced-form parameters, ϕ_k, $k = 1, \dots, K$, from $CS_{\phi,\alpha}$, let $q_{j^*,m}(\phi_k)$, $m = 1, \dots, M(\phi_k)$ be observationally equivalent q_{j^*} vectors labeled according to the ordering of $\eta^*\big(\phi, q_{j^*}\big)$, i.e., $\eta^*\big(\phi_k, q_{j^*,1}(\phi_k)\big) < \eta^*\big(\phi_k, q_{j^*,2}(\phi_k)\big) < \cdots < \eta^*\big(\phi_k, q_{j^*,M(\phi_k)}(\phi_k)\big)$. Similarly to the labeling procedure shown in Eq. (50), we assign cluster identifier $\tilde{m} = 1, \dots, \bar{M}$ to $q_{j^*,m}(\phi_k)$ by constructing $\hat{\rho}_{\phi_k}$ an increasing injective map from $\{1, \dots, M(\phi_k)\}$ to $\{1, \dots, \bar{M}\}$,

$$\hat{\rho}_{\phi_k} \in \arg\min_{\rho_{\phi_k}} \sum_{m=1}^{M(\phi_k)} \frac{\Big(\eta^*(\phi_k, q_{j^*,m}) - \mu_{\rho_{\phi_k}(m)}\Big)^2}{\sigma^2_{\rho_{\phi_k}(m)}},$$

where $\mu_{\tilde{m}} = \frac{1}{\tilde{K}} \sum_{\phi_k : M(\phi_k) = \bar{M}} \eta^*(\phi_k, q_{j^*,\tilde{m}})$ and $\sigma^2_{\tilde{m}} = \frac{1}{\tilde{K}-1} \sum_{\phi_k : M(\phi_k) = \bar{M}} \big(\eta^*(\phi_k, q_{j^*,\tilde{m}}) - \mu_{\tilde{m}}\big)^2$. We then construct $D(\phi_k)$ in the same way as the switching-label projection confidence set.

For each impulse response $\eta(\phi, q_{j^*}) = e'_i C_h(\phi) q_{j^*}$, $i = 1, \dots, n$, and $h = 0, 1, \dots,$ we construct

$$C_{\tilde{m}} = \left[\min_{\phi_k : D_{\tilde{m}}(\phi_k) = 1} \eta(\phi_k, q_{j^*, \hat{\rho}^{-1}_{\phi_k}(\tilde{m})}), \max_{\phi_k : D_{\tilde{m}}(\phi_k) = 1} \eta(\phi_k, q_{j^*, \hat{\rho}^{-1}_{\phi_k}(\tilde{m})}) \right]$$

and form confidence sets by taking the union over $\tilde{m}$ as in Eq. (52).

In contrast to the switching-label procedure, the fixed-label projection confidence sets keep the labeling of the observationally equivalent impulse responses $\hat{\rho}_{\phi_k}(m)$ fixed over variables $i = 1, \ldots, n$ and different horizons $h = 0, 1, \ldots$. If the impulse response $\eta^*(\phi, q_{j^*})$ chosen to anchor the labels can tie the observationally equivalent impulse responses to different economic models or hypotheses, the labels can be interpreted as indexing the underlying economic model or hypothesis and kept invariant throughout the impulse response analysis. The fixed-label projection confidence set approach is suitable in such a case, and allows us to track and compare the observationally equivalent impulse response functions across different models.

V.2.3 Robust Bayesian interpretation

If we obtain $\{\phi_k : k = 1, \ldots, K\}$ as draws from the credible region of the posterior distribution for ϕ, $\widehat{CS}^p_{\eta,\alpha}$ can be seen as an approximation of the set $C_{\eta,\alpha}$ satisfying

$$\pi_{\phi|Y}\big(IS_\eta(\phi) \subset C_{\eta,\alpha}\big) \geq \alpha.$$

In terms of the robust Bayesian procedure proposed in Giacomini and Kitagawa (2021), $C_{\eta,\alpha}$ can be interpreted as a robust credible region with credibility α; a set of η on which a posterior distribution for η assigns probability at least α irrespective of the choice of the unrevisable part of the prior $\pi_{Q|\phi}$. Our construction of the robust credible region can be conservative and is not guaranteed to provide the shortest one. We leave the construction of the shortest robust credible region for future research.

This link to robust Bayes inference also suggests that the *range* of posterior probabilities (lower and upper probabilities) spanned by arbitrary conditional priors for Q given ϕ can be computed straightforwardly if we can draw ϕ from the posterior. Let $\{\phi_\ell : \ell = 1, \ldots, L\}$ be Monte Carlo draws from $\pi_{\phi|Y}$ and $H_0 \subset \mathbb{R}$ be a hypothesis of interest. By applying Corollary A.1 of Giacomini and Kitagawa (2021), the range of posterior probabilities for $\{\eta \in H_0\}$ is given by the convex interval:

$$.\pi_{\eta|Y}(H_0) \in \big[\pi_{\eta|Y*}(H_0), \pi^*_{\eta|Y}(H_0)\big] \equiv \big[\pi_{\phi|Y}\big(IS_\eta(\phi) \subset H_0\big), \pi_{\phi|Y}\big(IS_\eta(\phi) \cap H_0 \neq \emptyset\big)\big] \tag{53}$$

Since the algorithms given in Section IV exhaust all the locally identified parameter values in $IS_\eta(\phi)$, we can approximate the lower and upper bounds of the posterior probabilities in Eq. (53) for each hypothesis of interest by the Monte Carlo frequencies for $\big\{IS_\eta(\phi) \subset A\big\}$ and $\big\{IS_\eta(\phi) \cap A \neq \emptyset\big\}$, respectively,

$$\big[\hat{\pi}_{\eta|Y*}(H_0), \hat{\pi}^*_{\eta|Y}(H_0)\big] \equiv \left[\frac{1}{L}\sum_{\ell=1}^{L} 1\big\{IS_\eta(\phi_\ell) \subset H_0\big\}, \frac{1}{L}\sum_{\ell=1}^{L} 1\big\{IS_\eta(\phi_\ell) \cap H_0 \neq \emptyset\big\}\right].$$

For a scalar impulse response, it is also straightforward to compute the range of posterior means. Let $\underline{\eta}(\phi) = \min\left\{\eta \in IS_\eta(\phi)\right\}$ and $\bar{\eta}(\phi) = \max\left\{\eta \in IS_\eta(\phi)\right\}$. Theorem 2 in Giacomini and Kitagawa (2021) shows that the range of posterior means is given by the connected interval $\left[E_{\phi|Y}\left(\underline{\eta}(\phi)\right), E_{\phi|Y}\left(\bar{\eta}(\phi)\right)\right]$, which can be approximated by Monte Carlo analogues based on draws $\{\phi_\ell : \ell = 1, \ldots, L\}$ from $\pi_{\phi|Y}$.

VI Empirical application

We illustrate how our approach works with an empirical application to the non-recursive New-Keynesian SVAR shown in Section III.2. We consider the small scale DSGE model presented in Eq. (27), which has the SVAR representation with sign normalizations and the zero restrictions Eq. (28). The vector of observables is inflation as measured by the GDP deflator (π_t), real GDP per capita as a deviation from a linear trend (x_t) and the federal funds rate (i_t).[16] The data are quarterly from 1965:1 to 2006:1.

As discussed in Section III.2, the imposed restrictions deliver local identification and, given the reduced form parameters, they can yield up to two admissible matrices, Q_1 and Q_2. To compute them, we apply Algorithm 1 at every draw of ϕ from its posterior, using the Matlab command `vpasolve` to solve the system of quadratic equations.

We specify the Jeffreys' prior for the reduced-form parameters. Its density function is proportional to $|\Sigma|^{-\frac{3+1}{2}}$. We draw from the posterior 2,000 times and, considering uniquely the zero restrictions in Eq. (28), obtain 2,000 realizations of $\mathcal{Q}_R(\hat{\phi})$, each of which is nonempty and consists of two orthogonal matrices Q_1 and Q_2. We label them by Q_1 and Q_2 according to the ordering of the contemporaneous inflation impulse response.

Figure 4 reports the impulse response to a contractionary monetary policy shock for the output gap (left panel) and inflation (right panel). It shows the posterior means and the highest posterior density regions with credibility 90% that would be obtained if the conditional prior for Q given ϕ assigned all probability mass to either Q_1 or Q_2. That is, reporting one of the inference outputs corresponds to the Bayesian approach that focuses only on one of the posterior modes, ignoring the other.[17]

The inference result based on Q_1 shows evidence for both price and output puzzles in the short run. In the medium term, on the other hand, a contractionary monetary policy shock triggers

[16] The data are used in Aruoba and Schorfheide (2011) and downloaded from Frank Schorfheide's website: https://web.sas.upenn.edu/schorf/. For details on the construction of the series, see Appendix D from Granziera et al. (2018) and Footnote 5 of Aruoba and Schorfheide (2011).

[17] Although not reported for saving space, the Bayesian credible intervals of Figure 4 are nearly identical to those obtained by the frequentist bootstrap-after-bootstrap approach of Kilian (1998)

a contraction of the output gap, leaving the price dynamics mostly unaffected. Inference based on the other orthogonal matrix Q_2, however, leads to a contrasting conclusion. The reaction of prices to the monetary policy shock is significantly negative, while the output gap responds positively and significantly, particularly in the medium-long run.

This example illustrates that different locally-identified observationally equivalent parameter values can lead to strikingly different conclusions, and ignoring this distorts the information contained in the data. A standard off-the-shelf econometric package could uncover just one of the two results. For instance, Gretl and Eviews both return results in line with those obtained through Q_1. These packages rely on algorithms that maximize the likelihood starting from some initial value without checking other local maxima. Thus, we recommend checking for the existence of other local maxima and, if any exist, addressing how the conclusions change among them by applying the methods proposed in this paper.

The inference approaches proposed in Section V produce the results reported in Figure 5. The left panels plot results for the output gap while the right panels plot those for inflation. The top panels show the draws of the impulse responses obtained based on the draws of ϕ from its posterior. For each draw, we highlight the two observationally equivalent impulse responses corresponding to admissible Q_1 (blue) and Q_2 (red) that are coherent with the zero restrictions in Eq. (28). The labels of Q_1 and Q_2 in the plots of impulse responses for inflation are maintained in the plots for output.

The middle and bottom panels present interval estimates based on the Bayesian and frequentist-valid inference procedures of Sections V.1 and V.2. The Bayesian posterior (whose highest density regions are reported both in the middle and bottom panels) is obtained by specifying the uniform conditional prior for Q given ϕ, i.e., equal weights are assigned to Q_1 and Q_2 conditional on ϕ. The highest posterior density regions are plotted with gradation in gray scale, where the credibility levels vary over 90%, 75%, 50%, 25%, and 10%, from the lightest to darkest.[18]

The middle left panel of Figure 5 shows the marginal posterior distributions for the output gap impulse response. These are unimodal up to $h = 4$, but become bimodal for longer horizons. While there is evidence for output gap puzzle at the shortest horizons, the probability density is tighter and higher for the negative impulse responses in the medium-long run: the darkest gray region (highest 25% and 10% of the distribution) appears mostly for the negative part of the responses. The middle right panel of Figure 5 shows the marginal posterior distribution for the inflation impulse response. This is bimodal up to $h = 10$ and becomes unimodal at longer horizons. Similarly to the output gap, for the horizons with bimodal distributions, the negative impulse responses have tighter and higher densities than the positive ones.

[18]The highest posterior density regions are computed by slicing the posterior density approximated through kernel smoothing of the posterior draws of the impulse responses.

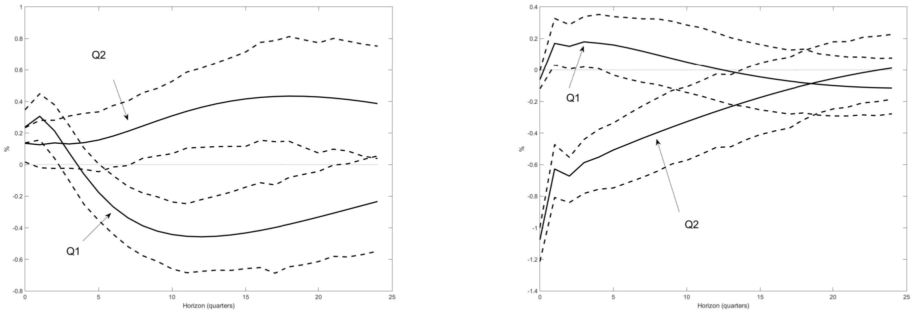
For both the output gap and inflation, we also present the frequentist-valid confidence intervals (in dotted-circle lines) proposed in Section V.2. These are obtained by retaining 90% of the draws of the reduced-form parameters with the highest value of the posterior density function. The middle panels show the fixed-label projection confidence sets of Section V.2.2, while the bottom panels show the switching-label projection confidence sets of Section V.2.1. In addition, for both the output gap and inflation, we show the range of posterior means obtained by the robust Bayesian approach (dotted lines). It is evident that the Bayesian approach gives the narrowest interval estimates, and the highest posterior density regions well visualize the bi-modal nature of the posterior distributions at some horizons. The wider confidence intervals of the frequentist-valid approach reflect a couple of their features. First, they are agnostic over the observationally equivalent parameters in the sense that they do not assign any weights over the observationally equivalent impulse responses. Second, our proposed frequentist-valid procedures rely on projecting the joint confidence intervals for the reduced-form parameters and do not optimize the width of the interval estimates for impulse responses. Concerning the results of the robust Bayesian approach, the bounds of the set of posterior means are in line with the two modes of the posterior distributions.

A useful strategy for reducing the number of locally-identified admissible solutions is to introduce sign restrictions. To refine the results reported in Figure 5, consider assuming no price puzzle by restricting the inflation responses to be non-positive for (a) the contemporaneous period, or (b) for the four quarters following a contractionary monetary policy shock. The results are reported in Figures 6 and 7, respectively.

The results with the contemporaneous non-positivity restriction (Figure 6) appear similar to those in Figure 5. A notable difference is in the upper bound of the frequentist-valid confidence intervals for the inflation response, which now excludes the positive responses shown in Figure 5 (top-right panel). For the first few quarters, both switching- and fixed-label projection confidence sets exclude a region between the two modes of the distribution.

Imposing the four sign restrictions (Figure 7), allows us to eliminate one of the admissible Q matrices for most of the draws of ϕ. In the top panels of Figure 7, the impulse responses plotted in black have a unique admissible Q under the imposed sign restrictions. Comparing Figure 5 and Figure 7 shows that in this latter the sign restrictions rule out the impulse responses corresponding to Q_1 matrices. The fixed-label and switching-label confidence intervals produce similar results. The only notable difference appears in the response of output gap, where the switching-label confidence intervals in the bottom-left panel have narrow "gaps" from $h = 11$ to $h = 18$, while the fixed-label confidence intervals in the middle-left panel do not.

Figure 4: Impulse response functions for the New-Keynesian non-recursive SVAR.



(a) Output gap to ε_t^{mp}: Single admissible Q_i.

(b) Inflation to ε_t^{mp}: Single admissible Q_i.

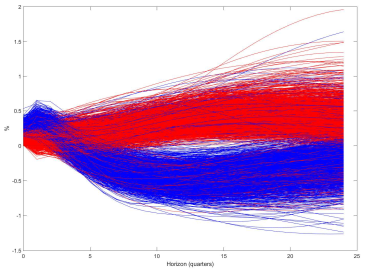
Notes: In the left column we report the impulse responses for the output gap obtained as the Bayesian posterior means with the upper and lower bounds of the highest posterior density regions with credibility 90% obtained through the admissible Q_1 and Q_2 matrices, considered separately. Similarly, in the right column we report the impulse responses for inflation.

VII Conclusion

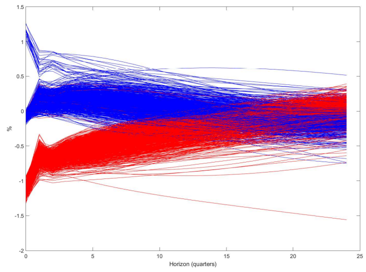
This paper analyzes SVARs that attain local identification but may fail to attain global identification. We identify the class of identifying restrictions that delivers local but non-global identification. This is characterized by non-homogeneous, non-recursive, and/or across-shock equality restrictions. Similar situations might appear also in SVARs identified through heteroskedasticity, non-normality or through external instruments. Exploiting the geometric structure of the identification problem, we propose a novel way to analyze and exhaustively compute the observationally equivalent impulse responses. The novel analytical and computational insights also contribute to the development of a posterior sampling algorithm for Bayesian inference and projection-based frequentist-valid inference in the presence of locally identified parameters.

Since local identification appears in other macroeconometric frameworks, among which the dynamic stochastic general equilibrium models, extending our computational and inference approaches to these models is a promising avenue for future research.

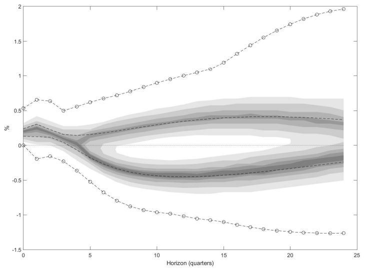
Figure 5: New-Keynesian non-recursive SVAR with zero restrictions only



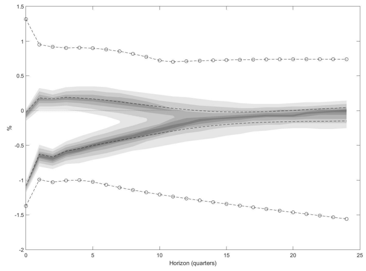
(a) Output gap to ε_t^{mp}: impulse response draws



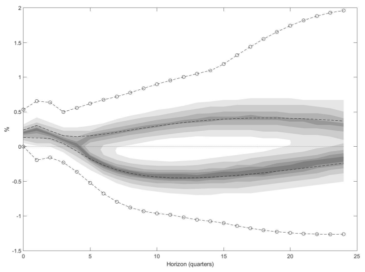
(b) Inflation to ε_t^{mp}: impulse response draws



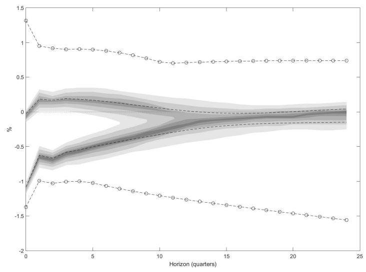
(c) Credible regions and fixed-label projection confidence sets



(d) Credible regions and fixed-label projection confidence sets.



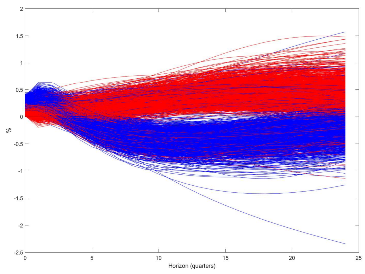
(e) Credible regions and switching-label projection confidence sets



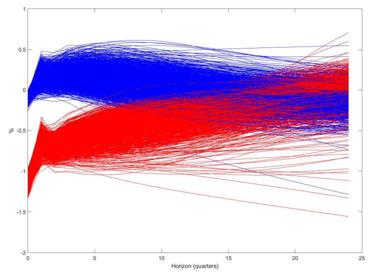
(f) Credible regions and switching-label projection confidence sets

Notes: The left column reports the output gap impulse responses and the right column reports the inflation impulse responses, both to a contractionary monetary policy shock. The middle and bottom panels report the posterior highest density regions at 90%, 75%, 50%, 25% and 10% in gray scale. The upper and loser bounds of the frequentist confidence sets are plotted by the dotted-circle lines. The dotted lines in the middle panels plot the set of posterior means.

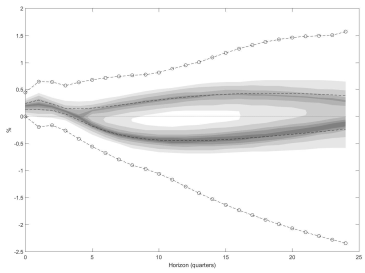
Figure 6: New-Keynesian non-recursive SVAR with zero restrictions and one sign restriction



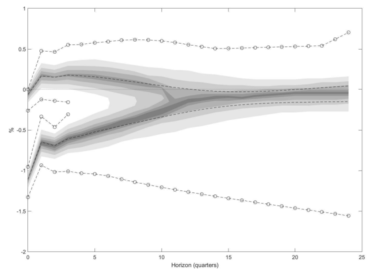
(a) Output gap to ε_t^{mp}: impulse response draws



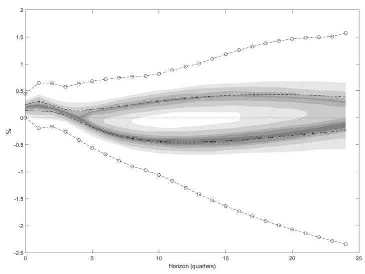
(b) Inflation to ε_t^{mp}: impulse response draws



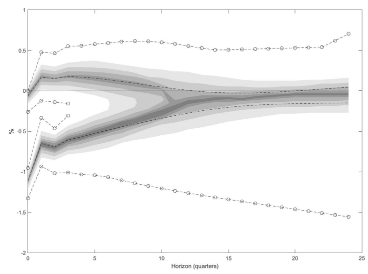
(c) Credible regions and fixed-label projection confidence sets



(d) Credible regions and fixed-label projection confidence sets.



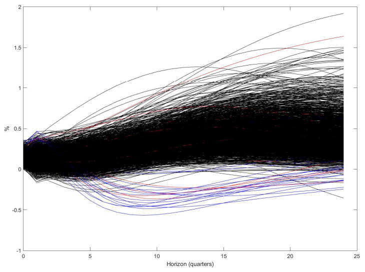
(e) Credible regions and switching-label projection confidence sets



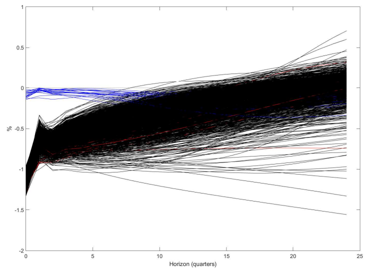
(f) Credible regions and switching-label projection confidence sets

Notes: The left column reports the output gap impulse responses and the right column reports the inflation impulse responses, both to a contractionary monetary policy shock. The middle and bottom panels report the posterior highest density regions at 90%, 75%, 50%, 25% and 10% in gray scale. The upper and loser bounds of the frequentist confidence sets are plotted by the dotted-circle lines. The dotted lines in the middle panels plot the set of posterior means.

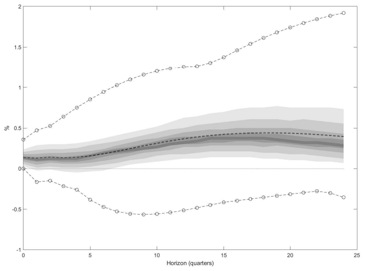
Figure 7: New-Keynesian non-recursive SVAR with zero and additional four sign restrictions



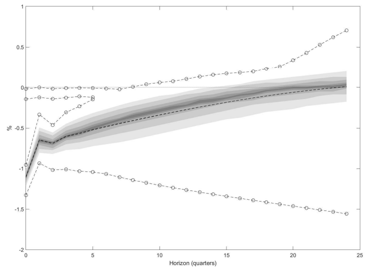
(a) Output gap to ε_t^{mp}: impulse response draws



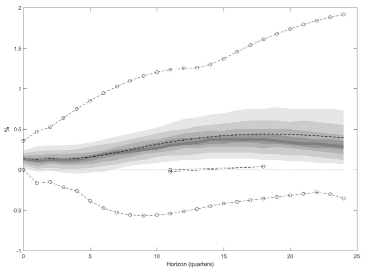
(b) Inflation to ε_t^{mp}: impulse response draws



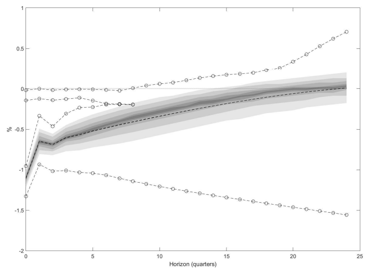
(c) Credible regions and fixed-label projection confidence sets



(d) Credible regions and fixed-label projection confidence sets.



(e) Credible regions and switching-label projection confidence sets



(f) Credible regions and switching-label projection confidence sets

Notes: The left column reports the output gap impulse responses and the right column reports the inflation impulse responses, both to a contractionary monetary policy shock. The middle and bottom panels report the posterior highest density regions at 90%, 75%, 50%, 25% and 10% in gray scale. The upper and loser bounds of the frequentist confidence sets are plotted by the dotted-circle lines. The dotted lines in the middle panels plot the set of posterior means.

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A Appendix: Some analytical results on the geometry of identification

Let the data generating process be the bivariate VAR defined in Section II.5 with the identifying restriction

$$(A_0)^{-1}_{[1,1]} = c \iff (e_1' \Sigma_{tr})\, q_1 = c \tag{54}$$

where $c > 0$ is a known (positive) scalar and e_1 is the first column of the (2×2) identity matrix. The non-homogeneous restriction in Eq. (54) affects the orthogonal matrix Q as $\sigma_1'\, q_1 = c$, with σ_1 denoting the first column of $\Sigma_{tr}' = \begin{pmatrix} \sigma_{1,1} & \sigma_{2,1} \\ 0 & \sigma_{2,2} \end{pmatrix}$.

The vector q_1 must satisfy the two equations

$$\begin{cases} \sigma_1'\, q_1 &= c \\ q_1'\, q_1 &= 1 \end{cases}$$

By simple algebra, the two solutions are

$$q_1^{(1)} = \begin{pmatrix} c/\sigma_{1,1} \\ \\ +\sqrt{\frac{\sigma_{1,1}^2 - c^2}{\sigma_{1,1}^2}} \end{pmatrix} \qquad \text{and} \qquad q_1^{(2)} = \begin{pmatrix} c/\sigma_{1,1} \\ \\ -\sqrt{\frac{\sigma_{1,1}^2 - c^2}{\sigma_{1,1}^2}} \end{pmatrix}. \tag{55}$$

These two possible solutions are represented in Figure 1. If $(\sigma_{1,1}^2 < c^2)$, the straight (vertical) line does not intersect the unit circle, and no real solution is admissible. The SVAR, although identified, does not admit any real solution given the reduced-form parameters ϕ. If, instead, $(\sigma_{1,1}^2 = c^2)$, i.e. the vertical red line is tangent to the unit circle, we continue to have global identification, although the imposed restriction is not coherent with those derived by RWZ. In all other situations, there will be two solutions that, *a priori*, can be admissible despite the sign normalization restriction. This is the case depicted in Figure 1.

Concerning the sign normalization restriction, for the first equation, the definition in Eq. (6) reduces to $q_1'\, \tilde{\sigma}_1 \geq 0$, where $\tilde{\sigma}_1$ is the first column of $\Sigma_{tr}^{-1} = 1/(\sigma_{1,1}\sigma_{2,2}) \begin{pmatrix} \sigma_{2,2} & 0 \\ -\sigma_{2,1} & \sigma_{1,1} \end{pmatrix}$. Through elementary algebra, we obtain that

$$q_1'\, \tilde{\sigma}_1 \geq 0 \qquad \iff \qquad \frac{q_{1,1}}{\sigma_{1,1}} \geq \frac{q_{1,2}\sigma_{2,1}}{\sigma_{1,1}\sigma_{2,2}} \tag{56}$$

where $q_{1,1}$ and $q_{1,2}$ are the two generic elements of q_1, i.e. $q_1 = (q_{1,1}\,,\, q_{1,2})'$. Suppose, first, that from the data we have $\sigma_{2,1} < 0$. In this case, if we substitute in the values of $q_{1,1}$ and $q_{1,2}$ obtained for $q_1^{(1)}$ in the left-hand side of Eq. (55), the sign normalization condition for the first equation becomes

$$\frac{c}{\sigma_{1,1}^2} \geq \frac{\sigma_{2,1}}{\sigma_{1,1}\sigma_{2,2}}\sqrt{\frac{\sigma_{1,1}^2 - c^2}{\sigma_{1,1}^2}} \tag{57}$$

As the left-hand side is always positive and the right-hand side always negative, this is always satisfied. If, instead, we substitute the values $q_{1,1}$ and $q_{1,2}$ obtained for $q_1^{(2)}$ in the right-hand side of Eq. (55), the sign normalization condition for the first equation becomes

$$\frac{c}{\sigma_{1,1}^2} \geq -\frac{\sigma_{2,1}}{\sigma_{1,1}\sigma_{2,2}}\sqrt{\frac{\sigma_{1,1}^2 - c^2}{\sigma_{1,1}^2}} \tag{58}$$

that is also satisfied when $c^2 \geq \frac{1}{2}\frac{\sigma_{1,1}^2\sigma_{2,1}^2}{\sigma_{2,2}^2}$. If this is the case, both solutions $q_1^{(1)}$ and $q_1^{(2)}$ are admissible, leading to local identification. The situation is very similar when $\sigma_{2,1} > 0$.

If, instead, $c = 0$ as in the standard RWZ setup, the two q_1 vectors in Eq. (55) become $q_1^{(1)} = (0\,,\,1)'$ and $q_1^{(2)} = (0\,,\,-1)'$. If, as before, we suppose $\sigma_{2,1} < 0$, the sign normalization for $q_1^{(1)}$ in Eq. (57) reduces to $0 \geq \sigma_{2,1}/(\sigma_{1,1}\sigma_{2,2})$, which is always true. The sign normalization for $q_1^{(2)}$ in Eq. (58) is $0 \geq -\sigma_{2,1}/(\sigma_{1,1}\sigma_{2,2})$ which, in contrast, is never true. The case where $\sigma_{2,1} > 0$, is exactly the same, but with inverted results. One of the two solutions, thus, will be always ruled out by the sign normalization, and global identification is guaranteed.

The second column of Q, the unit-length vector q_2, although not restricted, can be pinned down through the its orthogonality to q_1

$$\begin{cases} q_2'\,q_1 &= 0 \\ q_2'\,q_2 &= 1. \end{cases} \tag{59}$$

However, given that there are two admissible vectors $q_1^{(1)}$ and $q_1^{(2)}$, the system Eq. (59) must be solved for both. This can be done with simple algebra, yielding the two solutions

$$q_2^{(1)} = \begin{pmatrix} +\sqrt{\frac{(\sigma_{1,1}^2-c^2)(2c^2-\sigma_{1,1}^2)}{c^4}} \\ \\ -\sqrt{\frac{2c^2-\sigma_{1,1}^2}{c^2}} \end{pmatrix} \quad \text{and} \quad q_2^{(1)} = \begin{pmatrix} -\sqrt{\frac{(\sigma_{1,1}^2-c^2)(2c^2-\sigma_{1,1}^2)}{c^4}} \\ \\ +\sqrt{\frac{2c^2-\sigma_{1,1}^2}{c^2}} \end{pmatrix} \tag{60}$$

One of the two, precisely which depends on the reduced-form parameters, will be eliminated by the sign normalization restriction. This case is represented in the left panel of Figure 2, together with $q_1^{(1)}$.

The other possibility, represented in the right panel of Figure 2, is when we solve the system conditional on $q_1^{(2)}$, obtaining

$$q_2^{(2)} = \begin{pmatrix} +\sqrt{\frac{(\sigma_{1,1}^2 - c^2)(2c^2 - \sigma_{1,1}^2)}{c^4}} \\ \\ +\sqrt{\frac{2c^2 - \sigma_{1,1}^2}{c^2}} \end{pmatrix} \quad \text{and} \quad q_2^{(2)} = \begin{pmatrix} -\sqrt{\frac{(\sigma_{1,1}^2 - c^2)(2c^2 - \sigma_{1,1}^2)}{c^4}} \\ \\ -\sqrt{\frac{2c^2 - \sigma_{1,1}^2}{c^2}} \end{pmatrix} \tag{61}$$

where, as before, one of the two solutions is ruled out by the sign normalization restriction.

B Appendix: Further results on local identification

In this appendix we provide a new result on local identification for SVAR models. We consider a set of equality restrictions $\mathbf{F}(\phi, Q)$ satisfying the recursive identification scheme in Definition 3.

Proposition 5 (RWZ sufficient condition for checking local identification)**.** *Consider an SVAR with recursive identifying restrictions of the form Eq. (13). The SVAR is locally identified at $A = (A_0, A_+) \in \mathcal{A}_R$ if, for $i = 1, \ldots, n$,*

$$M_i(Q) \equiv \begin{pmatrix} \underset{(n-i)\times n}{F_{ii}(\phi)} \cdot \underset{n\times n}{Q} \\ \begin{pmatrix} \underset{i\times i}{I_i} & \underset{i\times(n-i)}{0} \end{pmatrix} \end{pmatrix} \tag{62}$$

is of rank n.

Proof. See Appendix C. □

Proposition 5 reconciles our condition for local identification of recursive SVARs with the general rank condition for global identification provided by RWZ (their Theorem 1). In particular, under a recursive identification scheme, the RWZ condition for global identification developed for the case of homogeneous restrictions implies local identification, even though we allow non-homogeneous and across shock restrictions.

C Appendix: Proofs

This appendix collects proofs for all propositions reported in this paper. We make use of the following matrices. K_n is the $n^2 \times n^2$ commutation matrix as defined in Magnus and Neudecker (2007) and $N_n = 1/2(I_{n^2} + K_n)$. Let $\tilde{D}_n$ be the $n^2 \times n(n-1)/2$ full-column rank matrix $\tilde{D}_n$ defined in Magnus (1988) such that for any $n(n-1)/2$-dimensional vector v,$\tilde{D}_n v \equiv \text{vec}\,(H)$ holds, where H is an $n \times n$ skew-symmetric matrix ($H = -H'$). See Appendix D for explicit constructions of $\tilde{D}_n$ for $n = 2, 3, 4$.

Proof of Proposition 2: necessary and sufficient condition for local identification

Fixing ϕ, a matrix Q satisfies the identifying restrictions if:

$$\mathbf{F}(\phi)\,\text{vec}\,Q = \mathbf{c} \tag{63}$$

$$Q'Q = I_n \tag{64}$$

which is a system of quadratic equations. Eq. (63) consists of $f = f_1 + \cdots + f_n$ linear and non-homogeneous equations. Eq. (64) is a set of quadratic equations stating that the columns of Q, the vectors $(q_1, \ldots, q_n)$, must be orthogonal and of unit length.

The system can be solved locally as:

$$\begin{aligned} \mathbf{F}(\phi)\,\text{vec}\,dQ &= 0 \\ dQ'\,Q + Q'dQ &= 0, \end{aligned}$$

which, using the Kronecker product and its properties, becomes

$$\begin{aligned} \mathbf{F}(\phi)\,\text{vec}\,dQ &= 0 \\ \Big[(Q' \otimes I_n) + (I_n \otimes Q')\Big]\text{vec}\,dQ &= 0. \end{aligned}$$

Moreover, using the commutation matrix K_n we have

$$\begin{aligned} \mathbf{F}(\phi)\,\text{vec}\,dQ &= 0 \\ \Big[K_n(I_n \otimes Q') + (I_n \otimes Q')\Big]\text{vec}\,dQ &= 0, \end{aligned}$$

and recalling $N_n = 1/2(I_{n^2} + K_n)$, we obtain

$$\begin{aligned} \mathbf{F}(\phi)\,\text{vec}\,dQ &= 0 \\ 2N_n(I_n \otimes Q')\text{vec}\,dQ &= 0. \end{aligned}$$

The Jacobian matrix can, thus, be defined as

$$J(Q) = \begin{pmatrix} \mathbf{F}(\phi) \\ 2N_n(I_n \otimes Q') \end{pmatrix} \tag{65}$$

Following Magnus and Neudecker (2007), a sufficient condition for local identification of Q at the point $Q = Q_0$ is that $J(Q_0)$ has full column rank. If there exists an admissible neighborhood of Q_0 such that $J(Q_0)$ is of full column rank, this condition becomes necessary too.

The condition regarding the rank of Eq. (65) can be further simplified. Given that Q is invertible (it is orthogonal), the rank of $J(Q)$ is unchanged if we post-multiply Eq. (65) by $(I_n \otimes Q^{-1\prime}) = I_n \otimes Q$. Checking whether $J(Q)$ is of full column rank, thus, corresponds to checking whether the system of equations

$$\begin{aligned} \mathbf{F}(\phi)(I_n \otimes Q)\, x &= 0 \\ 2N_n\, x &= 0 \end{aligned}$$

admits the null vector x as the unique solution. However, as in Magnus (1988), the second equation, can be solved as $x = \tilde{D}_n z$, with z a $n(n-1)/2 \times 1$ vector. Substituting this solution into the first equation leads to the rank condition in Eq. (20) of Proposition 2. Since $\tilde{D}_n$ is a matrix of full column rank $n(n-1)/2$, a necessary condition for the rank condition Eq. (20) is that the number of rows of $F(\phi)$, f, is greater than or equal to $n(n-1)/2$. This completes the proof of (i).

To show claim (ii), let $\bar{\mathcal{F}}$ be the set of matrices of dimension $f \times n(n-1)/2$ and denote by X a generic element of $\bar{\mathcal{F}}$. Viewing the space spanning the j-th column of X as V_j in Lemma 3 of RWZ, and defining the set S in Lemma 3 of RWZ to be the set of matrices with deficient rank $S = \{X \in \bar{\mathcal{F}} : \text{rank}(X) < n(n-1)/2\}$, Lemma 3 in RWZ shows that either $S = \bar{\mathcal{F}}$, or S is a set of measure zero in $\bar{\mathcal{F}}$.

Define

$$\mathcal{F} \equiv \{\mathbf{F}(\phi)(I_n \otimes Q)\tilde{D}_n : \mathbf{F}(\phi)\text{vec}\,(Q) = c, \;\; (\phi, Q) \in \Phi \times \mathcal{O}(n)\}. \tag{66}$$

Since $\mathcal{F} \subset \bar{\mathcal{F}}$, $S \cap \mathcal{F}$ is either equal to $\mathcal{F}$ or is a set of measure zero in $\mathcal{F}$. Let $g : \mathcal{A} \to \Phi \times \mathcal{O}(n)$ be the function that reparametrizes the structural parameters A to (ϕ, Q), and $h : (\Phi \times \mathcal{O}(n)) \to \bar{\mathcal{F}}$ be the function that maps (ϕ, Q) to $\left[\mathbf{F}(\phi)(I_n \otimes Q)\tilde{D}_n\right] \in \mathcal{F}$. By applying Lemma 2 in RWZ (proved in Spivak, 1965) to the chain of inverse maps h^{-1} and g^{-1}, we conclude that either $(g^{-1} \circ h^{-1})(\mathcal{F}) = \mathcal{A}_R$ or it is of measure zero in $\mathcal{A}_R$. The conclusion then follows by noting $(g^{-1} \circ h^{-1})(\mathcal{F}) = \mathcal{K}^c$. □

Proof of Corollary 1: necessary and sufficient condition for local identification of a subset of shocks

Fixing ϕ, a matrix $Q = \left[\, Q_1 \mid Q_2 \,\right]$ satisfies the identifying restrictions if:

$$\begin{aligned} \mathbf{F}_{11}(\phi)\, \mathbf{q}_1 &= \mathbf{c}_1 \\ \mathbf{F}_{22}(\phi)\, \mathbf{q}_2 &= \mathbf{c}_2 \\ Q_1' Q_1 &= I_s \\ Q_2' Q_2 &= I_{n-s} \\ Q_1' Q_2 &= 0_{s \times (n-s)} \end{aligned}$$

which is a system of quadratic equations, with $\mathbf{q}_1 = \text{vec}\ Q_1$ and $\mathbf{q}_2 = \text{vec}\ Q_2$. Similarly to the steps followed in the proof of Proposition 2, differentiating and using a bit of algebra, we obtain the system of equations

$$\begin{pmatrix} \mathbf{F}_{11}(\phi) & 0 \\ 2N_{ns}\left(I_n \otimes Q_1'\right) & \\ K_{s(n-s)}\left(I_s \otimes Q_2'\right) & \left(I_{(n-s)} \otimes Q_1'\right) \\ 0 & \mathbf{F}_{11}(\phi) \\ 0 & 2N_{n(n-s)}\left(I_n \otimes Q_2'\right) \end{pmatrix} \begin{pmatrix} d\mathbf{q}_1 \\ d\mathbf{q}_2 \end{pmatrix} = 0. \tag{67}$$

However, as the interest is only on $\mathbf{q}_1$, the part of the Jacobian that matters is

$$J_1(Q_1) = \begin{pmatrix} \mathbf{F}_{11}(\phi) \\ 2N_{ns}\left(I_n \otimes Q_1'\right) \end{pmatrix}$$

that, in order for $\mathbf{q}_1$ to be locally identified, must have full column rank equal to ns. □

Proof of Proposition 3: local identification in recursive SVARs

Assume that the rank condition of Proposition 2 holds at parameter point $A = (A_0, A_+) \in \mathcal{A}_R$, and let ϕ be the corresponding reduced-form parameter. Since local identification holds at A, there is no observationally equivalent parameter point in a neighborhood of A. In other words, no infinitesimal rotation of the orthogonal matrix Q generates observationally equivalent and admissible structural parameters in the neighborhood of A. Any infinitesimal rotation can be represented by $(I_n + H)$, where H is an $n \times n$ skew-symmetric matrix (see Lucchetti 2006) whose i-th column we denote by h_i.

Projecting on q_1, an admissible structural parameter lying in a local neighborhood of A has to satisfy

$$F_{11}(\phi)\Big[Q\left(I_n + H\right)\Big]e_1 = c_1 \quad \Longrightarrow \quad F_{11}(\phi)q_1 + F_{11}QHe_1 = c_1 \quad \Longrightarrow \quad F_{11}(\phi)Qh_1 = 0,$$

where e_i is the i-th column of the identity matrix I_n, and the last equation follows from the fact that $F_{11}(\phi)q_1 = c_1$. The system $F_{11}(\phi)Qh_1 = 0$ is characterized by $n-1$ equations and an n-dimensional vector of unknowns h_1. The first element of h_1 is zero by definition (the elements on the main diagonal of a skew-symmetric matrix are equal to zero). Hence, we have

$$\begin{pmatrix} F_{11}(\phi)Q \\ e_1' \end{pmatrix} h_1 = 0. \tag{68}$$

This linear equation system has $h_1 = 0$ as its unique solution if and only if $\begin{pmatrix} F_{11}(\phi)Q \\ e_1' \end{pmatrix}$ is of rank n, or, equivalently, $F_{11}(\phi)(q_2 \ldots q_n)$ is of full rank (equal to $n-1$). Since the model is locally identified by assumption, $h_1 = 0$ has to be the only solution of Eq. (68). Hence,

$\text{rank}\left(F_{11}(\phi)(q_2 \; \dots \; q_n)\right) = n-1$ must hold, implying that $\text{rank}(F_{11}(\phi)) = n-1$.

For q_2, given a q_1 vector solving $F_{11}(\phi)q_1 = 0$, we have the following system:

$$\begin{cases} F_{21}(\phi)q_1 + F_{22}(\phi)q_2 &= c_2 \\ q_1'q_2 &= 0, \end{cases}$$

Considering again an infinitesimal rotation

$$\begin{cases} F_{21}(\phi)q_1 + F_{22}(\phi)Q(I_n + H)e_2 &= c_2 \\ q_1'Q(I_n + H)e_2 &= 0 \end{cases} \implies \begin{cases} F_{21}(\phi)q_1 + F_{22}(\phi)Qe_2 + F_{22}(\phi)Qh_2 &= c_2 \\ q_1'Qh_2 &= 0 \end{cases},$$

but, given the restrictions, $F_{21}(\phi)q_1 + F_{22}(\phi)q_2 = c_2 \implies F_{22}(\phi)Qh_2 = 0$, which allows the system to be written as

$$\begin{cases} F_{22}(\phi)Qh_2 &= 0 \\ q_1'Qh_2 &= 0 \end{cases} \implies \begin{pmatrix} F_{22}(\phi)Q \\ q_1'Q \end{pmatrix} h_2 = 0. \tag{69}$$

Similarly to the argument for h_1 above, and noting that the first two entries of h_2 are zero, we can represent the linear equations as

$$\begin{pmatrix} F_{22}(\phi)Q \\ q_1'Q \\ e_2' \\ e_1' \end{pmatrix} h_2 = 0.$$

Since $q_1'Q = e_1'$, the last equation in this system is redundant. Thus, in order for $h_2 = 0$ to be the unique solution, $\begin{pmatrix} F_{22}(\phi) \\ q_1' \end{pmatrix}$ must have full raw rank (equal to $n-1$).

To obtain the sequential rank conditions of Proposition 3, we repeat this argument further for $i = 3, 4$

Next, we show the reverse implication. For each column of Q, we consider a system of equations of the form,

$$\begin{cases} \tilde{F}_{ii}(\phi)\, q_i &= (c_i', 0, \dots, 0)' \\ q_i'\, q_i &= 1, \end{cases}$$

sequentially for $i = 1, \dots, n$, where $\tilde{F}_{ii}(\phi)$ is as defined in the statement of Proposition 3. If $\text{rank}\left(\tilde{F}_{ii}(\phi)\right) = n-1$, the system of equations represents the intersection between a straight line and the unit circle in $\mathbb{R}^n$, which has at most two distinct solutions. Hence, any admissible Q matrices are isolated points, so the SVAR is locally identified. The rank condition of Eq. (20) follows by Proposition 2. □

Proof of Proposition 4: number of admissible Q's

We split the proof into five cases based on the type of equality restrictions. The first three cases are recursive identification schemes. The remaining two are non-recursive.

We first consider cases with recursive restrictions. That is, the variables are ordered to satisfy

$$f_1 \geq f_2 \geq \ldots \geq f_n. \tag{70}$$

Case 1: Recursive and homogeneous restrictions but no restrictions across shocks

Under recursive restrictions, we have shown in Proposition 3 that the rank condition of Eq. (20) is equivalent to the sequential rank conditions of Eq. (24). If the sign normalization restrictions select either the admissible q_i or $-q_i$ at every $i = 1, \ldots, n$, the sequential determination procedure of RWZ pins down an admissible Q matrix. The sequential rank conditions do not guarantee that the sign normalizations select a unique Q matrix, but the number of solutions for each q_i is at most two. Hence, the number of admissible Q matrices is at most equal to the number of distinct selections of two vectors $\{q_i, -q_i\}$ over $i = 1, \ldots, n$, which amounts to 2^n.

Case 2: Recursive non-homogeneous restrictions but no restrictions across shocks

Under recursive and non-homogeneous restrictions, consider solving for the admissible Q matrices column by column by exploiting the sequential rank conditions Eq. (24). For the first column q_1, we have

$$\begin{cases} F_{11}(\phi)\, q_1 &= c_1 \\ \quad q_1'\, q_1 &= 1 \end{cases} \tag{71}$$

Given that $F_{11}(\phi)$ has full row rank, the set of solutions of q_1 for the first equations can be spanned by any $n \times 1$ vector $t_1 \in \mathbb{R}$,

$$\begin{aligned} q_1 &= F_{11}(\phi)'\Big(F_{11}(\phi)F_{11}(\phi)'\Big)^{-1} c_1 + \Big(I_n - F_{11}(\phi)'\Big(F_{11}(\phi)F_{11}(\phi)'\Big)^{-1} F_{11}(\phi)\Big)t_1 \\ &\equiv d_1 + B_1\, t_1 \end{aligned} \tag{72}$$

Since the $(n \times n)$ matrix B_1 has rank $n - f_1 = 1$, it can be decomposed as $B_1 = \alpha_1 \beta_1'$, where α_1 is a basis for span(B_1), i.e. the column space of B_1, and both α_1 and β_1 are non-zero $n \times 1$ vectors. We can hence write

$$q_1 = d_1 + \alpha_1\, z_1 \tag{73}$$

with $z_1 = \beta_1'\, t_1$, being any scalar. The second (quadratic) equation in system Eq. (71) becomes

$$\begin{aligned} q_1' q_1 &= (d_1 + \alpha_1\, z_1)'\,(d_1 + \alpha_1\, z_1) \\ &= d_1' d_1 + 2 d_1' \alpha_1 z_1 + \alpha_1' \alpha_1 z_1^2 = 1 \\ &\Rightarrow \lambda_1 + 2\xi_1 z_1 + \omega_1 z_1^2 = 0 \end{aligned}$$

where $\lambda_1 = d_1' d_1 - 1$, $\xi_1 = d_1' \alpha_1$ and $\omega_1 = \alpha_1' \alpha_1$ are all functions of the reduced form parameters. There are hence three possibilities:

1. If $\xi_1^2 - \lambda_1 \omega_1 > 0$, we have two real solutions. It may be that none, one, or both satisfy the sign normalization restriction for q_1.

2. If $\xi_1^2 - \lambda_1 \omega_1 = 0$, we have a single real solution. It may or may not satisfy the sign normalization restriciton.

3. If $\xi_1^2 - \lambda_1 \omega_1 < 0$, we have no real solution, implying that ϕ is not compatible with the imposed restrictions.

In summary, at most there are two admissible q_1's. Denote them by $q_1^{(1)}$ and $q_1^{(2)}$ (allowing $q_1^{(1)} = q_1^{(2)}$).

Given an admissible $q_1 \in \{q_1^{(1)}, q_1^{(2)}\}$, consider obtaining an admissible second column vector q_2 by solving

$$\left\{ \begin{array}{rcl} F_{22}(\phi)\, q_2 &=& c_2 \\ q_1'\, q_2 &=& 0 \\ q_2'\, q_2 &=& 1 \end{array} \right. \tag{74}$$

with $\operatorname{rank}\left((F_{22}(\phi)', q_1)\right) = n - 1$. This system can be transformed as

$$\left\{ \begin{array}{rcl} F_{22}(\phi)\, q_2 &=& c_2 \\ q_1'\, q_2 &=& 0 \\ q_2'\, q_2 &=& 1 \end{array} \right. \Longrightarrow \left\{ \begin{array}{rcl} \begin{pmatrix} F_{22}(\phi) \\ q_1' \end{pmatrix} q_2 &=& \begin{pmatrix} c_2 \\ 0 \end{pmatrix} \\ q_2'\, q_2 &=& 1 \end{array} \right. \Longrightarrow \left\{ \begin{array}{rcl} \tilde{F}_{22}(\phi)\, q_2 &=& \tilde{c}_2 \\ q_2'\, q_2 &=& 1 \end{array} \right. \tag{75}$$

where $\tilde{F}_{22}(\phi) = (F_{22}'(\phi), q_1)'$ and $\tilde{c}_2 = (c_2', 0)'$. Given the assumption $\operatorname{rank}(\tilde{F}_{22}(\phi)) = n - 1$, Eq. (75) can be solved in the same way as the system for q_1. We can hence obtain at most two admissible q_2 vectors for each of $q_1 = q_1^{(1)}$ and $q_1 = q_1^{(2)}$. So far there are at most four admissible vectors for the first two columns of Q.

We repeat this argument for $i = 3, \ldots, n$. Given that there are at most 2^{i-1} admissible constructions of $(q_1, \ldots, q_{i-1})$, and at each admissible $(q_1, \ldots, q_{i-1})$, we solve for q_i

$$\left\{ \begin{array}{rcl} \tilde{F}_{ii}(\phi)\, q_i &=& \tilde{c}_i \\ q_i'\, q_i &=& 1, \end{array} \right. \tag{76}$$

where

$$\tilde{F}_{ii}(\phi) = (F_{ii}(\phi)', q_1, \ldots, q_{i-1})' \qquad \text{and} \qquad \tilde{c}_i = (c_i', 0, \ldots, 0).$$

Again, finding an admissible q_i given $(q_1, \ldots, q_{i-1})$ boils down to solving a quadratic equation, so there are at most two solutions for q_i, implying that there are at most 2^i admissible constructions of $(q_1, \ldots, q_{i-1}, q_i)$. At $i = n$, we obtain at most 2^n admissible Q matrices.

Case 3: Recursive non-homogeneous restrictions and restrictions across shocks

The recursive restrictions imply that $\mathbf{F}(\phi)$ is lower block-triangular, i.e. $F_{ij} = 0$ for $j > i$, and $f_i = n - i$ for all $i = 1, \ldots, n$. The case where $i = 1$ is identical to the initial step in *Case 2* above, so we have at most two admissible q_1 vectors. For $i > 1$ we exploit the sequential structure of the restrictions and obtain each admissible q_i sequentially given $(q_1, \ldots, q_{i-1})$ obtained in the preceding steps. The only difference with respect to *case 2* is that, once $(q_1, \ldots, q_{i-1})$ is given, the system of equations in Eq. (76), will be characterized by

$$\tilde{F}_{ii}(\phi) = (F'_{ii}(\phi), q_1, \dots, q_{i-1})', \quad \text{and} \quad \tilde{c}_i = ((c_i - F_{i1}(\phi)q_1 - \cdots - F_{i,i-1}(\phi)q_{i-1})', 0, \dots, 0)'. \tag{77}$$

Repeating the argument of *Case 2*, we conclude there are at most 2^n admissible $Q \in \mathcal{O}(n)$.

We now move to the cases with non-recursive identifying restrictions.

Case 4: Non-recursive restrictions and no restrictions across shocks

If $f_1 = n - 1$, we can proceed as in *Case 2* and globally or locally identify q_1, depending on the restrictions at hand. If, instead, $f_1 < n - 1$, we can only identify the basis spanning a subspace in $\mathbb{R}^n$ of dimension $n - f_1$ containing q_1. The system of equations characterizing q_1 is given by

$$\begin{cases} F_{11}(\phi)\, q_1 &= c_1 \\ q'_1\, q_1 &= 1. \end{cases} \tag{78}$$

Following the analysis of *Case 2*, we can represent an admissible q_1 by $q_1 = d_1 + \alpha_1 z_1$, where $z_1 = \beta'_1\, t_1 \in \mathbb{R}^{n-f_1}$, α_1 is a nonzero $n \times (n - f_1)$ matrix, β_1 is a nonzero $(n - f_1) \times n$ matrix, and $t_1 \in \mathbb{R}^n$. Given this representation of q_1, the second (quadratic) equation in system Eq. (78) becomes

$$\begin{aligned} q'_1 q_1 &= d'_1 d_1 + 2d'_1\alpha_1 z_1 + z'_1\alpha'_1\alpha_1 z_1 = 1 \\ &\Rightarrow \lambda_1 + 2\xi'_1 z_1 + z'_1\omega_1 z_1 = 0, \end{aligned}$$

where $\lambda_1 = d'_1 d_1 - 1$, $\xi_1 = \alpha'_1 d_1$, and $\omega_1 = \alpha'_1\alpha_1$. The set of real roots of this quadratic equation in z_1, if nonempty, is a singleton or a hyper-ellipsoid in $\mathbb{R}^{n-f_1}$ with its radius given by the constant term in the completion of squares (if nonnegative).

Assuming an admissible q_1 exists, consider the equation system for q_2,

$$\begin{cases} q_2 &= d_2 + \alpha_2\, z_2 \\ q'_2\, q_2 &= 1, \end{cases} \tag{79}$$

whose set of roots, if nonempty, is again a singleton or a $n - f_2$-dimensional hyper-ellipsoid. In addition, we have the following orthogonality restriction between q_1 and q_2,

$$\begin{aligned} q'_1 q_2 &= d'_1 d_2 + d'_1\alpha_2 z_2 + z'_1\alpha'_1 d_2 + z'_1\alpha'_1\alpha_2 z_2 \\ &\equiv \lambda_{1,2} + \xi'_{1,2} z_2 + z'_1\xi_{2,1} + z'_1\omega_{1,2} z_2 = 0. \end{aligned}$$

Enumerating these equations for all $i = 1, \ldots, n$, we obtain the following system of equations:

$$\begin{cases} z_1'\omega_1 z_1 + 2\xi_1' z_1 + \lambda_1 &= 0 \\ z_2'\omega_2 z_2 + 2\xi_2' z_2 + \lambda_2 &= 0 \\ \vdots & \\ z_n'\omega_n z_n + 2\xi_n' z_n + \lambda_n &= 0 \\ z_1'\omega_{1,2} z_2 + \xi_{1,2}' z_2 + z_1'\xi_{2,1} + \lambda_{1,2} &= 0 \\ z_1'\omega_{1,3} z_3 + \xi_{1,3}' z_3 + z_1'\xi_{3,1} + \lambda_{1,3} &= 0 \\ \vdots & \\ z_{n-1}'\omega_{n-1,n} z_n + \xi_{n-1,n}' z_n + z_{n-1}'\xi_{n,n-1} + \lambda_{n-1,n} &= 0. \end{cases} \tag{80}$$

The number of equations is $n + n(n-1)/2 = n(n+1)/2$. The number of unknowns, contained in $z_1, z_2, \ldots, z_n$, is

$$(n - f_1) + (n - f_2) + \ldots + (n - f_n) \leq n^2 - n(n-1)/2 = n(n+1)/2, \tag{81}$$

where the inquality follows by the order condition stated in Proposition 2, $\sum_{i=1}^{n} f_i \geq n(n-1)/2$. Hence, we have a system of $n(n+1)/2$ equations with at most $n(n+1)/2$ unknowns. Moreover, each one is a quadratic equation and, importantly, given the rank condition for local identification is satisfied, each of the solutions has to be an isolated point. Bézout's theorem gives that the maximum number of solutions is the product of the polynomial degree of all the equations, so the number of solutions is at most $2^{n(n+1)/2}$.

Case 5: Non-recursive and across-shocks restrictions

In this case analysis of identification requires considering all equations jointly. We will have a system of equations of the form

$$\begin{cases} \mathbf{F}(\phi)\operatorname{vec} Q &= \mathbf{c} \\ q_1' q_1 &= 1 \\ q_2' q_2 &= 1 \\ \vdots & \\ q_n' q_n &= 1 \\ q_1' q_2 &= 0 \\ \vdots & \\ q_{n-1}' q_n &= 0. \end{cases} \tag{82}$$

This system consists of n^2 equations with n^2 unknowns (the elements in Q). The first $n(n-1)/2$ equations are linear and the latter $n(n+1)/2$ equations are all quadratic. By Bézout's theorem, the maximum number of solutions is $2^{n(n+1)/2}$.□

Proof of Proposition 5: RWZ sufficient condition for checking local identification

The result is a by-product of Proposition 3. As observed in Eq. (68), the first column q_1 is locally identified if and only if $\begin{pmatrix} F_{11}(\phi)Q \\ e_1' \end{pmatrix}$ has full column rank equal to n. When moving to the identification of q_2, from the system Eq. (69), and recalling that the first two elements of h_2 are zero, we have no admissible infinitesimal rotation (i.e. $h_2 = 0$) if

$$\text{rank}\begin{pmatrix} \underset{(n-2)\times n}{F_{22}(\phi)} \cdot \underset{n\times n}{Q} \\ \begin{pmatrix} \underset{2\times 2}{I_2} & \underset{2\times(n-2)}{0} \end{pmatrix} \end{pmatrix} = n.$$

Repeating this argument for the remaining of columns of Q, we obtain the proposition. □

D Appendix: The $\tilde{D}_n$ matrix

A skew-symmetric (square) matrix A satisfies $A' = -A$. Let $\tilde{v}(A)$ be a vector containing the $n(n-1)/2$ *essential* elements of A. When A is skew-symmetric, it is possible to expand the elements of $\tilde{v}(A)$ to obtain $\text{vec}\, A$. $\tilde{D}_n$, thus, can be defined to be the $n^2 \times n(n-1/2)$ matrix with the property that

$$\tilde{D}_n\, \tilde{v}(A) = \text{vec}\, A$$

for any skew symmetric $n \times n$ matrix A. For a formal definition and properties of $\tilde{D}_n$, see Magnus (1988). Here, we present $\tilde{D}_n$ for $n = 2$, $n = 3$ and $n = 4$:

$$
\tilde{D}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \qquad
\tilde{D}_3 = \left(\begin{array}{ccc}
0 & 0 & 0 \\
\textcircled{1} & 0 & 0 \\
0 & \textcircled{1} & 0 \\
\hdashline
-1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \textcircled{1} \\
\hdashline
0 & -1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{array}\right), \qquad
\tilde{D}_4 = \left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\textcircled{1} & 0 & 0 & 0 & 0 & 0 \\
0 & \textcircled{1} & 0 & 0 & 0 & 0 \\
0 & 0 & \textcircled{1} & 0 & 0 & 0 \\
\hdashline
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \textcircled{1} & 0 & 0 \\
0 & 0 & 0 & 0 & \textcircled{1} & 0 \\
\hdashline
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \textcircled{1} \\
\hdashline
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),
$$

where we have circled the elements selecting the last $n-i$ columns, $i = 1, \ldots, n$, of the $F_{ii}(\phi)Q$ matrix in the proof of Proposition 5.

Finally, as can be seen from $\tilde{D}_2$, $\tilde{D}_3$ and $\tilde{D}_4$, the matrix $\tilde{D}_n$ is always of full column rank $n(n-1)/2$.

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